

ON SOLUTIONS TO THE QUATERNION MATRIX EQUATION $AXB + CYD = E^*$

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Abstract. Expressions, as well as necessary and sufficient conditions are given for the existence of the real and pure imaginary solutions to the consistent quaternion matrix equation $AXB + CYD = E$. Formulas are established for the extreme ranks of real matrices $X_i, Y_i, i = 1, \dots, 4$, in a solution pair $X = X_1 + X_2i + X_3j + X_4k$ and $Y = Y_1 + Y_2i + Y_3j + Y_4k$ to this equation. Moreover, necessary and sufficient conditions are derived for all solution pairs X and Y of this equation to be real or pure imaginary, respectively. Some known results can be regarded as special cases of the results in this paper.

Key words. Quaternion matrix equation, Extreme rank, Generalized inverse.

AMS subject classifications. 15A03, 15A09, 15A24, 15A33.

1. Introduction. Throughout this paper, we denote the real number field by R , and the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in R\}$$

by $H^{m \times n}$. The symbols $I, A^T, R(A), N(A), \dim R(A)$ stand for the identity matrix of the appropriate size, the transpose, the column right space, the row left space of a matrix A over H , and the dimension of $R(A)$, respectively. By [6], for a quaternion matrix A , $\dim R(A) = \dim N(A)$, which is called the rank of A and denoted by $r(A)$. A generalized inverse of a matrix A which satisfies $AA^-A = A$ is denoted by A^- . Moreover, R_A and L_A stand for the two projectors $L_A = I - A^-A, R_A = I - AA^-$ induced by A . For an arbitrary quaternion matrix $M = M_1 + M_2i + M_3j + M_4k$, we define a map $\phi(\cdot)$ from $H^{m \times n}$ to $R^{4m \times 4n}$ by

$$(1.1) \quad \phi(M) = \begin{pmatrix} M_4 & M_3 & M_2 & M_1 \\ -M_3 & M_4 & M_1 & -M_2 \\ M_2 & M_1 & -M_4 & -M_3 \\ M_1 & -M_2 & M_3 & -M_4 \end{pmatrix}.$$

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By (1.1), it is easy to verify that $\phi(\cdot)$ satisfies the following properties:

- (a) $M = N \iff \phi(M) = \phi(N)$.
- (b) $\phi(M + N) = \phi(M) + \phi(N)$, $\phi(MN) = \phi(M)\phi(N)$, $\phi(kM) = k\phi(M)$, $k \in R$.
- (c) $\phi(M) = -T_m^{-1}\phi(M)T_n = -R_m^{-1}\phi(M)R_n = S_m^{-1}\phi(M)S_n$, where $t = m, n$,

$$R_t = \begin{pmatrix} 0 & -I_{2t} \\ I_{2t} & 0 \end{pmatrix}, T_t = \begin{pmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{pmatrix}, S_t = \begin{pmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{pmatrix}.$$

- (d) $r[\phi(M)] = 4r(M)$.

Tian [17] in 2003 gave the maximal and minimal ranks of two real matrices X_0 and X_1 in complex solution $X = X_0 + iX_1$ to the classical linear matrix equation

$$(1.2) \quad AXB = C.$$

Liu [11] in 2006 investigated the extreme ranks of solution pairs X and Y , and the extreme ranks of four real matrices X_0, X_1, Y_0 and Y_1 in a pair of a complex solutions $X = X_0 + iX_1$ and $Y = Y_0 + iY_1$ to the generalized Sylvester matrix equation

$$(1.3) \quad AX + YB = C,$$

which is widely studied (see, e.g., [4], [5], [7], [8], [11], [13], [22]-[24], [33], [34]). As an extension of (1.2) and (1.3), the matrix equation

$$(1.4) \quad AXB + CYD = E$$

has been well-studied in matrix theory (see, e.g., [2], [9], [10], [12], [14], [18]-[21], [35]). For instance, Baksalary and Kala [2] gave necessary and sufficient conditions for the existence and an representation of the general solution to (1.4). Özgüler [12] investigated (1.4) over a principal ideal domain. Huang and Zeng [9] considered (1.4) over a simple Artinian ring. Wang [20]-[21] investigated (1.4) over an arbitrary division ring and on any regular ring with identity in 1996 and 2004, respectively. Tian [19] in 2006 established formulas for extremal ranks of the solution of (1.4) over the complex number field. Note that, to our knowledge, the real and pure imaginary solutions to (1.4) over the quaternion algebra H have not been investigated so far in the literature. Motivated by the work mentioned above, and keeping the applications and interests of quaternion matrices in view (e.g., [1], [3], [25]-[32], [36]-[37]), in this paper we consider the real and pure imaginary solutions to (1.4) over H . We first derive the formulas for extremal ranks of real matrices $X_i, Y_i, i = 1, \dots, 4$, in a solution pair $X = X_1 + X_2i + X_3j + X_4k$ and $Y = Y_1 + Y_2i + Y_3j + Y_4k$ to (1.4), then use the results to derive necessary and sufficient conditions for (1.4) over H to have a real solution and a pure imaginary solution, respectively. Finally, we establish necessary and sufficient conditions for all solutions (X, Y) to (1.4) over H to be real or pure imaginary.

2. Main results. In this section, we consider (1.4) over H , where $A = A_1 + A_2i + A_3j + A_4k$, $B = B_1 + B_2i + B_3j + B_4k$, $C = C_1 + C_2i + C_3j + C_4k$, $D = D_1 + D_2i + D_3j + D_4k$, $E = E_1 + E_2i + E_3j + E_4k$ are known and $X = X_1 + X_2i + X_3j + X_4k \in H^{n \times s}$, $Y = Y_1 + Y_2i + Y_3j + Y_4k \in H^{l \times k}$ unknown; here $A_i, B_i, C_i, D_i, E_i, X_i$ and $Y_i, i = 1, \dots, 4$, are real matrices with suitable sizes.

The following lemmas are due to Tian (see [15], [16] and [18]) that can be generalized to H .

LEMMA 2.1. Let $A \in H^{m \times k}$, $B \in H^{l \times n}$, $C \in H^{m \times i}$, $D \in H^{j \times n}$, $E \in H^{m \times n}$ and $p(X, Y, Z) = E - AXB - CY - ZD$. Then

$$(2.1) \quad \max_{X, Y, Z} r[p(X, Y, Z)] = \min \left\{ m, n, r \begin{pmatrix} E & A & C \\ D & 0 & 0 \end{pmatrix}, r \begin{pmatrix} E & C \\ B & 0 \\ D & 0 \end{pmatrix} \right\};$$

$$(2.2) \quad \min_{X, Y, Z} r[p(X, Y, Z)] = r \begin{pmatrix} E & C \\ B & 0 \\ D & 0 \end{pmatrix} - r \begin{pmatrix} E & A & C \\ B & 0 & 0 \\ D & 0 & 0 \end{pmatrix} + r \begin{pmatrix} E & A & C \\ D & 0 & 0 \end{pmatrix} - r(C) - r(D).$$

LEMMA 2.2. Let $A \in H^{m \times n}$, $B \in H^{m \times k}$ and $C \in H^{l \times n}$. Then

$$r(B, AL_C) = r \begin{pmatrix} B & A \\ 0 & C \end{pmatrix} - r(C); \quad r \begin{pmatrix} C \\ R_B A \end{pmatrix} = r \begin{pmatrix} C & 0 \\ A & B \end{pmatrix} - r(B).$$

LEMMA 2.3. Suppose that matrix equation (1.4) is consistent over H . Then its general solutions can be expressed as

$$(2.3) \quad X = \widetilde{X}_0 + \widetilde{S}_1 L_G U R_H \widetilde{T}_1 + L_A V_1 + V_2 R_B,$$

$$(2.4) \quad Y = \widetilde{Y}_0 + \widetilde{S}_2 L_G U R_H \widetilde{T}_2 + L_C W_1 + W_2 R_D$$

where \widetilde{X}_0 and \widetilde{Y}_0 are a pair of special solutions of (1.4), $\widetilde{S}_1 = (I_p, 0)$, $\widetilde{S}_2 = (0, I_s)$, $\widetilde{T}_1 = (I_q, 0)^T$, $\widetilde{T}_2 = (0, I_t)^T$, $G = (A, C)$, $H = (B^T, -D^T)^T$, the quaternion matrices U, V_1, V_2, W_1 and W_2 are arbitrary with suitable sizes.

In the following theorems and corollaries, $\widetilde{X}_0, \widetilde{Y}_0, \widetilde{S}_1, \widetilde{S}_2, \widetilde{T}_1, \widetilde{T}_2, G$ and H are defined as in Lemma 2.3.

THEOREM 2.4. Matrix equation (1.4) is consistent over H if and only if the matrix equation

$$(2.5) \quad \phi(A) (X_{ij})_{4 \times 4} \phi(B) + \phi(C) (Y_{ij})_{4 \times 4} \phi(D) = \phi(E), i, j = 1, 2, 3, 4,$$

is consistent over R . In that case, the general solution of (1.4) over H can be written as

$$(2.6) \quad \begin{aligned} X &= X_1 + X_2i + X_3j + X_4k \\ &= \frac{1}{4}(X_{11} + X_{22} - X_{33} - X_{44}) + \frac{1}{4}(X_{12} - X_{21} + X_{43} - X_{34})i \\ &\quad + \frac{1}{4}(X_{13} + X_{31} - X_{24} - X_{42})j + \frac{1}{4}(X_{41} + X_{14} + X_{32} + X_{23})k, \end{aligned}$$

$$(2.7) \quad \begin{aligned} Y &= Y_1 + Y_2i + Y_3j + Y_4k \\ &= \frac{1}{4}(Y_{11} + Y_{22} - Y_{33} - Y_{44}) + \frac{1}{4}(Y_{12} - Y_{21} + Y_{43} - Y_{34})i \\ &\quad + \frac{1}{4}(Y_{31} + Y_{13} - Y_{24} - Y_{42})j + \frac{1}{4}(Y_{41} + Y_{14} + Y_{32} + Y_{23})k. \end{aligned}$$

Written in an explicit form, X_i and $Y_i, i = 1, \dots, 4$, in (2.6) are as follows.

$$(2.8) \quad \begin{aligned} X_1 &= \frac{1}{4}P_1\phi(X_0)Q_1 + \frac{1}{4}P_2\phi(X_0)Q_2 - \frac{1}{4}P_3\phi(X_0)Q_3 - \frac{1}{4}P_4\phi(X_0)Q_4 \\ &\quad + \frac{1}{4}(P_1R_1, P_2R_1, -P_3R_1, -P_4R_1)U \begin{pmatrix} L_1Q_1 \\ L_1Q_2 \\ L_1Q_3 \\ L_1Q_4 \end{pmatrix} \\ &\quad + (P_1L_{\phi(A)}, P_2L_{\phi(A)}, -P_3L_{\phi(A)}, -P_4L_{\phi(A)})V + \tilde{U} \begin{pmatrix} R_{\phi(B)}Q_1 \\ R_{\phi(B)}Q_2 \\ -R_{\phi(B)}Q_3 \\ -R_{\phi(B)}Q_4 \end{pmatrix}, \end{aligned}$$

$$(2.9) \quad \begin{aligned} X_2 &= \frac{1}{4}P_1\phi(X_0)Q_2 - \frac{1}{4}P_2\phi(X_0)Q_1 + \frac{1}{4}P_4\phi(X_0)Q_3 - \frac{1}{4}P_3\phi(X_0)Q_4 \\ &\quad + \frac{1}{4}(P_1R_1, -P_2R_1, -P_3R_1, P_4R_1)U \begin{pmatrix} L_1Q_2 \\ L_1Q_1 \\ L_1Q_4 \\ L_1Q_3 \end{pmatrix} \\ &\quad + (-P_2L_{\phi(A)}, P_1L_{\phi(A)}, P_4L_{\phi(A)}, -P_3L_{\phi(A)})V + \tilde{U} \begin{pmatrix} -R_{\phi(B)}Q_2 \\ R_{\phi(B)}Q_1 \\ R_{\phi(B)}Q_4 \\ -R_{\phi(B)}Q_3 \end{pmatrix}, \end{aligned}$$

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$$(2.10) \quad X_3 = \frac{1}{4}P_1\phi(X_0)Q_3 - \frac{1}{4}P_2\phi(X_0)Q_4 + \frac{1}{4}P_3\phi(X_0)Q_1 - \frac{1}{4}P_4\phi(X_0)Q_2$$

$$+ \frac{1}{4}(P_1R_1, -P_2R_1, P_3R_1, -P_4R_1)U \begin{pmatrix} L_1Q_3 \\ L_1Q_4 \\ L_1Q_1 \\ L_1Q_2 \end{pmatrix}$$

$$+ (P_3L_{\phi(A)}, -P_2L_{\phi(A)}, P_1L_{\phi(A)}, -P_4L_{\phi(A)})V + \tilde{U} \begin{pmatrix} R_{\phi(B)}Q_3 \\ -R_{\phi(B)}Q_2 \\ R_{\phi(B)}Q_1 \\ -R_{\phi(B)}Q_4 \end{pmatrix},$$

$$(2.11) \quad X_4 = \frac{1}{4}P_1\phi(X_0)Q_4 + \frac{1}{4}P_2\phi(X_0)Q_3 + \frac{1}{4}P_3\phi(X_0)Q_2 + \frac{1}{4}P_4\phi(X_0)Q_1$$

$$+ \frac{1}{4}(P_1R_1, P_2R_1, P_3R_1, P_4R_1)U \begin{pmatrix} L_1Q_4 \\ L_1Q_3 \\ L_1Q_2 \\ L_1Q_1 \end{pmatrix}$$

$$+ (P_4L_{\phi(A)}, P_3L_{\phi(A)}, P_2L_{\phi(A)}, P_1L_{\phi(A)})V + \tilde{U} \begin{pmatrix} R_{\phi(B)}Q_4 \\ R_{\phi(B)}Q_3 \\ R_{\phi(B)}Q_2 \\ R_{\phi(B)}Q_1 \end{pmatrix},$$

$$(2.12) \quad Y_1 = \frac{1}{4}S_1\phi(Y_0)T_1 + \frac{1}{4}S_2\phi(Y_0)T_2 - \frac{1}{4}S_3\phi(Y_0)T_3 - \frac{1}{4}S_4\phi(Y_0)T_4$$

$$+ \frac{1}{4}(S_1R_2, S_2R_2, -S_3R_2, -S_4R_2)U \begin{pmatrix} L_2T_1 \\ L_2T_2 \\ L_2T_3 \\ L_2T_4 \end{pmatrix}$$

$$+ (S_1L_{\phi(A)}, S_2L_{\phi(A)}, -S_3L_{\phi(A)}, -S_4L_{\phi(A)})\hat{U} + W \begin{pmatrix} R_{\phi(B)}T_1 \\ R_{\phi(B)}T_2 \\ -R_{\phi(B)}T_3 \\ -R_{\phi(B)}T_4 \end{pmatrix},$$

$$(2.13) \quad Y_2 = \frac{1}{4}S_1\phi(Y_0)T_2 - \frac{1}{4}S_2\phi(Y_0)T_1 - \frac{1}{4}S_3\phi(Y_0)T_4 + \frac{1}{4}S_4\phi(Y_0)T_3$$

$$+ \frac{1}{4}(S_1R_2, -S_2R_2, -S_3R_2, S_4R_2)U \begin{pmatrix} L_2T_2 \\ L_2T_1 \\ L_2T_4 \\ L_2T_3 \end{pmatrix}$$

$$+ (-S_2L_{\phi(A)}, S_1L_{\phi(A)}, S_4L_{\phi(A)}, -S_3L_{\phi(A)})\widehat{U} + W \begin{pmatrix} -R_{\phi(B)}T_2 \\ R_{\phi(B)}T_1 \\ R_{\phi(B)}T_4 \\ -R_{\phi(B)}T_3 \end{pmatrix},$$

$$(2.14) \quad Y_3 = \frac{1}{4}S_1\phi(Y_0)T_3 - \frac{1}{4}S_2\phi(Y_0)T_4 + \frac{1}{4}S_3\phi(Y_0)T_1 - \frac{1}{4}S_4\phi(Y_0)T_2$$

$$+ \frac{1}{4}(S_1R_2, -S_2R_2, S_3R_2, -S_4R_2)U \begin{pmatrix} L_2T_3 \\ L_2T_4 \\ L_2T_1 \\ L_2T_2 \end{pmatrix}$$

$$+ (S_3L_{\phi(A)}, -S_4L_{\phi(A)}, S_1L_{\phi(A)}, -S_2L_{\phi(A)})\widehat{U} + W \begin{pmatrix} R_{\phi(B)}T_3 \\ -R_{\phi(B)}T_4 \\ R_{\phi(B)}T_1 \\ -R_{\phi(B)}T_2 \end{pmatrix},$$

$$(2.15) \quad Y_4 = \frac{1}{4}S_1\phi(Y_0)T_4 + \frac{1}{4}S_2\phi(Y_0)T_3 + \frac{1}{4}S_3\phi(Y_0)T_2 + \frac{1}{4}S_4\phi(Y_0)T_1$$

$$+ \frac{1}{4}(S_1R_2, S_2R_2, S_3R_2, S_4R_2)U \begin{pmatrix} L_2T_4 \\ L_2T_3 \\ L_2T_2 \\ L_2T_1 \end{pmatrix}$$

$$+ (S_4L_{\phi(A)}, S_3L_{\phi(A)}, S_2L_{\phi(A)}, S_1L_{\phi(A)})\widehat{U} + W \begin{pmatrix} R_{\phi(B)}T_4 \\ R_{\phi(B)}T_3 \\ R_{\phi(B)}T_2 \\ R_{\phi(B)}T_1 \end{pmatrix},$$

where

$$P_1 = (I_p, 0, 0, 0), \quad P_2 = (0, I_p, 0, 0), \quad P_3 = (0, 0, I_p, 0), \quad P_4 = (0, 0, 0, I_p),$$

$$S_1 = (I_m, 0, 0, 0), \quad S_2 = (0, I_m, 0, 0), \quad S_3 = (0, 0, I_m, 0), \quad S_4 = (0, 0, 0, I_m),$$

$$Q_1 = (I_n, 0, 0, 0)^T, \quad Q_2 = (0, I_n, 0, 0)^T, \quad Q_3 = (0, 0, I_n, 0)^T, \quad Q_4 = (0, 0, 0, I_n)^T,$$

$$T_1 = (I_q, 0, 0, 0)^T, \quad T_2 = (0, I_q, 0, 0)^T, \quad T_3 = (0, 0, I_q, 0)^T, \quad T_4 = (0, 0, 0, I_q)^T,$$

$$R_1 = \phi(\widetilde{S}_1)L_{\phi(G)}, \quad L_1 = R_{\phi(H)}\phi(\widetilde{T}_1), \quad R_2 = \phi(\widetilde{S}_2)L_{\phi(G)}, \quad L_2 = R_{\phi(H)}\phi(\widetilde{T}_2),$$

and U, \tilde{U}, \hat{U}, V and W are arbitrary real matrices with compatible sizes.

Proof. Suppose that (1.4) has a solution pair X, Y over H . Applying properties (a) and (b) of $\phi(\cdot)$ above to (1.4) yields

$$\phi(A)\phi(X)\phi(B) + \phi(C)\phi(Y)\phi(D) = \phi(E),$$

which implies that $\phi(X), \phi(Y)$ is a solution pair to (2.5). Conversely, suppose that (2.5) has a solution pair

$$\hat{X} = (X_{ij})_{4 \times 4}, \hat{Y} = (Y_{ij})_{4 \times 4}, i, j = 1, 2, 3, 4.$$

i.e.

$$\phi(A)\hat{X}\phi(B) + \phi(C)\hat{Y}\phi(D) = \phi(E).$$

Then, applying property (c) of $\phi(\cdot)$ above, it yields

$$\begin{aligned} T_m^{-1}\phi(A)T_p\hat{X}T_q^{-1}\phi(B)T_n + T_m^{-1}\phi(C)T_p\hat{Y}T_q^{-1}\phi(D)T_n &= -T_m^{-1}\phi(E)T_n, \\ R_m^{-1}\phi(A)R_p\hat{X}R_q^{-1}\phi(B)R_n + R_m^{-1}\phi(C)R_p\hat{Y}R_q^{-1}\phi(D)R_n &= -R_m^{-1}\phi(E)R_n, \\ S_m^{-1}\phi(A)S_p\hat{X}S_q^{-1}\phi(B)S_n + S_m^{-1}\phi(C)S_p\hat{Y}S_q^{-1}\phi(D)S_n &= S_m^{-1}\phi(E)S_n. \end{aligned}$$

Hence,

$$\begin{aligned} \phi(A)T_p\hat{X}T_q^{-1}\phi(B) + \phi(C)T_p\hat{Y}T_q^{-1}\phi(D) &= \phi(E), \\ \phi(A)R_p\hat{X}R_q^{-1}\phi(B) + \phi(C)R_p\hat{Y}R_q^{-1}\phi(D) &= \phi(E), \\ \phi(A)S_p\hat{X}S_q^{-1}\phi(B) + \phi(C)S_p\hat{Y}S_q^{-1}\phi(D) &= \phi(E), \end{aligned}$$

which implies that $T_p\hat{X}T_n^{-1}, T_m\hat{Y}T_q^{-1}, R_p\hat{X}R_n^{-1}, R_m\hat{Y}R_q^{-1}$ and $S_p\hat{X}S_n^{-1}, S_m\hat{Y}S_q^{-1}$ are also solutions of (2.5). Thus,

$$\frac{1}{4}(\hat{X} - T_p\hat{X}T_n^{-1} - R_p\hat{X}R_n^{-1} + S_p\hat{X}S_n^{-1}), \frac{1}{4}(\hat{Y} - T_m\hat{Y}T_q^{-1} - R_m\hat{Y}R_q^{-1} + S_m\hat{Y}S_q^{-1})$$

is a solution pair of (2.5), where

$$\hat{X} - T_p\hat{X}T_n^{-1} - R_p\hat{X}R_n^{-1} + S_p\hat{X}S_n^{-1} = (\widetilde{X}_{ij})_{4 \times 4}, i, j = 1, 2, 3, 4$$

and

$$\begin{aligned} \widetilde{X}_{11} &= X_{11} + X_{22} - X_{33} - X_{44}, \widetilde{X}_{12} = X_{12} - X_{21} + X_{43} - X_{34}, \\ \widetilde{X}_{13} &= X_{13} + X_{31} - X_{24} - X_{42}, \widetilde{X}_{14} = X_{41} + X_{14} + X_{32} + X_{23}, \\ \widetilde{X}_{21} &= X_{21} - X_{12} + X_{34} - X_{43}, \widetilde{X}_{22} = X_{11} + X_{22} - X_{33} - X_{44}, \\ \widetilde{X}_{23} &= X_{41} + X_{14} + X_{32} + X_{23}, \widetilde{X}_{24} = X_{24} + X_{42} - X_{13} - X_{31}, \\ \widetilde{X}_{31} &= X_{13} + X_{31} - X_{24} - X_{42}, \widetilde{X}_{32} = X_{41} + X_{14} + X_{32} + X_{23}, \\ \widetilde{X}_{33} &= X_{33} + X_{44} - X_{11} - X_{22}, \widetilde{X}_{34} = X_{21} - X_{12} + X_{34} - X_{43}, \\ \widetilde{X}_{41} &= X_{41} + X_{14} + X_{32} + X_{23}, \widetilde{X}_{42} = X_{24} + X_{42} - X_{13} - X_{31}, \\ \widetilde{X}_{43} &= X_{12} - X_{21} + X_{43} - X_{34}, \widetilde{X}_{44} = X_{33} + X_{44} - X_{11} - X_{22}. \end{aligned}$$

$\hat{Y} - T_m \hat{Y} T_q^{-1} - R_m \hat{Y} R_q^{-1} + S_m \hat{Y} S_q^{-1}$ has a form similar to $(\widetilde{X}_{ij})_{4 \times 4}$. We omit it here for simplicity.

Let

$$\begin{aligned} \widetilde{X} &= \frac{1}{4}(X_{11} + X_{22} - X_{33} - X_{44}) + \frac{1}{4}(X_{12} - X_{21} + X_{43} - X_{34})i \\ &\quad + \frac{1}{4}(X_{13} + X_{31} - X_{24} - X_{42})j + \frac{1}{4}(X_{41} + X_{14} + X_{32} + X_{23})k, \\ \widetilde{Y} &= \frac{1}{4}(Y_{11} + Y_{22} - Y_{33} - Y_{44}) + \frac{1}{4}(Y_{12} - Y_{21} + Y_{43} - Y_{34})i \\ &\quad + \frac{1}{4}(Y_{31} + Y_{13} - Y_{24} - Y_{42})j + \frac{1}{4}(Y_{41} + Y_{14} + Y_{32} + Y_{23})k. \end{aligned}$$

Then, by (1.1),

$$\begin{aligned} \phi(\widetilde{X}) &= \frac{1}{4}(\hat{X} - T_p \hat{X} T_n^{-1} - R_p \hat{X} R_n^{-1} + S_p \hat{X} S_n^{-1}), \\ \phi(\widetilde{Y}) &= \frac{1}{4}(\hat{Y} - T_m \hat{Y} T_q^{-1} - R_m \hat{Y} R_q^{-1} + S_m \hat{Y} S_q^{-1}) \end{aligned}$$

is a solution pair of (2.5). Hence, by property (a) of $\phi(\cdot)$, we know that $\widetilde{X}, \widetilde{Y}$ is a solution pair of (1.4). The above discussion shows that the two matrix equations (1.4) and (2.5) have the same solvability condition and their solutions satisfy (2.6) and (2.7). Observe that X_{ij} and Y_{ij} , $i, j = 1, 2, 3, 4$ in (2.5) can be written as

$$X_{ij} = P_i \hat{X} Q_j, \quad Y_{ij} = S_i \hat{Y} T_j.$$

From Lemma 2.3, the general solution to (2.5) can be written as

$$\begin{aligned} \hat{X} &= \phi(X_0) + \phi(\widetilde{S}_1) L_{\phi(G)} U R_{\phi(H)} \phi(\widetilde{T}_1) + 4L_{\phi(A)} V + 4\widetilde{U}^T R_{\phi(B)}, \\ \hat{Y} &= \phi(Y_0) + \phi(\widetilde{S}_2) L_{\phi(G)} U R_{\phi(H)} \phi(\widetilde{T}_2) + 4L_{\phi(C)} \hat{U} + 4W^T R_{\phi(D)}. \end{aligned}$$

Hence,

$$\begin{aligned} X_{ij} &= P_i \phi(X_0) Q_j + P_i \phi(\widetilde{S}_1) L_{\phi(G)} U R_{\phi(H)} \phi(\widetilde{T}_1) Q_j + 4P_i L_{\phi(A)} V_j + 4U_i R_{\phi(B)} Q_j, \\ Y_{ij} &= S_i \phi(Y_0) T_j + S_i \phi(\widetilde{S}_2) L_{\phi(G)} U R_{\phi(H)} \phi(\widetilde{T}_2) T_j + 4S_i L_{\phi(C)} \hat{U}_j + 4W_i R_{\phi(D)} T_j, \end{aligned}$$

where $U, U_1, \dots, U_4 \in R^{p \times 4q}$, $V_1, \dots, V_4 \in R^{4p \times q}$, $\hat{U}_1, \dots, \hat{U}_4 \in R^{4p \times q}$, $W_1, \dots, W_4 \in R^{p \times 4q}$ are arbitrary, and

$$\begin{aligned} V &= (V_1, V_2, V_3, V_4), \quad \widetilde{U}^T = (U_1^T, U_2^T, U_3^T, U_4^T), \\ \hat{U} &= (\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{U}_4), \quad W^T = (W_1^T, W_2^T, W_3^T, W_4^T). \end{aligned}$$

Putting them into (2.6), (2.7) yields the eight real matrices X_1, \dots, X_4 and Y_1, \dots, Y_4 , in (2.8)-(2.15). \square

Now we consider the extreme ranks of real matrices X_1, \dots, X_4 and Y_1, \dots, Y_4 in the solution (X, Y) to (1.4) over H .

THEOREM 2.5. *Suppose that (1.4) over H is consistent, and for $i, j = 1, 2, 3, 4$,*

$$J_i = \{X_i \in R^{n \times s} \mid A(X_1 + X_2i + X_3j + X_4k)B + C(Y_1 + Y_2i + Y_3j + Y_4k)D = E\},$$

$$K_j = \{Y_j \in R^{l \times k} \mid A(X_1 + X_2i + X_3j + X_4k)B + C(Y_1 + Y_2i + Y_3j + Y_4k)D = E\}.$$

Then we have the following:

(a) *The maximal and minimal ranks of X_i in the solution $X = X_1 + X_2i + X_3j + X_4k$ to (1.4) are given by*

$$\max_{X_i \in J_i} r(X_i) = \min \left\{ p, q, p + q + r \begin{pmatrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix} - 4(r(B) - r(A, C)), \right. \\ \left. p + q + r \begin{pmatrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{pmatrix} - 4r(A) - 4r \begin{pmatrix} B \\ D \end{pmatrix} \right\},$$

$$\min_{X_i \in J_i} r(X_i) = r \begin{pmatrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix} + r \begin{pmatrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{pmatrix} - r \begin{pmatrix} 0 & 0 & \widehat{B}_i \\ 0 & 0 & \phi(D) \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix}$$

where

$$\widehat{A}_1 = \begin{pmatrix} A_2 & A_3 & -A_4 \\ -A_1 & -A_4 & -A_3 \\ A_4 & -A_1 & A_2 \\ A_3 & -A_2 & -A_1 \end{pmatrix}, \widehat{A}_2 = \begin{pmatrix} -A_1 & A_3 & -A_4 \\ -A_2 & -A_4 & -A_3 \\ -A_3 & -A_1 & A_2 \\ A_4 & -A_2 & -A_1 \end{pmatrix},$$

$$\widehat{A}_3 = \begin{pmatrix} -A_1 & A_2 & -A_4 \\ -A_2 & -A_1 & -A_3 \\ -A_3 & A_4 & A_2 \\ A_4 & A_3 & -A_1 \end{pmatrix}, \widehat{A}_4 = \begin{pmatrix} -A_1 & A_2 & A_3 \\ -A_2 & -A_1 & -A_4 \\ -A_3 & A_4 & -A_1 \\ A_4 & A_3 & -A_2 \end{pmatrix},$$

$$\widehat{B}_1 = \begin{pmatrix} -B_2 & -B_1 & -B_4 & -B_3 \\ -B_3 & -B_4 & -B_1 & B_2 \\ B_4 & B_3 & -B_2 & -B_1 \end{pmatrix}, \widehat{B}_2 = \begin{pmatrix} -B_1 & B_2 & B_3 & -B_4 \\ -B_3 & -B_4 & -B_1 & B_2 \\ B_4 & B_3 & -B_2 & -B_1 \end{pmatrix},$$

$$\widehat{B}_3 = \begin{pmatrix} -B_1 & B_2 & B_3 & -B_4 \\ -B_2 & -B_1 & -B_4 & -B_3 \\ B_4 & B_3 & -B_2 & -B_1 \end{pmatrix}, \widehat{B}_4 = \begin{pmatrix} -B_1 & B_2 & B_3 & -B_4 \\ -B_2 & -B_1 & -B_4 & -B_3 \\ -B_3 & B_4 & -B_1 & B_2 \end{pmatrix}.$$

(b) The maximal and minimal ranks of Y_j in the solution $Y = Y_1 + Y_2i + Y_3j + Y_4k$ to (1.4) are given by

$$\max_{Y_j \in K_j} r(Y_j) = \min \left\{ s, t, s+t+r \begin{pmatrix} 0 & 0 & \widehat{D}_j \\ \widehat{C}_j & \phi(A) & \phi(E) \end{pmatrix} - 4r(B) - 4r(C, A) \right. \\ \left. s+t+r \begin{pmatrix} 0 & \widehat{D}_j \\ 0 & \phi(B) \\ \widehat{C}_j & \phi(E) \end{pmatrix} - 4r(C) - 4r \begin{pmatrix} B \\ D \end{pmatrix} \right\},$$

$$\min_{Y_j \in K_j} r(Y_j) = r \begin{pmatrix} 0 & 0 & \widehat{D}_j \\ \widehat{C}_j & \phi(A) & \phi(E) \end{pmatrix} + r \begin{pmatrix} 0 & \widehat{D}_j \\ 0 & \phi(B) \\ \widehat{C}_j & \phi(E) \end{pmatrix} - r \begin{pmatrix} 0 & 0 & \widehat{D}_j \\ 0 & 0 & \phi(B) \\ \widehat{C}_j & \phi(A) & \phi(E) \end{pmatrix}$$

where

$$\widehat{C}_1 = \begin{pmatrix} C_2 & C_3 & -C_4 \\ -C_1 & -C_4 & -C_3 \\ C_4 & -C_1 & C_2 \\ C_3 & -C_2 & -C_1 \end{pmatrix}, \widehat{C}_2 = \begin{pmatrix} -C_1 & C_3 & -C_4 \\ -C_2 & -C_4 & -C_3 \\ -C_3 & -C_1 & C_2 \\ C_4 & -C_2 & -C_1 \end{pmatrix}, \\ \widehat{C}_3 = \begin{pmatrix} -C_1 & C_2 & -C_4 \\ -C_2 & -C_1 & -C_3 \\ -C_3 & C_4 & C_2 \\ C_4 & C_3 & -C_1 \end{pmatrix}, \widehat{C}_4 = \begin{pmatrix} -C_1 & C_2 & C_3 \\ -C_2 & -C_1 & -C_4 \\ -C_3 & C_4 & -C_1 \\ C_4 & C_3 & -C_2 \end{pmatrix}, \\ \widehat{D}_1 = \begin{pmatrix} -D_2 & -D_1 & -D_4 & -D_3 \\ -D_3 & D_4 & -D_1 & D_2 \\ D_4 & D_3 & -D_2 & -D_1 \end{pmatrix}, \widehat{D}_2 = \begin{pmatrix} -D_1 & D_2 & D_3 & -D_4 \\ -D_3 & D_4 & -D_1 & D_2 \\ D_4 & D_3 & -D_2 & -D_1 \end{pmatrix}, \\ \widehat{D}_3 = \begin{pmatrix} -D_1 & D_2 & D_3 & -D_4 \\ -D_2 & -D_1 & -D_4 & -D_3 \\ D_4 & D_3 & -D_2 & -D_1 \end{pmatrix}, \widehat{D}_4 = \begin{pmatrix} -D_1 & D_2 & D_3 & -D_4 \\ -D_2 & -D_1 & -D_4 & -D_3 \\ -D_3 & D_4 & -D_1 & D_2 \end{pmatrix}.$$

Proof. We only derive the maximal and minimal ranks of the matrix X_1 ; the other $r(X_i), r(Y_j)$ can be established similarly. Applying (2.1) and (2.2) to (2.8), we get the following

$$\max_{X_1 \in J_1} r(X_1) = \min \{p, q, r(M_1), r(M_2)\}, \\ \min_{X_1 \in J_1} r(X_1) = r(M_1) + r(M_2) - r(M_3) - r(P) - r(Q),$$

where

$$M_1 = \begin{pmatrix} \widetilde{X}_0 & P & \widehat{P} \\ Q & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} \widetilde{X}_0 & P \\ Q & 0 \\ \widehat{Q} & 0 \end{pmatrix}, M_3 = \begin{pmatrix} \widetilde{X}_0 & P & \widehat{P} \\ Q & 0 & 0 \\ \widehat{Q} & 0 & 0 \end{pmatrix},$$

$$\widetilde{X}_0 = \frac{1}{4}P_1\phi(X_0)Q_1 + \frac{1}{4}P_2\phi(X_0)Q_2 - \frac{1}{4}P_3\phi(X_0)Q_3 - \frac{1}{4}P_4\phi(X_0)Q_4,$$

$$P = (P_1L_{\phi(A)}, P_2L_{\phi(A)}, -P_3L_{\phi(A)}, -P_4L_{\phi(A)}),$$

$$\widehat{P} = (P_1\phi(\widetilde{S}_1)L_{\phi(G)}, P_2\phi(\widetilde{S}_1)L_{\phi(G)}, -P_3\phi(\widetilde{S}_1)L_{\phi(G)}, -P_4\phi(\widetilde{S}_1)L_{\phi(G)}),$$

$$Q = \begin{pmatrix} R_{\phi(B)}Q_1 \\ R_{\phi(B)}Q_2 \\ -R_{\phi(B)}Q_3 \\ -R_{\phi(B)}Q_4 \end{pmatrix}, \widehat{Q} = \begin{pmatrix} R_{\phi(H)}\phi(\widetilde{T}_1)Q_1 \\ R_{\phi(H)}\phi(\widetilde{T}_1)Q_2 \\ R_{\phi(H)}\phi(\widetilde{T}_1)Q_3 \\ R_{\phi(H)}\phi(\widetilde{T}_1)Q_4 \end{pmatrix}.$$

By Lemma 2.2, block Gaussian elimination and $AX_0B + CY_0D = E$, we have that $r(L_A) = p - r(A)$, $r(R_B) = q - r(B)$,

$$r(M_1) = \begin{bmatrix} \widetilde{X}_0 & \widetilde{P} & \overline{P} & 0 \\ \widetilde{Q} & 0 & 0 & \Psi(B) \\ 0 & \Psi(A) & 0 & 0 \\ 0 & 0 & \Psi(G) & 0 \end{bmatrix} - 4r(\phi(A)) - 4r(\phi(B)) - 4r(\phi(G))$$

$$= r \begin{bmatrix} 0 & 0 & \phi(\widehat{B}_1) \\ \phi(\widehat{A}_1) & \phi(C) & \phi(E) \end{bmatrix} + p + q - 4r(B) - 4r(G)$$

where

$$\widetilde{P} = (P_1, P_2, -P_3, -P_4), \overline{P} = (P_1\phi(\widetilde{S}_1), P_2\phi(\widetilde{S}_1), -P_3\phi(\widetilde{S}_1), -P_4\phi(\widetilde{S}_1)),$$

$$\widetilde{Q} = \begin{pmatrix} Q_1 \\ Q_2 \\ -Q_3 \\ -Q_4 \end{pmatrix}, \Psi(Z) = \begin{pmatrix} \phi(Z) & 0 & 0 & 0 \\ 0 & \phi(Z) & 0 & 0 \\ 0 & 0 & \phi(Z) & 0 \\ 0 & 0 & 0 & \phi(Z) \end{pmatrix}, Z = A, B, G.$$

In the same manner, we can simplify $r(M_2)$ and $r(M_3)$ as follows.

$$r(M_2) = r \begin{pmatrix} 0 & \widehat{B}_1 \\ 0 & \phi(D) \\ \widehat{A}_1 & \phi(E) \end{pmatrix} + p + q - 4r(A) - 4r(H),$$

$$r(M_3) = r \begin{pmatrix} 0 & 0 & \widehat{B}_1 \\ 0 & 0 & \phi(D) \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{pmatrix} + p + q - 4r(G) - 4r(H).$$

Thus, we have the results for the extreme ranks of the matrix X_1 in (a). Similarly, applying (2.1) and (2.2) to (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15) yields the other results in (a), (b). \square

In the following corollaries, $\widehat{A}_i, \widehat{B}_i, \widehat{C}_j$ and \widehat{D}_j ($i, j = 1, 2, 3, 4$) are defined as in Theorem 2.5.

COROLLARY 2.6. *Suppose that (1.4) over H is consistent. Then*

(a) (1.4) has a real solution for X if and only if

$$r \begin{pmatrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix} + r \begin{pmatrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{pmatrix} = r \begin{pmatrix} 0 & 0 & \widehat{B}_i \\ 0 & 0 & \phi(D) \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix}, i = 2, 3, 4.$$

In that case, the real solution X can be expressed as $X = X_1$ in (2.8).

(b) All the solutions of (1.4) for X are real if and only if

$$p + q + r \begin{pmatrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix} = 4r(B) + 4r(A, C)$$

or

$$p + q + r \begin{pmatrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{pmatrix} = 4r(A) + 4r \begin{pmatrix} B \\ D \end{pmatrix}, i = 2, 3, 4.$$

In that case, the real solutions X can be expressed as $X = X_1$ in (2.8).

(c) (1.4) has a pure imaginary solution for X if and only if

$$r \begin{pmatrix} 0 & 0 & \widehat{B}_1 \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{pmatrix} + r \begin{pmatrix} 0 & \widehat{B}_1 \\ 0 & \phi(D) \\ \widehat{A}_1 & \phi(E) \end{pmatrix} = r \begin{pmatrix} 0 & 0 & \widehat{B}_1 \\ 0 & 0 & \phi(D) \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{pmatrix}.$$

In that case, the pure imaginary solution can be expressed as $X = X_2i + X_3j + X_4k$ where X_2, X_3 and X_4 are expressed as (2.9), (2.10) and (2.11), respectively.

(d) All the solutions of (1.4) for X are pure imaginary if and only if

$$p + q + r \begin{pmatrix} 0 & 0 & \widehat{B}_1 \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{pmatrix} = 4r(B) + 4r(A, C)$$

or

$$p + q + r \begin{pmatrix} 0 & \widehat{B}_1 \\ 0 & \phi(D) \\ \widehat{A}_1 & \phi(E) \end{pmatrix} = 4r(A) + 4r \begin{pmatrix} B \\ D \end{pmatrix}.$$

In that case, the pure imaginary solutions can be expressed as $X = X_2i + X_3j + X_4k$, where X_2, X_3 and X_4 are expressed as (2.9), (2.10) and (2.11), respectively.

Using the same method, we can get the corresponding results on Y .

We now consider all solution pairs X and Y to (1.4) over H to be real or pure imaginary, respectively.

THEOREM 2.7. *Suppose that (1.4) over H is consistent, $q(X_1, Y_1) = E - AX_1B - CY_1D$, and J_1, K_1 as in Theorem 2.2 are two independent sets. Then*

$$(2.16) \quad \max_{\substack{X_1 \in J_1, \\ Y_1 \in K_1}} r[q(X_1, Y_1)] = \min \left\{ r(A) + r(C) - r(A, C), r(B) + r(D) - r \begin{pmatrix} B \\ D \end{pmatrix} \right\}.$$

In particular:

(a) *The solutions (X_1, Y_1) of (1.4) over H are independent, that is, for any $X_1 \in J_1$ and $Y_1 \in K_1$ the solutions X_1 and Y_1 satisfy (1.4) over H if and only if*

$$(2.17) \quad r(A, C) = r(A) + r(C) \text{ or } r \begin{pmatrix} B \\ D \end{pmatrix} = r(B) + r(D).$$

(b) *Under (2.17), the general solution of (1.4) over H can be written as the two independent forms*

$$(2.18) \quad X = X_0 + \widetilde{S}_1 L_G U_1 R_H \widetilde{T}_1 + L_A V_1 + V_2 R_B,$$

$$(2.19) \quad Y = Y_0 + \widetilde{S}_2 L_G U_2 R_H \widetilde{T}_2 + L_C W_1 + W_2 R_D,$$

where (X_0, Y_0) is a particular real solution of (1.4), U_1, U_2, V_1, V_2, W_1 and W_2 are arbitrary.

Proof. Writing (2.3) and (2.4) as two independent matrix expressions, that is, replacing U in (2.3) and (2.4) by U_1 and U_2 respectively, then taking them into $E - AX_1B - CY_1D$ yields

$$\begin{aligned} q(X_1, Y_1) &= E - AX_0B - CY_0D - A\widetilde{S}_1 L_G U_1 R_H \widetilde{T}_1 B - C\widetilde{S}_2 L_G U_2 R_H \widetilde{T}_2 D \\ &= A\widetilde{S}_1 L_G (-U_1 + U_2) R_H \widetilde{T}_1 B, \end{aligned}$$

where U_1 and U_2 are arbitrary. Then by (2.1), it follows that

$$\begin{aligned} \max_{\substack{X_1 \in J_1, \\ Y_1 \in K_1}} r[q(X_1, Y_1)] &= \max_{U_1, U_2} r \left(A\widetilde{S}_1 L_G (-U_1 + U_2) R_H \widetilde{T}_1 B \right) \\ &= \min \left\{ r \left(A\widetilde{S}_1 L_G \right), r \left(R_H \widetilde{T}_1 B \right) \right\} \end{aligned}$$

where

$$r \left(\begin{matrix} A\widetilde{S}_1 L_G \\ G \end{matrix} \right) = r \left(\begin{matrix} A\widetilde{S}_1 \\ G \end{matrix} \right) - r(G) = r(A) + r(C) - r(G),$$

$$r \left(\begin{matrix} R_H \widetilde{T}_1 B \\ \widetilde{T}_1 B, H \end{matrix} \right) = r \left(\begin{matrix} \widetilde{T}_1 B, H \end{matrix} \right) - r(H) = r(B) + r(D) - r(H).$$

Therefore, we have (2.16). The result in (2.17) follows directly from (2.16) and the solutions in (2.18) and (2.19) follow from (2.3) and (2.4). \square

COROLLARY 2.8. *Suppose that (1.4) over H is consistent, and (2.17) holds. Then*

(a) *All solution pairs X and Y to (1.4) over H are real if and only if*

$$p + q + r \left(\begin{matrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{matrix} \right) = 4r(B) + 4r(A, C)$$

or

$$p + q + r \left(\begin{matrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{matrix} \right) = 4r(A) + 4r \left(\begin{matrix} B \\ D \end{matrix} \right)$$

and

$$s + t + r \left(\begin{matrix} 0 & 0 & \widehat{D}_j \\ \widehat{C}_j & \phi(A) & \phi(E) \end{matrix} \right) = 4r(B) + 4r(A, C)$$

or

$$s + t + r \left(\begin{matrix} 0 & \widehat{D}_j \\ 0 & \phi(B) \\ \widehat{C}_j & \phi(E) \end{matrix} \right) = 4r(C) + 4r \left(\begin{matrix} B \\ D \end{matrix} \right), i, j = 2, 3, 4.$$

(b) *All solution pairs X and Y to (1.4) over H are imaginary if and only if*

$$p + q + r \left[\begin{matrix} 0 & 0 & \widehat{B}_1 \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{matrix} \right] = 4r(B) + 4r(A, C)$$

or

$$p + q + r \left(\begin{matrix} 0 & \widehat{B}_1 \\ 0 & \phi(D) \\ \widehat{A}_1 & \phi(E) \end{matrix} \right) = 4r(A) + 4r \left(\begin{matrix} B \\ D \end{matrix} \right)$$

and

$$s + t + r \left(\begin{matrix} 0 & 0 & \widehat{D}_1 \\ \widehat{C}_1 & \phi(A) & \phi(E) \end{matrix} \right) = 4r(B) + 4r(C, A)$$

or

$$s + t + r \begin{pmatrix} 0 & \widehat{D}_1 \\ 0 & \phi(B) \\ \widehat{C}_1 & \phi(E) \end{pmatrix} = 4r(C) + 4r \begin{pmatrix} B \\ D \end{pmatrix}.$$

REMARK 2.9. The main results of [11], [17] and [19] can be regarded as special cases of results in this paper.

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