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NORMAL MATRIX POLYNOMIALS WITH NONSINGULAR LEADING COEFFICIENTS*

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Abstract. In this paper, the notions of weakly normal and normal matrix polynomials with nonsingular leading coefficients are introduced. These matrix polynomials are characterized using orthonormal systems of eigenvectors and normal eigenvalues. The conditioning of the eigenvalue problem of a normal matrix polynomial is also studied, thereby constructing an appropriate Jordan canonical form.

Key words. Matrix polynomial, Normal eigenvalue, Weighted perturbation, Condition number.

AMS subject classifications. 15A18, 15A22, 65F15, 65F35.

1. Introduction.

In pure and applied mathematics, normality of matrices (or operators) arises in many concrete problems. This is reflected in the fact that there are numerous ways to describe a normal matrix (or operator). A list of about ninety conditions on a square matrix equivalent to being normal can be found in [5, 7].

The study of matrix polynomials has also a long history, especially in the context of spectral analysis, leading to solutions of associated linear systems of higher order; see [6, 10, 11] and references therein. Surprisingly, it seems that the notion of normality has been overlooked by people working in this area. Two exceptions are the work of Adam and Psarrakos [1], as well as Lancaster and Psarrakos [9].

Our present goal is to take a comprehensive look at normality of matrix polynomials. To avoid infinite eigenvalues, we restrict ourselves to matrix polynomials with nonsingular leading coefficients. The case of singular leading coefficients and infinite eigenvalues will be considered in future work. The presentation is organized as follows. In the next section, we provide the necessary theoretical background on the spectral analysis of matrix polynomials. In Section 3, we introduce the notions of weakly normal and normal matrix polynomials, and obtain necessary and sufficient conditions for a matrix polynomial to be weakly normal. In Section 4, we consider the normal eigenvalues of matrix polynomials and use them to provide sufficient conditions for a matrix polynomial to be normal. Finally, in Section 5, we investigate

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the Jordan structure of a normal matrix polynomial, and study the conditioning of the associated eigenvalue problem.

2. Spectral analysis of matrix polynomials. Consider an $n \times n$ matrix polynomial

$$P(\lambda) \,=\, A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \tag{2.1}$$

where λ is a complex variable and $A_j \in \mathbb{C}^{n\times n}$ $(j = 0, 1, \ldots, m)$ with $\det A_m \neq 0$. If the leading coefficient A_m coincides with the identity matrix I, then $P(\lambda)$ is called *monic*. A scalar $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $P(\lambda)$ if $P(\lambda_0)x_0 = 0$ for some nonzero $x_0 \in \mathbb{C}^n$. This vector x_0 is known as a *(right) eigenvector* of $P(\lambda)$ corresponding to λ_0. A nonzero vector $y_0 \in \mathbb{C}^n$ that satisfies $y_0^* P(\lambda_0) = 0$ is called a *left eigenvector* of $P(\lambda)$ corresponding to λ_0.

The set of all eigenvalues of $P(\lambda)$ is the *spectrum* of $P(\lambda)$, namely, $\sigma(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}$, and since $\det A_m \neq 0$, it contains no more than nm distinct (finite) elements. The *algebraic multiplicity* of an eigenvalue $\lambda_0 \in \sigma(P)$ is the multiplicity of λ_0 as a zero of the (scalar) polynomial $\det P(\lambda)$, and it is always greater than or equal to the *geometric multiplicity* of λ_0, that is, the dimension of the null space of matrix $P(\lambda_0)$. A multiple eigenvalue of $P(\lambda)$ is called *semisimple* if its algebraic multiplicity is equal to its geometric multiplicity.

Let $\lambda_1, \lambda_2, \ldots, \lambda_r \in \sigma(P)$ be the eigenvalues of $P(\lambda)$, where each λ_i appears k_i times if and only if the geometric multiplicity of λ_i is k_i $(i = 1, 2, \ldots, r)$. Suppose also that for a $\lambda_i \in \sigma(P)$, there exist $x_{i,1}, x_{i,2}, \ldots, x_{i,s_i} \in \mathbb{C}^n$ with $x_{i,1} \neq 0$, such that

$$\begin{aligned}
P(\lambda_i)\, x_{i,1} &= 0 \\
\frac{P'(\lambda_i)}{1!}\, x_{i,1} + P(\lambda_i)\, x_{i,2} &= 0 \\
\vdots \qquad\qquad \vdots \qquad & \;\;\vdots \\
\frac{P^{(s_i-1)}(\lambda_i)}{(s_i-1)!}\, x_{i,1} + \frac{P^{(s_i-2)}(\lambda_i)}{(s_i-2)!}\, x_{i,2} + \cdots + \frac{P'(\lambda_i)}{1!}\, x_{i,s_i-1} + P(\lambda_i)\, x_{i,s_i} &= 0,
\end{aligned}$$

where the indices denote the derivatives of $P(\lambda)$ and s_i cannot exceed the algebraic multiplicity of λ_i. Then the vector $x_{i,1}$ is an eigenvector of λ_i, and the vectors $x_{i,2}, x_{i,3}, \ldots, x_{i,s_i}$ are known as *generalized eigenvectors.* The set $\{x_{i,1}, x_{i,2}, \ldots, x_{i,s_i}\}$ is called a *Jordan chain of length* s_i of $P(\lambda)$ corresponding to the eigenvalue λ_i. Any eigenvalue of $P(\lambda)$ of geometric multiplicity k has k maximal Jordan chains associated to k linearly independent eigenvectors, with total number of eigenvectors and generalized eigenvectors equal to the algebraic multiplicity of this eigenvalue.

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We consider now the $n \times nm$ matrix

$$X = [\, x_{1,1} \;\; x_{1,2} \;\; \cdots \;\; x_{1,s_1} \;\; x_{2,1} \;\; \cdots \;\; x_{r,1} \;\; x_{r,2} \;\; \cdots \;\; x_{r,s_r} \,]$$

formed by maximal Jordan chains of $P(\lambda)$ and the associated $nm \times nm$ Jordan matrix $J = J_1 \oplus J_2 \oplus \cdots \oplus J_r$, where each J_i is the Jordan block that corresponds to the Jordan chain $\{x_{i,1}, x_{i,2}, \ldots, x_{i,s_i}\}$ of λ_i. Then the $nm \times nm$ matrix $Q = \begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{m-1} \end{bmatrix}$ is invertible, and we can define $Y = Q^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_m^{-1} \end{bmatrix}$. The set (X, J, Y) is called a *Jordan triple* of $P(\lambda)$, and satisfies

$$P(\lambda)^{-1} = X(\lambda I - J)^{-1} Y \; ; \quad \lambda \notin \sigma(P). \tag{2.2}$$

The set $\{x_{1,1}, x_{1,2}, \ldots, x_{1,s_1}, x_{2,1}, \ldots, x_{r,1}, x_{r,2}, \ldots, x_{r,s_r}\}$ is known as a *complete system* of eigenvectors and generalized eigenvectors of $P(\lambda)$.

3. Weakly normal and normal matrix polynomials. In [1], the term "normal matrix polynomial" has been used for the matrix polynomials that can be diagonalized by a unitary similarity. For matrix polynomials of degree $m \geq 2$, this definition does not ensure the semisimplicity of the eigenvalues, and hence it is necessary to modify it. Consider, for example, the diagonal matrix polynomials

$$P(\lambda) = \begin{bmatrix} (\lambda-2)(\lambda-1) & 0 & 0 \\ 0 & \lambda(\lambda-1) & 0 \\ 0 & 0 & (\lambda+1)(\lambda+2) \end{bmatrix} \tag{3.1}$$

and

$$R(\lambda) = \begin{bmatrix} (\lambda-1)^2 & 0 & 0 \\ 0 & \lambda(\lambda-2) & 0 \\ 0 & 0 & (\lambda+1)(\lambda+2) \end{bmatrix}, \tag{3.2}$$

which have exactly the same eigenvalues (counting multiplicities): -2, -1, 0, 1 (double) and 2. The eigenvalue $\lambda = 1$ is semisimple as an eigenvalue of $P(\lambda)$ with algebraic and geometric multiplicities equal to 2. On the other hand, $\lambda = 1$ is not semisimple as an eigenvalue of $R(\lambda)$ since its algebraic multiplicity is 2 and its geometric multiplicity is 1.

DEFINITION 3.1. The matrix polynomial $P(\lambda)$ in (2.1) is called *weakly normal* if there is a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^* P(\lambda) U$ is diagonal for all $\lambda \in \mathbb{C}$. If,

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in addition, every diagonal entry of $U^*P(\lambda)U$ is a polynomial with exactly m distinct zeros, or equivalently, all the eigenvalues of $P(\lambda)$ are semisimple, then $P(\lambda)$ is called *normal*.

Clearly, the matrix polynomial $P(\lambda)$ in (3.1) is normal, and the matrix polynomial $R(\lambda)$ in (3.2) is weakly normal (but not normal). Note also that the notions of weakly normal and normal matrix polynomials coincide for matrices and for *linear pencils* of the form $P(\lambda) = A_1\lambda + A_0$.

The next two lemmas are necessary to characterize weakly normal matrix polynomials.

LEMMA 3.2. *Let $A, B \in \mathbb{C}^{n\times n}$ be normal matrices such that $AB^* = B^*A$. Then the matrices $A + B$ and AB are also normal.*

LEMMA 3.3. *Suppose that for every $\mu \in \mathbb{C}$, the matrix $P(\mu)$ is normal. Then for every $i, j = 0, 1, \ldots, m$, it holds that $A_iA_j^* = A_j^*A_i$. In particular, all coefficient matrices $A_0, A_1, \ldots, A_m$ are normal.*

Proof. Let $P(\mu)$ be a normal matrix for every $\mu \in \mathbb{C}$. Then $P(0) = A_0$ is normal, i.e. $A_0A_0^* = A_0^*A_0$. From the proof of [14, Lemma 16], we have that $A_0A_i^* = A_i^*A_0$ for every $i = 1, 2, \ldots, m$. Thus, $P(\mu)A_0^* = A_0^*P(\mu)$ for every $\mu \in \mathbb{C}$. By Lemma 3.2, it follows that for the matrix polynomials $P_0(\lambda) = A_m\lambda^{m-1} + \cdots + A_2\lambda + A_1$ and $P(\lambda) - A_0 = \lambda P_0(\lambda)$, the matrices $P_0(\mu)$ and $P(\mu) - A_0 = \mu P_0(\mu)$ are normal for every $\mu \in \mathbb{C}$.

Similarly, by [14, Lemma 16] and the fact that $P_0(\mu)$ is normal for every $\mu \in \mathbb{C}$, we have that $P_0(0) = A_1$ is also normal and $A_1A_i^* = A_i^*A_1$ for every $i = 2, 3, \ldots, m$. Hence, as before, $P_1(\mu) = A_m\mu^{m-2} + \cdots + A_3\mu + A_2$ and $P_0(\mu) - A_1 = \mu P_1(\mu)$ are normal matrices for every $\mu \in \mathbb{C}$. Repeating the same process, completes the proof. □

THEOREM 3.4. *The matrix polynomial $P(\lambda)$ in* (2.1) *is weakly normal if and only if for every $\mu \in \mathbb{C}$, the matrix $P(\mu)$ is normal.*

Proof. If the matrix polynomial $P(\lambda)$ is weakly normal, then it is apparent that for every $\mu \in \mathbb{C}$, the matrix $P(\mu)$ is normal.

For the converse, suppose that for every $\mu \in \mathbb{C}$, the matrix $P(\mu)$ is normal. The next assumption is necessary.

Assumption. Suppose that there is a coefficient matrix A_i with $s \geq 2$ distinct eigenvalues. Then, without loss of generality, we may assume that

$$A_i = \lambda_{i1}I_{k_1} \oplus \lambda_{i2}I_{k_2} \oplus \cdots \oplus \lambda_{is}I_{k_s}$$

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and

$$A_j \;=\; A_{j1} \oplus A_{j2} \oplus \cdots \oplus A_{js}\;;\quad j \neq i,$$

where the eigenvalues $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{is}$ of A_i are distinct and nonzero, with multiplicities $k_1, k_2, \ldots, k_s$, respectively, and $A_{j1} \in \mathbb{C}^{k_1 \times k_1}$, $A_{j2} \in \mathbb{C}^{k_2 \times k_2}$, $\ldots$, $A_{js} \in \mathbb{C}^{k_s \times k_s}$.

Justification of the Assumption. Since A_i is normal, there is a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that

$$U^* A_i U \;=\; \lambda_{i1} I_{k_1} \oplus \lambda_{i2} I_{k_2} \oplus \cdots \oplus \lambda_{is} I_{k_s}.$$

We observe that for any $\mu, a \in \mathbb{C}$, the matrix $P(\mu)$ is normal if and only if the matrix $U^* P(\mu) U + aI\mu^i$ is normal. Thus, without loss of generality, we may assume that all λ_{il}'s are nonzero. By Lemma 3.3, it follows that for every $j \neq i$,

$$A_i A_j^* \;=\; A_j^* A_i,$$

or equivalently,

$$A_j^* \;=\; A_i^{-1} A_j^* A_i.$$

By straightforward calculations, the justification of the assumption is complete.

We proceed now with the proof of the converse, which is by induction on the order n of $P(\lambda)$. Clearly, for $n = 1$, there is nothing to prove.

If $n = 2$, and there is a coefficient matrix with distinct eigenvalues, then by the Assumption, all $A_0, A_1, \ldots, A_m$ are diagonal. If there is no coefficient matrix of $P(\lambda)$ with distinct eigenvalues, then each A_i $(i = 0, 1, \ldots, m)$ is normal with a double eigenvalue, and hence, it is scalar, i.e., $A_i = a_i I$. As a consequence, $P(\lambda)$ is diagonal.

Assume now that for any $n = 3, 4, \ldots, k-1$, every $n \times n$ matrix polynomial $P(\lambda)$ such that the matrix $P(\mu)$ is normal for any $\mu \in \mathbb{C}$, is weakly normal.

Let $P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \cdots + A_1 \lambda + A_0$ be a $k \times k$ matrix polynomial, and suppose that there is a A_i with $s \geq 2$ distinct eigenvalues. By the Assumption,

$$A_i \;=\; \lambda_{i1} I_{k_1} \oplus \lambda_{i2} I_{k_2} \oplus \cdots \oplus \lambda_{is} I_{k_s},$$

and for every $j \neq i$,

$$A_j \;=\; A_{j1} \oplus A_{j2} \oplus \cdots \oplus A_{js}.$$

Then

$$P(\lambda) \;=\; P_1(\lambda) \oplus P_2(\lambda) \oplus \cdots \oplus P_s(\lambda),$$

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where the matrix polynomials $P_1(\lambda), P_2(\lambda), \ldots, P_s(\lambda)$ are weakly normal. Hence, there are unitary matrices $U_t \in \mathbb{C}^{k_t \times k_t}$, $t = 1, 2, \ldots, s$, such that $U_t^* P_t(\lambda) U_t$, $t = 1, 2, \ldots, s$, are diagonal. Thus, for the $k \times k$ unitary matrix $U = U_1 \oplus U_2 \oplus \cdots \oplus U_s$, the matrix polynomial $U^* P(\lambda) U$ is diagonal.

Suppose that there is no A_i with at least two distinct eigenvalues. Then each coefficient matrix A_i is normal with exactly one eigenvalue (of algebraic multiplicity k), and hence, it is scalar, i.e., $A_i = a_i I$. As a consequence, $P(\lambda)$ is diagonal. □

By the above theorem, it follows that the matrix polynomial $P(\lambda)$ is weakly normal if and only if $P(\lambda)[P(\lambda)]^* - [P(\lambda)]^* P(\lambda) = 0$ for every $\lambda \in \mathbb{C}$. We observe that each entry of the matrix function $P(\lambda)[P(\lambda)]^* - [P(\lambda)]^* P(\lambda)$ is of the form $\chi(\alpha, \beta) + \mathrm{i}\,\psi(\alpha, \beta)$, where α and β are the real and imaginary parts of variable λ, respectively, and $\chi(\alpha, \beta)$ and $\psi(\alpha, \beta)$ are real polynomials in $\alpha, \beta \in \mathbb{R}$ of total degree at most $2m$. As a consequence, Lemma 3.1 of [12] yields the following corollary.

Corollary 3.5. *The matrix polynomial $P(\lambda)$ in* (2.1) *is weakly normal if and only if for any distinct real numbers $s_1, s_2, \ldots, s_{4m^2+2m+1}$, the matrices $P(s_j + \mathrm{i}\, s_j^{2m+1})$ $(j = 1, 2, \ldots, 4m^2 + 2m + 1)$ are normal.*

By Theorem 3.4, Corollary 3.5 and [16] (see also the references therein), the next corollary follows readily.

Corollary 3.6. *Let $P(\lambda) = A_m \lambda^m + \cdots + A_1 \lambda + A_0$ be an $n \times n$ matrix polynomial as in* (2.1). *Then the following are equivalent:*

(i) *The matrix polynomial $P(\lambda)$ is weakly normal.*
(ii) *For every $\mu \in \mathbb{C}$, the matrix $P(\mu)$ is normal.*
(iii) *For any distinct real numbers $s_1, s_2, \ldots, s_{4m^2+2m+1}$, the matrices $P(s_j + \mathrm{i}\, s_j^{2m+1})$ $(j = 1, 2, \ldots, 4m^2 + 2m + 1)$ are normal.*
(iv) *The coefficient matrices $A_0, A_1, \ldots, A_m$ are normal and mutually commuting, i.e., $A_i A_j = A_j A_i$ for $i \neq j$.*
(v) *All the linear combinations of the coefficient matrices $A_0, A_1, \ldots, A_m$ are normal matrices.*
(vi) *The coefficient matrices $A_0, A_1, \ldots, A_m$ are normal and satisfy property L, that is, there exists an ordering of the eigenvalues $\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_n^{(j)}$ of A_j $(j = 0, 1, \ldots, m)$ such that for all scalars $t_0, t_1, \ldots, t_m \in \mathbb{C}$, the eigenvalues of $t_0 A_0 + t_1 A_1 + \cdots + t_m A_m$ are $t_0 \lambda_i^{(0)} + t_1 \lambda_i^{(1)} + \cdots + t_m \lambda_i^{(m)}$ $(i = 1, 2, \ldots, n)$.*
(vii) *There exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that $U^* A_j U$ is diagonal for every $j = 0, 1, \ldots, m$.*

4. Normal eigenvalues. In the matrix case, it is well known that normality (or diagonalizability) is equivalent to the orthogonality (respectively, linear independence) of eigenvectors. In the matrix polynomial case, it is clear (by definition) that any $n \times n$

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normal matrix polynomial of degree m has an orthogonal system of n eigenvectors such that each one of these eigenvectors corresponds to exactly m distinct eigenvalues.

PROPOSITION 4.1. *Consider a matrix polynomial $P(\lambda)$ as in* (2.1) *with all its eigenvalues semisimple. Suppose also that $P(\lambda)$ has a complete system of eigenvectors, where each vector of a basis $\{g_1, g_2, \ldots, g_n\}$ of $\mathbb{C}^n$ appears exactly m times. Then there exists a diagonal matrix polynomial $D(\lambda)$ such that*

$$P(\lambda) \;=\; A_m G\, D(\lambda)\, G^{-1},$$

where $G = [\, g_1 \; g_2 \; \cdots \; g_n \,] \in \mathbb{C}^{n\times n}$.

Proof. Each vector g_i $(i = 1, 2, \ldots, n)$ appears exactly m times as an eigenvector of m distinct eigenvalues of $P(\lambda)$, say $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{im}$. By [8, Theorem 1], we have that

$$P(\lambda)\, g_i \;=\; \prod_{j=1}^{m} (\lambda - \lambda_{ij})\, g_i \;; \quad i = 1, 2, \ldots, n.$$

Thus,

$$P(\lambda)\, [\, g_1 \; g_2 \; \cdots \; g_n \,] \;=\; A_m \left[\prod_{j=1}^{m} (\lambda - \lambda_{1j})\, g_1 \;\; \prod_{j=1}^{m} (\lambda - \lambda_{2j})\, g_2 \;\; \cdots \;\; \prod_{j=1}^{m} (\lambda - \lambda_{nj})\, g_n \right]$$

$$=\; A_m\, [\, g_1 \; g_2 \; \cdots \; g_n \,]\, diag \left\{ \prod_{j=1}^{m} (\lambda - \lambda_{1j})\, g_1,\, \prod_{j=1}^{m} (\lambda - \lambda_{2j})\, g_2,\, \ldots,\, \prod_{j=1}^{m} (\lambda - \lambda_{nj})\, g_n \right\}.$$

Consequently,

$$P(\lambda) \;=\; A_m\, G\; diag \left\{ \prod_{j=1}^{m} (\lambda - \lambda_{1j}),\, \prod_{j=1}^{m} (\lambda - \lambda_{2j}),\, \ldots,\, \prod_{j=1}^{m} (\lambda - \lambda_{nj}) \right\} G^{-1},$$

and the proof is complete. ◻

COROLLARY 4.2. *Under the assumptions of Proposition* 4.1*, the following hold:*

(i) *If the matrix $G^{-1} A_m G$ is diagonal, then the matrix polynomial $G^{-1} P(\lambda)\, G$ is diagonal.*

(ii) *If G is unitary, then there exists a diagonal matrix polynomial $D(\lambda)$ such that $P(\lambda) = A_m G\, D(\lambda)\, G^*$.*

(iii) *If G is unitary and the matrix $G^* A_m G$ is diagonal, then the matrix polynomial $G^* P(\lambda)\, G$ is diagonal, i.e., $P(\lambda)$ is normal.*

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Note that if $P(\lambda) = I\lambda - A$, then in Corollary 4.2, $G^{-1}A_mG = G^{-1}I\,G = I$ for nonsingular G, and $G^*A_mG = G^*I\,G = I$ for unitary G. This means that all the parts of the corollary are direct generalizations of standard results on matrices.

The following definition was introduced in [9].

DEFINITION 4.3. Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (2.1). An eigenvalue $\lambda_0 \in \sigma(P)$ of algebraic multiplicity k is said to be *normal* if there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$U^*P(\lambda)\,U \;=\; [(\lambda - \lambda_0)\,D(\lambda)] \oplus Q(\lambda),$$

where the matrix polynomial $D(\lambda)$ is $k \times k$ diagonal and $\lambda_0 \notin \sigma(D) \cup \sigma(Q)$.

By this definition and Definition 3.1, it is obvious that every normal matrix polynomial has all its eigenvalues normal. In the sequel, we obtain the converse.

PROPOSITION 4.4. *Suppose that all the eigenvalues of $P(\lambda)$ in (2.1) are semisimple, and that $\lambda_1, \lambda_2, \ldots, \lambda_s$ are normal eigenvalues of $P(\lambda)$ with multiplicities $m_1, m_2, \ldots, m_s$, respectively, such that $m_1 + m_2 + \cdots + m_s = n$. If $P(\lambda)$ satisfies*

$$U_j^*P(\lambda)\,U_j \;=\; [(\lambda - \lambda_j)D_j(\lambda)] \oplus Q_j(\lambda)\,; \quad j = 1, 2, \ldots, s,$$

where for each j, the matrix $U_j \in \mathbb{C}^{n\times n}$ is unitary, the matrix polynomial $D_j(\lambda)$ is $m_j \times m_j$ diagonal and $\lambda_i \in \sigma(Q_j)\backslash\sigma(D_j)$ $(i = j+1, \ldots, s)$, then the matrix polynomial $P(\lambda)$ is normal.

Proof. For the eigenvalue λ_1, by hypothesis, we have that

$$U_1^*P(\lambda)U_1 \;=\; [(\lambda - \lambda_1)D_1(\lambda)] \oplus Q_1(\lambda),$$

where U_1 is unitary and $D_1(\lambda)$ is $m_1 \times m_1$ diagonal. The first m_1 columns of U_1 are an orthonormal system of eigenvectors of λ_1. From the hypothesis we also have that $\det A_m \neq 0$ and all the eigenvalues of $P(\lambda)$ are semisimple. As a consequence, each one of the m_1 eigenvectors of λ_1 is an eigenvector for m exactly eigenvalues of $P(\lambda)$ (counting λ_1). Moreover, since $\lambda_i \in \sigma(Q_1)\backslash\sigma(D_1)$, $i = 2, 3, \ldots, s$, these m_1 eigenvectors of λ_1, are orthogonal to the eigenspaces of the eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_s$.

Similarly, for the eigenvalue λ_2, we have

$$U_2^*P(\lambda)U_2 \;=\; [(\lambda - \lambda_2)D_2(\lambda)] \oplus Q_2(\lambda),$$

where U_2 is unitary and $D_2(\lambda)$ is $m_2 \times m_2$ diagonal. As before, the first m_2 columns of U_2 are an orthonormal system of eigenvectors of λ_2. In addition, each one of these m_2 eigenvectors of λ_2 is an eigenvector for m exactly eigenvalues of $P(\lambda)$ (counting λ_2). Since $\lambda_i \in \sigma(Q_2)\backslash\sigma(D_2)$, $i = 3, 4, \ldots, s$, these m_2 eigenvectors of λ_2 are orthogonal to the eigenspaces of the eigenvalues $\lambda_3, \lambda_4, \ldots, \lambda_s$.

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Repeating this process for the eigenvalues $\lambda_3, \lambda_4, \ldots, \lambda_s$, we construct an orthonormal basis of $\mathbb{C}^n$,

$$\underbrace{u_1, u_2, \ldots, u_{m_1}}_{\text{of } \lambda_1}, \underbrace{u_{m_1+1}, u_{m_1+2}, \ldots, u_{m_1+m_2}}_{\text{of } \lambda_2}, \ldots, \underbrace{u_{n-m_s+1}, u_{n-m_s+2}, \ldots, u_n}_{\text{of } \lambda_s},$$

where each vector is an eigenvector for m distinct eigenvalues of $P(\lambda)$ and an eigenvector of the leading coefficient A_m. By Corollary 4.2, $P(\lambda)$ is a normal matrix polynomial. □

The next lemma is needed in our discussion and follows readily.

Lemma 4.5. *If an $n \times n$ matrix polynomial $P(\lambda)$ has a normal eigenvalue of multiplicity n or $n-1$, then it is weakly normal.*

Theorem 4.6. *Consider a matrix polynomial $P(\lambda)$ as in (2.1). If all its eigenvalues are normal, then $P(\lambda)$ is normal.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the distinct eigenvalues of $P(\lambda)$ with corresponding multiplicities $k_1, k_2, \ldots, k_s$ $(k_1+k_2+\cdots+k_s = nm)$, and suppose that they are normal. It is easy to see that $s \geq m$. If $s = m$ then all the eigenvalues have multiplicity n and by Lemma 4.5, the theorem follows.

Suppose that $s > m$ and for every $j = 0, 1, \ldots, s$, there is a unitary matrix $U_j = [\, u_{j1} \; u_{j2} \; \ldots \; u_{jn} \,]$ such that

$$U_j^* P(\lambda) U_j \; = \; [(\lambda - \lambda_j) D_j(\lambda)] \oplus Q_j(\lambda),$$

where $D_j(\lambda)$ is $k_j \times k_j$ diagonal and $\lambda_j \notin \sigma(D_j) \cup \sigma(Q_j)$. Then the first k_j columns of U_j $(j = 1, 2, \ldots, s)$ are right and left eigenvectors of $P(\lambda)$, and also of $A_0, A_1, \ldots, A_m$. The set of all vectors

$$u_{11},\, u_{12},\, \ldots,\, u_{1k_1},\, u_{21},\, u_{22},\, \ldots,\, u_{2k_2},\, \ldots\, u_{s1},\, u_{s2},\, \ldots,\, u_{sk_s}$$

form a complete system of eigenvectors of $P(\lambda)$. So, by [13], there is a basis of $\mathbb{C}^n$

$$\{u_1, u_2, \ldots, u_n\} \subseteq \{u_{11}, u_{12}, \ldots, u_{1k_1},\; u_{21}, u_{22}, \ldots, u_{2k_2},\; \ldots, u_{s1}, u_{s2}, \ldots, u_{sk_s}\}.$$

We also observe that the vectors $u_1, u_2, \ldots, u_n$ are linearly independent right and left common eigenvectors of the coefficient matrices $A_0, A_1, \ldots, A_m$. Keeping in mind that any left and right eigenvectors (of the same matrix) corresponding to distinct eigenvalues are orthogonal, [7, Condition 13] implies that all A_j's are normal. Moreover, any two vectors u_i, u_j $(i \neq j)$ that correspond to distinct eigenvalues of a coefficient matrix are also orthogonal. Hence, it is straightforward to see that there exists a unitary matrix U such that all $U^* A_j U$'s are diagonal. As a consequence, the matrix polynomial $P(\lambda)$ is weakly normal, and since all its eigenvalues are semisimple, $P(\lambda)$ is normal. □

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5. Weighted perturbations and condition numbers. Let $P(\lambda)$ be a matrix polynomial as in (2.1). We are interested in perturbations of $P(\lambda)$ of the form

$$Q(\lambda) \,=\, P(\lambda) + \Delta(\lambda) \,=\, \sum_{j=0}^{m} (A_j + \Delta_j)\lambda^j, \tag{5.1}$$

where the matrices $\Delta_0, \Delta_1, \ldots, \Delta_m \in \mathbb{C}^{n\times n}$ are arbitrary. For a given parameter $\varepsilon > 0$ and a given set of nonnegative weights $\mathrm{w} = \{w_0, w_1, \ldots, w_m\}$ with $w_0 > 0$, we define the set of admissible perturbed matrix polynomials

$$\mathcal{B}(P, \varepsilon, \mathrm{w}) = \{Q(\lambda) \text{ as in (5.1)} : \|\Delta_j\| \leq \varepsilon\, w_j,\; j = 0, 1, \ldots, m\},$$

where $\|\cdot\|$ denotes the spectral matrix norm (i.e., that norm subordinate to the euclidean vector norm). The weights $w_0, w_1, \ldots, w_m$ allow freedom in how perturbations are measured, and the set $\mathcal{B}(P, \varepsilon, \mathrm{w})$ is convex and compact [3] with respect to the max norm $\|P(\lambda)\|_\infty = \max_{0\leq j\leq m}\|A_j\|$.

In [15], motivated by (2.2) and the work of Chu [4], for a Jordan triple (X, J, Y) of $P(\lambda)$, the authors introduced the *condition number of the eigenproblem of* $P(\lambda)$, that is,[1]

$$k(P) \,=\, \|X\|\, \|Y\|\,.$$

Furthermore, they applied the Bauer-Fike technique [2, 4] and used $k(P)$, to bound eigenvalues of perturbations of $P(\lambda)$. Denoting $w(\lambda) = w_m\lambda^m + \cdots + w_1\lambda + w_0$, one of the results of [15] is the following.

PROPOSITION 5.1. *Let* (X, J, Y) *be a Jordan triple of* $P(\lambda)$*, and let* $Q(\lambda) \in \mathcal{B}(P, \varepsilon, \mathrm{w})$ *for some* $\varepsilon > 0$*. If the Jordan matrix* J *is diagonal, then for any* $\mu \in \sigma(Q)\backslash\sigma(P)$*,*

$$\min_{\lambda\in\sigma(P)} |\mu - \lambda| \,\leq\, k(P)\,\varepsilon\, w(|\mu|).$$

As in the matrix case, we say that a matrix polynomial eigenvalue problem is *well-conditioned* (or *ill-conditioned*) if its condition number is sufficiently small (respectively, sufficiently large).

In the remainder of this section, we confine our discussion to normal matrix polynomials. Recall that for an $n \times n$ normal matrix polynomial $P(\lambda)$, there is a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^*P(\lambda)U$ is diagonal and all its diagonal entries

[1] Note that the definition of $k(P)$ clearly depends on the choice of the triple (X, J, Y), but to keep things simple, the Jordan triple will not appear explicitly in the notation.

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are polynomials of degree exactly m, with distinct zeros. Moreover, for any Jordan triple (X, J, Y) of $P(\lambda)$, the Jordan matrix J is diagonal. In the sequel, we derive some bounds for the condition number $k(P)$.

The following lemma is a simple exercise.

LEMMA 5.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be m distinct scalars. Then it holds that*

$$\frac{1}{\prod_{j=1}^{m}(\lambda-\lambda_j)} = \sum_{j=1}^{m}\frac{1}{(\lambda-\lambda_j)\prod_{i\neq j}(\lambda_i-\lambda_j)}.$$

Next we compute a Jordan triple of a monic normal matrix polynomial $P(\lambda)$ and the associated condition number $k(P)$.

PROPOSITION 5.3. *Let $P(\lambda)$ be an $n\times n$ monic normal matrix polynomial of degree m, and let $U^*P(\lambda)U = (I\lambda - J_1)(I\lambda - J_2)\cdots(I\lambda - J_m)$ for some unitary $U\in\mathbb{C}^{n\times n}$ and diagonal matrices $J_1, J_2, \ldots, J_m$. Then a Jordan triple (X, J, Y) of $P(\lambda)$ is given by*

$$X = U\left[\, I \quad (J_2-J_1)^{-1} \quad \cdots \quad \prod_{i=1}^{m-2}(J_{m-1}-J_i)^{-1} \quad \prod_{i=1}^{m-1}(J_m-J_i)^{-1}\,\right],$$

$$J = J_1\oplus J_2\oplus\cdots\oplus J_m \quad \textit{and} \quad Y = \begin{bmatrix} \prod_{i=2}^{m}(J_1-J_i)^{-1} \\ \prod_{i=3}^{m}(J_2-J_i)^{-1} \\ \vdots \\ (J_{m-1}-J_m)^{-1} \\ I \end{bmatrix} U^*.$$

Proof. [2] By Lemma 5.2 and straightforward calculations, we see that

$$\prod_{j=1}^{m}(I\lambda-J_j)^{-1} = \sum_{j=1}^{m}\left[(I\lambda-J_j)\prod_{i\neq j}(J_i-J_j)\right]^{-1},$$

or equivalently,

$$P(\lambda)^{-1} = X(I\lambda-J)^{-1}Y.$$

[2]Our original proof of this proposition is constructive and justifies the choice of matrices X and Y. It is also inductive on the degree m of $P(\lambda)$, uses Corollary 3.3 of [6] and requires some computations. As a consequence, we decided to present this short proof.

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Theorem 2.6 of [6] completes the proof. ☐

THEOREM 5.4. *Let $P(\lambda)$ be an $n \times n$ monic normal matrix polynomial of degree m, and let (X, J, Y) be the Jordan triple given by Proposition* 5.3. *If we denote*

$$J_i \;=\; diag\{\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_n^{(i)}\}\;; \quad i = 1, 2, \ldots, m,$$

then the condition number of the eigenproblem of $P(\lambda)$ is

$$k(P) = \left(1 + \max_{s=1,2,\ldots,n}\left\{\frac{1}{|\lambda_s^{(2)} - \lambda_s^{(1)}|^2} + \cdots + \prod_{i=1}^{m-1}\frac{1}{|\lambda_s^{(m)} - \lambda_s^{(i)}|^2}\right\}\right)^{1/2}$$
$$\times\left(1 + \max_{s=1,2,\ldots,n}\left\{\frac{1}{|\lambda_s^{(m)} - \lambda_s^{(m-1)}|^2} + \cdots + \prod_{i=2}^{m}\frac{1}{|\lambda_s^{(i)} - \lambda_s^{(1)}|^2}\right\}\right)^{1/2}.$$

Proof. Since $P(\lambda)$ is normal, it follows $\lambda_s^{(i)} \neq \lambda_s^{(j)}$, $i \neq j$, $s = 1, 2, \ldots, n$. Recall that

$$X \;=\; U\left[\; I \quad (J_2 - J_1)^{-1} \quad \cdots \quad \prod_{i=1}^{m-2}(J_{m-1} - J_i)^{-1} \quad \prod_{i=1}^{m-1}(J_m - J_i)^{-1}\;\right],$$

and observe that

$$XX^* \;=\; U\left(I + (J_2 - J_1)^{-1}\,\overline{(J_2 - J_1)^{-1}} + \cdots + \prod_{i=1}^{m-1}(J_m - J_i)^{-1}\,\overline{(J_m - J_i)^{-1}}\right)U^*.$$

If we denote by $\lambda_{\max}(\cdot)$ the largest eigenvalue of a square matrix, then

$$\|X\|^2 \;=\; \lambda_{\max}(XX^*) \;=\; 1 + \max_{s=1,2,\ldots,n}\left\{\frac{1}{|\lambda_s^{(2)} - \lambda_s^{(1)}|^2} + \cdots + \prod_{i=1}^{m-1}\frac{1}{|\lambda_s^{(m)} - \lambda_s^{(i)}|^2}\right\}.$$

Similarly, we verify that

$$\|Y\|^2 \;=\; \lambda_{\max}(Y^*Y) \;=\; 1 + \max_{s=1,2,\ldots,n}\left\{\frac{1}{|\lambda_s^{(m)} - \lambda_s^{(m-1)}|^2} + \cdots + \prod_{i=2}^{m}\frac{1}{|\lambda_s^{(i)} - \lambda_s^{(1)}|^2}\right\},$$

and the proof is complete. ☐

It is worth noting that since a (monic) normal matrix polynomial is "essentially diagonal", the condition number of its eigenproblem depends on the eigenvalues and not on the eigenvectors. Furthermore, by the above theorem, it is apparent that if $m \geq 2$ and the mutual distances of the eigenvalues of the monic matrix polynomial $P(\lambda)$ are sufficiently large, then the condition number $k(P)$ is relatively close to 1, i.e., the eigenproblem of $P(\lambda)$ is well-conditioned. On the other hand, if $m \geq 2$ and

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the mutual distances of the eigenvalues are sufficiently small, then $k(P)$ is relatively large, i.e., the eigenproblem of $P(\lambda)$ is ill-conditioned.

Theorem 5.4 implies practical lower and upper bounds for the condition number $k(P)$. For convenience, we denote

$$\Theta \;=\; \max_{\substack{\lambda,\,\hat{\lambda}\,\in\,\sigma(P)\\ \lambda\neq\hat{\lambda}}} \;|\lambda-\hat{\lambda}| \quad \text{and} \quad \theta \;=\; \min_{\substack{\lambda,\,\hat{\lambda}\,\in\,\sigma(P)\\ \lambda\neq\hat{\lambda}}} \;|\lambda-\hat{\lambda}|\,,$$

and assume that $\Theta, \theta \neq 1$.

COROLLARY 5.5. *Let $P(\lambda)$ be an $n \times n$ monic normal matrix polynomial of degree m, and let (X, J, Y) be the Jordan triple given by Proposition* 5.3. *Then the condition number $k(P)$ satisfies*

$$\frac{\Theta^{2m}-1}{\Theta^{2m}-\Theta^{2(m-1)}} \;\leq\; k(P) \;\leq\; \frac{\theta^{2m}-1}{\theta^{2m}-\theta^{2(m-1)}}\,.$$

Consider the matrix polynomial $P(\lambda)$ in (2.1), and recall that its leading coefficient A_m is nonsingular. By [6, 10], (X, J, Y) is a Jordan triple of the monic matrix polynomial $\hat{P}(\lambda) = A_m^{-1}P(\lambda) = I\lambda^m + A_m^{-1}A_{m-1}\lambda^{m-1} + \cdots + A_m^{-1}A_1\lambda + A_m^{-1}A_0$ if and only if (X, J, YA_m^{-1}) is a Jordan triple of $P(\lambda)$. This observation and the proof of Theorem 5.4 yield the next results.

COROLLARY 5.6. *Let $P(\lambda) = A_1\lambda + A_0$ be an $n \times n$ normal linear pencil, and let $U^*P(\lambda)U$ be diagonal for some unitary $U \in \mathbb{C}^{n\times n}$. Then a Jordan triple of $P(\lambda)$ is $(X, J, Y) = (U, J, UA_1^{-1})$, and $k(P) = \|A_1^{-1}\|$.*

THEOREM 5.7. *Let $P(\lambda)$ in* (2.1) *be normal, and let (X, J, Y) be the Jordan triple of the monic matrix polynomial $\hat{P}(\lambda) = A_m^{-1}P(\lambda)$ given by Proposition* 5.3. *Then for the condition number $k(P) = \|X\|\,\left\|YA_m^{-1}\right\|$, we have*

$$\|A_m\|^{-1}\;\frac{\Theta^{2m}-1}{\Theta^{2m}-\Theta^{2(m-1)}} \;\leq\; k(P) \;\leq\; \left\|A_m^{-1}\right\|\;\frac{\theta^{2m}-1}{\theta^{2m}-\theta^{2(m-1)}}\,.$$

Proof. As mentioned above, (X, J, Y) is a Jordan triple of the monic matrix polynomial $\hat{P}(\lambda)$ if and only if (X, J, YA_m^{-1}) is a Jordan triple of $P(\lambda)$. Notice also that $\hat{P}(\lambda)$ is normal, and by the proof of Theorem 5.4,

$$\sqrt{\frac{\Theta^{2m}-1}{\Theta^{2m}-\Theta^{2(m-1)}}} \;\leq\; \|X\|\,,\,\|Y\| \;\leq\; \sqrt{\frac{\theta^{2m}-1}{\theta^{2m}-\theta^{2(m-1)}}}\,.$$

Furthermore, there is an $n \times n$ unitary matrix U such that $D(\lambda) = U^*P(\lambda)U$ and $D_m = U^*A_mU$ are diagonal. As a consequence, $\|A_m\| = \|D_m\|$ and $\left\|A_m^{-1}\right\| = \left\|D_m^{-1}\right\|$.

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Since the matrix $YU \in \mathbb{C}^{nm\times n}$ is a block-column of m diagonal matrices of order n, it is straightforward to see that

$$\left\|YA_m^{-1}\right\| = \left\|YUD_m^{-1}U^*\right\| = \left\|YUD_m^{-1}\right\| = \left\|YD_m^{-1}\right\|.$$

The matrix Y^*Y is also diagonal, and thus,

$$\left\|YD_m^{-1}\right\| = \left\|\left(D_m^{-1}\right)^* Y^*YD_m^{-1}\right\|^{1/2} = \left\|Y^*Y\left(D_m^{-1}\right)^* D_m^{-1}\right\|^{1/2}.$$

Hence, it follows that

$$\|Y\|\,\|D_m\|^{-1} \le \left\|YD_m^{-1}\right\| \le \|Y\|\,\left\|D_m^{-1}\right\|.$$

By Corollary 5.5, the proof is complete. □

Proposition 5.1 implies directly the following.

Corollary 5.8. *Let $P(\lambda)$ in (2.1) be normal, and let $Q(\lambda) \in \mathcal{B}(P, \varepsilon, \mathrm{w})$ for some $\varepsilon > 0$. Then for any $\mu \in \sigma(Q)\backslash\sigma(P)$, it holds that*

$$\min_{\lambda\in\sigma(P)} |\mu-\lambda| \le \varepsilon\, w(|\mu|)\,\left\|A_m^{-1}\right\| \frac{\theta^{2m}-1}{\theta^{2m}-\theta^{2(m-1)}}.$$

Remark 5.9. In the construction of the above bounds, we have assumed that Θ and θ are different than 1. Suppose that this assumption fails for a normal matrix polynomial $P(\lambda)$. Then, keeping in mind the Jordan triple (X, J, Y) of $\hat{P}(\lambda) = A_m^{-1}P(\lambda)$ given by Proposition 5.3 and the proofs of Theorems 5.4 and 5.7, one can easily see that $k(P) \ge m\,\|A_m\|^{-1}$ when $\Theta = 1$, and $k(P) \le m\,\left\|A_m^{-1}\right\|$ when $\theta = 1$.

Finally, as an example, recall the monic normal matrix polynomial $P(\lambda)$ in (3.1). For the weights $w_0 = w_1 = w_2 = 1$, Theorem 5.4 yields $k(P) = 2$, i.e., the eigenproblem of $P(\lambda)$ is well-conditioned. Note also that $\theta = 1$ and the value 2 coincides with the upper bound given in Remark 5.9.

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