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Semilinear heat equation with singular terms

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Abstract. The main goal of this paper is to analyze the existence and nonexistence as well as the regularity of positive solutions for the following initial parabolic problem

$$\begin{cases} \partial_t u - \Delta u = \mu \dfrac{u}{|x|^2} + \dfrac{f}{u^\sigma} & \text{in } \Omega_T := \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded open, $\sigma \geq 0$ and $\mu > 0$ are real constants and $f \in L^m(\Omega_T)$, $m \geq 1$, and u_0 are nonnegative functions. The study we lead shows that the existence of solutions depends on σ and the summability of the datum f as well as on the interplay between μ and the best constant in the Hardy inequality. Regularity results of solutions, when they exist, are also provided. Furthermore, we prove uniqueness of finite energy solutions.

Keywords: heat equation, existence and regularity, Hardy potential, singular terms.

2020 Mathematics Subject Classification: 35K20, 35K91, 35K67, 35B65.

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Set $\Omega_T := \Omega \times (0,T)$ where $T > 0$ is a real constant. In this paper we investigate the existence and regularity as well as the uniqueness of solutions to the following initial parabolic problem

$$\begin{cases} \partial_t u - \Delta u = \mu \dfrac{u}{|x|^2} + \dfrac{f}{u^\sigma} & \text{in } \Omega_T, \\ u = 0 & \text{on } \Gamma := \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\sigma \geq 0$ and $\mu \geq 0$. The source terms f and u_0 satisfy

$$f \geq 0, \quad f \in L^m(\Omega_T), \quad m \geq 1 \tag{1.2}$$

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and $u_0 \in L^\infty(\Omega)$ such that

$$\forall w \subset\subset \Omega \quad \exists d_w > 0 : u_0 \geq d_w \quad \text{in } w. \tag{1.3}$$

It is clear that problem (1.1) is strongly related to the following classical Hardy inequality which asserts that

$$\Lambda_{N,2} \int_\Omega \frac{|u|^2}{|x|^2} dx \leq \int_\Omega |\nabla u|^2 dx, \tag{1.4}$$

for all $u \in C_0^\infty(\Omega)$, where $\Lambda_{N,2} = (\frac{N-2}{2})^2$ is optimal and not achieved (see for instance [20, 50] and [11] when $\Omega = \mathbb{R}^N$). The presence of a term with negative exponent generally induces a difficulty in defining the notion of solution for the problem (1.1).

In the literature, singular problems like (1.1) are considered and intensively studied in various situations depending on σ or μ. If $\sigma = 0$ and $\mu > 0$, the problem (1.1) is reduced to the following heat equation involving the Hardy potential

$$\begin{cases} \partial_t u - \Delta u = \mu \dfrac{u}{|x|^2} + f & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

and is studied first by Baras and Goldstein in their pioneering work [15]. When the data $0 \leq f \in L^1(\Omega_T)$ and u_0 is a positive L^1-function or a positive Radon measure on Ω are not both identically zero (otherwise the result is false since $u \equiv 0$ is a solution), Baras and Goldstein [15] have proved that if $0 \leq \mu \leq \Lambda_{N,2}$ then there exists a positive global solution for the problem (1.5), while if $\mu > \Lambda_{N,2}$ there is no solution.

Problem (1.5) with $-\text{div} a(x, t, \nabla u)$ instead of $-\Delta$ was studied in [45], where the author proved that all the solutions have the same asymptotic behaviour, that is they all tend to the solution of the original problem which satisfies a zero initial condition. In [46] the authors studied the influence of the presence of the Hardy potential and the summability of the datum f on the regularity of the solutions of problem (1.5) with the nonlinear operator $-\text{div} a(x, t, u, \nabla u)$ in the principal part.

The singular Hardy potential appears in the context of combustion theory (see [50] and references therein) and quantum mechanics (see [15] and [50] and references therein). There is a wide literature about problems involving the Hardy potential where the existence and regularity of solutions as well as nonexistence of solutions are analyzed, for instance, we refer to [2–7, 10, 17, 18, 32, 36–38, 54].

Problems involving singularities (like (1.1) with $\mu = 0$) describe naturally several physical phenomena. Stationary cases include the semilinear equation $-\Delta u = f(x)u^{-\sigma}$, $x \in \Omega \subset \mathbb{R}^N$, that can be obtained as a generalization to the higher dimension from a one dimensional ODE ($N = 1$) by some transformations of boundary layer equations for the class of non-Newtonian fluids called pseudoplastic (see [29, 39]). As far as we know, semilinear equations with singularities arise in various contexts of chemical heterogeneous catalysts [9], non-Newtonian fluids as well as heat conduction in electrically conducting materials (the term u^σ describes the resistivity of the material), see for instance, [31, 39]. In view of this physical interpretation various generalizations of this later equation considered in the framework of partial differential equations ($N \geq 2$) has been the subject of study in many papers. For the mathematical analysis account, the seminal papers [23, 49] constitute the starting point of a wide literature about singular semilinear elliptic equations. Far from being complete we quote the list [8, 17, 19, 21, 26, 27, 33, 34, 40, 42, 43, 52, 53, 56].

It is worth recalling that due to the meaning of the unknowns (concentrations, populations,...), only the positive solutions are relevant in most cases.

As far as the parabolic setting is concerned for problems as in (1.1) with $\mu = 0$, the literature is not rich enough. For problems like (1.1) with p-Laplacian operator, existence results of nonnegative solutions are obtained in [25] for data with higher summability while in [41] the authors proved the existence of nonnegative distributional solutions for non regular data (L^1 and measure) and the uniqueness is proved for energy solutions. Other related problems with singular terms can be found in [12–14].

In the case where $\sigma \neq 0$ and $\mu = 0$, problem (1.1) with a quite more general diffusion operator including the Laplacian one was studied in [24]. The authors considered nonnegative data having suitable Lebesgue-type summabilities and assumed the strict positivity on the initial data inside the parabolic cylinder. They have shown, via Harnack's inequality, that this strict positivity is inherited by the constructed solution to the problem, thus giving a meaning to the notion of solution considered. Some regularity results are obtained according to the regularity of f and the values of σ.

Our main goal in this paper is to study the problem (1.1) in the presence of the two singular terms, that is $\mu > 0$ and $\sigma \geq 0$ extending to the evolution case some results obtained for the elliptic problem (with the Δ_p operator instead of Laplacian one) studied in [1]. Abdellaoui and Attar [1] investigated the interplay between the summability of f and σ providing the largest class of the datum f for which the problem admits a solution in the sense of distributions. Uniqueness and regularity results on the distributional solutions are also established. In the same spirit, the parabolic case with $\mu = 0$ was investigated in [24]. Our work fits in the context of recent work on equations involving the Hardy potential in the case of nonexistence of solutions. We start by studying first the case $\mu < \Lambda_{N,2} := \frac{(N-2)^2}{4}$ distinguishing two cases where $\sigma \geq 1$ and $f \in L^1(\Omega_T)$ and the case where $\sigma < 1$ with $f \in L^{m_1}(\Omega_T)$, $m_1 = \frac{2N}{2N+(\sigma-1)(N-2)}$. Then we investigate the case $\mu = \Lambda_{N,2}$ and $\sigma = 1$ with data $f \in L^1(\Omega_T)$. In both cases we prove the existence of a weak solution obtained as limit of approximations that belongs to a suitable Sobolev space. The approach we use consists in approximating the singular equation with a regular problem, where the standard techniques (e.g., fixed point argument) can be applied and then passing to the limit to obtain the weak solution of the original problem. The regularity of weak solutions is analyzed according to the Lebesgue summability of f and σ. Furthermore, we prove the uniqueness of finite energy solutions when the source term f has a compact support by extending the formulation of weak solutions to a large class of test functions. Finally, in the case where $\mu > \Lambda_{N,2}$ we prove a nonexistence result.

The paper is presented as follows. Section 2 contains all the main results (existence, regularity and uniqueness) and also graphic presentations allowing to better locate the obtained results. In Section 3 we first prove an existence result for approximate regular problems of the problem (1.1) and then we give the proof of all the main results Theorem 2.2, Theorem 2.4, Theorem 2.5, Theorem 2.6, Theorem 2.8 and Theorem 2.10. At the end, some results that are necessary for the accomplishment of the work are given in an appendix to make the paper quite self contained.

2 Main results

We begin by stating the definition of weak solution and finite energy solution of the problem (1.1) and then we state and comment the main results.

Definition 2.1.

1) By a weak solution of the problem (1.1) we mean a function $u \in L^1(0,T;W^{1,1}_{loc}(\Omega))$ satisfying

$$\forall \Omega' \subset\subset \Omega \quad \exists C_{\Omega'} > 0 : u \geq C_{\Omega'} \quad \text{in } \Omega'$$

and

$$-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u\partial_t\phi dxdt + \int_{\Omega_T} \nabla u \cdot \nabla\phi dxdt = \int_{\Omega_T} \left(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\right)\phi dxdt, \quad (2.1)$$

for every $\phi \in C_0^\infty(\Omega \times [0,T))$.

2) We call a finite energy solution of the problem (1.1) a weak solution u that satisfies $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;H^{-1}(\Omega)) + L^1(0,T;L^1_{loc}(\Omega))$.

In Definition 2.1 above all the integrals make sense. Generated by the singular terms, the only difficulty is raised in the right-hand side. Indeed, by Hardy's inequality the integral $\int_{\Omega_T} \frac{u\phi}{|x|^2}dxdt$ is finite while we make use of a comparison result with a solution of a problem in [24, Proposition 2.2], where the hypothesis (1.3) is used, for the integral $\int_{\Omega_T} |\frac{f\phi}{u^\sigma}|dxdt$ to be finite. Thus one has $\frac{f}{u^\sigma} \in L^1(0,T;L^1_{loc}(\Omega))$.

Throughout this paper, we will use the two real auxiliary truncation functions T_k and G_k defined for $k > 0$ respectively as $T_k(s) = \max(-k, \min(s,k))$ and $G_k(s) = (|s| - k)^+\text{sign}(s)$. We also define

$$m_1 := \frac{2N}{2N - (1-\sigma)(N-2)}.$$

Observe that $m_1 \geq 1$ if and only if $\sigma \leq 1$. We will prove the existence of solution for the problem (1.1) under the assumption that the datum f satisfies

$$\begin{cases} f \in L^{m_1}(\Omega_T) & \text{if } 0 \leq \sigma \leq 1, \\ f \in L^1(\Omega_T) & \text{if } \sigma \geq 1. \end{cases} \quad (2.2)$$

2.1 The case $\mu < \Lambda_{N,2}$: existence of weak solutions

The first existence result is the following.

Theorem 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Assume that u_0 and f are nonnegative functions satisfying (1.3) and (2.2) respectively. If $\mu < \Lambda_{N,2}$ then the problem (1.1) has a positive weak solution u such that*

1. *if $0 \leq \sigma \leq 1$ then u is a finite energy solution,*
2. *if $\sigma > 1$ then $u \in L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with $G_k(u) \in L^2(0,T;H_0^1(\Omega))$. Moreover, if $\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$ then we have $u^{\frac{\sigma+1}{2}} \in L^2(0,T;H_0^1(\Omega))$,*
3. *if $\sigma > 1$ and $\text{supp}(f) \subset\subset \Omega$ then u is a finite energy solution.*

Remark 2.3. Let us notice that in absence of the Hardy potential (i.e. $\mu = 0$), the result corresponding to the case $\sigma \leq 1$ is already obtained in [24, Theorem 1.3 (i)], when $p = 2$ and the source term f belongs to $L^{m_2}(\Omega_T)$, $m_2 := \frac{2(N+2)}{2(N+2)-N(1-\sigma)}$. Note that since $m_1 < m_2$, the result we prove here is a refinement of that in [24, Theorem 1.3 (i)]. While in the case $\sigma > 1$

we obtain the same result to that in [24, Theorem 1.3 (ii)]. Note that if $\sigma = 1$ the above results coincide.

Observe that $1 \le m_1 \le \frac{2N}{N+2}$ for any $0 \le \sigma \le 1$. We point out that in the case where $\sigma = 0$, which yields $m_1 = \frac{2N}{N+2}$, we find the result already established in [46, Theorem 1.2] for data $f \in L^r(0,T;L^q(\Omega))$ with $r = q \ge \frac{2N}{N+2}$. It is worth recalling here that $\frac{2N}{N+2}$ is the Hölder conjugate exponent of the Sobolev exponent $\frac{2N}{N-2}$ and by duality argument, data belonging to the Lebesgue space of exponent $\frac{2N}{N+2}$ are in force in the dual space $L^2(0,T;H^{-1}(\Omega))$.

2.2 The case $\mu = \Lambda_{N,2}$: existence of infinite energy solutions

In the following result we deal with the case where $\mu = \Lambda_{N,2}$. The weak solutions found do not generally belong to the energy space.

Theorem 2.4. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \ge 3$, containing the origin. Suppose that (1.3) is fulfilled and assume that $\sigma = 1$ and $f \in L^1(\Omega_T)$. If $\mu = \Lambda_{N,2}$ then the problem (1.1) has a weak solution u such that $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, for every $q < 2$.*

2.3 The case $\mu > \Lambda_{N,2}$: nonexistence of weak solutions

If we assume $\mu > \Lambda_{N,2}$ then the problem (1.1) has no weak solution. This is stated in the following theorem.

Theorem 2.5. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \ge 3$, containing the origin. Assume that (1.3) and (2.2) hold. If $\mu > \Lambda_{N,2}$ then the problem (1.1) has no positive weak solution.*

The following Figure 2.1 summarizes the different existence results according to the interactions between the singularities.

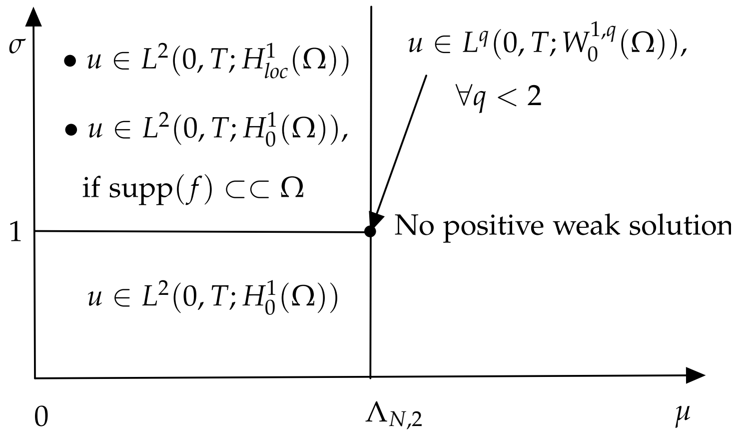

Figure 2.1: Existence and nonexistence results

2.4 Regularity of weak solutions

In the following theorem we give some regularity results for the weak solution u of the problem (1.1) obtained in Theorem 2.2.

Theorem 2.6. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \ge 3$, containing the origin. Assume that (1.2) and (1.3) hold and suppose that $\sigma \ge 0$ and $\mu < \Lambda_{N,2}$. Then* *(i) if $\sigma \geq 1$ and $m \geq 1$ one has*

(a) if $m > \frac{N}{2} + 1$ then $u \in L^\infty(\Omega_T)$,

(b) if $1 \leq m < \frac{N}{2} + 1$, then $u^{\frac{\gamma+1}{2}} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ where $\gamma = \frac{Nm(1+\sigma)-N+2m-2}{N-2m+2}$ provided that $\frac{4\gamma}{(\gamma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$.

(ii) If $0 \leq \sigma \leq 1$ one has

(c) if $m > \frac{N}{2} + 1$ then $u \in L^\infty(\Omega_T)$,

(d) if $m_1 \leq m < \frac{N}{2} + 1$ then $u^{\frac{\gamma+1}{2}} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ where $\gamma = \frac{Nm(1+\sigma)-N+2m-2}{N-2m+2}$ provided that $\frac{4\gamma}{(\gamma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$.

Remark 2.7.

1. Observe that since $0 < \sigma \leq 1$ and $N \geq 3$ one has $1 \leq m_1 := \frac{2N}{2N-(1-\sigma)(N-2)} < \frac{N}{2} + 1$.

2. If $\sigma \geq 1$ and $1 \leq m < \frac{N}{2} + 1$ then $\gamma \geq m\sigma \geq 1$.

3. If $0 \leq \sigma \leq 1$ and $m_1 \leq m < \frac{N}{2} + 1$ then $\gamma \geq m\sigma \geq 0$.

4. Notice that $0 \leq \frac{4\gamma}{(\gamma+1)^2} \leq 1$ and since $\mu < \Lambda_{N,2}$ the assumption $\frac{4\gamma}{(\gamma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$ is necessary in order to get the results stated in Theorem 2.6.

In the case where $0 < \sigma \leq 1$, the regularity results obtained in the previous Theorem 2.6 concerns the weak solutions corresponding to data $f \in L^m(\Omega_T)$, with $m \geq m_1$. When we decrease the summability of the data, that is $f \in L^m(\Omega_T)$ with $1 < m < m_1$, we obtain solutions lying in a bigger space than the energy one. Actually, we have the following result.

Theorem 2.8. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Assume that (1.3) holds and $f \in L^m(\Omega_T)$, with $1 < m < m_1$ and suppose that $0 \leq \sigma \leq 1$ and $\mu < \Lambda_{N,2}$. Then if $\frac{mN(1+\sigma)}{2N-4(m-1)} > \frac{\Lambda_{N,2}}{\mu}\left(1 - \sqrt{1 - \frac{\mu}{\Lambda_{N,2}}}\right)$, the problem (1.1) has a weak solution u such that $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ with $q = \frac{m(N+2)(1+\sigma)}{N+2-m(1-\sigma)}$ and $\gamma = \frac{m(1+\sigma)(N+2)}{N-2m+2}$.*

Remark 2.9. We point out that for the particular case $\sigma = 0$ we obtain that the solution u belongs to $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ with $q = \frac{m(N+2)}{N+2-m}$ and $\gamma = \frac{m(N+2)}{N-2m+2}$. These are exactly the same exponents as those obtained in nonsingular case in [16, Theorem 1.9] when $f \in L^{m_3}(\Omega_T)$, $m_3 := \frac{2(N+2)}{2(N+2)-N}$. Observe that since for $\sigma = 0$ we have $m_1 = \frac{2N}{N+2} < m_3$, the result we prove is a refinement of the one in [16, Theorem 1.9]. This is not surprising since the effect of Hardy's potential vanishes for $\mu < \Lambda_{N,2}$ as it is shown in the proof of Theorem 2.8. Remark that we cannot the consider case where $\sigma = 0$ and $m = 1$, since the test functions we use in order to obtain the regularity stated in Theorem 2.8 cannot be chosen.

The following Figure 2.2 summarizes the previous regularity results considering the interplay between the singularity and the summability of the source term f.

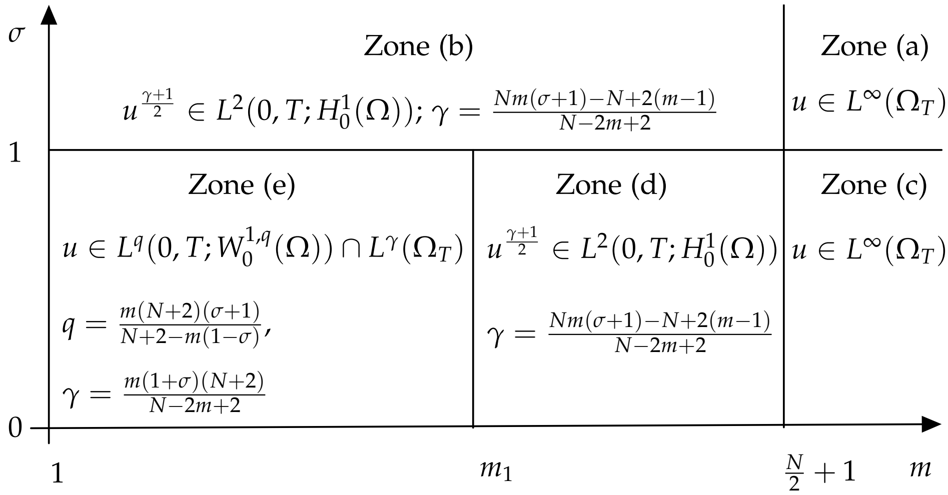

Figure 2.2: Regularity results for $\mu < \Lambda_{N,2}$. Zone (e) corresponds to the result in Theorem 2.8

2.5 Uniqueness of finite energy solutions

As far as the uniqueness is concerned, we give the following result for the finite energy solutions in the case of data with compact support.

Theorem 2.10. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Suppose that* (1.3) *is fulfilled, $\mu < \Lambda_{N,2}$ and $\sigma \geq 0$. If $f \in L^m(\Omega_T)$, with $m \geq 1$ and $\operatorname{supp}(f) \subset\subset \Omega_T$ then the energy solution $u \in L^2(0,T;H_0^1(\Omega))$ of the problem* (1.1) *is unique.*

3 Proofs of the results

3.1 Approximate problems

Let us consider the following sequence of approximate initial-boundary value problems

$$
\begin{cases}
\partial_t u_n - \Delta u_n = \mu \dfrac{T_n(u_n)}{|x|^2 + \frac{1}{n}} + \dfrac{f_n}{(|u_n| + \frac{1}{n})^\sigma} & \text{in } \Omega \times (0,T), \\
u_n(x,t) = 0 & \text{in } \partial\Omega \times (0,T), \\
u_n(x,0) = u_0(x) & \text{in } \Omega,
\end{cases}
\tag{3.1}
$$

where $f_n = T_n(f) = \min(f,n)$. The case $\sigma = 0$ leads to the variational framework since $m_1 = \frac{2N}{N+2}$ is the Hölder conjugate exponent of the Sobolev exponent $2^* := \frac{2N}{N-2}$ and then by the Sobolev embedding and a duality argument we obtain $f \in L^{m_1}(\Omega_T) \hookrightarrow L^2(0,T;H^{-1}(\Omega))$ and the existence of u_n can be found in [30, Theorem 3] on page 356. If $0 < \sigma \leq 1$, the proof of the existence of a solution u_n to the approximate problem (3.1), which is based on the Schauder's fixed point theorem, is now classical. For the convenience of the reader we give it here.

Lemma 3.1. *Assume that $0 < \sigma \leq 1$ and $\mu \leq \Lambda_{N,2}$. For each integer $n \in \mathbb{N}$ the approximate problem* (3.1) *has a solution $u_n \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ such that $\partial_t u_n \in L^2(0,T;H^{-1}(\Omega))$ satisfying* *for every* $\phi \in L^2(0,T;H_0^1(\Omega))$

$$\int_0^T\int_\Omega \partial_t u_n \phi dxdt + \int_0^T\int_\Omega \nabla u_n \nabla\phi dxdt \\ = \mu\int_0^T\int_\Omega \frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}dxdt + \int_0^T\int_\Omega \frac{f_n\phi}{(|u_n|+\frac{1}{n})^\sigma}dxdt \tag{3.2}$$

Moreover, u_n *is such that for every* $\Omega' \subset\subset \Omega$ *there exists* $C_{\Omega'} > 0$ *(not depending on* n*), such that* $u_n \geq C_{\Omega'}$ *in* $\Omega' \times [0,T]$*.*

Proof. Let $v \in L^2(\Omega_T)$ and let $n \in \mathbb{N}$ be fixed. We consider $w \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)) \cap L^\infty(\Omega_T)$ with $\partial_t w \in L^2(0,T;H^{-1}(\Omega))$ the unique weak solution (depending on v and n) of the following problem

$$\begin{cases} \partial_t w - \Delta w = \mu\frac{T_n(w)}{|x|^2+\frac{1}{n}} + \frac{f_n}{\left(|v|+\frac{1}{n}\right)^\sigma} & \text{in } \Omega_T \\ w(x,t) = 0 & \text{in } \partial\Omega \times (0,T) \\ w(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.3}$$

which satisfies for every $\phi \in L^2(0,T;H_0^1(\Omega))$

$$\int_0^T\int_\Omega \partial_t u_n \phi dxdt + \int_0^T\int_\Omega \nabla u_n \cdot \nabla\phi dxdt \\ = \mu\int_0^T\int_\Omega \frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}dxdt + \int_0^T\int_\Omega \frac{f_n\phi}{(|v|+\frac{1}{n})^\sigma}dxdt$$

The existence of w can be found in [30, Theorem 3] on page 356 (see also [35]). Let us consider the map S defined by $S(v) = w$. Taking w as test function in (3.3) we get

$$\|\nabla w\|^2_{L^2(\Omega_T)} \leq \mu\int_{\Omega_T} \frac{T_n(w)w}{|x|^2+\frac{1}{n}}dxdt + \int_{\Omega_T} \frac{f_n w}{\left(|v|+\frac{1}{n}\right)^\sigma}dxdt + \|u_0\|^2_{L^2(\Omega)}.$$

Thus, by the Hölder inequality we arrive at

$$\|\nabla w\|^2_{L^2(\Omega_T)} \leq |\Omega_T|^{\frac{1}{2}}\left(\mu n^2 + n^{\sigma+1}\right)\left(\int_{\Omega_T} w^2 dxdt\right)^{\frac{1}{2}} + \|u_0\|^2_{L^2(\Omega)},$$

so that by the Poincaré inequality one has

$$\|w\|^2_{L^2(\Omega_T)} \leq C_1\|w\|_{L^2(\Omega_T)} + C_2,$$

where $C_1 = C_P^2|\Omega_T|^{\frac{1}{2}}(\mu n^2 + n^{\sigma+1})$, $C_2 = C_P^2\|u_0\|^2_{L^2(\Omega)}$ and C_p is the constant in the Poincaré inequality. Therefore by the Young inequality we obtain

$$\|w\|_{L^2(\Omega_T)} \leq C := \sqrt{C_1^2 + 2C_2}. \tag{3.4}$$

Defining the ball $B := \left\{v \in L^2(\Omega_T) : \|v\|_{L^2(\Omega_T)} \leq C\right\}$ of $L^2(\Omega_T)$ we have proved that the map $S : B \to B$ is well defined. In order to apply Schauder's fixed point theorem over S to guarantee the existence of a solution for (3.1) in the sense of (3.2), we need to check that the map S is continuous and compact on B.

Let us first prove the continuity of S. In order to do this, let $\{v_k\}_k \subset B$ be a sequence such that

$$\lim_{k\to+\infty} \|v_k - v\|_{L^2(\Omega_T)} = 0.$$

Denote by $w_k := S(v_k)$ and $w := S(v)$. Then w_k is the solution of the problem

$$\begin{cases} \partial_t w_k - \Delta w_k = \mu \frac{T_n(w_k)}{|x|^2+\frac{1}{n}} + \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} & \text{in } \Omega_T \\ w_k = 0 & \text{on } \partial\Omega \times (0,T), \\ w_k(\cdot,0) = u_0(\cdot) & \text{in } \Omega. \end{cases} \tag{3.5}$$

We shall prove that

$$\lim_{k\to+\infty} \|w_k - w\|_{L^2((\Omega_T)} = 0.$$

Observe that up to a subsequence, we can assume that $v_k \to v$ a.e. in Ω_T. So that one has $\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}}$ converges to $\frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}}$ a.e. in Ω_T. Furthermore, since

$$\frac{|f_n|}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} \le n^{\sigma+1},$$

by the dominated convergence theorem we have

$$\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} \to \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}} \quad \text{in } L^2(\Omega_T). \tag{3.6}$$

Thus, testing by $w_k - w$ in the difference equations solved by w_k and w and using the fact that $w_k(x,0) = w(x,0) = u_0$ and the Hölder inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega ((w_k(x,T) - w(x,T)))^2 dx + \int_{\Omega_T} |\nabla(w_k - w)|^2 dxdt - \mu \int_{\Omega_T} \frac{(w_k-w)^2}{|x|^2+\frac{1}{n}} dxdt \\ &\quad \le \left(\int_{\Omega_T} \left| \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} - \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}} \right|^2 dxdt \right)^{\frac{1}{2}} \|w_k - w\|_{L^2(\Omega_T)}. \end{aligned}$$

If $\mu < \Lambda_{N,2}$ then by the Poincaré inequality we obtain

$$\left(1 - \frac{\mu}{\Lambda_{N,2}}\right) \|w_k - w\|_{L^2(\Omega_T)} \le C_p^2 \left(\int_{\Omega_T} \left| \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} - \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}} \right|^2 dxdt \right)^{\frac{1}{2}},$$

where C_p is the Poincaré constant. While if $\mu = \Lambda_{N,2}$ then by [50, Theorem 2.1] there exists a constant $C(\Omega) > 0$ such that

$$C(\Omega)\|w_k - w\|_{L^2(\Omega_T)} \le \left(\int_{\Omega_T} \left| \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} - \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}} \right|^2 dxdt \right)^{\frac{1}{2}}.$$

Having in mind (3.6) we conclude that the sequence $\{w_k\}_k$ converges to w in $L^2(\Omega_T)$ and so S is continuous.

We turn now to prove that S is compact on B. Let $\{v_k\}_{k\in\mathbb{N}}$ be a bounded sequence in B. We shall prove that there exists a subsequence of $w_k := S(v_k)$ that converges in norm in $L^2(\Omega_T)$.

Taking $w_k = S(v_k)$ as a test function in (3.5) solved by w_k and using the Hölder inequality we obtain

$$\|w_k\|^2_{L^2(0,T;H^1_0(\Omega))} \le |\Omega_T|^{\frac{1}{2}} \left(\mu n^2 + n^{\sigma+1}\right) \left(\int_{\Omega_T} w_k^2 dxdt\right)^{\frac{1}{2}} + \|u_0\|^2_{L^2(\Omega)}.$$

Thus, from the Poincaré and Young inequalities it follows

$$\|w_k\|_{L^2(0,T;H^1_0(\Omega))} \le C, \tag{3.7}$$

where C is a positive constant not depending on k. Hence, by (3.7) the sequence $\{w_k\}_k$ is uniformly bounded in $L^2(0,T;H^1_0(\Omega))$. Now, testing by an arbitrary $\phi \in L^2(0,T;H^1_0(\Omega))$ in (3.5) we obtain

$$\int_{\Omega_T} \partial_t w_k \phi dxdt + \int_{\Omega_T} \nabla w_k \cdot \nabla \phi dxdt \le (\mu n^2 + n^{\sigma+1}) \int_{\Omega_T} \phi dxdt.$$

Then,

$$\int_{\Omega_T} \partial_t w_k \phi dxdt \le \int_{\Omega_T} |\nabla w_k \cdot \nabla \phi| dxdt + (\mu n^2 + n^{\sigma+1}) \int_{\Omega_T} \phi dxdt.$$

By Hölder's inequality we have

$$\int_{\Omega_T} \partial_t w_k \phi dxdt \le \left(\left(\int_{\Omega_T} |\nabla w_k|^2 dxdt\right)^{\frac{1}{2}} + C(n,\Omega,T)\right) \left(\int_{\Omega_T} |\phi|^2 dxdt\right)^{\frac{1}{2}},$$

so that since the sequence $\{w_k\}_k$ is uniformly bounded in $L^2(0,T;H^1_0(\Omega))$ then so is $\{\partial_t w_k\}_k$ in $L^1(0,T;H^{-1}(\Omega))$. Therefore, by [47, Corollary 4] there exists a subsequence of $\{w_k\}_{k\in\mathbb{N}}$ which converges in norm in $L^2(\Omega_T)$. So $S : B \to B$ is compact. Given these conditions on S, Schauder's fixed point theorem provides the existence of a function $u_n \in B$ such that $u_n = S(u_n)$ that is u_n solves (3.1) in the sense of (3.2). In particular we have $u_n \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(\Omega_T)$. The last assertion follows from Lemma A.5 (in Appendix). □

We also observe that from Lemma A.6 (in Appendix) the sequence $\{u_n\}_n$ is increasing.

3.2 Proof of Theorem 2.2

The main argument is to get a priori estimates on $\{u_n\}_n$ and then to pass to the limit as $n \to +\infty$. We divide the proof in four cases, the case where $\sigma = 1$, the case $\sigma < 1$, the case $\sigma > 1$ and the case $\sigma > 1$ with $\mathrm{supp}(f) \subset\subset \Omega_T$.

Case 1 : $\sigma = 1$.

Taking $u_n \chi(0,\tau)(t)$ as test function in (3.2), with $0 \le \tau \le T$, we get

$$\frac{1}{2}\int_\Omega (u_n(x,\tau))^2 dx + \int_0^\tau \int_\Omega |\nabla u_n|^2 dxdt \le \mu \int_0^\tau \int_\Omega \frac{u_n^2}{|x|^2 + \frac{1}{n}} dxdt \int_{\Omega_T} f dxdt + \|u_0\|^2_{L^2(\Omega)}.$$

Then, by using (1.4) we obtain

$$\frac{1}{2}\int_\Omega (u_n(x,\tau))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right) \int_0^\tau \int_\Omega |\nabla u_n|^2 dxdt \le \|f\|_{L^1(\Omega_T)} + \|u_0\|^2_{L^2(\Omega)}.$$

Passing to the supremum in $\tau \in [0,T]$, we obtain

$$\frac{1}{2}\sup_{0\le\tau\le T}\int_\Omega (u_n(x,\tau))^2dx + \left(1-\frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_T}|\nabla u_n|^2dxdt \le \|f\|_{L^1(\Omega_T)} + \|u_0\|^2_{L^2(\Omega)}.$$

This shows that the sequence $\{u_n\}_n$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega))$. Then, there exist a subsequence of $\{u_n\}_n$ still indexed by n and a function $u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^2(0,T;H^1_0(\Omega))$. Moreover, the boundedness of $\{\partial_t u_n\}_n$ in the dual space $L^2(0,T;H^{-1}(\Omega))$ implies that the sequence $\{u_n\}_n$ is relatively compact in $L^1(\Omega_T)$ (see [47, Corollary 4]) and hence for a subsequence, indexed again by n, we have $u_n \to u$ a.e. in Ω_T.

Let $\phi \in C^\infty_0(\Omega\times[0,T))$. Using ϕ as test function in (3.2) we obtain

$$\begin{aligned} -\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n\partial_t\phi dtdx + \int_{\Omega_T}\nabla u_n\cdot\nabla\phi dxdt \\ = \mu\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}dxdt + \int_{\Omega_T}\frac{f_n\phi}{|u_n|+\frac{1}{n}}dxdt. \end{aligned} \tag{3.8}$$

Notice that since $u_n \rightharpoonup u$ weakly in $L^2(0,T;H^1_0(\Omega))$, we immediately have

$$\lim_{n\to+\infty}\int_{\Omega_T}\nabla u_n\cdot\nabla\phi dxdt = \int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt$$

and

$$\lim_{n\to+\infty}\int_{\Omega_T}u_n\partial_t\phi dtdx = \int_{\Omega_T}u\partial_t\phi dtdx.$$

As regards the first integral in the right-hand side of (3.8), we know that the sequence $\{u_n\}$ is increasing to its limit u so we have

$$\left|\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}\right| \le \frac{|u\phi|}{|x|^2}.$$

Applying Hölder's and Hardy's inequalities we obtain

$$\int_{\Omega_T}\frac{|u\phi|}{|x|^2}dxdt \le \|\phi\|_\infty(\Lambda_{N,2})^{-\frac{1}{2}}\left(\int_{\Omega_T}|\nabla u|^2dxdt\right)^{\frac{1}{2}}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac{1}{2}}.$$

As $N\ge 3$ and Ω bounded, a straightforward calculation yields the existence of a positive constant C_1 such that

$$\int_\Omega\frac{dx}{|x|^2}\le C_1. \tag{3.9}$$

Therefore, the function $\frac{|u\phi|}{|x|^2}$ lies in $L^1(\Omega_T)$ and since $\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}} \to \frac{u\phi}{|x|^2}$ a.e. in Ω_T the Lebesgue dominated convergence theorem gives

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2}dxdt = \int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt.$$

On the other hand, the support $\mathrm{supp}(\phi)$ of the function ϕ is a compact subset of Ω_T and so by Lemma A.5 (in Appendix) there exists a constant $C_{\mathrm{supp}(\phi)}>0$ such that $u_n \ge C_{\mathrm{supp}(\phi)}$ in $\mathrm{supp}(\phi)$. Then,

$$\left|\frac{f_n\phi}{u_n+\frac{1}{n}}\right| \le \frac{\|\phi\|_\infty}{C_{\mathrm{supp}(\phi)}}|f| \in L^1(\Omega_T).$$

So that by the Lebesgue dominated convergence theorem we can get

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{f_n\phi}{u_n+\frac{1}{n}}dxdt=\int_{\Omega_T}\frac{f\phi}{u}dxdt.$$

Now passing to the limit as n tends to ∞ in (3.8) we obtain

$$\begin{aligned}-\int_\Omega u_0\phi(x,0)dx-\int_{\Omega_T}u\partial_t\phi dtdx+\int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt\\=\mu\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt+\int_{\Omega_T}\frac{f\phi}{u}dxdt\end{aligned}$$

for all $\phi\in C_0^\infty(\Omega_T)$, namely u is a finite energy solution to (1.1).

Case 2 : $\sigma<1$.

The function $u_n\chi_{(0,\tau)}\in L^2(0,T;H_0^1(\Omega))$, $\tau\in(0,T)$, is an admissible test function in (3.2). Taking it so and using Hölder's inequality and (1.4) we arrive at

$$\begin{aligned}\frac{1}{2}\int_\Omega|u_n(x,\tau)|^2dx+\left(1-\frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega|\nabla u_n|^2dxdt\\\le\|f\|_{L^{m_1}(\Omega_T)}\left(\int_0^\tau\int_\Omega|u_n|^{(1-\sigma)m_1'}dxdt\right)^{\frac{1}{m_1'}}+\frac{1}{2}\|u_0\|_{L^2(\Omega)},\end{aligned}$$

where $m_1:=\frac{2N}{2N-(1-\sigma)(N-2)}$ and $m_1':=\frac{m_1}{m_1-1}$. Setting $2^*:=\frac{2N}{N-2}$ one has

$$(1-\sigma)m_1'=2^*.$$

Then, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega|u_n(x,\tau)|^2dx+\left(1-\frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega|\nabla u_n|^2dxdt\\\le\|f\|_{L^{m_1}(\Omega_T)}\left(\int_0^\tau\int_\Omega|u_n|^{2^*}dxdt\right)^{\frac{1-\sigma}{2^*}}+\frac{1}{2}\|u_0\|_{L^2(\Omega)}.\end{aligned}$$

By Sobolev's inequality there exists a positive constant C such that

$$\begin{aligned}\frac{1}{2}\int_\Omega|u_n(x,\tau)|^2dx+\left(1-\frac{\mu}{\Lambda_{N,2}}\right)\|u_n\|^2_{L^2(0,\tau;H_0^1(\Omega))}\\\le C\|f\|_{L^{m_1}(\Omega_T)}\|u_n\|^{1-\sigma}_{L^2(0,\tau;H_0^1(\Omega))}+\frac{1}{2}\|u_0\|_{L^2(\Omega)}.\end{aligned}\tag{3.10}$$

For every real numbers a, $b\ge0$ and for every Let $\epsilon>0$ be arbitrary. For every positive real numbers a and b, the Young inequality yields

$$ab\le\epsilon a^p+C_\epsilon b^q,\tag{3.11}$$

where $p>1$, $q=\frac{p}{p-1}$ and $C_\epsilon=\frac{p-1}{p(p\epsilon)^{\frac{1}{p-1}}}$. Since $\frac{2^*}{m_1'}=1-\sigma<2$ we apply (3.11) with $p=\frac{2m_1'}{2^*}$ in the first term on the right hand side of (3.10) obtaining

$$\begin{aligned}\frac{1}{2}\int_\Omega|u_n(x,\tau)|^2dx+\left(1-\frac{\mu}{\Lambda_{N,2}}-\epsilon\right)\|u_n\|^2_{L^2(0,\tau;H_0^1(\Omega))}\\\le C_\epsilon(C\|f\|_{L^{m_1}(\Omega_T)})^{\frac{2m_1'}{2m_1'-2^*}}+\frac{1}{2}\|u_0\|_{L^2(\Omega)}.\end{aligned}$$

Choosing ϵ such that $1 - \frac{\mu}{\Lambda_{N,2}} - \epsilon > 0$ and passing to the supremum in $\tau \in [0, T]$ we obtain

$$\frac{1}{2} \sup_{0 \le \tau \le T} \int_\Omega (u_n(x,\tau))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}} - \epsilon\right) \int_{\Omega_T} |\nabla u_n|^2 dxdt \le C_3,$$

with $C_3 = C_\epsilon (C\|f\|_{L^{m_1}(\Omega_T)})^{\frac{2m_1'}{2m_1' - 2^*}} + \frac{1}{2}\|u_0\|_{L^2(\Omega)}$. Therefore, the sequence $\{u_n\}_n$ is uniformly bounded in $L^2(0,T; H_0^1(\Omega))$ and $L^\infty(0,T; L^2(\Omega))$. Thus there exist a subsequence of $\{u_n\}_n$, still labelled by n, and a function $u \in L^2(0,T; H_0^1(\Omega))$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(0,T; H_0^1(\Omega)).$$

Now we shall prove that u is a weak solution of (1.1). For this, let us insert as a test function in (3.2) an arbitrary function $\phi \in C_0^\infty(\Omega \times [0,T))$.

$$\begin{aligned} -\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n \partial_t \phi dtdx + \int_{\Omega_T} \nabla u \cdot \nabla \phi dxdt \\ = \mu \int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2 + \frac{1}{n}} dxdt + \int_{\Omega_T} \frac{f_n \phi}{(u_n + \frac{1}{n})^\sigma} dxdt. \end{aligned}$$

As in the first case, we can pass to the limit in the above equality to conclude that u is a finite energy solution of (1.1).

Case 3 : $\sigma > 1$.

In order to prove that $\{u_n\}_n$ is uniformly bounded in $L^2(0,T; H_{loc}^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$, we will prove that the sequence $G_k(u_n)$ is uniformly bounded in $L^2(0,T; H_0^1(\Omega))$ $\cap L^\infty(0,T; L^2(\Omega))$ and $T_k(u_n)$ is uniformly bounded in $L^2(0,T; H_{loc}^1(\Omega)) \cap L^\infty(0,T; L^{\sigma+1}(\Omega))$. Let us first prove that $G_k(u_n)$ is uniformly bounded in $L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$. Inserting $G_k(u_n)\chi_{(0,\tau)}$, with $0 \le \tau \le T$, as a test function in (3.2) we obtain

$$\begin{aligned} \int_0^\tau \int_\Omega \partial_t u_n G_k(u_n) dxdt + \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \\ = \mu \int_{\Omega_\tau} \frac{T_n(u_n) G_k(u_n)}{|x|^2 + \frac{1}{n}} dxdt + \int_{\Omega_\tau} \frac{f_n G_k(u_n)}{(u_n + \frac{1}{n})^\sigma} dxdt \\ \le \mu \int_{\Omega_\tau} \frac{u_n G_k(u_n)}{|x|^2} dxdt + \int_{\Omega_T} \frac{f_n G_k(u_n)}{(u_n + \frac{1}{n})^\sigma} dxdt. \end{aligned} \tag{3.12}$$

Observe that the function $G_k(u_n)$ is different from zero only on the set $B_{n,k} := \{(x,t) \in \Omega_\tau : u_n(x,t) > k\}$, and so we have

$$\begin{aligned} \int_0^\tau \int_\Omega \partial_t u_n G_k(u_n) dxdt = \frac{1}{2} \int_{B_{n,k}} \partial_t (u_n - k)^2 dxdt = \frac{1}{2} \int_{\Omega_\tau} \partial_t (G_k(u_n(x,\tau)))^2 dxdt \\ = \frac{1}{2} \int_\Omega (G_k(u_n(x,\tau)))^2 dx - \frac{1}{2} \int_\Omega (G_k(u_n(x,0)))^2 dx. \end{aligned}$$

Since $\int_\Omega (G_k(u_n(x,0)))^2 dx \le \int_\Omega (u_0(x))^2 dx$ and $u_n + \frac{1}{n} \ge k$ on $B_{n,k}$ inequality (3.12) becomes

$$\frac{1}{2} \int_\Omega (G_k(u_n(x,\tau)))^2 dx + \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \le \mu \int_{\Omega_\tau} \frac{u_n G_k(u_n)}{|x|^2} dxdt + C_4,$$

with $C_4 = \|u_0\|^2_{L^2(\Omega)} + \frac{1}{k^{\sigma-1}}\|f\|_{L^1(\Omega_T)}$. Moreover, since $u_n G_k(u_n) = (G_k(u_n))^2 + kG_k(u_n)$ on the set $B_{n,k}$ we get

$$\begin{aligned}\frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx + \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt - \mu \int_{\Omega_\tau} \frac{(G_k(u_n))^2}{|x|^2} dxdt \\ \leq \mu k \int_{\Omega_\tau} \frac{G_k(u_n)}{|x|^2} dxdt + C_4.\end{aligned}$$

Taking into account that $\mu < \Lambda_{N,2}$ by (1.4) we have

$$\begin{aligned}\frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right) \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \\ \leq \mu k \int_{\Omega_\tau} \frac{G_k(u_n(x,t))}{|x|^2} dxdt + C_4.\end{aligned} \tag{3.13}$$

We shall now estimate the term $\mu k \int_{\Omega_\tau} \frac{G_k(u_n(x,t))}{|x|^2} dxdt$. Let us fix α such that $1 < \alpha < 2$ and set $\beta = \frac{\alpha}{\alpha - 1}$. By Young's inequality we can write

$$\mu k \int_{\Omega_\tau} \frac{G_k(u_n)}{|x|^2} dxdt \leq \frac{\mu}{\alpha} \int_{\Omega_\tau} \frac{(G_k(u_n))^\alpha}{|x|^2} dxdt + \frac{\mu}{\beta} \int_{\Omega_\tau} \frac{k^\beta}{|x|^2} dxdt.$$

Having in mind (3.9) we get

$$\mu k \int_{\Omega_\tau} \frac{G_k(u_n)}{|x|^2} dxdt \leq \mu \int_{\Omega_\tau} \frac{(G_k(u_n))^\alpha}{|x|^2} dxdt + C_5,$$

where $C_5 = \frac{C_1 \mu k^\beta}{\beta}$. Then the Hölder inequality yields

$$\begin{aligned}\mu k \int_{\Omega_\tau} \frac{G_k(u_n)}{|x|^2} dxdt &\leq \mu \left(\int_{\Omega_\tau} \frac{(G_k(u_n))^2}{|x|^2} dxdt\right)^{\frac{\alpha}{2}} \left(\int_{\Omega_\tau} \frac{dxdt}{|x|^2}\right)^{\frac{2-\alpha}{2}} + C_5 \\ &\leq C_6 \left(\int_{\Omega_\tau} \frac{(G_k(u_n))^2}{|x|^2} dxdt\right)^{\frac{\alpha}{2}} + C_5,\end{aligned}$$

where $C_6 = \mu C_1^{\frac{2-\alpha}{2}}$ and by (1.4) we obtain

$$\mu k \int_{\Omega_\tau} \frac{G_k(u_n(x,t))}{|x|^2} dxdt \leq C_7 \left(\int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt\right)^{\frac{\alpha}{2}} + C_5,$$

where $C_7 = \frac{C_6}{\Lambda_{N,2}}$. For arbitrary $\epsilon > 0$, applying the Young inequality (3.11) with $a = \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt$, $b = C_7$ and $p = \frac{2}{\alpha}$, we get

$$\mu k \int_{\Omega_\tau} \frac{G_k(u_n(x,t))}{|x|^2} dxdt \leq \epsilon \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt + C_8, \tag{3.14}$$

where $C_8 = C_5 + C_\epsilon C_7^{\frac{2-\alpha}{2}}$. Choosing ϵ such that $1 - \frac{\mu}{\Lambda_{N,2}} - \epsilon > 0$ and gathering (3.13) and (3.14), we deduce that

$$\frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}} - \epsilon\right) \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \leq C_9, \tag{3.15}$$

where $C_9 = C_8 + C_4$. Passing to the supremum in $\tau \in [0,T]$, we conclude that the sequence $\{G_k(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

We now turn to prove that the sequence $\{T_k(u_n)\}_n$ is uniformly bounded in $L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^{\sigma+1}(\Omega))$. Using $T_k^\sigma(u_n)\chi_{(0,\tau)}$, $0 \le \tau \le T$, as a test function in (3.2) we obtain

$$\begin{aligned}\frac{1}{\sigma+1}\int_\Omega (T_k(u_n(x,\tau)))^{\sigma+1}dx + \int_{\Omega_\tau}(T_k(u_n))^{\sigma-1}|\nabla T_k(u_n)|^2 dxdt \\ \le k^{\sigma-1}\mu\int_{\Omega_T}\frac{u_n^2}{|x|^2}dxdt + \int_{\Omega_T} f dxdt + \frac{1}{\sigma+1}\|u_0\|_{L^{\sigma+1}(\Omega)},\end{aligned} \tag{3.16}$$

where we have dropped $\sigma > 1$ in the second integral on the left-hand side and written $T_k^\sigma(u_n) = T_k^{\sigma-1}(u_n)T_k(u_n)$ in the first integral on the right-hand side of the inequality. As $u_n = T_k(u_n) + G_k(u_n)$, the first term on the right-hand side of the above inequality can be estimated as

$$\begin{aligned}\int_{\Omega_T}\frac{u_n^2}{|x|^2}dxdt &= \int_{\Omega_T}\frac{(T_k(u_n))^2}{|x|^2}dxdt + \int_{\Omega_T}\frac{(G_k(u_n))^2}{|x|^2}dxdt + 2\int_{\Omega_T}\frac{T_k(u_n)G_k(u_n)}{|x|^2}dxdt \\ &\le k^2\int_{\Omega_T}\frac{dxdt}{|x|^2} + \int_{\Omega_T}\frac{(G_k(u_n))^2}{|x|^2}dxdt + 2k\int_{\Omega_T}\frac{G_k(u_n)}{|x|^2}dxdt.\end{aligned}$$

So that by (1.4), (3.9), (3.14) and (3.15) there exists a real constant $C_{10} > 0$ such that

$$\int_{\Omega_T}\frac{u_n^2}{|x|^2}dxdt \le C_{10}.$$

Then, it follows that the inequality (3.16) reads as

$$\tfrac{1}{\sigma+1}\int_\Omega (T_k(u_n(x,\tau)))^{\sigma+1}dx + \int_{\Omega_\tau}(T_k(u_n))^{\sigma-1}|\nabla T_k(u_n)|^2 dxdt \le C_{11}, \tag{3.17}$$

with $C_{11} = k^{\sigma-1}\mu C_{10} + \|f\|_{L^1(\Omega_T)} + \frac{1}{\sigma+1}\|u_0\|_{L^{\sigma+1}(\Omega)}$. On the other hand, let $\Omega' \subset\subset \Omega$. By Lemma A.5 (in Appendix) there exists $C_{\Omega'} > 0$ such that

$$T_k(u_n(x,t)) \ge C_0 := \min\{k, C_{\Omega'}\}, \tag{3.18}$$

for all $(x,t) \in \Omega' \times [0,T]$. Thus, by (3.17) and (3.18) we get

$$\frac{1}{\sigma+1}\int_\Omega (T_k(u_n(x,\tau)))^{\sigma+1}dx + C_0^{\sigma-1}\int_0^\tau\int_{\Omega'}|\nabla T_k(u_n)|^2 dxdt \le C_{11}.$$

Passing to the supremum in $\tau \in [0,T]$, we get that the sequence $\{T_k(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Therefore, we conclude that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. As a consequence, there exist a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, relabelled again by n, and a function $u \in L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^2(0,T;H^1_{loc}(\Omega))$.

On the other hand, let us assume that $\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$. Taking $u_n^\sigma\chi_{(0,\tau)}(t)$, $0 \le \tau \le T$, as a test function in (3.2) and using the Hardy inequality (1.4) we arrive at

$$\begin{aligned}\frac{1}{\sigma+1}\int_\Omega (u_n(x,\tau))^{\sigma+1}dx + \left(\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega |\nabla u_n^{\frac{\sigma+1}{2}}|^2 dxdt \\ \le \int_{\Omega_T} f dxdt + \frac{1}{\sigma+1}\int_\Omega (u_0(x))^{\sigma+1}dx.\end{aligned}$$

This shows that $u_n^{\frac{\sigma+1}{2}}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and so by the Poincaré inequality the sequence u_n is uniformly bounded in $L^{\sigma+1}(\Omega_T)$ and hence for a subsequence, labelled again by n, we have $u_n \to u$ a.e. in Ω_T.

Testing by an arbitrary function $\phi \in C_0^\infty(\Omega \times [0,T))$ in (3.2) we obtain

$$\begin{aligned} &-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n\partial_t\phi dtdx + \int_{\Omega_T} \nabla u_n \cdot \nabla\phi dxdt \\ &= \mu\int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}dxdt + \int_{\Omega_T}\frac{f_n\phi}{|u_n|+\frac{1}{n}}dxdt. \end{aligned} \tag{3.19}$$

We shall now pass of the limit in each term of (3.19). Notice that since $u_n \rightharpoonup u$ weakly in $L^2(0,T;H^1_{loc}(\Omega))$ we have

$$\lim_{n\to+\infty}\int_{\Omega_T} \nabla u_n \cdot \nabla\phi dxdt = \int_{\Omega_T} \nabla u \cdot \nabla\phi dxdt$$

and

$$\lim_{n\to+\infty}\int_{\Omega_T} u_n\partial_t\phi dtdx = \int_{\Omega_T} u\partial_t\phi dtdx.$$

For the first integral in the right-hand side of (3.19), we know that the sequence $\{u_n\}$ is increasing to its limit u so we obtain

$$\left|\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}\right| \le \frac{|u\phi|}{|x|^2}.$$

By Hölder's and Hardy's inequalities we get

$$\begin{aligned} \int_{\Omega_T}\frac{|u\phi|}{|x|^2}dxdt &\le \|\phi\|_\infty\int_{\text{supp}(\phi)}\frac{|u|}{|x|^2}dxdt = \|\phi\|_\infty\int_{\text{supp}(\phi)}\frac{|u|}{|x|}\times\frac{1}{|x|}dxdt \\ &\le \|\phi\|_\infty\left(\int_{\text{supp}(\phi)}\frac{|u|^2}{|x|^2}dxdt\right)^{\frac{1}{2}}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac{1}{2}} \\ &\le \|\phi\|_\infty(\Lambda_{N,2})^{-\frac{1}{2}}\left(\int_{\text{supp}(\phi)}|\nabla u|^2dxdt\right)^{\frac{1}{2}}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac{1}{2}}. \end{aligned}$$

Since $u \in L^2(0,T;H^1_{loc}(\Omega))$, a calculation as in (3.9) allows us conclude that the function $\frac{|u\phi|}{|x|^2}$ lies in $L^1(\Omega_T)$. Moreover, $\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}} \to \frac{u\phi}{|x|^2}$ a.e. in Ω_T, so that by the Lebesgue dominated convergence theorem one has

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2}dxdt = \int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt.$$

As regards the last term in (3.19), by Lemma A.5 (in Appendix) there exists a constant $C_{\text{supp}(\phi)} > 0$ such that $u_n \ge C_{\text{supp}(\phi)}$ in supp(ϕ). Then,

$$\int_{\Omega_T}\left|\frac{f_n\phi}{u_n+\frac{1}{n}}\right|dxdt \le \frac{\|\phi\|_\infty}{C_{\text{supp}(\phi)}}\int_{\Omega_T}|f|dxdt < +\infty.$$

So that by the Lebesgue dominated convergence theorem we get

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{f_n\phi}{u_n+\frac{1}{n}}dxdt = \int_{\Omega_T}\frac{f\phi}{u}dxdt.$$

Finally passing to the limit as n tends to ∞ in (3.19) we obtain

$$-\int_\Omega u_0\phi(x,0)dx - \int_{\Omega_T} u\partial_t\phi dtdx + \int_{\Omega_T} \nabla u\cdot\nabla\phi dxdt = \mu\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt + \int_{\Omega_T}\frac{f\phi}{u}dxdt$$

for all $\phi \in C_0^\infty(\Omega_T)$. Furthermore, by Lemma A.5 there exists a constant $C_{\Omega'} > 0$ such that $u \geq C_{\Omega'}$ in $\Omega' \times [0,T]$ which shows that u is a weak solution of (1.1).

Now assume that $\sigma > 1$ is such that $\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$. For $0 \leq \tau \leq T$ let us use $u_n^\sigma\chi_{(0,\tau)}$ as a test function in (3.2). By the Hardy inequality (1.4) we arrive at

$$\frac{1}{\sigma+1}\int_\Omega (u_n(x,\tau))^{\sigma+1}dx + \left(\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega |\nabla u_n^{\frac{\sigma+1}{2}}|^2 dxdt \leq C,$$

where $C = \|f\|_{L^1(\Omega_T)} + \frac{1}{\sigma+1}\|u_0\|_{L^{\sigma+1}(\Omega)}$. Therefore, we deduce that $u^{\frac{\sigma+1}{2}} \in L^2(0,T;H_0^1(\Omega))$.

Case 4 :

Suppose that $\sigma > 1$ and $\mathrm{supp}(f) \subset\subset \Omega_T$. Taking $u_n\chi(0,\tau)$, $0 \leq \tau \leq T$, as a test function in (3.2) and using (1.4) we get

$$\frac{1}{2}\int_\Omega (u_n(x,\tau))^2dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega |\nabla u_n|^2dxdt \leq \int_{\Omega_T}\frac{f}{u_n^{\sigma-1}}dxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

Applying Lemma A.5 (in Appendix) there exists $C > 0$ such that $u_n \geq C$ in $\mathrm{supp}(f)$. Whence, passing to the supremum in $\tau \in [0,T]$ we obtain

$$\frac{1}{2}\sup_{0\leq\tau\leq T}\int_\Omega (u_n(x,\tau))^2dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right)\int_\Omega |\nabla u_n|^2dxdt \leq \frac{1}{C^{\sigma-1}}\int_{\mathrm{supp}(f)} fdxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

Thus, the sequence $\{u_n\}_n$ is bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Therefore, there exist a function $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and a subsequence of $\{u_n\}_n$, still indexed by n, such that $u_n \rightharpoonup u$ in $L^2(0,T;H_0^1(\Omega))$ and then u is a finite energy solution of the problem (1.1). □

3.3 Proof of Theorem 2.4

Let $0 \leq \tau \leq T$. Taking $u_n\chi_{(0,\tau)}(t)$ as a test function in (3.2), we get

$$\frac{1}{2}\int_\Omega (u_n(x,\tau))^2dx + \int_0^\tau\int_\Omega |\nabla u_n|^2dxdt - \Lambda_{N,2}\int_0^\tau\int_\Omega \frac{u_n^2}{|x|^2}dxdt \leq \|f\|_{L^1(\Omega_T)} + \frac{1}{2}\|u_0\|_{L^2(\Omega)}.$$

Passing to the supremum in $\tau \in [0,T]$ and using Theorem A.1 (in Appendix) we conclude that the sequence $\{u_n\}_n$ is uniformly bounded in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, for all $q < 2$. As a consequence, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$. Arguing in a similar way as in the case 1, we conclude that u is a weak solution of the problem (1.1). □

3.4 Proof of Theorem 2.5

Suppose that $\mu > \Lambda_{N,2}$. Arguing by contradiction, assume that (1.1) admits a positive weak solution u. Thus u is also a weak solution to the problem

$$\begin{cases} \partial_t u - \Delta u - \Lambda_{N,2}\frac{u}{|x|^2} = (\mu - \Lambda_{N,2})\frac{u}{|x|^2} + \frac{f}{u^\sigma} & \text{in } \Omega_T, \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \partial\Omega \times (0,T). \end{cases}$$

By virtue of Lemma A.3 (in Appendix) we have

$$\left((\mu - \Lambda_{N,2})\frac{u}{|x|^2} + \frac{f}{u^\sigma} \right) |x|^{-\alpha_1} \in L^1(B_{r_1}(0) \times (t_1, t_2)),$$

for any small enough parabolic cylinder $B_{r_1}(0) \times (t_1, t_2) \subset\subset \Omega_T$ where α_1 is defined in (A.1). As in our equation $\lambda = \Lambda_{N,2}$ we have $\alpha_1 = \frac{N-2}{2}$. Since $u > 0$ and $f \geq 0$ we have in particular

$$(\mu - \Lambda_{N,2})\frac{u}{|x|^2}|x|^{-\frac{N-2}{2}} \in L^1(B_{r_1}(0) \times (t_1, t_2)). \tag{3.20}$$

On the other hand, since

$$\partial_t u - \Delta u - \Lambda_{N,2}\frac{u}{|x|^2} = (\mu - \Lambda_{N,2})\frac{u}{|x|^2} + \frac{f}{u^\sigma} \geq 0$$

by Lemma A.2 (in Appendix) there exists a constant $C > 0$ such that

$$u \geq C|x|^{-\frac{N-2}{2}}. \tag{3.21}$$

Gathering (3.20) and (3.21) we obtain

$$|x|^{-N} \in L^1(B_{r_1}(0) \times (t_1, t_2))$$

which is a contradiction. Therefore, if $\mu > \Lambda_{N,2}$ the problem (1.1) has no positive weak solution. □

3.5 Proof of Theorem 2.6

The proofs of (i) and (ii) are similar. We only give the proof of (i).

• **Proof of (a)** – We shall establish an a priori L^∞-estimate for the solution u_n of (3.2). To do so, we use standard ideas that can be found in several nonsingular cases as for instance in [22,28,48,51,55,57]. Despite being classic, we give the proof for the convenience of the reader. Let $k \geq k_0 := \max(1, \|u_0\|_\infty)$. We choose $G_k(u_n)\chi_{(0,\tau)}$, $0 \leq \tau \leq T$, as a test function in (3.2), we get

$$\int_0^\tau \int_\Omega \partial_t u_n G_k(u_n) dxdt + \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt$$
$$\leq \mu \int_{A_{k,n}} \frac{u_n G_k(u_n)}{|x|^2} dxdt + \int_{A_{k,n}} \frac{f G_k(u_n)}{(u_n + \frac{1}{n})^\sigma} dxdt,$$

where we have set $A_{k,n} = \{(x,t) \in \Omega_\tau : u_n(x,t) > k\}$. Observe that since $G_k(u_n)$ is different from zero only on the set $A_{k,n}$ and according to the choice of k, one has

$$\int_0^\tau \int_\Omega \partial_t u_n G_k(u_n) dxdt = \frac{1}{2} \int_\Omega G_k(u_n(x,\tau))^2 dx.$$

Note that the Hölder inequality implies

$$\int_{A_{k,n}} \frac{u_n G_k(u_n)}{|x|^2} dxdt \le \left(\int_{A_{k,n}} \frac{u_n^2}{|x|^2} dxdt \right)^{\frac{1}{2}} \left(\int_{A_{k,n}} \frac{G_k(u_n)^2}{|x|^2} dxdt \right)^{\frac{1}{2}}.$$

Taking into account that on the subset $A_{k,n}$ one has $\nabla G_k(u_n) = \nabla u_n$ a.e. in Ω, so that Hardy's inequality yields

$$\int_{A_{k,n}} \frac{u_n G_k(u_n)}{|x|^2} dxdt \le \frac{1}{\Lambda_{N,2}} \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt.$$

Since $u_n + \frac{1}{n} > k_0$ on the subset $A_{k,n}$ we get

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} G_k(u_n(x,\tau))^2 dx + \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt \\ &\quad \le \frac{\mu}{\Lambda_{N,2}} \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt + \frac{1}{k_0^{\sigma}} \int_{A_{k,n}} f G_k(u_n) dxdt. \end{aligned}$$

Then passing to the supremum in $\tau \in (0,T)$ we obtain

$$\frac{1}{2} \|G_k(u_n)\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right) \|G_k(u_n)\|^2_{L^2(0,T;H_0^1(\Omega))} \le \frac{1}{k_0^{\sigma}} \int_{\Omega_T} f G_k(u_n) dxdt. \tag{3.22}$$

On the other hand, since $G_k(u_n) \in L^\infty(\Omega_T) \cap L^2(0,T;H_0^1(\Omega))$ then $G_k(u_n) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Therefore, by [28, Proposition 3.1] there exists a positive constant c such that

$$\int_{\Omega_T} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le c^{\frac{2N+4}{N}} \left(\int_{\Omega_T} |\nabla G_k(u_n)|^2 dxdt \right) \left(\|G_k(u_n)\|^2_{L^\infty(0,T;L^2(\Omega))} \right)^{\frac{2}{N}}.$$

Setting $\Gamma_{N,2} := 1 - \frac{\mu}{\Lambda_{N,2}}$ and $C_1 := \frac{c^{\frac{2N+4}{N}} 2^{\frac{2}{N}}}{\Gamma_{N,2} k_0^{\sigma(1+\frac{2}{N})}}$, we obtain using (3.22)

$$\int_{\Omega_T} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le C_1 \left(\int_{\Omega_T} f G_k(u_n) dxdt \right)^{1+\frac{2}{N}}.$$

Observe that both integrals are on the subset $A_{k,n}$. Using Hölder's inequality in the right-hand side term with exponents $\frac{2N+4}{N}$ and $\frac{2N+4}{N+4}$, we get

$$\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le C_1 \left(\int_{A_{k,n}} f^{\frac{2N+4}{N+4}} dxdt \right)^{\frac{N+4}{2N}} \left(\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \right)^{\frac{1}{2}},$$

from which it follows

$$\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le C_1^2 \left(\int_{A_{k,n}} f^{\frac{2N+4}{N+4}} dxdt \right)^{\frac{N+4}{N}}.$$

Since $f \in L^m(\Omega_T)$ with $m > \frac{N}{2} + 1 > \frac{2N+4}{N+4}$, we use again Hölder's inequality obtaining

$$\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le C_1^2 \|f\|^{\frac{2N+4}{N}}_{L^m(\Omega_T)} |A_{k,n}|^{\frac{N+4}{N} - \frac{2N+4}{mN}}.$$

Now let $h > k$. It's easy to see that $A_{h,n} \subset A_{k,n}$ and $G_k(u_n) \ge h - k$ on $A_{h,n}$, so that one has

$$|A_{h,n}|(h-k)^{\frac{2N+4}{N}} \le C_1^2 \|f\|^{\frac{2N+4}{N}}_{L^m(\Omega_T)} |A_{k,n}|^{\frac{N+4}{N} - \frac{2N+4}{mN}}.$$

Setting $\psi(k) = |A_{k,n}|$, we get

$$\psi(h) \leq \frac{C_2}{(h-k)^{\alpha}} \psi(k)^{\beta},$$

where $C_2 = C_1^2 \|f\|_{L^m(\Omega_T)}^{\frac{2N+4}{N}}$, $\alpha = \frac{2N+4}{N}$ and $\beta = \frac{N+4}{N} - \frac{2N+4}{mN}$. Since $m > \frac{N}{2}+1$ we have $\beta > 1$ and then we can apply the first item of [48, Lemma 4.1] to conclude that there exists a constant C_∞, such that $\psi(C_\infty) = 0$, that is

$$\|u_n\|_\infty \leq C_\infty.$$

□

• **Proof of (b)** – Using $u_n^{\gamma}\chi_{(0,\tau)}$, $0 < \tau < T$, as a test function in (3.2) and applying the Hölder's inequality and (1.4) we arrive at

$$\begin{aligned} &\frac{1}{\gamma+1}\int_\Omega (u_n(x,\tau))^{\gamma+1}dx + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq \|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_T} u_n^{(\gamma-\sigma)m'} dxdt\right)^{\frac{1}{m'}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned} \tag{3.23}$$

Note that $1 \leq \sigma \leq \gamma = \frac{Nm(\sigma+1)-N+2m-2}{N-2m+2}$. Since we have supposed that $\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}} > 0$, we discuss the two cases $\sigma = \gamma$ and $\sigma < \gamma$. Thus, if $\sigma = \gamma$ we immediately have

$$\begin{aligned} &\frac{1}{\gamma+1}\int_\Omega (u_n(x,\tau))^{\gamma+1}dx + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq |\Omega_T|^{\frac{1}{m'}}\|f\|_{L^m(\Omega_T)} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned}$$

While If $\sigma < \gamma$, we compute $(\gamma-\sigma)m' = (\gamma+1)\frac{N+2}{N} < (\gamma+1)\frac{N}{N-2}$. Therefore, by (3.23) there exists a positive constant C such that

$$\begin{aligned} &\frac{1}{\gamma+1}\int_\Omega (u_n(x,\tau))^{\gamma+1}dx + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq C\|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_T} (u_n^{\frac{\gamma+1}{2}})^{2^*} dxdt\right)^{\frac{2(\gamma-\sigma)}{2^*(\gamma+1)}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned}$$

Using the Sobolev inequality in the first term on the right hand side of the above inequality, we conclude that there exists a positive constant C_1 such that

$$\begin{aligned} &\frac{1}{\gamma+1}\int_\Omega (u_n(x,\tau))^{\gamma+1}dx + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq C_1\|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt\right)^{\frac{\gamma-\sigma}{\gamma+1}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned}$$

Applying (3.11) with $a = \left(\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt\right)^{\frac{\gamma-\sigma}{\gamma+1}}$, $b = C_1\|f\|_{L^m(\Omega_T)}$, $p = \frac{\gamma+1}{\gamma-\sigma}$ and $q = \frac{\gamma+1}{\sigma+1}$ to obtain

$$\begin{aligned} &\frac{1}{\gamma+1}\|u_n^{\frac{\gamma+1}{2}}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}} - \epsilon\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq C_\epsilon (C\|f\|_{L^m})^{\frac{\gamma+1}{\sigma+1}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned}$$

Finally we choose ϵ such that $\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}} - \epsilon > 0$. Consequently, in both cases the sequence $\{u_n^{\frac{\gamma+1}{2}}\}_n$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Whence, there exist a subsequence of $\{u_n^{\frac{\gamma+1}{2}}\}_n$, still indexed by n, and a function $v \in L^2(0,T;H_0^1(\Omega))$ such that $u_n^{\frac{\gamma+1}{2}} \rightharpoonup v$ weakly in $L^2(0,T;H_0^1(\Omega))$. Now according to the proof of the second item of Theorem 2.2, we know that $u_n \rightharpoonup u$ weakly in $L^2(0,T;H_{loc}^1(\Omega))$ so that identifying almost everywhere the limits one has $v = u^{\frac{\gamma+1}{2}} \in L^2(0,T;H_0^1(\Omega))$. □

3.6 Proof of Theorem 2.8

The ideas we use are standard and we follow the lines of [24, Theorem 4.1, (i)-(b)]. Let us choose $u_n^{2\delta-1}\chi_{(0,\tau)}$, $0 < \tau < T$, as a test function in (3.2) where δ is a positive real constant satisfying $\frac{\Lambda_{N,2}}{\mu}(1 - \sqrt{1 - \frac{\mu}{\Lambda_{N,2}}}) < \delta < 1$. This choice made possible by the fact that $\mu < \Lambda_{N,2}$ implies $\frac{1}{2} < \delta$ and $\frac{2\delta-1}{\delta^2} - \frac{\mu}{\Lambda_{N,2}} > 0$ that will be chosen after few lines. We get

$$
\begin{aligned}
&\frac{1}{2\delta}\int_\Omega (u_n(x,\tau))^{2\delta}dx + \frac{(2\delta-1)}{\delta^2}\int_{\Omega_\tau} |\nabla u_n^\delta|^2 dxdt \\
&\quad \le \mu \int_{\Omega_\tau} \frac{u_n^{2\delta}}{|x|^2}dxdt + \int_{\Omega_\tau} f u_n^{(2\delta-1-\sigma)}dxdt + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Passing to the supremum in $\tau \in (0,T)$ and applying Hardy's inequality (1.4) and then Hölder's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2\delta}\|u_n^\delta\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(\frac{2\delta-1}{\delta^2} - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_T} |\nabla u_n^\delta|^2 dxdt \\
&\quad \le \int_{\Omega_T} f u_n^{2\delta-1-\sigma}dxdt + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)} \\
&\quad \le \|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_T} u_n^{(2\delta-1-\sigma)m'}dxdt\right)^{\frac{1}{m'}} + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.24}
$$

Since $u_n \in L^\infty(\Omega_T) \cap L^2(0,T;H_0^1(\Omega))$ then $u_n \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Thus, by [28, Proposition 3.1] there exists a positive constant c such that

$$
\int_{\Omega_T} (u_n^\delta)^{\frac{2N+4}{N}}dxdt \le c^{\frac{2N+4}{N}}\left(\int_{\Omega_T} |\nabla u_n^\delta|^2 dxdt\right)\left(\|u_n^\delta\|^2_{L^\infty(0,T;L^2(\Omega))}\right)^{\frac{2}{N}}.
$$

Then, using (3.24) we obtain

$$
\begin{aligned}
\int_{\Omega_T} (u_n^\delta)^{\frac{2N+4}{N}}dxdt &\le \frac{(2\delta)^{\frac{2}{N}} c^{\frac{2N+4}{N}}}{\Lambda_\delta}\left(\|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_T} u_n^{(2\delta-1-\sigma)m'}dxdt\right)^{\frac{1}{m'}} + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}\right)^{1+\frac{2}{N}} \\
&\le \frac{(4\delta)^{\frac{2}{N}} c^{\frac{2N+4}{N}}}{\Lambda_\delta}\left(\|f\|^{1+\frac{2}{N}}_{L^m(\Omega_T)}\left(\int_{\Omega_T} u_n^{(2\delta-1-\sigma)m'}dxdt\right)^{\frac{N+2}{Nm'}} + \frac{1}{(2\delta)^{1+\frac{2}{N}}}\|u_0^\delta\|^{\frac{2N+4}{N}}_{L^2(\Omega)}\right),
\end{aligned}
$$

where $\Lambda_\delta = \frac{2\delta-1}{\delta^2} - \frac{\mu}{\Lambda_{N,2}}$. Now we choose δ to be such that $\delta\frac{2N+4}{N} = (2\delta-1-\sigma)m'$, that is $\delta = \frac{mN(1+\sigma)}{2N-4(m-1)}$. Observe that since $1 < m < m_1 < \frac{N}{2}+1$ one has $N - 2(m-1) > 0$ and $\delta > \frac{1+\sigma}{2} \ge \frac{1}{2}$. We point out that $\frac{\mu}{\Lambda_{N,2}} > 0$ implies $\frac{\Lambda_{N,2}}{\mu}(1 - \sqrt{1 - \frac{\mu}{\Lambda_{N,2}}}) > \frac{1}{2}$ and the choice

$\delta > \frac{\Lambda_{N,2}}{\mu}(1-\sqrt{1-\frac{\mu}{\Lambda_{N,2}}})$ ensures that $\Lambda_\delta > 0$. To check the upper bound on δ, we notice that $\delta < 1$ is equivalent to $m < \frac{2N+4}{N(1+\sigma)+4}$. Such an inequality is always satisfied since for $\sigma \le 1$ we have $m < m_1 \le \frac{2N+4}{N(1+\sigma)+4}$. Therefore, with this choice of δ we obtain

$$\|u_n\|^{(2\delta-1-\sigma)m'}_{L^{(2\delta-1-\sigma)m'}(\Omega_T)} \le \frac{(4\delta)^{\frac{2}{N}}c^{\frac{2N+4}{N}}}{\Lambda_\delta}\|f_n\|^{\frac{2}{N}+1}_{L^m(\Omega_T)}\|u_n\|^{\frac{(N+2)(2\delta-1-\sigma)}{N}}_{L^{(2\delta-1-\sigma)m'}(\Omega_T)} + \frac{(4\delta)^{\frac{2}{N}}c^{\frac{2N+4}{N}}}{\Lambda_\delta}\frac{1}{(2\delta)^{1+\frac{2}{N}}}\|u_0^\delta\|^{\frac{2N+4}{N}}_{L^2(\Omega)}.$$

Since $m < \frac{N}{2}+1$ we have

$$(2\delta-1-\sigma)m' > \frac{(N+2)(2\delta-1-\sigma)}{N}$$

and so by virtue of Young's inequality the sequence $\{u_n\}_n$ is uniformly bounded in $L^\gamma(\Omega_T)$ with

$$\gamma = (2\delta-1-\sigma)m' = \frac{m(N+2)(1+\sigma)}{N-2m+2} > 1.$$

Now we shall obtain an estimation on ∇u_n. Notice that from (3.24) we get

$$\Lambda_\delta \delta^2 \int_{\Omega_T} \frac{|\nabla u_n|^2}{u_n^{2(1-\delta)}}dxdt \le \|f_n\|^{\frac{2}{N}+1}_{L^m(\Omega_T)}\|u_n\|^{(2\delta-1-\sigma)}_{L^\gamma(\Omega_T)} + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}$$

and since $\{u_n\}_n$ is uniformly bounded in $L^\gamma(\Omega_T)$, we deduce the existence of a positive constant C, not depending on n, such that

$$\int_{\Omega_T} \frac{|\nabla u_n|^2}{u_n^{2(1-\delta)}}dxdt \le C.$$

Let now $q \ge 1$ be such that $q < 2$. An application of Hölder's inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$ yields

$$\begin{aligned}\int_{\Omega_T} |\nabla u_n|^q dxdt &= \int_{\Omega_T} \frac{|\nabla u_n|^q}{u_n^{q(1-\delta)}} u_n^{q(1-\delta)} dxdt \\ &\le \left(\int_{\Omega_T} \frac{|\nabla u_n|^2}{u_n^{2(1-\delta)}}dxdt\right)^{\frac{q}{2}} \left(\int_{\Omega_T} u_n^{\frac{(1-\delta)2q}{2-q}}dxdt\right)^{\frac{2-q}{2}} \\ &\le C^{\frac{q}{2}}\left(\int_{\Omega_T} u_n^{\frac{(1-\delta)2q}{2-q}}dxdt\right)^{\frac{2-q}{2}}.\end{aligned}$$

Now we impose the condition $\gamma = \frac{(1-\delta)2q}{2-q}$ that gives $q = \frac{m(N+2)(\sigma+1)}{N+2-m(1-\sigma)}$. Observe that $q \ge m(\sigma+1) > 1$ and since $\sigma \le 1$ we have $m < m_1 \le \frac{2N+4}{N(1+\sigma)+4}$ which implies $q < 2$. Thus, the sequence $\{u_n\}_n$ is uniformly bounded in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$. Therefore, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ such that $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ and $u_n \to u$ a.e. in Ω_T. Using $\phi \in C_0^\infty(\Omega \times [0,T))$ as test function in (3.2) we obtain

$$\begin{aligned}&-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n\partial_t\phi dtdx + \int_{\Omega_T}\nabla u_n\cdot\nabla\phi dxdt \\ &\quad = \mu\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}dxdt + \int_{\Omega_T}\frac{f_n\phi}{|u_n|+\frac{1}{n}}dxdt.\end{aligned} \tag{3.25}$$

Notice that since $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$, we immediately have

$$\lim_{n\to+\infty}\int_{\Omega_T}\nabla u_n\cdot\nabla\phi dxdt=\int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt$$

and

$$\lim_{n\to+\infty}\int_{\Omega_T}u_n\partial_t\phi dtdx=\int_{\Omega_T}u\partial_t\phi dtdx.$$

As regards the first integral in the right-hand side of (3.25), we know that the sequence $\{u_n\}$ is increasing to its limit u so we have

$$\Big|\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}\Big|\le\frac{|u\phi|}{|x|^2}.$$

Applying Hölder's and Hardy's inequalities with exponents 2δ and $\frac{2\delta}{2\delta-1}$ we obtain

$$\begin{aligned}\int_{\Omega_T}\frac{|u\phi|}{|x|^2}dxdt&\le\|\phi\|_\infty\int_{\Omega_T}\frac{|u|}{|x|^{\frac{1}{\delta}}}\times\frac{1}{|x|^{\frac{2\delta-1}{\delta}}}dxdt\\&\le\|\phi\|_\infty\left(\int_{\Omega_T}\frac{|u|^{2\delta}}{|x|^2}dxdt\right)^{\frac{1}{2\delta}}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac{2\delta-1}{2\delta}}\\&\le\|\phi\|_\infty(\Lambda_{N,2})^{\frac{-1}{2\delta}}\left(\int_{\Omega_T}|\nabla u^\delta|^2dxdt\right)^{\frac{1}{2\delta}}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac{2\delta-1}{2\delta}}.\end{aligned}$$

From (3.9) and (3.24) we deduce that the sequence $\{u_n^\delta\}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and thus there exist a subsequence of $\{u_n^\delta\}$, still indexed by n, and a function $v\in L^2(0,T;H_0^1(\Omega))$ such that $u_n^\delta\rightharpoonup v$ weakly in $L^2(0,T;H_0^1(\Omega))$ and $u_n^\delta\to v$ a.e. in Ω_T. But we also have $u_n^\delta\rightharpoonup v$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$ and hence follows $v=u^\delta\in L^2(0,T;H_0^1(\Omega))$. Which shows that the function $\frac{|u\phi|}{|x|^2}$ lies in $L^1(\Omega_T)$. Furthermore, since $\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}\to\frac{u\phi}{|x|^2}$ a.e. in Ω_T, the Lebesgue dominated convergence theorem gives

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2}dxdt=\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt.$$

On the other hand, the support $\mathrm{supp}(\phi)$ of the function ϕ is a compact subset of Ω_T and so by Lemma A.5 (in Appendix) there exists a constant $C_{\mathrm{supp}(\phi)}>0$ such that $u_n\ge C_{\mathrm{supp}(\phi)}$ in $\mathrm{supp}(\phi)$. Then,

$$\left|\frac{f_n\phi}{u_n+\frac{1}{n}}\right|\le\frac{\|\phi\|_\infty}{C_{\mathrm{supp}(\phi)}}|f|\in L^1(\Omega_T).$$

So that by the Lebesgue dominated convergence theorem we get

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{f_n\phi}{u_n+\frac{1}{n}}dxdt=\int_{\Omega_T}\frac{f\phi}{u}dxdt.$$

We point out that we also have $u\ge C_{\mathrm{supp}(\phi)}$ in $\mathrm{supp}(\phi)$. Now passing to the limit as n tends to ∞ in (3.25) we obtain

$$-\int_\Omega u_0\phi(x,0)dx-\int_{\Omega_T}u\partial_t\phi dtdx+\int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt=\mu\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt+\int_{\Omega_T}\frac{f\phi}{u}dxdt$$

for all $\phi\in C_0^\infty(\Omega\times[0,T))$. Namely u is a finite energy solution of the problem (1.1). □

3.7 Proof of Theorem 2.10

Let $u, v \in L^2(0,T;H_0^1(\Omega))$ be two energy solutions of the problem (1.1) corresponding to the same data u_0 satisfying (1.3) and $f \in L^m(\Omega_T)$, $m \geq 1$. Since the datum f is compactly supported in Ω_T, then $\partial_t u \in L^2(0,T;H^{-1}(\Omega)) + L^1(\Omega_T)$. Let $k > 0$ and $r > k$. The function $T_k((u-v)_+) \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ is an admissible test function in the formulation of solution (A.8) in Lemma A.7 (in Appendix). Taking it so in the difference of formulations (A.8) solved by u and v, we obtain

$$\begin{aligned}\int_{\Omega_T} \partial_t(u-v)_+ T_k((u-v)_+)dxdt + \int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \\ \leq \int_{\{(u-v)_+\leq k\}} \frac{(T_k((u-v)_+))^2}{|x|^2} dxdt + k\mu \int_{\{(u-v)_+>k\}} \frac{(u-v)_+}{|x|^2} dxdt \\ + \int_{\Omega_T} f\left(\frac{1}{u^\sigma} - \frac{1}{v^\sigma}\right) T_k((u-v)_+)dxdt\end{aligned}$$

Setting $\Theta_k(s) = \int_0^s T_k(\nu)d\nu$ and dropping the negative term, we get

$$\begin{aligned}\int_{\Omega} \Theta_k((u-v)_+(x,T))dx + \int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \\ \leq \int_{\{(u-v)_+\leq k\}} \frac{(T_k((u-v)_+))^2}{|x|^2} dxdt + k\mu \int_{\{(u-v)_+>k\}} \frac{(u-v)_+}{|x|^2} dxdt \\ + \int_{\Omega} \Theta_k((u-v)_+(x,0))dx.\end{aligned}$$

Using $\int_\Omega \Theta_k((u-v)_+(x,T))dx \geq 0$, the fact that $u(x,0) = v(x,0) = u_0(x)$, Hardy's inequality (1.4) and Hölder's inequality, we arrive at

$$\begin{aligned}\int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \\ + k\mu \left(\int_{\{(u-v)_+>k\}} \frac{((u-v)_+)^2}{|x|^2} dxdt\right)^{\frac{1}{2}} \left(\int_{\{(u-v)_+>k\}} \frac{dxdt}{|x|^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Having in mind (3.9) and using again (1.4) we reach that

$$\begin{aligned}\int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \\ + \frac{k\mu T^{\frac{1}{2}} C_1^{\frac{1}{2}}}{\Lambda_{N,2}^{\frac{1}{2}}} \left(\int_{\{(u-v)_+>k\}} |\nabla(u-v)_+|^2 dxdt\right)^{\frac{1}{2}}.\end{aligned} \tag{3.26}$$

On the other hand, taking $T_r(G_k((u-v)_+)) \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ as a test function in the problems solved by u and v and subtracting the two equations we obtain

$$\begin{aligned}\int_{\Omega_T} \partial_t(u-v)_+ T_r(G_k((u-v)_+))dxdt + \int_{\{k<(u-v)_+<k+r\}} |\nabla(u-v)_+|^2 dxdt \\ \leq \mu \int_{\{(u-v)_+>k\}} \frac{(u-v)_+^2}{|x|^2} dxdt + \int_{\Omega_T} f\left(\frac{1}{u^\sigma} - \frac{1}{v^\sigma}\right) T_r(G_k((u-v)_+))dxdt.\end{aligned}$$

Setting $\Theta_{k,r}(s) = \int_0^s T_r(G_k(\nu))d\nu$ and dropping the negative term, the above inequality becomes

$$\int_\Omega \Theta_{k,r}((u-v)_+(x,T))dx + \int_{\{k<(u-v)_+<k+r\}} |\nabla(u-v)_+|^2 dxdt$$
$$\leq \mu \int_{\{(u-v)_+>k\}} \frac{(u-v)_+^2}{|x|^2} dxdt + \int_\Omega \Theta_{k,r}((u-v)_+(x,0))dx.$$

Note that $\int_\Omega \Theta_{k,r}((u-v)_+(x,T))dx \geq 0$ and $\int_\Omega \Theta_{k,r}((u-v)_+(x,0))dx = 0$. Whence, by (1.4) we obtain

$$\int_{\{k<(u-v)_+<k+r\}} |\nabla(u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\{(u-v)_+>k\}} |\nabla(u-v)_+|^2 dxdt.$$

Then, passing to the limit as r tends to $+\infty$ we get

$$\int_{\{k<(u-v)_+\}} |\nabla(u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\{k<(u-v)_+\}} |\nabla(u-v)_+|^2 dxdt. \tag{3.27}$$

Therefore, gathering (3.26) and (3.27) we obtain

$$\int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt$$
$$+ \frac{k\mu C_1}{\Lambda_{N,2}} \left(\int_{\{(u-v)_+>k\}} |\nabla(u-v)_+|^2 dxdt \right)^{\frac{1}{2}}.$$

Passing now to the limit as k tends to 0 we obtain

$$\int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt,$$

which, recalling that $u - v \in C([0,T]; L^1(\Omega))$ (see [44, Theorem 1.1]) implies $(u-v)_+(\cdot,\tau) = 0$ for any $\tau \in [0,T]$ and for almost every $x \in \Omega$. Since u and v play symmetrical roles we conclude that $u = v$ a.e. in Ω_T. □

A Appendix

We give here some important lemmas that are necessary for the accomplishment of the proofs of the previous results.

Theorem A.1 ([50, Theorem 2.2])**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$. Then for every $1 \leq q < 2$ there exists a positive constant $C = C(\Omega, q)$ such that for all $u \in H_0^1(\Omega)$ we have*

$$C \left(\int_\Omega |\nabla u|^q dx \right)^{\frac{2}{q}} \leq \int_\Omega |\nabla u|^2 dx - \Lambda_{N,2} \int_\Omega \frac{u^2}{|x|^2} dx.$$

Let

$$\alpha_1 := \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda} \tag{A.1}$$

be the smallest root of $\alpha^2 - (N-2)\alpha + \lambda = 0$. It is well known that this root yields the radial solution $|x|^{-\alpha_1}$ to the homogeneous equation

$$-\Delta v - \lambda \frac{v}{|x|^2} = 0.$$

The following lemma provides a local comparison result with this radial solution.

Lemma A.2 ([5, Lemma 2.2])**.** *Assume that u is a non-negative function defined in Ω such that $u \not\equiv 0$, $u \in L^1_{loc}(\Omega_T)$. If u satisfies*

$$\partial_t u - \Delta u - \lambda \frac{u}{|x|^2} \geq 0, \quad in\ \mathcal{D}'(\Omega_T)$$

with $\Omega_T := \Omega \times (0,T)$, $\lambda \leq \Lambda_{N,2}$ and $B_r(0) \subset\subset \Omega$, then there exists a constant $C = C(N, r, t_1, t_2)$ such that for each cylinder $B_{r_1}(0) \times (t_1, t_2) \subset \Omega \times (0,T)$, $0 < r_1 < r$,

$$u \geq C|x|^{-\alpha_1} \quad in\ B_{r_1}(0) \times (t_1, t_2),$$

where α_1 is the constant defined in (A.1)*.*

Lemma A.3. *Let $0 < \lambda \leq \Lambda_{N,2}$ and $g \in L^1(0,T; L^1_{loc}(\Omega))$, $g \geq 0$. If u is a weak solution of the problem*

$$\begin{cases} \partial_t u - \Delta u = \lambda \dfrac{u}{|x|^2} + g & \text{in } \Omega_T := \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{A.2}$$

where $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, then g satisfies

$$\int_{t_1}^{t_2} \int_{B_{r_1}(0)} |x|^{-\alpha_1} g dx dt < +\infty,$$

for any ball $B_{r_1}(0) \subset\subset \Omega$, where α_1 is defined in (A.1)*.*

Proof. We use similar arguments as in [5, Remark 2.4]. Let $B_r(0) \subset\subset \Omega$ and $\phi \in L^2(0,T; H^1_0(\Omega)) \cap L^\infty(\Omega_T)$ be a weak solution of the problem

$$\begin{cases} \partial_t \phi - \Delta \phi - \lambda \frac{\phi}{|x|^2} = 1 & \text{in } \Omega_T, \\ \phi = 0 & \text{in } \partial\Omega \times (0,T), \\ \phi(x,0) = 1 & \text{in } \Omega. \end{cases} \tag{A.3}$$

Multiplying (A.2) by $T_n(\phi)$ and integrating over $B_r(0) \times (0,T)$ we obtain

$$\int_0^T \int_{B_r(0)} \partial_t u T_n(\phi) dx dt - \int_0^T \int_{B_r(0)} \Delta u T_n(\phi) dx dt - \lambda \int_0^T \int_{B_r(0)} \frac{u}{|x|^2} T_n(\phi) dx dt$$
$$= \int_0^T \int_{B_r(0)} g T_n(\phi) dx dt.$$

Since u is a weak solution of (A.2) the above integrals make sense for each integer n. By the classical by-parts integration formula, one has

$$\begin{aligned} &\int_{B_r(0)} u(x,T) T_n(\phi(x,T)) dx - \int_{B_r(0)} u(x,0) dx - \int_0^T \int_{B_r(0)} u \partial_t (T_n(\phi)) dx dt \\ &\quad - \int_0^T \int_{B_r(0)} u \Delta(T_n(\phi)) dx dt - \lambda \int_0^T \int_{B_r(0)} \frac{u}{|x|^2} T_n(\phi) dx dt \\ &= \int_0^T \int_{B_r(0)} g T_n(\phi) dx dt. \end{aligned} \tag{A.4}$$

Since $T_n(\phi) \to \phi$ in $L^1(\Omega_T)$ and a.e. in Ω_T and $\phi \in L^\infty(\Omega_T)$, we can apply the Lebesgue dominated convergence theorem in the (A.4) to get

$$\int_{B_r(0)} u(x,T)\phi(x,T)dx - \int_{B_r(0)} u_0(x)dx - \int_0^T \int_{B_r(0)} u\partial_t\phi dxdt$$
$$- \int_0^T \int_{B_r(0)} u\Delta\phi dxdt - \lambda \int_0^T \int_{B_r(0)} \frac{u}{|x|^2}\phi dxdt = \int_0^T \int_{B_r(0)} g\phi dxdt.$$

As ϕ is a solution of (A.3), we get

$$\int_{B_r(0)} u(x,T)\phi(x,T)dx - \int_{B_r(0)} u_0 dx - 2\int_0^T \int_{B_r(0)} u\partial_t\phi dxdt + \int_0^T \int_{B_r(0)} udxdt$$
$$= \int_0^T \int_{B_r(0)} g\phi dxdt.$$

Applying again the by-parts integration formula we obtain

$$- \int_{B_r(0)} u(x,T)\phi(x,T)dx + \int_{B_r(0)} u_0(x)dx + 2\int_0^T \int_{B_r(0)} \partial_t u\phi dxdt + \int_0^T \int_{B_r(0)} udxdt$$
$$= \int_0^T \int_{B_r(0)} g\phi dxdt.$$

By Lemma A.2, for every cylinder $B_{r_1}(0) \times (t_1, t_2) \subset B_r(0) \times (0, T)$, $0 < r_1 < r$ there exists a constant $C > 0$ such that

$$\int_{t_1}^{t_2} \int_{B_{r_1}(0)} |x|^{-\alpha_1} g dxdt \leq \int_{B_r(0)} u(x,T)\phi(x,T)dx + \int_{B_r(0)} u_0 dx$$
$$+ 2\int_0^T \int_{B_r(0)} |\partial_t u\phi| dxdt + \int_0^T \int_{B_r(0)} udxdt.$$

Since $u \in L^1(0,T; L^1_{loc}(\Omega))$, $u_0 \in L^\infty(\Omega)$, $\phi \in L^\infty(\Omega_T)$ and $\partial_t u \in L^2(0,T; H^{-1}_{loc}(\Omega)) + L^1(0,T; L^1_{loc}(\Omega))$ conclude that

$$\int_{t_1}^{t_2} \int_{B_{r_1}(0)} |x|^{-\alpha_1} g dxdt < +\infty.$$ □

We will now compare the solution u_n of (3.1) with the solution w_n of the problem

$$\begin{cases} \partial_t w_n - \Delta w_n = \dfrac{f_n}{(w_n + \frac{1}{n})^\sigma} & \text{in } \Omega \times (0,T), \\ w_n(x,t) = 0 & \text{in } \partial\Omega \times (0,T), \\ w_n(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{A.5}$$

where $f = \min(f, n)$ and u_0 satisfies (1.3). Recall that (A.5) has a weak solution w_n (see [24, Lemma 2.1]).

Lemma A.4. *Let u_n be a solution of (3.1) and w_n be a solution of (A.5). Then, $w_n \leq u_n$ a.e. in Ω_T.*

Proof. Consider the problems solved by w_n and u_n, subtracting the two equations, we get

$$\begin{aligned} \partial_t(w_n - u_n) - \Delta(w_n - u_n) &= -\mu \frac{T_n(u_n)}{|x|^2 + \frac{1}{n}} + f_n \left(\frac{1}{(w_n + \frac{1}{n})^\sigma} - \frac{1}{(u_n + \frac{1}{n})^\sigma} \right) \\ &\leq f_n \left(\frac{1}{(w_n + \frac{1}{n})^\sigma} - \frac{1}{(u_n + \frac{1}{n})^\sigma} \right). \end{aligned} \tag{A.6}$$

Using $(w_n - u_n)_+\chi_{(0,\tau)}$, $0 \le \tau \le T$, as test function in (A.6) it follows that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (w_n - u_n)_+^2(x,\tau)dx + \int_{\Omega_\tau} |\nabla(w_n - u_n)_+|^2 dxdt \\
&\le \int_{\Omega_\tau} f_n \left(\frac{(u_n + \frac{1}{n})^\sigma - (w_n + \frac{1}{n})^\sigma}{(u_n + \frac{1}{n})^\sigma (w_n + \frac{1}{n})^\sigma} \right) (w_n - u_n)_+ dxdt \\
&\le 0,
\end{aligned}$$

where we have used $w_n(x,0) = u_n(x,0) = u_0(x)$. Hence we conclude that

$$\int_{\Omega_T} |\nabla(w_n - u_n)_+|^2(x,\tau)dx = 0.$$

Recalling that $w_n - u_n \in \mathcal{C}([0,T];L^1(\Omega))$ (see [44, Theorem 1.1]) implies $(w_n - u_n)_+(\cdot,\tau) = 0$ for every $0 \le \tau \le T$ and for almost every $x \in \Omega$. Thus, $w_n \le u_n$ a.e. in Ω_T. □

Lemma A.5. *Let u_n be the solution of (3.1) given by Lemma 3.1. Then for every $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} > 0$ (not depending on n), such that $u_n \ge C_{\Omega'}$ in $\Omega' \times [0,T]$.*

Proof. The proof follows by combining [24, Proposition 2.2] and Lemma A.4. □

Lemma A.6. *Assume that $\mu \le \Lambda_{N,2}$ and let u_n be a solution of (3.1). The sequence $\{u_n\}_{n\in\mathbb{N}}$ is nonnegative and increasing with respect to $n \in \mathbb{N}$.*

Proof. Writing (3.2) with u_n and u_{n+1} and then subtracting the two corresponding equations, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t(u_n - u_{n+1})\phi dxdt + \int_0^T \int_\Omega \nabla(u_n - u_{n+1})\nabla\phi dxdt \\
&\le \mu \int_0^T \int_\Omega \frac{T_{n+1}(u_n) - T_{n+1}(u_{n+1})}{|x|^2 + \frac{1}{n+1}} \phi dxdt \\
&+ \int_0^T \int_\Omega f_{n+1} \left(\frac{1}{(u_n + \frac{1}{n+1})^\sigma} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^\sigma} \right) \phi dxdt
\end{aligned} \tag{A.7}$$

for every $\phi \in L^2(0,T;H_0^1(\Omega))$. Inserting $(u_n - u_{n+1})_+ \in L^2(0,T;H_0^1(\Omega))$ as a test function in (A.7) and using the fact that T_{n+1} is a 1-Lipschitzian function, we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_T} \partial_t(u_n - u_{n+1})_+^2 dxdt + \int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^2 dxdt \\
&\le \int_{\Omega_T} f_{n+1}(u_n - u_{n+1})_+ \left(\frac{1}{(u_n + \frac{1}{n+1})^\sigma} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^\sigma} \right) dxdt \\
&+ \mu \int_{\Omega_T} \frac{(u_n - u_{n+1})_+^2}{|x|^2} dxdt.
\end{aligned}$$

Dropping the non-negative parabolic term and using the fact that

$$(u_n - u_{n+1})_+ \left(\frac{1}{(u_n + \frac{1}{n+1})^\sigma} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^\sigma} \right) \le 0,$$

we obtain

$$\int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^2 dxdt \le \mu \int_{\Omega_T} \frac{(u_n - u_{n+1})_+^2}{|x|^2} dxdt.$$

Thus, if $\mu < \Lambda_{N,2}$ the Hardy inequality (1.4) yields

$$\int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^2 dxdt = 0,$$

while if $\mu = \Lambda_{N,2}$ we can apply Theorem A.1 obtaining

$$\int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^q dxdt = 0,$$

for all $q < 2$. Therefore, in both cases we get $(u_n - u_{n+1})_+ = 0$ a.e. in Ω_T, that is $u_n \le u_{n+1}$ a.e. in Ω_T. In addition, as $u_n \ge u_0$ we infer that u_n is nonnegative. □

Lemma A.7. *Let $u \in L^2(0,T;H_0^1(\Omega))$ be a finite energy solution of (1.1) with a datum $f \in L^1(\Omega_T)$ such that $\operatorname{supp}(f) \subset\subset \Omega_T$. Then u satisfies $\frac{u\phi}{|x|^2} \in L^1(\Omega_T)$, $\frac{f\phi}{u^\sigma} \in L^1(\Omega_T)$ and*

$$\int_{\Omega_T} \partial_t u\phi dxdt + \int_{\Omega_T} \nabla u \cdot \nabla\phi dxdt = \int_{\Omega_T} \left(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\right)\phi dxdt, \tag{A.8}$$

for every $\phi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$.

Proof. Let $\phi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ be a nonnegative function. A direct application of Hardy's inequality yields $\mu\frac{u\phi}{|x|^2} \in L^1(\Omega_T)$, while since f is compactly supported in Ω_T, by Lemma A.5 there exists a constant $C_{\operatorname{supp}(f)} > 0$ such that $u \ge C_{\operatorname{supp}(f)}$ in $\operatorname{supp}(f)$ so that one has

$$\int_{\Omega_T} \frac{|f\phi|}{u^\sigma} dxdt \le C^\sigma_{\operatorname{supp}(f)} \|\phi\|_\infty \|f\|_{L^1(\Omega_T)} < \infty.$$

We argue as in [41, Lemma 4.2] considering a sequence of function $\phi_n \in C_0^\infty(\Omega_T)$, with $\phi_n \ge 0$ and $\phi_n \to \phi$ in $L^2(0,T;H_0^1(\Omega))$, with $\|\phi_n\|_\infty \le \|\phi\|_\infty$. Inserting ϕ_n as a test function in (2.1) and integrating by parts, we obtain

$$\int_{\Omega_T} \partial_t u\phi_n dxdt + \int_{\Omega_T} \nabla u \cdot \nabla\phi_n dxdt = \int_{\Omega_T} \left(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\right)\phi_n dxdt. \tag{A.9}$$

Since $\phi_n \to \phi$ in $L^2(\Omega_T)$ then, for a subsequence still indexed by n, we may assume that $\phi_n \to \phi$ a.e. in Ω_T. As f is compactly supported in Ω_T we have

$$\left(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\right)\phi_n \le \|\phi\|_\infty \left(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\right) \in L^1(\Omega_T).$$

Thus, by the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\to\infty} \int_{\Omega_T} \left(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\right)\phi_n dxdt = \int_{\Omega_T} \left(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\right)\phi dxdt.$$

Since $\partial_t u \in L^2(0,T;H^{-1}(\Omega)) + L^1(\Omega_T)$ we use the convergence $\phi_n \to \phi$ in $L^2(0,T;H_0^1(\Omega))$ and again the Lebesgue dominated convergence theorem in (A.9) obtaining

$$\int_{\Omega_T} \partial_t u\phi dxdt + \int_{\Omega_T} \nabla u \cdot \nabla\phi dxdt = \int_{\Omega_T} \left(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\right)\phi dxdt.$$ □

Conflict of interest

The authors declare no potential conflict of interests.

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