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DISJOINT UNIONS OF COMPLETE GRAPHS CHARACTERIZED BY THEIR LAPLACIAN SPECTRUM*

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Abstract. A disjoint union of complete graphs is in general not determined by its Laplacian spectrum. It is shown in this paper that if one only considers the family of graphs without isolated vertex, then a disjoint union of complete graphs is determined by its Laplacian spectrum within this family. Moreover, it is shown that the disjoint union of two complete graphs with a and b vertices, $\frac{a}{b} > \frac{5}{3}$ and $b > 1$ is determined by its Laplacian spectrum. A counter-example is given when $\frac{a}{b} = \frac{5}{3}$.

Key words. Graphs, Laplacian, Complete graph, Graph determined by its spectrum, Strongly regular graph.

AMS subject classifications. 05C50, 68R10.

1. Introduction and basic results. The Laplacian of a graph G is the matrix L defined by $L = D - A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G. The Laplacian spectrum gives some information about the structure of the graph but determining graphs characterized by their Laplacian spectrum remains a difficult problem [2].

In this paper we focus on the disjoint union of complete graphs. A complete graph on n vertices is denoted by K_n and the disjoint union of the graphs G and G' is denoted by $G \cup G'$. The Laplacian spectrum of $K_{k_1} \cup K_{k_2} \cup \ldots \cup K_{k_n}$ is

$$\{k_1^{(k_1-1)}, k_2^{(k_2-1)}, \ldots, k_n^{(k_1-1)}, 0^{(n)}\}$$

but in general the converse is not true: a disjoint union of complete graphs is not in general determined by its Laplacian spectrum. For instance [2], the disjoint union of the Petersen graph with 5 isolated vertices is L-cospectral with the disjoint union of the complete graph with five vertices and five complete graphs with two vertices. These graphs are depicted in figure 1.1.

In this paper we show in Section 2 that the disjoint union of complete graphs without isolated vertex is determined by its Laplacian spectrum in the family of graphs without isolated vertex. Then in Section 3, we study the disjoint union of two

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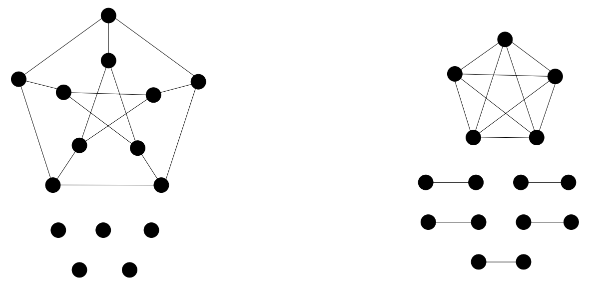

FIG. 1.1. *The graph drawn on the left is a graph non-isomorphic to a disjoint union of complete graphs but Laplacian cospectral with the disjoint union of complete graphs drawn on the right*

complete graphs K_a and K_b and show that if $\frac{a}{b} > \frac{5}{3}$, then $K_a \cup K_b$ is determined by its Laplacian spectrum.

To fix the notation, the set of vertices of a graph G is denoted by $V(G)$ and the set of edges is denoted by $E(G)$; for $v \in V(G)$, $d(v)$ denotes the degree of v. The complement of a graph G is denoted by $\overline{G}$. Concerning the spectrum, $\mathrm{Sp}(G) = \{\mu_1^{(m_1)}, \cdots \mu_k^{(m_k)}\}$ means that μ_i is m_i times an eigenvalue of L (the multiplicity of μ_i is at least m_i; we may allow $\mu_i = \mu_j$ for $i \neq j$).

We end this introduction with some known results about the Laplacian spectrum and strongly regular graphs.

THEOREM 1.1. [6] *The multiplicity of the Laplacian eigenvalue* 0 *is the number of connected components of the graph.*

THEOREM 1.2. [4, 6] *Let G be a graph on n vertices whose Laplacian spectrum is $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$. Then:*

1. $\mu_{n-1} \leq \frac{n}{n-1} \min\{d(v), v \in V(G)\}$.
2. *If G is not a complete graph then* $\mu_{n-1} \leq \min\{d(v), v \in V(G)\}$.
3. $\mu_1 \leq \max\{d(u) + d(v), uv \in E(G)\}$.
4. $\mu_1 \leq n$.
5. $\sum_i \mu_i = 2|E(G)|$.
6. $\mu_1 \geq \frac{n}{n-1} \max\{d(v), v \in V(G)\} > \max\{d(v), v \in V(G)\}$.

THEOREM 1.3. *Let G be a graph on n vertices, the Laplacian spectrum of $\overline{G}$ is:*

$$\mu_i(\overline{G}) = n - \mu_{n-i}(G),\ 1 \leq i \leq n-1$$

COROLLARY 1.4. *Let G be a graph on n vertices, we have $\mu_1(G) \leq n$ with equality if and only if $\overline{G}$ is a non-connected graph.*

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THEOREM 1.5. [2] *A complete graph is determined by its Laplacian spectrum.*

DEFINITION 1.6. [5] A graph G is *strongly regular* with parameters n, k, α, γ if

- G is not the complete graph or the graph without edges
- G is k-regular
- Every two adjacent vertices have exactly α common neighbors
- Every two non-adjacent vertices have exactly γ common neighbors

THEOREM 1.7. [5] *A regular connected graph is strongly regular if and only if it has exactly three distinct adjacency eigenvalues.*
A strongly regular non-connected graph is the disjoint union of r complete graphs K_{k+1} for a given r.

THEOREM 1.8. [5] *Let G be a connected strongly regular graph with parameters n, k, α, γ and let k, θ, τ the eigenvalues of its adjacency matrix. Then:*

$$\theta = \frac{\alpha - \gamma + \sqrt{\Delta}}{2}$$
$$\tau = \frac{\alpha - \gamma - \sqrt{\Delta}}{2}$$

where

$$\Delta = (\alpha - \gamma)^2 + 4(k - \gamma) = (\theta - \tau)^2$$

Moreover, let m_θ (resp. m_τ) the multiplicity of θ (resp. τ), then:

$$m_\theta = -\frac{(n-1)\tau + k}{\theta - \tau}$$
$$m_\tau = \frac{(n-1)\theta + k}{\theta - \tau}$$

That is:

$$m_\theta = \frac{1}{2}\left(n - 1 - \frac{2k + (n-1)(\alpha - \gamma)}{\sqrt{\Delta}}\right)$$
$$m_\tau = \frac{1}{2}\left(n - 1 + \frac{2k + (n-1)(\alpha - \gamma)}{\sqrt{\Delta}}\right)$$

2. Disjoint union of complete graphs. The aim of this section is to show that if we consider graphs without isolated vertex, then the disjoint union of complete graphs is determined by its Laplacian spectrum.

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We first state some results about disjoint union of complete graphs (including isolated vertices).

PROPOSITION 2.1. *The Laplacian spectrum of a graph G with one and only one positive Laplacian eigenvalue a is $\{a^{(ra-r)}, 0^{(r+p)}\}$ and G is isomorphic to* $\underbrace{K_a \cup K_a \cdots \cup K_a}_{r \text{ times}} \cup \underbrace{K_1 \cup K_1 \cup \cdots \cup K_1}_{p \text{ times}}$.

Proof. Let G be a graph with one and only one positive Laplacian eigenvalue a and let H be a connected component of G different from K_1. The graph H has one and only one positive eigenvalue a. If H is not a complete graph, then by theorem 1.2 we have $a \leq \min\{d(v), v \in V(G)\} \leq \max\{d(v), v \in V(G)\} < a$, contradiction. As a result H is a complete graph and H is isomorphic to K_a and there exists $r \in \mathbb{N}^*$, $p \in \mathbb{N}$ such that G is isomorphic to $\underbrace{K_a \cup K_a \cdots \cup K_a}_{r \text{ times}} \cup \underbrace{K_1 \cup K_1 \cup \cdots \cup K_1}_{p \text{ times}}$. □

THEOREM 2.2. *There is no cospectral non-isomorphic disjoint union of complete graphs.*

Proof. Let $G = K_{k_1} \cup \cdots \cup K_{k_n}$ and $G' = K_{k'_1} \cup \cdots \cup K_{k'_{n'}}$, we have $n = n'$ (same number of connected components). If G and G' are not isomorphic, then there exists $\lambda \in \mathbb{N} \setminus \{0, 1\}$ such that the number of connected components of G isomorphic to K_λ is different from the number of connected components of G' isomorphic to K_λ. Therefore, the multiplicity of λ as an eigenvalue of the Laplacian spectrum of G is different from the multiplicity of λ as an eigenvalue of the Laplacian spectrum of G' and so G and G' are not cospectral. □

THEOREM 2.3. *Let G be a graph without isolated vertex. If the Laplacian spectrum of G is $\{k_1^{(k_1-1)}, k_2^{(k_2-1)}, \ldots, k_n^{(k_n-1)}, 0^{(n)}\}$ with $k_i \in \mathbb{N}\backslash\{0, 1\}$ then G is a disjoint union of complete graphs of order $k_1, \ldots, k_n$.*

Proof. The graph G has n connected components (Theorem 1.1) $G_1, \ldots, G_n$ of order $l_1, \ldots, l_n$. We denote by N the number of vertices of G. We have

$$N = \sum_{i=1}^{n} l_i = \sum_{i=1}^{n} k_i$$

Let k_j be an eigenvalue of G, there exists i such that k_j is an eigenvalue of G_i, so $l_i \geq k_j$ (Theorem 1.2).

Let G_i be a connected component, as G does not have isolated vertices we have $l_i > 1$ and G_i possesses at least one eigenvalue different from 0, let k_j be this eigenvalue, we have $l_i \geq k_j$.

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As a result

$$\forall j\ \exists i : l_i \geq k_j$$
$$\forall i\ \exists j : l_i \geq k_j$$

We assume $k_1 \geq k_2 \geq \ldots \geq k_n > 0$ and $l_1 \geq l_2 \geq \ldots \geq l_n > 1$. We now show by induction on j that $k_{n-j} \leq l_{n-j},\ \forall j = 0 \ldots n-1$.

- $j = 0$: we know that there exists j such that $k_j \leq l_n$, so $k_n \leq l_n$.

- Let $j_0 > 0$. We assume that $\forall j < j_0,\ k_{n-j} \leq l_{n-j}$ et let us show that $k_{n-j_0} \leq l_{n-j_0}$ by contradiction. If $k_{n-j_0} > l_{n-j_0}$ then $k_{n-j_0} > l_{n-j},\ \forall j < j_0$ so $k_{n-j},\ j \geq j_0$, cannot be an eigenvalue of $G_{n-j},\ j < j_0$. So

$$\bigcup_{j<j_0} (Sp(G_{n-j}) \setminus \{0\}) \subset \bigcup_{j<j_0} \{k_{n-j}^{(k_{n-j}-1)}\}.$$

But

$$\left|\bigcup_{j<j_0} Sp(G_{n-j}) \setminus \{0\}\right| = \sum_{j<j_0} l_{n-j} - j_0 \geq \sum_{j<j_0} k_{n-j} - j_0 = \left|\bigcup_{j<j_0} \{k_{n-j}^{(k_{n-j}-1)}\}\right|$$

so

$$\bigcup_{j<j_0} (Sp(G_{n-j}) \setminus \{0\}) = \bigcup_{j<j_0} \{k_{n-j}^{(k_{n-j}-1)}\}.$$

As a result $k_{n-j},\ j < j_0$ cannot be an eigenvalue of G_{n-j_0} and as $k_{n-j_0} > l_{n-j_0}$, $k_{n-j},\ j \geq j_0$ cannot be an eigenvalue of G_{n-j_0}. That implies that G_{n-j_0} does not have any positive eigenvalue which contradicts that G is without isolated vertex. So $k_{n-j_0} \leq l_{n-j_0}$ which conclude this induction.

As $\sum l_i = \sum k_i$ and $\forall j = 1, \cdots, n,\ l_j \geq k_j$ we have $\forall j = 1, \cdots, n,\ l_j = k_j$.

We now show by induction on j that $Sp(G_{n-j}) \setminus \{0\} = \{k_{n-j}^{(k_{n-j}-1)}\},\ \forall j = 0, \cdots, n-1$.

- $j = 0$. Let k_r be an eigenvalue of G_n then $k_r \leq l_n = k_n$ and as $k_r \geq k_n$ we have $k_r = k_n$. So the $k_n - 1$ positive eigenvalues of G_n are the k_n's.

- Let $j_0 > 0$. We assume that $\forall j < j_0\ Sp(G_{n-j}) \setminus \{0\} = \{k_{n-j}^{(k_{n-j}-1)}\}$. Then $k_{n-j}^{(k_{n-j}-1)}$ for $j < j_0$ are not eigenvalues of G_{n-j_0} and as $k_{n-j} = l_{n-j} \geq l_{n-j_0}$ for $j \geq j_0$ the positive eigenvalues of G_{n-j_0} are necessarily l_{n-j_0} *i.e.* $k_{n-j_0}^{(k_{n-j_0}-1)}$.

By Theorem 2.1 we have that $G_i,\ i = 1, \cdots, n$ are the complete graphs on k_i vertices. $\square$

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3. Disjoint union of two complete graphs. In this section we consider the disjoint union of two complete graphs and we aim to replace the condition "without isolated vertex" of Theorem 2.3 (this condition cannot be deduced from the spectrum) by a condition on the eigenvalues.

The spectrum of $K_a \cup K_b$, $a \geq b > 1$ is $\{a^{(a-1)}, b^{(b-1)}, 0^{(2)}\}$. The disjoint union of two complete graphs is not in general determined by its spectrum. Here is a counter-example: The Laplacian spectrum of the line graph of K_6 (which is a strongly regular graph with parameters $15, 8, 4, 4$) is $\{10^{(9)}, 6^{(5)}, 0\}$, so the Laplacian spectrum of $\mathcal{L}(K_6) \cup K_1$ is $\{10^{(9)}, 6^{(5)}, 0^{(2)}\}$ which is also the spectrum of $K_{10} \cup K_6$.

As the disjoint union $K_a \cup K_a \cup \ldots \cup K_a$ is determined by its spectrum [2], we assume $a \neq b$ The aim of this section is to show that a graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0^{(2)}\}$ with $a > \frac{5}{3}b$ is the disjoint union of two complete graphs.

The paper [3] and the thesis [1] study graphs with few eigenvalues. We can in particular mention the following results:

THEOREM 3.1. [3, Theorem 2.1 and Corollary 2.4] *A k-regular connected graph with exactly two positive Laplacian eigenvalues a and b is strongly regular with parameters n, k, α, γ with $\gamma = \frac{ab}{n}$. Moreover k verifies $k^2 - k(a + b - 1) - \gamma + \gamma n = 0$.*

REMARK 3.2. In [3], the relation $\gamma = \frac{ab}{n}$ and the equation $k^2 - k(a + b - 1) - \gamma + \gamma n = 0$ are given in the proof of Theorem 2.1.

THEOREM 3.3. [3] *Let G be a non-regular graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0\}$ with $a > b > 1$, $a, b \in \mathbb{N}^*$. Then G possesses exactly two different degrees k_1 and k_2 ($k_1 \geq k_2$) verifying: $\begin{cases} k_1 + k_2 = a + b - 1 \\ k_1 k_2 = ab - \frac{ab}{n} \end{cases}$ and $k_2 \geq b$ with $k_2 = b$ if and only if G or $\overline{G}$ is not connected.*

LEMMA 3.4. *A regular graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0\}$ is a strongly regular graph with parameters n, $\frac{n+1}{2}$, $\frac{n+1}{4}$, $\frac{n+1}{4}$.*
Moreover we have $(a - b)^2 = a + b$.

Proof. Let G be a regular graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0\}$, then according to Theorem 3.1 G is strongly regular with parameters n, k, α, γ. The spectrum of the adjacency matrix of G is $\{(k-a)^{(a-1)}, (k-b)^{(b-1)}, k\}$. By Theorem 1.8 we have $k-b = \frac{\alpha-\gamma+\sqrt{\Delta}}{2}$ and $k-a = \frac{\alpha-\gamma-\sqrt{\Delta}}{2}$ where $\Delta = (\alpha-\gamma)^2 + 4(k-\gamma) = (a-b)^2$ and we have $\alpha - \gamma = 2k - a - b$ so (remind that $a + b - 1 = n$):

$$\alpha - \gamma = 2k - n - 1 \tag{3.1}$$

Moreover Theorem 1.8 gives $b - 1 = \frac{1}{2}\left(n - 1 - \frac{2k+(n-1)(\alpha-\gamma)}{\sqrt{\Delta}}\right)$ and

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$a-1=\frac{1}{2}\left(n-1+\frac{2k+(n-1)(\alpha-\gamma)}{\sqrt{\Delta}}\right)$ so $a-b=\frac{2k+(n-1)(\alpha-\gamma)}{\sqrt{\Delta}}$ and so

$$(a-b)^2=2k+(n-1)(\alpha-\gamma) \tag{3.2}$$

But $ab=\gamma n$ (Theorem 3.1) so

$$(a-b)^2=2k+n\alpha-ab-(\alpha-\gamma) \tag{3.3}$$

Equations 3.1 and 3.3 give:

$$(a-b)^2=1+n(\alpha+1)-ab \tag{3.4}$$

As the mean of the degrees is k, we have $k=\frac{2|E|}{n}$ and $2|E|$ is the sum of the Laplacian eigenvalues, so $k=\frac{a(a-1)+b(b-1)}{n}$ *i.e.*

$$nk=a^2+b^2-n-1 \tag{3.5}$$

Equation 3.4 gives $a^2+b^2-ab-n-1=n\alpha$, using Equation 3.5 we have $nk-ab=n\alpha$ *i.e.* $n(k-\alpha)=ab$. But $ab=\gamma n$ so

$$\alpha+\gamma=k \tag{3.6}$$

Using $\Delta=(\alpha-\gamma)^2+4(k-\gamma)$ and $\Delta=(a-b)^2=2k+(n-1)(\alpha-\gamma)$ (Equation 3.2) we obtain $(\alpha-\gamma)^2+4(k-\gamma)=2k+(n-1)(\alpha-\gamma)$ but $n-1=-\alpha+\gamma+2k-2$ (Equation 3.1) and $2k=2\alpha+2\gamma$ (Equation 3.6), so $(\alpha-\gamma)^2+4\alpha=2\alpha+2\gamma+(\alpha+3\gamma-2)(\alpha-\gamma)$ that is

$$(\alpha-\gamma)(4-4\gamma)=0 \tag{3.7}$$

As a result, we have $\alpha=\gamma$ or $\gamma=1$. Let us show that $\gamma=1$ is impossible: $\gamma=1$ implies $\alpha=k-1$ and Equation 3.1 becomes $n=k+1$ and so G is the complete graph with n vertices, which is impossible because $\{a^{(a-1)},b^{(b-1)},0\}$ with $a>b>1$ is not the spectrum of a complete graph.

Finally $\alpha=\gamma=\frac{k}{2}$ (Equation 3.6) and (Equation 3.1) $k=\frac{n+1}{2}$ and (Equation 3.2) $(a-b)^2=2k=n+1=a+b$. ∎

Theorem 3.5. *Let G be a graph whose Laplacian spectrum is $\{a^{(a-1)},b^{(b-1)},0\}$ with $a>b>1$, $a,b\in\mathbb{N}\setminus\{0,1\}$ and $a>\frac{5}{3}b$. Then G is not regular.*

Proof. Proof by contradiction. Let G be a graph whose Laplacian spectrum is $\{a^{(a-1)},b^{(b-1)},0\}$ with $a>b>1$, $a,b\in\mathbb{N}\setminus\{0,1\}$ and $a>\frac{5}{3}b$ and we assume that G

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is a k-regular graph. Then by the previous lemma we have that G is strongly regular and $(a-b)^2 = a+b$.

This implies $a^2 - 2ab - a = b - b^2 < 0$ so $a - 2b - 1 < 0$ and $a \leq 2b$. Then $3b \geq a + b = (a-b)^2 > \frac{4}{9}b^2$ so $b(\frac{4}{9}b - 3) < 0$ which gives $b \leq 6$. We have $\frac{5}{3}b < a \leq 2b$ and $b \leq 6$, by listing the different cases we show that $(a-b)^2 \neq a+b$ for $b \neq 3$:

b	a	$a+b$	$(a-b)^2$
3	6	9	9
4	7	11	9
	8	12	16
5	9	14	16
	10	15	25
6	11	17	25
	12	18	36

If $b = 3$ then $a = 6$ and $n + 1 = a + b = 9$ but, according to Lemma 3.4, $n+1$ is even, a contradiction. □

Remark 3.6. When $b < a \leq \frac{5}{3}b$, the equation $(a-b)^2 = a+b$ admits an infinity of solutions; it is not difficult to show that the couples $(a,b) = (u_i, u_{i-1})$ where $u_0 = 3$ and $u_i = u_{i-1} + i + 2$ are solutions.

Lemma 3.7. *There is no graph with Laplacian spectrum* $\{a^{(a-1)}, b^{(b-1)}, 0\}$ *with* $a, b \in \mathbb{N} \setminus \{0, 1\}$ *and* $a > 2b$.

Proof. Let G be a graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0\}$ with $a, b \in \mathbb{N} \setminus \{0,1\}$ and $a > 2b$, then G is not regular (Theorem 3.5) and by Theorem 3.3 we have that G possesses exactly two different degrees k_1 and k_2 verifying:

$$\begin{cases} k_1 + k_2 = a + b - 1 \\ k_1 k_2 = ab - \frac{ab}{n} \end{cases}$$

with $k_2 \geq b + 1$ because G and $\overline{G}$ are connected ($\overline{G}$ is disconnected if and only if the greatest eigenvalue of G is $|G|$, but here $|G| = a + b - 1 \neq a$).

We have $\frac{ab}{n} \in \mathbb{N}^*$, but $ab \neq n$ because $a + b = n + 1$ and $a, b \geq 2 \Rightarrow ab \geq a + b$. So $\frac{ab}{n} \geq 2$.
The integers k_1 and k_2 are solutions of the equation $x^2 - (a+b-1)x + ab - \frac{ab}{n} = 0$ whose discriminant is $\Delta = (a+b-1)^2 - 4ab + 4\frac{ab}{n}$ and so $k_1 = \frac{a+b-1+\sqrt{\Delta}}{2}$, $k_2 = \frac{a+b-1-\sqrt{\Delta}}{2}$.
On one hand we have:

$$\begin{aligned} \Delta = (a+b)^2 - 2(a+b) + 1 - 4ab + 4\frac{ab}{n} &= (a-b)^2 - 2(a+b) + 1 + 4\frac{ab}{n} \\ &\geq (a-b)^2 - 2(a+b) + 9 \end{aligned}$$

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and on the other hand we have (remind that $k_2 \geq b+1$):

$$\begin{aligned} \Delta = (a+b-1-2k_2)^2 &\leq (a+b-1-2(b+1))^2 \\ &\leq (a-b-3)^2 = (a-b)^2 - 6a + 6b + 9 \\ &= (a-b)^2 - 2a - 4a + 6b + 9 \ \text{ but } a > 2b \text{ i.e. } -4a < -8b \\ &< (a-b)^2 - 2a - 2b + 9 = (a-b)^2 - 2(a+b) + 9 \end{aligned}$$

Contradiction. □

THEOREM 3.8. *There is no graph with Laplacian spectrum* $\{a^{(a-1)}, b^{(b-1)}, 0\}$ *with* $a, b \in \mathbb{N} \setminus \{0,1\}$ *and* $\frac{5}{3}b < a$.

Proof. Let G be a graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0\}$ with $a, b \in \mathbb{N} \setminus \{0,1\}$ and $\frac{5}{3}b < a$. By Lemma 3.7 we can assume $a \leq 2b$. The graph G is not regular (Theorem 3.5) and by Theorem 3.3 we have that G possesses exactly two different degrees k_1 and k_2 verifying:

$$\begin{cases} k_1 + k_2 = a + b - 1 \\ k_1 k_2 = ab - \frac{ab}{n} \end{cases}$$

with $k_2 \geq b+1$.

First we show that we have $b \geq 6$; for that aim we use the relation $\frac{5}{3}b < a \leq 2b$ and list the possible values of a if $b < 6$ and we show that n does not divide ab. This is summed up into the following table.

b	a	n	ab	Does n divide ab?
3	6	8	18	no
4	7	10	28	no
	8	11	32	no
5	9	13	45	no
	10	14	50	no

Henceforth we assume $b \geq 6$. Let us show that the case $k_2 = b+1$ is impossible. If $k_2 = b+1$ then $k_1 = a-2$. We denote by n_1 (resp. n_2) the number of vertices of degree k_1 (resp. k_2). The sum of the degrees is on one hand $k_1n_1 + k_2n_2$ and on the other hand $a(a-1)+b(b-1)$ (sum of the eigenvalues) *i.e.* $k_1(k_1+3)+k_2(k_2-3)+4$. So $k_1n_1 + k_2n_2 = k_1(k_1+3)+k_2(k_2-3)+4$ *i.e.* $k_1(n_1-k_1-3)+k_2(n_2-k_2+3) = 4$. But $(n_1-k_1-3)+(n_2-k_2+3) = 0$ because $n = k_1+k_2 = n_1+n_2$ so $(n_1-k_1-3)(k_1-k_2) = 4$ *i.e.* $(n_1-k_1-3)(a-b-3) = 4$. As a result $a-b-3$ divides 4.

- If $a-b-3 = 1$ then $a = b+4$ but $a > \frac{5}{3}b = b + \frac{2}{3}b \geq b+4$ (because $b \geq 6$). This case is impossible.

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- If $a-b-3=2$ then $a=b+5$ and $a>\frac{5}{3}b=b+\frac{2}{3}b\geq b+5$ as soon as $b\geq 8$.
  - If $b=6$ then $a=11$ and $n=16$, $ab=66$ and 16 does not divide 66. This case is impossible.
  - If $b=7$ then $a=12$ and $n=18$, $ab=84$ and 18 does not divide 84. This case is impossible.
- If $a-b-3=4$ then $a=b+7$ and $a>\frac{5}{3}b=b+\frac{2}{3}b\geq b+7$ as soon as $b\geq 11$. The cases $6\leq b\leq 10$ are considered in the following table:

b	a	n	ab	Does n divide ab?
6	13	18	78	no
7	14	20	98	no
8	15	22	120	no
9	16	24	144	yes
10	17	26	170	no

  For the case $b=9$, $a=16$ we have $k_1=14$ and $k_2=10$, $k_1k_2=140$ and $ab-\frac{ab}{n}=138$, this case is impossible.

As a result we have $k_2\geq b+2$.

We have that k_1 and k_2 are solutions of the equation $x^2-(a+b-1)x+ab-\frac{ab}{n}=0$ whose discriminant is $\Delta=(a+b-1)^2-4ab+4\frac{ab}{n}$.

Let $\overline{d}$ be the mean degree of G, we have $\overline{d}=\frac{a(a-1)+b(b-1)}{n}=\frac{-2ab+(a+b)(a+b-1)}{n}=-2\frac{ab}{n}+a+b$ so $\frac{ab}{n}=\frac{1}{2}(b+a-\overline{d})\geq 4$ because $b\geq 6$ and $a>k_1>\overline{d}$ gives $a-\overline{d}\geq 2$.

We have on one hand:

$$\begin{aligned}\Delta=(a+b)^2-2(a+b)+1-4ab+4\frac{ab}{n}&=(a-b)^2-2(a+b)+1+4\frac{ab}{n}\\&\geq(a-b)^2-2(a+b)+17\end{aligned}$$

and on the other hand (remind that $a+b-1-2k_2=n-2k_2>0$):

$$\begin{aligned}\Delta=(a+b-1-2k_2)^2&\leq(a+b-1-2(b+2))^2\\&\leq(a-b-5)^2=(a-b)^2-10a+10b+25\\&=(a-b)^2-2(a+b)-8a+12b+25\quad\text{but } -8a<-\tfrac{40}{3}b\\&<(a-b)^2-2(a+b)-\frac{4}{3}b+25\leq(a-b)^2-2(a+b)+17.\end{aligned}$$

Contradiction. ∎

Remark 3.9. If $b<a<\frac{5}{3}b$ then the system $\begin{cases}k_1+k_2=a+b-1\\ k_1k_2=ab-\frac{ab}{n}\end{cases}$ with $k_1,k_2,a,b,n\in\mathbb{N}^*$ admits solutions. If $(a-b)^2=a+b$ then $\Delta=1$ and the system admits a solution if $a+b$ is even. The following table shows solutions with $a,b\leq 1000$ and $(a-b)^2\neq a+b$.

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a	b	$\sqrt{\Delta}$	k_1	k_2
51	35	13	49	36
81	64	12	78	66
190	153	32	187	155
290	204	83	288	205
260	222	31	256	225
469	403	59	465	406
595	528	58	590	532
784	638	141	781	640
936	833	94	931	837

THEOREM 3.10. *The graph $K_a \cup K_b$ with $a, b \in \mathbb{N}\backslash\{0,1\}$ and $\frac{5}{3}b < a$ is determined by its Laplacian spectrum.*

Proof. Let G be a graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0^{(2)}\}$ with $a, b \in \mathbb{N}^*$ and $\frac{5}{3}b < a$ then G has two connected components. If G has an isolated vertex then the Laplacian spectrum of a connected component of G is $\{a^{(a-1)}, b^{(b-1)}, 0\}$, which is impossible (Theorem 3.8). Consequently G does not have isolated vertex and we apply Theorem 2.3. □

The following corollary is straightforward thanks to Theorem 1.3:

COROLLARY 3.11. *The complete bipartite graph $K_{a,b}$ with $a, b \in \mathbb{N} \setminus \{0,1\}$ and $\frac{5}{3}b < a$ is determined by its Laplacian spectrum.*

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