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AN INVARIANT OF 2×2 MATRICES*

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Abstract. Let W be the space of 2×2 matrices over a field K. Let f be any linear function on W that kills scalar matrices. Let $A \in W$ and define $f_k(A) = f(A^k)$. Then the quantity $f_{k+1}(A)/f(A)$ is invariant under conjugation and moreover $f_{k+1}(A)/f(A) = \operatorname{trace} S^kA$, where S^kA is the k-th symmetric power of A, that is, the matrix giving the action of A on homogeneous polynomials of degree k.

Key words. Matrix invariants, Power of a matrix, Trace, Symmetric Power of a Matrix.

AMS subject classifications. 15A72, 15A69.

1. Introduction.

Given a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $b \neq 0$, denote its k-th power by $A^k = \left(\begin{smallmatrix} a_k & b_k \\ c_k & d_k \end{smallmatrix}\right)$. The present paper proves that the quantity b_{k+1}/b is invariant under conjugation showing that it is equal to the invariant $\operatorname{trace} S^kA$, where S^kA is the k-th symmetric power of A, that is, the matrix giving the action of A on homogeneous polynomials of degree k. This observation, although elementary, seems not to be in the literature or to be known.

The author originally proved this result by direct combinatorial computation of both quantities in terms of the coefficients $a, \dots, d$. This proof does not give any *a priori* reason why b_{k+1}/b is invariant.

Several people, after showing them the result, have given different proofs, in particular, Robert Guralnick and Alastair King. Robert Guralnick also pointed out to me that the result was also true for any linear function that kills the scalar matrices, as stated in Theorem 2.1 and in the abstract.

The natural question is to ask if this result can be generalized to $n \times n$ matrices, that is, given an $n\times n$ matrix A, can $\operatorname{trace} S^kA$ be written in terms of some coefficient of A^{k+1} and the corresponding coefficient in A? For an $n \times n$ matrix A, are there other invariant quantities given by coefficients of A^{k+1}?

In Section 2 we give some notation and state the main theorem (Theorem 2.1). We also state the result for the particular case when the function is the coordinate function on the 1,2 entry (Proposition 2.2) and we show that it has as a corollary the general case. In Section 3 we present several proofs of Proposition 2.2, the original one and the other proofs communicated to me, to show the different approaches. Finally in the last section we give an application.

2. The main result.

Let W be the vector space of 2 x 2 matrices over a field K. Let $A \in W$, the matrix A acts naturally by matrix multiplication on K^2. The k-symmetric power S^kK^2 is isomorphic to the space of homogeneous polynomials of degree k in two variables x and y. The k-th symmetric power S^kA of A is the matrix

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of the linear action of A on S^kK^2 given by

$$(A \cdot P)(z) = P(zA), \tag{2.1}$$

where

$$P \in S^kK^2, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z = (x, y) \quad \text{and} \quad zA = (ax + cy, bx + dy).$$

The monomials

$$\mathrm{P}_j(x, y) = x^{k-j}y^j, \quad 0 \le j \le k,$$

give a basis for the space S^kK^2.

Let $K[W]$ be the ring of polynomial functions on W. Denote by S the subset of $K[W]$ consisting of linear functions that kill scalar matrices, that is, $f \in S$ if and only if it is linear and $f(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)) = 0$ for any $\lambda \in K$. The group $G = P\mathrm{GL}(2, K)$ acts on W by conjugation and this action induces an action on $K[W]$ given by

$$(g \cdot f)(A) = f(g^{-1}Ag), \qquad A \in W \text{ and } g \in G.$$

The main result of the paper is the following

THEOREM 2.1. *Let $A \in W$ and $f \in S$. Put $f_k(A) = f(A^k)$. If $f(A) \neq 0$ then*
1. *$f_k(A)/f(A)$ is G-invariant. In other words, $f_k(A)/f(A)$ belongs to the invariant ring $K[W]^G$.*
2. *$f_{k+1}(A)/f(A) = \operatorname{trace} S^kA$.*

Theorem 2.1 is an immediate consequence of the following proposition for the case when f is the coordinate function on the 1,2 entry or on the 2,1 entry.

PROPOSITION 2.2. *Let b be the coordinate function on W on the 1,2 entry and $b_k(A) = b(A^k)$. Let c and c_k be the corresponding functions for the 2,1 entry. Then*
1. *$b_k/b = c_k/c$ is G-invariant.*
2. *$b_{k+1}(A)/b(A) = \operatorname{trace} S^kA$.*

Proof of Theorem 2.1. Once the proposition is proved for the coordinate functions b and c, the theorem is clearly true for any G-translate (and this is a linear condition), whence true for the span of that orbit that is precisely S. □

3. Four proofs. In this section we give four proofs of Proposition 2.2. The first one is the most efficient, is due to Jeremy Rickard and it was communicated to me by Alastair King. The second one is the original combinatorial proof. These two proofs show that b_{k+1}/b is equal to $\operatorname{trace} S^kA$ and therefore invariant, but they do not give any *a priori* reason why b_{k+1}/b is invariant.

The third one is an algebraic proof by Robert Guralnick, which is valid in all characteristics; the fourth one is by Alastair King, which is of a more geometrical nature. In both of the latter two proofs, it is first shown that b_{k+1}/b is invariant and afterward that $b_{k+1}/b = \operatorname{trace} S^kA$.

In the second part of the fourth proof it is not necessary to know that b_{k+1}/b is invariant, so it is in itself a proof of Proposition 2.2. It is due to M.S. Narasimhan and was communicated to me by Alastair King.

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First proof of Proposition 2.2. Observe that by continuity is enough to prove the result for diagonalizable matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{xw - zy} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} w & -y \\ -z & x \end{pmatrix}.$$

Then, since A^{k+1} has eigenvalues p^{k+1} and q^{k+1} one easily computes that

$$b_{k+1}/b = \frac{q^{k+1} - p^{k+1}}{q - p},$$

which is well known to be trace S^kA and therefore invariant.

Second proof of Proposition 2.2. In order to prove Proposition 2.2 we need the following lemmas. The first lemma, given the matrix A, expresses trace S^kA in terms of the entries of A.

LEMMA 3.1. *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. *Then*

$$\operatorname{trace} S^kA = \sum_{j=0}^{k} \sum_{i=0}^{\min\{k-j,j\}} \binom{k-j}{i} \binom{j}{j-i} a^{k-j-i} b^i c^i d^{j-i}.$$

Proof. Consider the basis of S^kK^2 given by the monomials P_j, $0 \le j \le k$. Use the action of A on P_j defined in (2.1) to compute the matrix of the automorphism of S^kK^2 given by the action of A and then take the trace. □

The second lemma expresses the n-th power of A in terms of its entries.

LEMMA 3.2. *Consider the matrix* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. *Denote by* a_n, b_n, c_n *and* d_n *the corresponding entries of the matrix* A^n, *i.e.* $A^n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$. *Then*

$$a_n = a^n + \sum_{s=1}^{[\frac{n}{2}]} \sum_{m=0}^{n-2s} \binom{n-s-m}{s} \binom{m+s-1}{m} a^{n-2s-m} b^s c^s d^m,$$

$$b_n = \sum_{s=0}^{[\frac{n-1}{2}]} \sum_{m=0}^{n-2s-1} \binom{n-s-m-1}{s} \binom{m+s}{m} a^{n-2s-m-1} b^{s+1} c^s d^m,$$

$$c_n = \sum_{s=0}^{[\frac{n-1}{2}]} \sum_{m=0}^{n-2s-1} \binom{n-s-m-1}{s} \binom{m+s}{m} a^{n-2s-m-1} b^s c^{s+1} d^m,$$

$$d_n = d^n + \sum_{s=1}^{[\frac{n}{2}]} \sum_{m=0}^{n-2s} \binom{n-s-m-1}{s-1} \binom{m+s}{m} a^{n-2s-m} b^s c^s d^m,$$

where $[x]$ *denotes the integral part of* x.

Proof. Since $A^n = A^{n-1}A = \left(\begin{smallmatrix} a_{n-1} & b_{n-1} \\ c_{n-1} & d_{n-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ one gets the recursive equations

$$a_n = aa_{n-1} + cb_{n-1}, \qquad c_n = ac_{n-1} + cd_{n-1},$$
$$b_n = ba_{n-1} + db_{n-1}, \qquad d_n = bc_{n-1} + dd_{n-1}.$$

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Using this equations, one can find which kind of terms appear in the entries of A^n. Next, using elementary combinatorics one can count how many times each term appears and this is given by the binomial coefficients in the formulae. □

Proof of Proposition 2.2. Combining the formula in Lemma 3.1 and some of the formulae in Lemma 3.2 we show that $b_{k+1}/b = c_{k+1}/c = \operatorname{trace} S^k A$. Just put $n = k+1$, $s = i$ and $m = j - i$ in the expression of b_n (or c_n) in Lemma 3.2 and compare with the formula in Lemma 3.1. To see that in both cases i and j take the same values, from the expression of b_n (or c_n) in Proposition 3.2 and taking $n = k + 1$, $s = i$ and $m = j - i$ we have that

$$0 \le i \le [\frac{k}{2}], \tag{3.1}$$

$$0 \le j - i \le k - 2i, \tag{3.2}$$

From (3.1) we have

$$0 \le i. \tag{3.3}$$

From (3.2) we have

$$i \le j, \tag{3.4}$$

$$i \le k - j. \tag{3.5}$$

From (3.3), (3.4) and (3.5) we have that

$$0 \le i \le \min\{j, k - j\}.$$

From (3.3) and (3.5) we have that

$$j \le i + j \le k. \tag{3.6}$$

Finally, by (3.3), (3.4) and (3.6)

$$0 \le j \le k. \qquad \square$$

Third proof of Proposition 2.2. Let W, b, b_k, c, c_k and G as in Section 2.

1. Let B the Borel subgroup of G of lower triangular matrices with U the unipotent radical, i.e., matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$. It is easy to see by direct computation that $b_k(uAu^{-1}) = b_k(A)$ for every $u \in U$ and therefore b_k and b are each U-invariant.
   Let T be the diagonal torus, i.e. matrices of the form $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$. Then both b and b_k are eigenfunctions with eigenvalue $r^{-1}s$ (for the diagonal matrix $diag(r, s)$) and so b_k/b is T-invariant. Since b_k/b is both U-invariant and T-invariant it is also B-invariant.
   Similarly, c_k/c is invariant under the opposite Borel (upper triangular matrices). So it suffices to show that $b_k/b = c_k/c$ (for then these are invariant under both Borels which generate G).

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We use induction on k. If $k = 1$, this is clear.
Write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$A^j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

By induction, $b_{k-1}/b = c_{k-1}/c$ or $b_{k-1}c = bc_{k-1}$. So $A^k = A^{k-1}A$, hence $b_k = a_{k-1}b + b_{k-1}d$.
Also, $A^k = AA^{k-1}$, therefore $c_k = a_{k-1}c + c_{k-1}d$ so $b_k/b = a_{k-1} + d(b_{k-1}/b)$ and $c_k/c = a_{k-1} + d(c_{k-1}/c)$, whence the result by induction.

2. $b_{k+1}(A)/b(A) = \text{trace}(S^k(A))$ when $b(A)$ is nonzero.
Consider A with $b(A)$ nonzero. Since b_k/b is invariant, we can conjugate A and assume that it is upper triangular with $b(A) = 1$ and diagonal entries r, s say. It is easy to prove by induction that

$$\begin{pmatrix} r & 1 \\ 0 & s \end{pmatrix}^{k+1} = \begin{pmatrix} r^{k+1} & \sum_{i=0}^{k} r^i s^{k-i} \\ 0 & s^{k+1} \end{pmatrix}.$$

Hence we have that $b_{k+1}(A) = tr(S^k(A)) = \sum_{i=0}^{k} r^i s^{k-i}$. □

Fourth proof of Proposition 2.2.

1. Let V be a 2-dimensional vector space and identify W with $End(V)$. Let $A \in End(V)$. Then the natural interpretation of the off-diagonal entry b is as follows:
Let S be a subspace of V (spanned by the second basis element) $j\colon S \to V$ the inclusion and $q\colon V \to V/S$ the quotient. Then $b = qAj\colon S \to V/S$. Likewise $b_{k+1} = qA^{k+1}j\colon S \to V/S$, so the ratio is at least a well-defined scalar, but *a priori* depending on S.
Now globalize over the projective line $P(V)$ that parametrizes all such S.
Then j and q become $J\colon O(-1) \to V \otimes O$ and $Q\colon V \otimes O \to O(1)$, where $O(-1)$ is the tautological line bundle, O the trivial bundle and $O(1)$ the hyperplane bundle (dual to $O(-1)$).
Considering $A \in End(V \otimes O)$, we have sections $B = QAJ$ and $B_{k+1} = QA^{k+1}J$ of $O(2) \cong Hom(O(-1), O(1))$, which give b and b_{k+1} for each S. The zeros of these sections occur at the eigenspaces of A and A^{k+1}, but these are the same, thus B_{k+1} is a constant multiple of B. In other words, b_{k+1} is a constant multiple of b, independent of S. This shows that $b_k(A)/b(A)$ is invariant.
2. $b_{k+1}(A)/b(A) = \text{trace}(S^k A)$ when $b(A)$ is nonzero.
From Cayley-Hamilton one has that $A^{k+2} - (\text{trace}\, A)A^{k+1} + (\det A)A^k = 0$ hence

$$b_{k+2}(A)/b(A) - (\text{trace}\, A)b_{k+1}(A)/b(A) + (\det A)b_k(A)/b(A) = 0.$$

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On the other hand, the symmetric powers of V satisfy the 'fusion rules'

$$V \otimes S^k V = S^{k+1} V \oplus \Lambda^2 V \otimes S^{k-1} V,$$

taking traces and remembering that $\det A = \mathrm{trace}(\Lambda^2 A)$, show that the quantity $\mathrm{trace}(S^k A)$ satisfies precisely the same recurrence as $b_{k+1}(A)/b(A)$; since they both start with $b_2(A)/b(A) = \mathrm{trace}\, A$ and $b_1(A)/b(A) = 1$, they must be equal. □

4. Application. The main result of the present paper was found computing the characters of some representations of finite subgroups of SU(2).

For each $k = 0, 1, \ldots$, there is a complex irreducible representation E_k of SU(2) of dimension $k + 1$. We can describe this representations as follows. Firstly, $E_0 = \mathbb{C}$ is the trivial representation and E_1 is the standard representation on $\mathbb{C}^2$ given by matrix multiplication. For $k \geq 2$, the representation space of the representation E_k is the k-th symmetric power $S^k E_1$. Let Γ be a finite subgroup of SU(2). Consider the restriction of E_k to Γ which we shall denote by $E_k|_\Gamma$ and let be $\chi_{E_k} : \Gamma \to \mathbb{C}$ its character given by

$$\chi_k(g) = \mathrm{trace}(S^k g)$$

for every $g \in \Gamma \subset \mathrm{SU}(2)$.

Let c_Γ be the least common multiple of the different orders of the elements of Γ. The number c_Γ is called the **exponent** of the group Γ. Theorem 2.1 can be used to prove the following result.

PROPOSITION 4.1. *Let Γ be a finite subgroup of* SU(2). *Let $g \in \Gamma$ with $g \neq \pm I$, where I is the identity. If $k \equiv l \mod c_\Gamma$, then*

$$\chi_{E_k}(g) = \chi_{E_l}(g).$$

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SU}(2)$ of finite order $|g|$ with $g \neq \pm I$. Without loss of generality suppose that g is not diagonal (if it is, conjugate by a non-diagonal matrix to get a non-diagonal matrix and since characters are constant on conjugacy classes we get the same result). If $k \equiv l \mod c_\Gamma$ we have that $k \equiv l \mod |g|$ and then $b_{k+1} = b_{l+1}$. Thus by Theorem 2.1

$$\chi_k(g) = b_{k+1}/b = b_{l+1}/b = \chi_l(g). \qquad \square$$

REMARK 4.2. Note that Proposition 4.1 is also a consequence of the following well-known formula $\chi E_k(g) = \sum_{l=0}^{k} e^{it(k-2l)}$ for the characters χ_{E_k} [1, p. 86] where $e^{\pm it}$ are the eigenvalues of g.

Some of the applications of Proposition 4.1 are the following. In first place, it was used in [2] to find an explicit formula for the inner product $\langle \chi_{E_k}, \chi_\alpha \rangle$ of χ_{E_k} with the character of any finite dimensional representation α of Γ. Such formula [2, Prop. 4.1] was used to compute the multiplicities of the eigenvalues of the Dirac operator of S^3/Γ twisted by α [2, Thm. 3.2]. On the other hand, the aforementioned formula

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can also be used for the question mentioned by Kostant [3, 4], in relation with the McKay correspondence, of decomposing $E_K|_\Gamma$ into Γ-irreducibles. More specifically, if $\{\alpha_1, \dots, \alpha_s\}$ is the set of equivalence classes of irreducible representations of Γ, then $\mu_{tk} = \langle \chi_{E_k}, \chi_{\alpha_t} \rangle$ is the multiplicity of α_t in $E_k|_\Gamma$.

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