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ON A CONJECTURE REGARDING CHARACTERISTIC POLYNOMIAL OF A MATRIX PAIR*

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Abstract. For n-by-n Hermitian matrices $A(>0)$ and B, define

$$\eta(A,B)=\sum_S \det A(S)\det B(S')\ ,$$

where the summation is over all subsets of $\{1,\dots,n\}$, S' is the complement of S, and by convention $\det A(\emptyset)=1$. Bapat proved for $n=3$ that the zeros of $\eta(\lambda A,-B)$ and the zeros of $\eta(\lambda A(23),-B(23))$ interlace. This result is generalized to a broader class of matrices.

Key words. Symmetric matrices, Cycles, Characteristic polynomial, Interlacing.

AMS subject classifications. 15A15, 15A42.

1. Introduction. Let $A=(a_{ij})$ and $B=(b_{ij})$ be matrices of order n. For index sets $S\subset\{1,\dots,n\}$, we denote by $A(S)$ the $|S|\times|S|$ principal submatrix lying in the rows and columns indexed by S. We may also denote $A(S')$ by A_S, with S' indexed the complement of S.

Define

$$\eta(A,B):=\sum_S \det A(S)\det B(S') \tag{1.1}$$

where the summation is over all subsets of $\{1,\dots,n\}$ and, by convention, $\det A(\emptyset)=\det B(\emptyset)=1$. Notice that

$$\eta(\lambda I_n,-B)=\det(\lambda I_n-B)\ , \tag{1.2}$$

i.e., $\eta(\lambda I_n,-B)$ is the characteristic polynomial of B. It is well-known that, if B is Hermitian, then the roots of (1.2), the eigenvalues of B, are all real. Motivated by this result, Johnson [3] considered the polynomial (of degree n)

$$\eta(\lambda A,-B)=\sum_{k=0}^{n}\sum_{|S|=k}(-1)^{n-k}\det A(S)\det B(S')\lambda^k\ , \tag{1.3}$$

and stated the conjecture:

CONJECTURE 1.1 (Johnson [3]). *If A and B are Hermitian and A is positive semidefinite, then the polynomial $\eta(\lambda A,-B)$ has only real roots.*

For a square matrix A, we write $A>0$ to denote that A is positive definite. If all roots of the polynomial (1.3), say $\lambda_\ell^A(B)$, for $\ell=1,\dots,n$, are real, we assume that

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they have been arranged in increasing order $\lambda_1^A(B) \leq \cdots \leq \lambda_n^A(B)$. Bapat in [1] and Johnson in [4] conjectured:

CONJECTURE 1.2. *If $A > 0$ and B are Hermitian, then $\lambda_\ell^{A_1}(B_1)$, for $\ell = 1, \ldots, n-1$, interlace $\lambda_\ell^A(B)$, for $\ell = 1, \ldots, n$, i.e.,*

$$\lambda_\ell^A(B) \leq \lambda_\ell^{A_1}(B_1) \leq \lambda_{\ell+1}^A(B)\ , \quad \ell = 1, \ldots, n-1\ .$$

Conjecture 1.1 has been verified for the case $n = 3$ by Rublein in [5] in a very complicated way. On the other hand, Bapat in [1] gave concise solutions for the cases $n \leq 3$. Bapat also verified that Conjectures 1.1 and 1.2 are true when both A and B are tridiagonal. Recently, the author generalized these results for matrices whose graph is a tree [2].

In this note we generalize the result of Bapat when $n = 3$ to matrices whose graph is a cycle.

For sake of simplicity we consider only symmetric matrices throughout. All the results can be easily generalized to Hermitian matrices.

2. Results on tridiagonal matrices. We define the weights of a symmetric matrix A as $w_{ij}(A) = -a_{ij}^2$ if $i \neq j$, and $w_{ii}(A) = a_{ii}$. Sometimes we abbreviate to w_{ij}, with no mention of A.

LEMMA 2.1 (Bapat [1]). *Let A and B be symmetric tridiagonal matrices and let $S = \{1, 2\}$. Then*

$$\eta(A, B) = \sum_{\ell \in S} (w_{1\ell}(A) + w_{1\ell}(B))\eta(A_{1\ell}, B_{1\ell})\ . \tag{2.1}$$

For tridiagonal matrices we also state the following result, which can be proved by induction.

LEMMA 2.2. *Let A and B be symmetric tridiagonal matrices of order n. Then*

$$\eta(A, B)\eta(A_{1n}, B_{1n}) = \eta(A_1, B_1)\eta(A_n, B_n) - (a_{12}^2 + b_{12}^2) \cdots (a_{n-1,n}^2 + b_{n-1,n}^2). \tag{2.2}$$

Notice that (2.2) holds up to permutation similarity.

3. An interlacing theorem. Bapat proved the veracity of Conjectures 1.1 and 1.2 in the case $n \leq 3$.

THEOREM 3.1 (Bapat [1]). *Let A and B be Hermitian matrices of order 3 with $A > 0$ and B has all nonzero subdiagonal entries. Then $\eta(\lambda A, -B)$ has three real roots, say $\lambda_1 < \lambda_2 < \lambda_3$. Furthermore, if $\mu_1 < \mu_2$ are the roots of $\eta(\lambda A_1, -B_1)$, then $\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3$.*

Consider symmetric matrices A and B such that $a_{ij} = b_{ij} = 0$ for $|i - j| > 1$ and $(i, j) \neq (1, n)$. We say, for obvious reasons, that A and B are matrices whose graph is a cycle. The next result generalizes Lemma 2.1, since if $w_{1,n} = 0$, we get (2.1) with $i = 1$.

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LEMMA 3.2. *Let A and B be symmetric matrices whose graph is a cycle and set $S = \{i-1, i, i+1\}$. Then*

$$\eta(A,B) = \sum_{\ell \in S}(w_{i\ell}(A) + w_{i\ell}(B))\eta(A_{i\ell}, B_{i\ell}) + 2(-1)^{n-1}(\prod_{\ell=1}^{n} a_{\ell,\ell+1} + \prod_{\ell=1}^{n} b_{\ell,\ell+1}), \tag{3.1}$$

with the convention $(n, n+1) = (1, n)$.

Proof. Let $C = \{i-1, i+1\}$. Considering the partition of all subsets of $\{1, \ldots, n\}$, define

$$\mathcal{A}_P = \{S \mid i \in S,\ P \subset S,\ P' \cap S = \emptyset\}$$

and

$$\mathcal{C}_P = \{S \mid i \notin S,\ P \subset S,\ P' \cap S = \emptyset\}\ ,$$

for each subset P of C, where P' is the complement of P with respect to C. Evaluating $\det A(S)$ for each $S \in \mathcal{A}_P$, and $\det B(S')$ for each $S \in \mathcal{C}_P$, substituting in (1.3) the expressions obtained and finally rearranging the terms we get (3.1). ☐

Without loss of generality, set $i = 1$. Notice that A_i and B_i are permutation similar to tridiagonal matrices. Suppose that B_1 is nonsingular and the subdiagonal entries of B are nonzero. From (3.1) we have

$$\begin{aligned} \eta(\lambda A, -B) &= (\lambda a_{11} - b_{11})\eta(\lambda A_1, -B_1) \\ &\quad + (a_{12}, a_{1n})P(\lambda)(a_{12}, a_{1n})^t + (b_{12}, b_{1n})Q(\lambda)(b_{12}, b_{1n})^t\ , \end{aligned} \tag{3.2}$$

where

$$P(\lambda) = \begin{pmatrix} -\eta(\lambda A_{12}, -B_{12}) & (-)^{n-1}\lambda^{n-2}a_{23}\cdots a_{n-1,n} \\ (-)^{n-1}\lambda^{n-2}a_{23}\cdots a_{n-1,n} & -\eta(\lambda A_{1n}, -B_{1n}) \end{pmatrix} \tag{3.3}$$

and

$$Q(\lambda) = \begin{pmatrix} -\eta(\lambda A_{12}, -B_{12}) & -b_{23}\cdots b_{n-1,n} \\ -b_{23}\cdots b_{n-1,n} & -\eta(\lambda A_{1n}, -B_{1n}) \end{pmatrix}\ . \tag{3.4}$$

Suppose that the conjectures are true for such matrices of order less than $n-1$ in the conditions above, and proceed by induction on n. By hypothesis, $\eta(\lambda A_1, -B_1)$ has $n-1$ real roots, say $\mu_1 < \mu_2 < \cdots < \mu_{n-1}$, which strictly interlace the $n-2$ real roots of $\eta(\lambda A_{12}, -B_{12})$ and the $n-2$ real roots of $\eta(\lambda A_{1n}, -B_{1n})$. Since $\eta(\lambda A_{12},\ -B_{12}), \eta(\lambda A_{1n}, -B_{1n}) \to \infty$ as $\lambda \to \infty$, the sign of $\eta(\mu_k A_{12}, -B_{12})$ and of $\eta(\mu_k A_{1n}, -B_{1n})$ must be $(-)^{n-k-1}$, for $k = 1, \ldots, n-1$. Setting $\lambda = \mu_k$ in (3.3), we have

$$\det P(\mu_k) = \eta(\mu_k A_{12} - B_{12})\eta(\mu_k A_{1n} - B_{1n}) - \mu_k^{2n-4}a_{23}^2 \cdots a_{n-1,n}^2\ .$$

According to (2.2), since $\eta(\mu_k A_1, -B_1) = 0$, we have

$$\text{sign}\ \det P(\mu_k) = +\ .$$

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Analogously we can prove that sign $\det Q(\mu_k) = +$. Therefore, $P(\mu_k)$ and $Q(\mu_k)$ are positive definite if $n-k-1$ is odd, and negative definite if $n-k-1$ is even. Hence

$$\begin{aligned} \operatorname{sign} \eta(\mu_k A, -B) &= (-)^{n-k} + (-)^{n-k} \\ &= (-)^{n-k}, \qquad k = 1, \dots, n-1 . \end{aligned}$$

Since $\eta(\lambda A, -B) \to (\pm)^n \infty$ as $\lambda \to \pm\infty$, it follows that $\eta(\lambda A, -B)$ has a root in each of the intervals

$$(-\infty, \mu_1),\ (\mu_2, \mu_3), \dots,\ (\mu_{n-2}, \mu_{n-1}),\ (\mu_{n-1}, \infty) ,$$

and therefore $\eta(\lambda A, -B)$ has n distinct real roots, which strictly interlace $\mu_1, \mu_2, \dots, \mu_{n-1}$. Relaxing now by a continuity argument the nondegeneracy of the nonsingularity B_1, we have:

THEOREM 3.3. *Let A and B be Hermitian matrices whose graph is a given cycle, with $A > 0$ and $b_{ij} \neq 0$ for $|i-j| = 1$. Then $\eta(\lambda A, -B)$ has n distinct real roots, say*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n .$$

Furthermore, if

$$\mu_1 < \mu_2 < \cdots < \mu_{n-1}$$

are the roots of $\eta(\lambda A_i, -B_i)$, $i = 1, \dots, n$, then

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{n-1} < \lambda_n .$$

4. Example.

Let us consider the Hermitian matrices

$$A = \begin{pmatrix} 3 & i & 0 & 1-i \\ -i & 2 & 1 & 0 \\ 0 & 1 & 4 & -2 \\ 1+i & 0 & -2 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 2 & 0 & 1 \\ 2 & 0 & i & 0 \\ 0 & -i & 0 & -2 \\ 1 & 0 & -2 & -1 \end{pmatrix} .$$

The matrix A is positive definite and

$$\eta(\lambda A, -B) = 16 - 32\lambda - 97\lambda^2 + 44\lambda^3 + 47\lambda^4 ,$$

with roots

$$\begin{aligned} \lambda_1 &= -1.8109 \\ \lambda_2 &= -0.5646 \\ \lambda_3 &= 0.2895 \\ \lambda_4 &= 1.1498 \end{aligned} .$$

On the other hand

$$\eta(\lambda A_2, -B_2) = -4 - 12\lambda + 28\lambda^2 + 40\lambda^3 ,$$

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with roots

$$\begin{array}{rcr} \mu_1 & = & -0.9090 \\ \mu_2 & = & -0.2432 \\ \mu_3 & = & 0.4522 \end{array} .$$

Hence

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \mu_3 \leq \lambda_4 .$$

Finally, note that $\eta(\lambda A, -B)$ has as many positive and negative roots as the inertia of B $(2, 2, 0)$; see [4, (2)].

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