

SCHUR COMPLEMENTS AND BANACHIEWICZ-SCHUR FORMS*

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Abstract. Through the matrix rank method, this paper gives necessary and sufficient conditions for a partitioned matrix to have generalized inverses with Banachiewicz-Schur forms. In addition, this paper investigates the idempotency of generalized Schur complements in a partitioned idempotent matrix.

Key words. Banachiewicz-Schur form, Generalized inverse, Generalized Schur complement, Idempotent matrix, Matrix rank method, Maximal rank, Minimal rank, Moore-Penrose inverse, Partitioned matrix.

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1. Introduction. Let $\mathbb{C}^{m \times n}$ denote the set of all $m \times n$ matrices over the field of complex numbers. The symbols A^* , $r(A)$ and $\mathcal{R}(A)$ stand for the conjugate transpose, the rank and the range (column space) of a matrix $A \in \mathbb{C}^{m \times n}$, respectively; $[A, B]$ denotes a row block matrix consisting of A and B .

A matrix $X \in \mathbb{C}^{n \times m}$ is called a g -inverse of $A \in \mathbb{C}^{m \times n}$, denoted by $X = A^-$, if it satisfies $AXA = A$. The collection of all A^- is denoted by $A\{1\}$. When A is nonsingular, $A^- = A^{-1}$. In addition to A^- , there are also other g -inverses satisfying some additional equations. The well-known Moore-Penrose inverse A^\dagger of $A \in \mathbb{C}^{m \times n}$ is defined to be the unique matrix $X \in \mathbb{C}^{n \times m}$ satisfying the following four Penrose equations

- (i) $AXA = A$,
- (ii) $XAX = X$,
- (iii) $(AX)^* = AX$,
- (iv) $(XA)^* = XA$.

An X is called an $\{i, \dots, j\}$ -inverse of A , denoted by $A^{(i, \dots, j)}$, if it satisfies the i, \dots, j th equations, while the collection of all $\{i, \dots, j\}$ -inverses of A is denoted by $A\{i, \dots, j\}$. In particular, $\{1, 2\}$ -inverse of A is called reflexive g -inverse of A ; $\{1, 3\}$ -inverse of A is called least squares g -inverse of A ; $\{1, 4\}$ -inverse of A is called minimum norm g -inverse of A . For simplicity, let $P_A = I_m - AA^-$ and $Q_A = I_n - A^-A$.

Generalized inverses of block matrices have been a main concern in the theory of generalized inverses and applications. Let M be a 2×2 block matrix

$$(1.1) \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{l \times n}$ and $D \in \mathbb{C}^{l \times k}$. The generalized Schur

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complement of A in M is defined by

$$(1.2) \quad S = D - CA^{-1}B.$$

If both M and A in (1.1) are nonsingular, then $S = D - CA^{-1}B$ is nonsingular, too, and M can be decomposed as

$$(1.3) \quad M = \begin{bmatrix} I_m & 0 \\ CA^{-1} & I_l \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I_m & A^{-1}B \\ 0 & I_l \end{bmatrix},$$

where I_t is the identity matrix of order t . In this case, the inverse of M can be written as

$$(1.4) \quad \begin{aligned} M^{-1} &= \begin{bmatrix} I_m & -A^{-1}B \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. \end{aligned}$$

Result (1.4) is well known and has extensively been used in dealing with inverses of block matrices.

Motivated by (1.4), the Banachiewicz-Schur form of M in (1.1) is defined by

$$(1.5) \quad \begin{aligned} N(A^-) &= \begin{bmatrix} I_n & -A^-B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^- & 0 \\ 0 & S^- \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^- & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^- + A^-BS^-CA^- & -A^-BS^- \\ -S^-CA^- & S^- \end{bmatrix}, \end{aligned}$$

where S is defined as in (1.2). The matrix $N(A^-)$ in (1.5) may vary with respect to the choice of A^- and S^- . In this case, the collection of all $N(A^-)$ is denoted by $\{N(A^-)\}$. It should be pointed out that $N(A^-)$ is not necessarily a g -inverse of M for some given A^- and S^- although $N(A^-)$ is an extension of (1.4). Many authors have investigated the relations between M^- and $N(A^-)$, see, e.g., [1, 3, 5, 6, 7, 11] among others. Because M^- , A^- and S^- can be taken as $\{i, \dots, j\}$ -inverses, the equality $M^- = N(A^-)$ has a variety of different expressions. In these cases, it is of interest to give necessary and sufficient conditions for each equality to hold. In this paper, we shall establish a variety of formulas for the ranks of the difference

$$(1.6) \quad M^- - N(A^-),$$

and then use the rank formulas to characterize the corresponding equality.

Some useful rank formulas for partitioned matrices and g -inverses are given in the following lemma.

LEMMA 1.1 ([10]). *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{l \times n}$ and $D \in \mathbb{C}^{l \times k}$. Then:*

(a) $r \begin{bmatrix} A & B \end{bmatrix} = r(A) + r(B - AA^-B) = r(B) + r(A - BB^-A).$

(b) $r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C - CA^-A) = r(C) + r(A - AC^-C).$

$$(c) \ r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[(I_m - BB^-)A(I_n - C^-C)].$$

$$(d) \ r \begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r \begin{bmatrix} 0 & B - AA^-B \\ C - CA^-A & D - CA^-B \end{bmatrix}.$$

The formulas in Lemma 1.1 can be used to simplify various matrix expressions involving g -inverses. For example,

$$(1.7) \quad r \begin{bmatrix} P_{B_1}A_1 \\ P_{B_2}A_2 \end{bmatrix} = r \begin{bmatrix} B_1 & 0 & A_1 \\ 0 & B_2 & A_2 \end{bmatrix} - r(B_1) - r(B_2),$$

$$(1.8) \quad r[D_1Q_{C_1}, D_2Q_{C_2}] = r \begin{bmatrix} D_1 & D_2 \\ C_1 & 0 \\ 0 & C_2 \end{bmatrix} - r(C_1) - r(C_2),$$

$$(1.9) \quad r \begin{bmatrix} A & BQ_{B_1} \\ P_{C_1}C & 0 \end{bmatrix} = r \begin{bmatrix} A & B & 0 \\ C & 0 & C_1 \\ 0 & B_1 & 0 \end{bmatrix} - r(B_1) - r(C_1),$$

$$(1.10) \quad r \begin{bmatrix} P_{B_1}AQ_{C_1} & P_{B_1}BQ_{B_2} \\ P_{C_2}CQ_{C_1} & 0 \end{bmatrix} = r \begin{bmatrix} A & B & B_1 & 0 \\ C & 0 & 0 & C_2 \\ C_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \end{bmatrix} - r(B_1) - r(B_2) - r(C_1) - r(C_2).$$

The generalized Schur complement $D - CA^-B$ in (1.2) may vary with respect to the choice of A^- . In this case, the following results give the maximal and minimal ranks of $D - CA^-B$ and its properties.

LEMMA 1.2 ([15]). *Let M and S be as given in (1.1) and (1.2). Then*

$$(1.11) \quad \max_{A^-} r(D - CA^-B) = \min \left\{ r[C, D], \ r \begin{bmatrix} B \\ D \end{bmatrix}, \ r(M) - r(A) \right\},$$

$$(1.12) \quad \min_{A^-} r(D - CA^-B) = r(A) + r(M) + r[C, D] + r \begin{bmatrix} B \\ D \end{bmatrix} - r \begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} \\ - r \begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix},$$

$$(1.13) \quad \max_{A^{(1,2)}} r(D - CA^{(1,2)}B) = \min \left\{ r(A) + r(D), \ r[C, D], \ r \begin{bmatrix} B \\ D \end{bmatrix}, \ r(M) - r(A) \right\},$$

$$(1.14) \quad \max_{A^{(1,3)}} r(D - CA^{(1,3)}B) = \min \left\{ r \begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} - r(A), \ r \begin{bmatrix} B \\ D \end{bmatrix} \right\},$$

$$(1.15) \quad \max_{A^{(1,4)}} r(D - CA^{(1,4)}B) = \min \left\{ r[C, D], \ r \begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} - r(A) \right\}.$$

COROLLARY 1.3. *Let M be as given in (1.1). Then there is A^- such that*

$$(1.16) \quad r(M) = r(A) + r(D - CA^-B)$$

if and only if

$$(1.17) \quad r(M) \leq r(A) + \min \left\{ r[C, D], \quad r \begin{bmatrix} B \\ D \end{bmatrix} \right\}.$$

Equality (1.16) holds for any A^- if and only if

$$(1.18) \quad r \begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} = r(A) + r[C, D] \text{ and } r \begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} = r(A) + r \begin{bmatrix} B \\ D \end{bmatrix},$$

that is, $\mathcal{R}[(PAB)^*] \subseteq \mathcal{R}[(PCD)^*]$ and $\mathcal{R}(CQ_A) \subseteq \mathcal{R}(DQ_B)$.

Proof. Note from (1.11) that $r(M) - r(A)$ is an upper bound for $r(D - CA^-B)$. Thus, there is A^- such that (1.16) holds if and only if

$$\max_{A^-} r(D - CA^-B) = r(M) - r(A).$$

Substituting (1.11) into this and simplifying yield (1.17). Equality (1.16) holds for any A^- if and only if

$$\min_{A^-} r(D - CA^-B) = r(M) - r(A).$$

Substituting (1.12) into this yields

$$\left(r \begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r[C, D] - r(A) \right) + \left(r \begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} - r \begin{bmatrix} B \\ D \end{bmatrix} - r(A) \right) = 0.$$

Note that both terms are nonnegative. Hence this equality is equivalent to (1.18). Applying Lemma 1.1(a) and (b) to the block matrices in (1.18) yields

$$r \begin{bmatrix} PAB \\ PCD \end{bmatrix} = r(PCD) \text{ and } r[CQ_A, DQ_B] = r(DQ_B).$$

These two rank equalities are obviously equivalent to $\mathcal{R}[(PAB)^*] \subseteq \mathcal{R}[(PCD)^*]$ and $\mathcal{R}(CQ_A) \subseteq \mathcal{R}(DQ_B)$. \square

Equality (1.16) is in fact a rank equation for the partitioned matrix M . For given A, A^-, B and C , it is of interest to find D such that (1.16) holds. Other two basic rank equations for the partitioned matrix M are given by

$$r \begin{bmatrix} A & B \\ C & X \end{bmatrix} = r(A), \quad r \begin{bmatrix} A & B \\ C & X \end{bmatrix} = r \begin{bmatrix} A & B \\ C & CA^-B \end{bmatrix} + r(X - CA^-B).$$

The first one has been investigated in [9, 13].

LEMMA 1.4 ([15]). Let $A \in \mathbb{C}^{m \times n}$ and $G \in \mathbb{C}^{n \times m}$. Then

$$(1.19) \quad \min_{A^-} r(A^- - G) = r(A - AGA),$$

$$(1.20) \quad \min_{A^{(1,2)}} r(A^{(1,2)} - G) = \max\{r(A - AGA), r(G) + r(A) - r(GA) - r(AG)\},$$

$$(1.21) \quad \min_{A^{(1,3)}} r(A^{(1,3)} - G) = r(A^*AG - A^*),$$

$$(1.22) \quad \min_{A^{(1,4)}} r(A^{(1,4)} - G) = r(GAA^* - A^*).$$

When A and G are some given block matrices, the rank formulas on the right-hand sides of (1.19)–(1.22) can possibly be simplified by elementary matrix operations. In these cases, some necessary and sufficient conditions for G to be $\{1\}$ -, $\{1,2\}$ -, $\{1,3\}$ - and $\{1,4\}$ -inverses of A can be derived from the rank formulas. We shall use this rank method to establish necessary and sufficient conditions for $N(A^-)$ in (1.5) to be $\{1\}$ -, $\{1,2\}$ -, $\{1,3\}$ - and $\{1,4\}$ -inverses of M in (1.1).

2. Generalized inverses of partitioned matrices. In this section, we show a group of formulas for the rank of the difference in (1.6) and then use the formulas to characterize the equality $M^- = N(A^-)$.

THEOREM 2.1. *Let $N(A^-)$ be as given in (1.5). Then*

$$(2.1) \quad r[N(A^-)] = r(A^-) + r[(D - CA^-B)^-].$$

Proof. It follows from the first equality in (1.5). \square

THEOREM 2.2. *Let M and $N(A^-)$ be as given in (1.1) and (1.5), respectively. Then*

$$(2.2) \quad \min_{M^-} r[M^- - N(A^-)] = r(M) - r(A) - r(D - CA^-B).$$

Hence, the following statements are equivalent:

- (a) $N(A^-) \in M\{1\}$.
- (b) The g -inverse A^- in $N(A^-)$ satisfies $r(M) = r(A) + r(D - CA^-B)$.
- (c) The g -inverse A^- in $N(A^-)$ satisfies

$$r \begin{bmatrix} 0 & B - AA^-B \\ C - CA^-A & D - CA^-B \end{bmatrix} = r(D - CA^-B).$$

Proof. It follows from (1.19) that

$$(2.3) \quad \min_{M^-} r[M^- - N(A^-)] = r[M - MN(A^-)M].$$

It is easy to verify that

$$M - MN(A^-)M = \begin{bmatrix} -P_A BS^-CQ_A & P_A BQ_S \\ P_S CQ_A & 0 \end{bmatrix},$$

where $S = D - CA^-B$. Recall that elementary block matrix operations do not change the rank of matrix. Applying (1.10) and elementary block matrix operations to the matrix on the right-hand side leads to

$$\begin{aligned} r[M - MN(A^-)M] &= r \begin{bmatrix} P_A BS^-CQ_A & P_A BQ_S \\ P_S CQ_A & 0 \end{bmatrix} \\ &= r \begin{bmatrix} BS^-C & B & A & 0 \\ C & 0 & 0 & S \\ A & 0 & 0 & 0 \\ 0 & S & 0 & 0 \end{bmatrix} - 2r(A) - 2r(S) \end{aligned}$$

$$\begin{aligned}
 &= r \begin{bmatrix} 0 & B & A & 0 \\ C & 0 & 0 & S \\ A & 0 & 0 & 0 \\ -SS^{-1}C & S & 0 & 0 \end{bmatrix} - 2r(A) - 2r(S) \\
 &= r \begin{bmatrix} 0 & B & A & 0 \\ C & 0 & 0 & S \\ A & 0 & 0 & 0 \\ 0 & S & 0 & S \end{bmatrix} - 2r(A) - 2r(S) \\
 &= r \begin{bmatrix} 0 & B & A \\ C & -S & 0 \\ A & 0 & 0 \end{bmatrix} - 2r(A) - r(S) \\
 &= r \begin{bmatrix} 0 & B & A \\ C & -D + CA^{-1}B & 0 \\ A & 0 & 0 \end{bmatrix} - 2r(A) - r(S) \\
 &= r \begin{bmatrix} 0 & B & A \\ C & -D & 0 \\ A & 0 & A \end{bmatrix} - 2r(A) - r(S) \\
 &= r \begin{bmatrix} -A & B & 0 \\ C & -D & 0 \\ 0 & 0 & A \end{bmatrix} - 2r(A) - r(S) \\
 &= r(M) - r(A) - r(S).
 \end{aligned}$$

Substituting this into (2.3) gives (2.2). The equivalence of (a) and (b) follows from (2.2). The equivalence of (b) and (c) follows from Lemma 1.1(d). \square

Theorem 2.2 gives necessary and sufficient conditions for a given $N(A^-)$ to be a generalized inverse of M . From Theorem 2.2, we are also able to give the existence of A^- so that $N(A^-) \in M\{1\}$.

THEOREM 2.3. *Let M and $N(A^-)$ be as given in (1.1) and (1.5), respectively. Then:*

- (a) *The following statements are equivalent:*
 - (i) *There is A^- such that $N(A^-) \in M\{1\}$.*
 - (ii) *There is A^- such that $r(M) = r(A) + r(D - CA^{-1}B)$.*
 - (iii) *The rank of M satisfies the following inequality*

$$(2.4) \quad r(M) \leq \min \left\{ r(A) + r \begin{bmatrix} B \\ D \end{bmatrix}, r(A) + r[C, D] \right\}.$$

- (b) *The following statements are equivalent:*
 - (i) *The set inclusion $\{N(A^-)\} \subseteq M\{1\}$ holds.*
 - (ii) *$r(M) = r(A) + r(D - CA^{-1}B)$ holds for any A^- .*
 - (iii) *(1.18) holds.*

Proof. Substituting (1.11) and (1.12) into (2.2) gives

$$\min_{A^-, M^-} r[M^- - N(A^-)]$$

$$\begin{aligned}
 &= \max \left\{ r(M) - r(A) - r[C, D], \quad r(M) - r(A) - r \begin{bmatrix} B \\ D \end{bmatrix}, \quad 0 \right\}, \\
 &\max_{A^-} \min_{M^-} r[M^- - N(A^-)] \\
 &= \left(r \begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r[C, D] - r(A) \right) + \left(r \begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} - r \begin{bmatrix} B \\ D \end{bmatrix} - r(A) \right).
 \end{aligned}$$

Combining these two equalities with Corollary 1.3 results in the equivalences in (a) and (b). \square

THEOREM 2.4. *Let M and $N(A^-)$ be as given in (1.1) and (1.5), respectively. Then the following statements are equivalent:*

- (a) $N(A^-) \in M\{1, 2\}$.
- (b) *The g -inverses A^- and S^- in $N(A^-)$ satisfy $A^- \in A\{1, 2\}$, $S^- \in S\{1, 2\}$ and $r(M) = r(A) + r(D - CA^-B)$.*

Proof. Recall that $X \in A\{1, 2\}$ if and only if $X \in A\{1\}$ and $r(X) = r(A)$. Hence, $N(A^-) \in M\{1, 2\}$ if and only if $N(A^-) \in M\{1\}$ and $r[N(A^-)] = r(M)$. From Theorem 2.2(a) and (b), $N(A^-) \in M\{1\}$ is equivalent to

$$(2.5) \quad r(M) = r(A) + r(D - CA^-B).$$

From Theorem 2.1, $r[N(A^-)] = r(M)$ is equivalent to

$$(2.6) \quad r(M) = r(A^-) + r[(D - CA^-B)^-].$$

Also note that $r(A^-) \geq r(A)$ and $r[(D - CA^-B)^-] \geq r(D - CA^-B)$. Hence, (2.5) and (2.6) imply that $r(A^-) = r(A)$ and $r[(D - CA^-B)^-] = r(D - CA^-B)$. These two rank equalities show that $A^- \in A\{1, 2\}$ and $S^- \in S\{1, 2\}$. Thus (a) implies (b). Conversely, the third equality in (b) implies $N(A^-) \in M\{1\}$ by Theorem 2.2(a) and (b). If $A^- \in A\{1, 2\}$ and $S^- \in S\{1, 2\}$, then we see from Theorem 2.1 that

$$r[N(A^-)] = r(A^{(1,2)}) + r(S^{(1,2)}) = r(A) + r(S) = r(M).$$

Thus $N(A^-) \in M\{1, 2\}$. \square

THEOREM 2.5. *Let M and $N(A^-)$ be as given in (1.1) and (1.5), respectively. Then there are $A^- \in A\{1, 2\}$ and $S^- \in S\{1, 2\}$ such that $N(A^-) \in M\{1, 2\}$ if and only if*

$$(2.7) \quad r(M) \leq \min \left\{ 2r(A) + r(D), \quad r(A) + r \begin{bmatrix} B \\ D \end{bmatrix}, \quad r(A) + r[C, D] \right\}$$

holds.

Proof. It follows from (1.20) that

$$(2.8) \quad \min_{M^- \in M\{1,2\}} r[M^- - N(A^-)] = \max\{r_1, r_2\},$$

where

$$r_1 = r[M - MN(A^-)M], \quad r_2 = r(M) + r[N(A^-)] - r[N(A^-)M] - r[MN(A^-)].$$

When $A^- \in A\{1, 2\}$ and $S^- \in S\{1, 2\}$,

$$r[M - MN(A^-)M] = r(M) - r(A) - r(S)$$

holds by (2.2). It is easy to verify that

$$(2.9) \quad MN(A^-) = \begin{bmatrix} AA^- & P_A BS^- \\ CA^- & SS^- \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^- & I_l \end{bmatrix},$$

$$(2.10) \quad N(A^-)M = \begin{bmatrix} I_n & -A^-B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^-A & A^-B \\ S^-CQ_A & S^-S \end{bmatrix}.$$

When $A^- \in A\{1, 2\}$ and $S^- \in S\{1, 2\}$, we can find by (2.1) and elementary block matrix operations that

$$r[N(A^-)] = r[MN(A^-)] = r[N(A^-)M] = r(A) + r(S).$$

Hence,

$$(2.11) \quad \min_{M^- \in M\{1,2\}} r[M^- - N(A^-)] = r(M) - r(A) - r(D - CA^-B).$$

Substituting (1.13) into (2.11) gives

$$(2.12) \quad \min_{M^- \in M\{1,2\}, A^- \in A\{1,2\}} r[M^- - N(A^-)] = \max\{r_1, r_2, r_3, 0\},$$

where

$$r_1 = r(M) - 2r(A) - r(D), \quad r_2 = r(M) - r(A) - r[C, D], \quad r_3 = r(M) - r(A) - r \begin{bmatrix} B \\ D \end{bmatrix}.$$

Let the right-hand side of (2.12) be zero, we obtain (2.7). \square

THEOREM 2.6. *Let M be given in (1.1) and $N(A^-)$ in (1.5). Then*

$$(2.13) \quad \min_{M^- \in M\{1,3\}} r[M^- - N(A^-)] = r \begin{bmatrix} A^*P_A & C^*P_S \\ B^*P_A & D^*P_S \end{bmatrix}.$$

Hence the following statements are equivalent:

- (a) $N(A^-) \in M\{1, 3\}$.
- (b) The g -inverses A^- and S^- in $N(A^-)$ satisfy $A^*P_A = 0$, $C^*P_S = 0$, $B^*P_A = 0$ and $D^*P_S = 0$.
- (c) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, $\mathcal{R}(C) \subseteq \mathcal{R}(S)$ and the g -inverses A^- and S^- in $N(A^-)$ satisfy $A^- \in A\{1, 3\}$ and $S^- \in S\{1, 3\}$.

Proof. From (1.21)

$$\min_{M^- \in M\{1,3\}} r[M^- - N(A^-)] = r[M^*MN(A^-) - M^*].$$

From (2.9)

$$\begin{aligned} M^*MN(A^-) - M^* &= -M^*[I_{m+l} - MN(A^-)] \\ &= -M^* \begin{bmatrix} P_A + P_A BS^- CA^- & -P_A BS^- \\ -P_S CA^- & P_S \end{bmatrix}. \end{aligned}$$

Also by elementary block matrix operations

$$\begin{aligned} r[M^*MN(A^-) - M^*] &= r\left(M^* \begin{bmatrix} P_A + P_ABS^-CA^- & -P_ABS^- \\ -P_SCA^- & P_S \end{bmatrix}\right) \\ &= r\left(M^* \begin{bmatrix} P_A & 0 \\ 0 & P_S \end{bmatrix}\right). \end{aligned}$$

Thus we have (2.13). The equivalence of (a) and (b) is derived from (2.13). Note $A^*P_A = 0$ is $A^*AA^- = A^*$. This is equivalent to $A^- \in A\{1, 3\}$, this is, $AA^- = AA^\dagger$. In this case, $B^*P_A = 0$ is $B^*AA^\dagger = A^*$, which is equivalent to $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. The two equalities $C^*P_S = 0$ and $D^*P_S = 0$ imply that $(D - CA^-B)^*P_S = S^* - S^*SS^- = 0$, which is equivalent to $S^- \in S\{1, 3\}$. In this case, $C^*P_S = 0$ is equivalent to $\mathcal{R}(C) \subseteq \mathcal{R}(S)$. Conversely, if (c) holds, then the first three equalities in (b) follow immediately. Also note that $\mathcal{R}[C, S] = \mathcal{R}[C, D]$ and that $\mathcal{R}(C) \subseteq \mathcal{R}(S)$ implies $\mathcal{R}[C, S] = \mathcal{R}(S)$. Hence $\mathcal{R}[C, D] = \mathcal{R}(S)$. This shows that $\mathcal{R}(D) \subseteq \mathcal{R}(S)$. Thus $D^*P_S = 0$. \square

THEOREM 2.7. *Let M and $N(A^-)$ be as given in (1.1) and (1.5), respectively. Then there are $A^- \in A\{1, 3\}$ and $S^- \in S\{1, 3\}$ such that $N(A^-) \in M\{1, 3\}$ if and only if*

$$(2.14) \quad \mathcal{R}(B) \subseteq \mathcal{R}(A), \mathcal{R} \begin{bmatrix} A^*A \\ B^*A \end{bmatrix} \cap \mathcal{R} \begin{bmatrix} C^* \\ D^* \end{bmatrix} = \{0\} \text{ and } r[C, D] \leq r \begin{bmatrix} B \\ D \end{bmatrix}$$

hold.

Proof. If $A^- \in A\{1, 3\}$ and $S^- \in A\{1, 3\}$, then $AA^{(1,3)} = AA^\dagger$ and $SS^{(1,3)} = SS^\dagger$. In these cases,

$$\begin{aligned} r\left(M^* \begin{bmatrix} P_A & 0 \\ 0 & P_S \end{bmatrix}\right) &= r \begin{bmatrix} P_AA & P_AB \\ P_SC & P_SD \end{bmatrix} \\ &= r \begin{bmatrix} A & 0 & A & B \\ 0 & S & C & D \end{bmatrix} - r(A) - r(S) \quad (\text{by (1.7)}) \\ &= r \begin{bmatrix} A & 0 & 0 & B \\ 0 & D & C & 0 \end{bmatrix} - r(A) - r(S) \\ &= r[A, B] + r[C, D] - r(A) - r(D - CA^{(1,3)}B). \end{aligned}$$

Substituting this into (2.13) results in

$$(2.15) \quad \min_{M^- \in M\{1,3\}} r[M^- - N(A^-)] = r[A, B] + r[C, D] - r(A) - r(D - CA^-B).$$

Substituting (1.14) into (2.15) gives

$$\min_{A^- \in A\{1,3\}, M^- \in M\{1,3\}} r[M^- - N(A^-)] = \max\{r_1, r_2\},$$

where

$$r_1 = r[A, B] + r[C, D] - r \begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix}, \quad r_2 = r[A, B] + r[C, D] - r(A) - r \begin{bmatrix} B \\ D \end{bmatrix}.$$

Let the right-hand side be zero, we see that there are $A^- \in A\{1, 3\}$ and $S^- \in S\{1, 3\}$ such that $N(A^-) \in M\{1, 3\}$ if and only if

$$(2.16) \quad r[A, B] + r[C, D] = r \begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} \text{ and } r[A, B] + r[C, D] \leq r(A) + r \begin{bmatrix} B \\ D \end{bmatrix}.$$

Note that

$$r[A, B] + r[C, D] \geq r(A) + r[C, D] \geq r \begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix}.$$

Hence, the first equality in (2.16) is equivalent to

$$\mathcal{R}(B) \subseteq \mathcal{R}(A) \text{ and } r \begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} = r[AA^*, A^*B] + r[C, D],$$

and the second inequality in (2.16) is equivalent to $r[C, D] \leq r \begin{bmatrix} B \\ D \end{bmatrix}$. Thus (2.16) is equivalent to (2.14). \square

Similarly, the following two theorems can be derived from (1.22) and (1.15).

THEOREM 2.8. *Let M and $N(A^-)$ be as given in (1.1) and (1.5), respectively. Then*

$$(2.17) \quad \min_{M^- \in M\{1,4\}} r[M^- - N(A^-)] = r \begin{bmatrix} Q_A A^* & Q_A C^* \\ Q_S B^* & Q_S D^* \end{bmatrix}.$$

Hence the following statements are equivalent:

- (a) $N(A^-) \in M\{1, 4\}$.
- (b) The g -inverses A^- and S^- in $N(A^-)$ satisfy $Q_A A^* = 0$, $Q_A C^* = 0$, $Q_S B^* = 0$ and $Q_S D^* = 0$.
- (c) $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, $\mathcal{R}(B^*) \subseteq \mathcal{R}(S^*)$ and the g -inverses A^- and S^- in $N(A^-)$ satisfy $A^- \in A\{1, 4\}$ and $S^- \in S\{1, 4\}$.

THEOREM 2.9. *Let M and $N(A^-)$ be as given in (1.1) and (1.5), respectively. Then there are $A^- \in A\{1, 4\}$ and $S^- \in S\{1, 4\}$ such that $N(A^-) \in M\{1, 4\}$ if and only if*

$$\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*), \mathcal{R} \begin{bmatrix} AA^* \\ CA^* \end{bmatrix} \cap \mathcal{R} \begin{bmatrix} B \\ D \end{bmatrix} = \{0\} \text{ and } r \begin{bmatrix} B \\ D \end{bmatrix} \leq r[C, D]$$

hold.

Note that $X = A^\dagger$ if and only if $X \in A\{1, 2\}$, $X \in A\{1, 3\}$ and $X \in A\{1, 4\}$. Hence the following result is derived from Theorems 2.4, 2.6 and 2.8.

THEOREM 2.10 ([1]). *Let M and $N(A^-)$ be as given in (1.1) and (1.5). Then the following statements are equivalent:*

- (a) $N(A^-) = M^\dagger$.
- (b) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, $\mathcal{R}(C) \subseteq \mathcal{R}(S)$, $\mathcal{R}(B^*) \subseteq \mathcal{R}(S^*)$ and the g -inverses A^- and S^- in $N(A^-)$ satisfy $A^- = A^\dagger$ and $S^- = S^\dagger$.

3. The Generalized Schur complement in idempotent matrix. Consider the square block matrix

$$(3.1) \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathbb{C}^{m \times m}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{k \times m}$ and $D \in \mathbb{C}^{k \times k}$. Then this matrix is idempotent, namely, $M^2 = M$, if and only if

$$(3.2) \quad A = A^2 + BC, \quad B = AB + BD, \quad C = CA + DC, \quad D = CB + D^2.$$

From (3.2), we find that the square of the generalized Schur complement $S = D - CA^-B$ can be written as

$$(3.3) \quad \begin{aligned} S^2 &= D^2 - DCA^-B - CA^-BD + CA^-BCA^-B \\ &= D - CB - (C - CA)A^-B - CA^-(B - AB) + CA^-(A - A^2)A^-B \\ &= S - C(I_m - A^-A)(I_m - AA^-)B + C(A^-AA^- - A^-)B. \end{aligned}$$

Hence,

$$(3.4) \quad S^2 = S \Leftrightarrow C(I_m - A^-A)(I_m - AA^-)B = C(A^-AA^- - A^-)B.$$

The right-hand side is a quadratic equation with respect to A^- . Hence it is difficult to show the existence of A^- satisfying this equation. Instead, we first consider the existence of A^- satisfying

$$C(I_m - A^-A)(I_m - AA^-)B = 0 \quad \text{and} \quad C(A^-AA^- - A^-)B = 0.$$

Notice that if there is A^- satisfying $C(I_m - A^-A)(I_m - AA^-)B = 0$, then the product $G = A^-AA^-$ satisfies

$$G \in A\{1, 2\} \quad \text{and} \quad C(I_m - GA)(I_m - AG)B = C(GAG - G)B = 0.$$

In such a case, $D - CGB$ is idempotent by (3.4). In view of this, we only consider the existence of $A^{(1,2)}$ such that $C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B = 0$.

LEMMA 3.1. *Let M in (3.1) be idempotent. Then:*

(a) *There is $A^{(1,2)}$ such that $C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B = 0$ if and only if*

$$(3.5) \quad r \begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} \leq r \begin{bmatrix} A^2 \\ CA \end{bmatrix} + r[A^2, AB].$$

(b) *$C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B = 0$ holds for any $A^{(1,2)}$ if and only if*

$$(3.6) \quad \mathcal{R}(B) \subseteq \mathcal{R}(A), \quad \text{or} \quad \mathcal{R}(C^*) \subseteq \mathcal{R}(A^*), \quad \text{or} \quad \begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} = 0.$$

Proof. It is well known that the general expression of $A^{(1,2)}$ can be expressed as

$$A^{(1,2)} = (A^\dagger + F_A V)A(A^\dagger + W E_A),$$

where $E_A = I_m - AA^\dagger$ and $F_A = I_m - A^\dagger A$, the two matrices V and W are arbitrary; see [4, 8, 12]. Correspondingly,

$$(3.7) \quad C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B = (CF_A - CF_AVA)(E_AB - AWE_AB).$$

It is easy to see

$$r[(CF_A - CF_AVA)(P_AB - AWP_AB)] = r \begin{bmatrix} I_m & E_AB - AWE_AB \\ CF_A - CF_AVA & 0 \end{bmatrix} - m,$$

where

$$(3.8) \quad \begin{bmatrix} I_m & E_AB - AWE_AB \\ CF_A - CF_AVA & 0 \end{bmatrix} = \begin{bmatrix} I_m & E_AB \\ CF_A & 0 \end{bmatrix} - \begin{bmatrix} A \\ 0 \end{bmatrix} W[0, E_AB] \\ - \begin{bmatrix} 0 \\ CF_A \end{bmatrix} V[A, 0].$$

Applying the following rank formula in [14]

$$(3.9) \quad \min_{X_1, X_2} r(A - B_1X_1C_1 - B_2X_2C_2) = r \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + r[A, B_1, B_2] + \max\{r_1, r_2\},$$

where

$$r_1 = r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}, \\ r_2 = r \begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}$$

to (3.8) and simplifying yield

$$\min_{V, W} r \begin{bmatrix} I_m & E_AB - AWE_AB \\ CF_A - CF_AVA & 0 \end{bmatrix} \\ = \max \left\{ m, m + r \begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} - r \begin{bmatrix} A^2 \\ CA \end{bmatrix} - r[A^2, AB] \right\}.$$

Hence

$$\min_{A^{(1,2)}} r[C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B] \\ = \max \left\{ 0, r \begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} - r \begin{bmatrix} A^2 \\ CA \end{bmatrix} - r[A^2, AB] \right\}.$$

Letting the right-side be zero yields (a). Applying the following rank formula in [14]

$$\max_{X_1, X_2} r(A - B_1X_1C_1 - B_2X_2C_2) \\ = \min \left\{ r \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix}, r[A, B_1, B_2], r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix}, r \begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} \right\}$$

to (3.8) and simplifying yield

$$\begin{aligned} & \max_{V,W} r \begin{bmatrix} I_m & E_{AB} - AWE_{AB} \\ CF_A - CF_AVA & 0 \end{bmatrix} \\ &= m + \min \left\{ r \begin{bmatrix} A \\ C \end{bmatrix} - r(A), \quad r[A, B] - r(A), \quad r \begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} & \max_{A^{(1,2)}} r[C(I_m - A^{(1,2)}A)(I_m - AA^{(1,2)})B] \\ &= \min \left\{ r \begin{bmatrix} A \\ C \end{bmatrix} - r(A), \quad r[A, B] - r(A), \quad r \begin{bmatrix} A^2 & AB \\ CA & CB \end{bmatrix} \right\}. \end{aligned}$$

Letting the right-side be zero leads to (b). \square

Applying Theorem 3.1 to (3.4) gives us the following result.

THEOREM 3.2. *Let M in (3.1) be idempotent. Then:*

- (a) *There is $A^{(1,2)}$ such that the generalized Schur complement $D - CA^{(1,2)}B$ is idempotent if and only if (3.5) holds.*
- (b) *The generalized Schur complement $D - CA^{(1,2)}B$ is idempotent for any $A^{(1,2)}$ if and only if (3.6) holds.*

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