Ground state sign-changing solutions for critical Choquard equations with steep well potential

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Abstract. In this paper, we study sign-changing solution of the Choquard type equation

$$-\Delta u + (\lambda V(x) + 1) u = (I_\alpha * |u|^{2^*_{\alpha}})|u|^{2^*_{\alpha} - 2} u + \mu |u|^{p-2} u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in ((N - 4)^+, N)$, $I_\alpha$ is a Riesz potential, $p \in [2^*_\alpha, \frac{2N}{N-4})$, $2^*_\alpha := \frac{N+\alpha}{N-2}$ is the upper critical exponent in terms of the Hardy–Littlewood–Sobolev inequality, $\mu > 0$, $\lambda > 0$, $V \in C(\mathbb{R}^N, \mathbb{R})$ is nonnegative and has a potential well. By combining the variational methods and sign-changing Nehari manifold, we prove the existence and some properties of ground state sign-changing solution for $\lambda, \mu$ large enough. Further, we verify the asymptotic behaviour of ground state sign-changing solutions as $\lambda \to +\infty$ and $\mu \to +\infty$, respectively.

Keywords: Choquard equation, upper critical exponent, steep well potential, ground state sign-changing solution, asymptotic behaviour.

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1 Introduction and main results

The Choquard equation has a physical prototype, namely the Hartree type evolution equation

$$-i\partial_t \psi = \Delta \psi + (I_2 * |\psi|^2) \psi, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}_+,$$

where $\mathbb{R}_+ = [0, +\infty)$, $I_2(x) = \frac{1}{4\pi|x|}$, $\forall \ x \in \mathbb{R}^3 \setminus \{0\}$, and $*$ is convolution in $\mathbb{R}^3$. Eq. (1.1) was firstly proposed by Pekar to describe a resting polaron in [24]. Two decades later, Choquard [16] introduced Eq. (1.1) as a certain approximation to Hartree–Fock theory of one component plasma, and used it to characterize an electron trapped in its own hole. Afterwards, viewing the quantum state reduction as a gravitational phenomenon in quantum gravity, Penrose et al. [20] proposed Eq. (1.1) in the form of Schrödinger–Newton system to model a single particle moving in its own gravitational field.

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As we know, standing wave solution of Eq. (1.1) corresponds to solution of the Choquard equation

\[-\Delta u + u = (I_2 * |u|^2) u \quad \text{in } \mathbb{R}^3. \tag{1.2}\]

In detail, with a suitable scaling, the wave function \( \psi(x,t) = e^{-it}u(x) \) is a solution of Eq. (1.1) once \( u \) is a solution of Eq. (1.2). Lieb demonstrated the seminal work on Eq. (1.2) in [16], in which he certified the existence and uniqueness (up to translations) of positive radial ground state solution by applying symmetrically decreasing rearrangement inequalities. After this, Lions [18] studied the same problem and further proved the existence of infinitely many radial solutions via the variational methods.

From mathematical perspective, scholars prefer to study the general Choquard equation

\[-\Delta u + W(x)u = \gamma (I_s * G(u)) g(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}\]

where \( N \geq 3, \gamma \in \mathbb{R}^+, I_s \) is the Riesz potential of order \( \alpha \in (0,N) \) defined for \( x \in \mathbb{R}^N \setminus \{0\} \) by

\[I_s(x) = \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{with} \quad A_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)^2 \pi^{\frac{N}{2}}}.\]

\( \Gamma \) is the Gamma function, \(*\) is convolution, \( W \in C(\mathbb{R}^N, \mathbb{R}), g \in C(\mathbb{R}, \mathbb{R}) \) and \( G(u) = \int_0^u g(s)ds \).

To establish the variational framework for Choquard equations, we need the following celebrated Hardy–Littlewood–Sobolev inequality.

**Proposition 1.1** ([17, Theorem 4.3]). Let \( r,s > 1, 0 < \alpha < N \) satisfy \( \frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N} \). Then there exists a sharp constant \( C(N,\alpha, r, s) > 0 \) such that, for all \( f \in L^r(\mathbb{R}^N) \) and \( h \in L^s(\mathbb{R}^N) \), there holds

\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\alpha}} dxdy \right| \leq C(N,\alpha, r, s) \|f\|_r \|h\|_s. \tag{1.4}\]

In particular, if \( r = s = \frac{2N}{N+\alpha} \) then the constant \( C(N,\alpha, r, s) \) admits a precise expression, namely,

\[C(N,\alpha) := C(N,\alpha, r, s) = \pi^{\frac{\alpha}{2}} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{N+\alpha}{2}\right)} \left[\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right]^{-\frac{\alpha}{N}}.\]

Thanks to (1.4), the integral \( \int_{\mathbb{R}^N} (I_s * |u|^p)|u|^p dx \) is well defined in \( H^1(\mathbb{R}^N) \) once \( p \in [2_+^*, 2^*_+], \)

where \( 2_+^* := \frac{N+\alpha}{N-2} \) and \( 2^*_+ := \frac{N+\alpha}{N} \) are usually called upper and lower critical exponents with respect to the Hardy–Littlewood–Sobolev inequality, respectively. It is easy to clarify that the critical terms

\[
\int_{\mathbb{R}^N} (I_s * |u|^{2_+^*})|u|^{2_+^*} dx \quad \text{and} \quad \int_{\mathbb{R}^N} (I_s * |u|^{2^*_+})|u|^{2^*_+} dx
\]

are invariant under the scaling actions \( \sigma^{-\frac{N-2}{2}} u(\sigma \cdot) \) and \( \sigma^{-\frac{N}{2}} u(\sigma \cdot) \) (\( \sigma > 0 \), respectively, and these two scaling actions served as group actions are noncompact on \( H^1(\mathbb{R}^N) \). Consequently, from the perspective of variational methods, the critical exponents \( 2_+^* \) and \( 2^*_+ \) may provoke two kinds of lack of compactness. However, fortunately, similar to the Sobolev critical case studied in [3], these two kinds of loss of compactness can be recovered to some extent by using the extremal functions of the Hardy–Littlewood–Sobolev inequality.

In [21], Moroz and Van Schaftingen studied the case of Eq. (1.3) that \( W(x) \equiv 1, \gamma = \frac{1}{p} \) and \( G(u) = |u|^p (p > 1) \), they proved the existence, regularity, radially symmetry and decaying property at infinity of ground state solution when \( p \in (2_+^*, 2^*_+) \). Meanwhile, based on the regularity of solutions, they established a Nehari–Pohožaev type identity and then showed
the nonexistence of nontrivial solutions for Eq. (1.3) when \( p \notin (2^*_a, 2^*_s) \). Afterwards, in [22], they extended the existence results in [21] to the case of Eq. (1.3) that \( g \) satisfies the so-called almost necessary conditions of Berestycki–Lions type. For the critical cases of Eq. (1.3), with the nonexistence result of [21] in hand, an increasing number of scholars devote to studying Eq. (1.3) with critical term and a noncritical perturbed term. We refer the interested readers to [4, 9, 14, 30] for upper critical case, [23, 26] for lower critical case and [15, 25, 31] for doubly critical case.

When it comes to the case \( W(x) \neq \text{const.} \), we focus our attention on steep well potential of the form \( \lambda V(x) + b \), where \( \lambda > 0, b \in \mathbb{R} \) and \( V \in C(\mathbb{R}^N, \mathbb{R}) \) satisfies the following hypotheses:

\[
\begin{align*}
(V_1) & \quad V \text{ is bounded from below, } \Omega := \text{int } V^{-1}(0) \text{ is nonempty and } \overline{\Omega} = V^{-1}(0), \\
(V_2) & \quad \text{there exists some constant } M > 0 \text{ such that } \left| \{ x \in \mathbb{R}^N : V(x) \leq M \} \right| < +\infty.
\end{align*}
\]

This type of potential was firstly introduced by Bartsch and Wang in [2] to study the existence and multiplicity of nontrivial solutions for subcritical Schrödinger equations in the case of \( b > 0 \). Later, Ding and Szulkin further considered the case \( b = 0 \) in [8]. Since \( |\Omega| < +\infty \), then \( -\Delta \) possesses a sequence of positive Dirichlet eigenvalues \( \mu_1 < \mu_2 < \cdots < \mu_n \to +\infty \).

Assuming \( b < 0 \) and \( b \neq -\mu_i \) for any \( i \in \mathbb{N}_+ \), Clapp and Ding [6], together with Tang [27], studied the existence and concentration of ground state solution for critical Schrödinger equation. Recently, the pre-existing results on Schrödinger equations have been extended to the Choquard equations, see e.g. [1, 14, 15, 19] and the references therein.

As we concerned here, sign-changing solution of elliptic equation is a focusing topic due to its wide application in biology and physics etc. In [7], Clapp and Salazar investigated the Choquard equation

\[-\Delta u + W(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \Omega,
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is an exterior domain, \( p \in (2, 2^*_a) \), \( \alpha \in ((N - 4)^+, N) \) and \( W \in C(\mathbb{R}^N, \mathbb{R}) \). Under symmetrical assumptions on \( \Omega \) and decaying properties on \( W \), they derived multiple sign-changing solutions. After this, many scholars considered the same topic in the whole Euclidean space, namely,

\[-\Delta u + W(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.5}\]

In [11], Ghimenti and Van Schaftingen studied the case that \( N \geq 1, \alpha \in ((N - 4)^+, N) \), \( W(x) \equiv 1 \) and \( p \in (2, 2^*_a) \) of Eq. (1.5). There, by introducing a new minimax principle and concentration-compactness lemmas for sign-changing Palais–Smale sequences, they obtained a ground state sign-changing solution. Also, they proved that the least energy in the sign-changing Nehari manifold has no minimizers when \( p \in (2^*_a, \max\{2, 2^*_s\}) \). Further, Ghimenti, Moroz and Van Schaftingen [10] constructed a ground state sign-changing solution of Eq. (1.5) when \( p = 2 \) by approaching the case \( p = 2 \) with the cases \( p \in (2, 2^*_a) \). Van Schaftingen and Xia [28] assumed that \( N \geq 1, \alpha \in ((N - 4)^+, N), p \in (2, 2^*_a) \) and \( W \in C(\mathbb{R}^N, \mathbb{R}) \) satisfies the coercive condition \( \lim_{|x| \to +\infty} W(x) = +\infty \). By using a constrained minimization argument in sign-changing Nehari manifold, they derived a ground state sign-changing solution of Eq. (1.5) (see the similar result in [32]). Moreover, Zhong and Tang [33] studied the following Choquard equation

\[-\Delta u + (\lambda V(x) + 1)u = (I_\alpha * (K|u|^p))K(x)|u|^{p-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,
\]

where \( N \geq 3, 2^* = \frac{2N}{N-2}, \alpha \in ((N - 4)^+, N), p \in (2, 2^*_a), \lambda < 0 \) and the functions \( V, K \) satisfy
(V3) $V \in L^N_+(\mathbb{R}^N) \setminus \{0\}$ is nonnegative,

(V4) there exist constants $\rho, \beta, C > 0$ such that $V(x) \geq C|x|^{-\beta}$ for all $|x| < \rho$,

(K1) $K \in L^r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \setminus \{0\}$ for some $r \in \left[\frac{2N}{N+a-2\beta}, +\infty\right)$ and $K$ is nonnegative.

It follows from (V3) that the first eigenvalue $\lambda_1$ of $-\Delta u + u = \lambda V(x)u$ in $H^1(\mathbb{R}^N)$ is positive. When $\lambda \in (-\lambda_1, 0)$ and $\beta \in (2 - \min\left\{\frac{N+a}{2p} - \frac{N-2}{2}, \frac{N-2}{2}\right\}, 2)$, following the ideas in [5], they derived a ground state sign-changing solution by using minimization arguments in sign-changing Nehari manifold.

Motivated by the above works, in the present paper, we study the Choquard equation

$$-\Delta u + (\lambda V(x) + 1) u = (I_\alpha * |u|^{2^*_\alpha}) |u|^{2^*_\alpha} - 2 u + \mu |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \quad (1.6)$$

where $\lambda > 0$, $\mu > 0$, $N \geq 3$, $\alpha \in ((N-4)^+, N)$, $p \in [2^*_\alpha, 2^*)$, and $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies the hypotheses

(V5) $V(x) \geq 0$ in $\mathbb{R}^N$ and there exists some $M > 0$ such that $\{|x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$,

(V6) $\Omega := \text{int } V^{-1}(0)$ is a nonempty set with smooth boundary and $\overline{\Omega} = V^{-1}(0)$.

Let $E_\lambda := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x)u^2 \, dx < +\infty\}$ be equipped with the inner product

$$(u, v)_\lambda := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + (\lambda V(x) + 1) uv \, dx, \quad \forall u, v \in E_\lambda,$$

and the norm $\| \cdot \|_\lambda = (\cdot, \cdot)^{\frac{1}{2}}_\lambda$ for any $\lambda > 0$. Since $V \geq 0$ in $\mathbb{R}^N$, it is easy to see that $E_\lambda \hookrightarrow H^1(\mathbb{R}^N)$ and, for any $s \in [2, 2^*)$, there is some constant $v_s > 0$ such that, for all $\lambda > 0$,

$$|u|_s \leq v_s \|u\| \leq v_s \|u\|_\lambda, \quad \forall u \in E_\lambda. \quad (1.7)$$

By (1.4) and (1.7), we deduce the energy functional $J_{\lambda, \mu}$ of Eq. (1.6) belongs to $C^1(E_{\lambda, \mathbb{R}})$, where

$$J_{\lambda, \mu}(u) = \frac{1}{2} \|u\|^2_\lambda - \frac{1}{2} \cdot \frac{2^*_\alpha}{2^*_\alpha} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_\alpha}) |u|^{2^*_\alpha} \, dx - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p \, dx.$$

Now we are prepared to state our main results.

**Theorem 1.2.** Assume that $N \geq 3$, $\alpha \in ((N-4)^+, N)$, $p \in [2^*_\alpha, 2^*)$ and (V5), (V6) hold. Then there exist $\Lambda > 0$ and $\mu_* > 0$ such that Eq. (1.6) admits a ground state sign-changing solution $u_{\lambda, \mu}$ for any $\lambda \geq \Lambda$ and $\mu \geq \mu_*$. Further, for any $\mu \geq \mu_*$ and sequence $\{\lambda_n\} \subset [\Lambda, +\infty)$ satisfying $\lambda_n \to +\infty$, the sequence $\{u_{\lambda_n, \mu}\}$ of ground state sign-changing solutions to Eq. (1.6) strongly converges to some $u_\mu$ in $H^1(\mathbb{R}^N)$ in the sense of subsequence, where $u_\mu$ is a ground state sign-changing solution of

$$\begin{cases} -\Delta u + u = A_\alpha \int_\Omega \frac{|u(y)|^{2^*_\alpha}}{|x-y|^{N-\alpha}} \, dy |u|^{2^*_\alpha} - 2 u + \mu |u|^{p-2} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial \Omega. \end{cases} \quad (1.8)$$

Moreover, for any $\lambda \geq \Lambda$ and sequence $\{\mu_n\} \subset [\mu_*, +\infty)$ with $\mu_n \to +\infty$, the sequence $\{u_{\lambda, \mu_n}\}$ of ground state sign-changing solutions to Eq. (1.6) strongly converges to $0$ in $H^1(\mathbb{R}^N)$ up to a subsequence.
Remark 1.3. Similar to the proof of Theorem 1.1 in [14], by minimizing \( J_{\lambda,\mu} \) on the Nehari manifold

\[
N_{\lambda,\mu} = \left\{ u \in E_{\lambda} \setminus \{0\}, \langle J'_{\lambda,\mu}(u), u \rangle = 0 \right\},
\]

we can demonstrate that Eq. (1.6) has a positive ground state solution \( v_{\lambda,\mu} \) for any \( \lambda, \mu > 0 \) large enough. It is easy to show \( J_{\lambda,\mu}(u_{\lambda,\mu}) > J_{\lambda,\mu}(v_{\lambda,\mu}) \). Indeed, if \( J_{\lambda,\mu}(u_{\lambda,\mu}) = J_{\lambda,\mu}(v_{\lambda,\mu}) \), then \( |u_{\lambda,\mu}| \in N_{\lambda,\mu} \) satisfies \( J_{\lambda,\mu}(|u_{\lambda,\mu}|) = \inf_{N_{\lambda,\mu}} J_{\lambda,\mu} \). Thereby, in a standard way, we may deduce \( J'_{\lambda,\mu}(|u_{\lambda,\mu}|) = 0 \). Whereas, the strong maximum principle implies \( |u_{\lambda,\mu}| > 0 \) in \( \mathbb{R}^N \), and the regular estimates for Choquard equations (see e.g. [21, 22]) implies \( u_{\lambda,\mu} \in C(\mathbb{R}^N, \mathbb{R}) \), thus \( u_{\lambda,\mu} \) has constant sign in \( \mathbb{R}^N \), which contradicts with \( u_{\lambda,\mu}^\pm \neq 0 \). Furthermore, due to the presence of the perturbed term \( \mu u^{p-2} u \), the methods introduced in [11, 32] to verify that the least energy of sign-changing solutions is less than twice the least energy of nontrivial solutions seem invalid here, we propose an open question whether \( J_{\lambda,\mu}(u_{\lambda,\mu}) < 2 J_{\lambda,\mu}(v_{\lambda,\mu}) \).

Remark 1.4. To our knowledge, there seem to be no results on (ground state) sign-changing solutions for Choquard equations with upper critical exponent, even on the bounded domain. Our present work extends and improves the existence results of sign-changing solutions verified in [7, 10, 11, 28, 33]. In [5], the authors studied the ground state sign-changing solutions for a class of critical Schrödinger equations

\[
\begin{cases}
-\Delta u - \lambda u = |u|^{2^*-2}u & \text{in } \mathcal{D}, \\
u = 0 & \text{on } \partial \mathcal{D},
\end{cases}
\]

where \( \mathcal{D} \subset \mathbb{R}^N \) (\( N \geq 6 \)) is a bounded domain and \( \lambda \in (0, \lambda_1) \), with \( \lambda_1 \) denoting the first eigenvalue of \( -\Delta \) on \( \mathcal{D} \). They proved that any sign-changing \((PS)_c\) sequence is relatively compact once \( c < c_0 + \frac{1}{\lambda} S_{\lambda}^2 \), where \( c_0 \) is the least energy of nontrivial solutions. As a counterpart for the work in [5], Zhong and Tang studied a class of Choquard equations with critical Sobolev exponent in [33], where they showed the relative compactness of sign-changing \((PS)_c\) sequence with \( c \) less than the similar threshold. However, in this paper, due to the presence of the upper critical nonlocal term \( (I_{\lambda} * |u|^2_s)|u|^{2^*-2}u \) in Eq. (1.6), the relative compactness of sign-changing \((PS)_c\) sequence with

\[
c \in \left[ \frac{2 + \alpha}{2(N + \alpha)} S_{\alpha}^{\frac{N+\alpha}{2}} \inf_{N_{\lambda,\mu}} J_{\lambda,\mu} + \frac{2 + \alpha}{2(N + \alpha)} S_{\alpha}^{\frac{N+\alpha}{2}}, \right]
\]

cannot be deduced as expected, where \( S_\alpha \) is defined by (2.12) hereinafter. Also, it seems intractable to search for sign-changing \((PS)_c\) sequence such that \( c < \frac{2^{\alpha} S_{\alpha}^{(N+\alpha)/(2+\alpha)}}{2(N+\alpha)} S_{\alpha}^{(N+\alpha)/(2+\alpha)} \) for small \( \mu > 0 \). Naturally, we attempt to construct a sign-changing \((PS)_c\) sequence with \( c < \frac{2^{\alpha} S_{\alpha}^{(N+\alpha)/(2+\alpha)}}{2(N+\alpha)} S_{\alpha}^{(N+\alpha)/(2+\alpha)} \) by assuming that \( \mu > 0 \) is sufficiently large. Therefrom, by applying the properties of steep well potential \( \lambda V \), we can standardly prove the relative compactness of this type of sign-changing \((PS)_c\) sequence and then obtain ground state sign-changing solution.

We will give the proof of Theorem 1.2 in the forthcoming section. Throughout this paper, we use the following notations:

\( \blacklozenge \) \( L^p(\mathbb{R}^N) \) is the usual Lebesgue space with the norm \( |u|_p = (\int_{\mathbb{R}^N} |u|^p dx)^{\frac{1}{p}} \) for \( p \in [1, +\infty) \).

\( \clubsuit \) \( L^\infty(\mathbb{R}^N) \) is the space of measurable functions with the norm \( |u|_\infty = \text{ess sup}_{x \in \mathbb{R}^N} |u(x)| \).
\begin{itemize}
\item $C_c^\infty(\mathbb{R}^N)$ consists of infinitely times differentiable functions with compact support in $\mathbb{R}^N$.
\item $H^1(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N) \}$ endowed with the inner product and norm 
\[(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + u v dx \quad \text{and} \quad \| u \| = (u, u)^{\frac{1}{2}}.
\item $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ with the norm $\| u \|_\Omega = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$.
\item $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\| u \|_D = |\nabla u|_2$.
\item The best Sobolev constant $S = \inf \{ \| u \|_D^2 : u \in D^{1,2}(\mathbb{R}^N) \}$.
\item $u^\pm(x) = \pm \max \{ \pm u(x), 0 \}$ and $(E^*, \| \cdot \|_*)$ is the dual space of Banach space $(E, \| \cdot \|)$.
\item $\mathcal{B}_r(y)$ is a quantity tending to 0 as $n \to +\infty$ and $|\Omega|$ is the Lebesgue measure of $\Omega \subset \mathbb{R}^N$.
\item $\mathcal{B}_r = \{ x \in \mathbb{R}^N : |x - y| < r \}$, $\mathcal{B}^c_r = \mathbb{R}^N \setminus \mathcal{B}_r(y)$ and $\mathcal{B}_r(0) = \mathcal{B}_r$ for $r > 0, y \in \mathbb{R}^N$.
\end{itemize}

\section{Proof of Theorem 1.2}

For the limiting problem of Eq. (1.6) as $\lambda \to +\infty$, namely Eq. (1.8), its energy functional is 
\[J_{\infty, \mu}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + u^2 dx - \frac{A_n}{2^{2\gamma}} \int_\Omega \int_\Omega \frac{|u(x)|^2 |u(y)|^{2\gamma}}{|x - y|^{N-2\gamma}} dy dx - \frac{\mu}{p} \int_\Omega |u|^p dx.\]

Due to (1.4) and $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, $J_{\infty, \mu} \in C^1(H_0^1(\Omega), \mathbb{R})$. Define the sign-changing Nehari manifolds 
\[M_{\lambda, \mu} = \{ u \in E_\lambda : u^\pm \neq 0, \langle J'_{\lambda, \mu}(u), u^\pm \rangle = 0 \}, \]
\[M_{\infty, \mu} = \{ u \in H_0^1(\Omega) : u^\pm \neq 0, \langle J'_{\infty, \mu}(u), u^\pm \rangle = 0 \}. \]

Clearly, $M_{\lambda, \mu}$ and $M_{\infty, \mu}$ contain all of the sign-changing solutions of Eqs. (1.6) and (1.8), respectively. To search for ground state sign-changing solutions, we consider the following minimization problems:

\[m_{\lambda, \mu} = \inf \{ J_{\lambda, \mu}(u) : u \in M_{\lambda, \mu} \}, \]
\[m_{\infty, \mu} = \inf \{ J_{\infty, \mu}(u) : u \in M_{\infty, \mu} \}. \]

Before completing the proof of Theorem 1.2, we establish several preliminary lemmas.

\textbf{Lemma 2.1.} For any $\lambda > 0$, $\mu > 0$ and $u \in E_\lambda$ with $u^\pm \neq 0$, there exists a unique pair $(s_{\lambda, \mu, u}, t_{\lambda, \mu, u})$ of positive numbers such that $s_{\lambda, \mu, u}^\frac{1}{2} u^+ + t_{\lambda, \mu, u}^\frac{1}{2} u^- \in M_{\lambda, \mu}$, also,

\[J_{\lambda, \mu}(s_{\lambda, \mu, u}^\frac{1}{2} u^+ + t_{\lambda, \mu, u}^\frac{1}{2} u^-) = \max_{s, t \geq 0} J_{\lambda, \mu}(s^\frac{1}{2} u^+ + t^\frac{1}{2} u^-).
\]

\textbf{Proof.} Firstly, we certify the existence of such pair of numbers. For any $\lambda > 0$, $\mu > 0$ and
$u \in E_\lambda$ with $u^\pm \neq 0$, define the function $F_{\lambda,\mu}(s, t)$ for any $(s, t) \in [0, +\infty)^2$ by

$$F_{\lambda,\mu}(s, t) = \mathcal{J}_{\lambda,\mu}(\frac{1}{\lambda} u^+ + \frac{1}{\mu} u^-)$$

$$= \frac{s^2}{2} \|u^+\|_{\lambda} - \frac{s^2}{2} \cdot \frac{1}{2} \int_{\mathbb{R}^N} (I_{\alpha} * |u^+|^2) |u^+|^2 dx - \frac{\mu s^2}{p} \int_{\mathbb{R}^N} |u^+|^p dx$$

$$+ \frac{t^2}{2} \|u^-\|_{\mu}^2 - \frac{t^2}{2} \cdot \frac{1}{2} \int_{\mathbb{R}^N} (I_{\alpha} * |u^-|^2) |u^-|^2 dx - \frac{\mu t^2}{p} \int_{\mathbb{R}^N} |u^-|^p dx$$

$$- st \int_{\mathbb{R}^N} (I_{\alpha} * |u^+|^2) |u^-|^2 dx.$$ 

It is easy to derive $\lim_{\|s, t\| \to 0} F_{\lambda,\mu}(s, t) = 0$ and $\lim_{\|s, t\| \to +\infty} F_{\lambda,\mu}(s, t) = -\infty$. Then there exists some point $(s_{\lambda,\mu}, t_{\lambda,\mu}) \in [0, +\infty)^2$ such that

$$F_{\lambda,\mu}(s_{\lambda,\mu}, t_{\lambda,\mu}) = \max_{(s, t) \in [0, +\infty)^2} F_{\lambda,\mu}(s, t).$$

Since $F_{\lambda,\mu}(s, t_{\lambda,\mu})$ is increasing in $s$ for $s > 0$ small enough, there results $s_{\lambda,\mu} \neq 0$. Similarly, we deduce $t_{\lambda,\mu} \neq 0$. Thereby, $(s_{\lambda,\mu}, t_{\lambda,\mu}) \in (0, +\infty)^2$. Then

$$\frac{\partial F_{\lambda,\mu}}{\partial s}(s_{\lambda,\mu}, t_{\lambda,\mu}) = \frac{\partial F_{\lambda,\mu}}{\partial t}(s_{\lambda,\mu}, t_{\lambda,\mu}) = 0.$$ 

Naturally, $\frac{1}{\lambda} s_{\lambda,\mu} u^+ + \frac{1}{\mu} t_{\lambda,\mu} u^- \in \mathcal{M}_{\lambda,\mu}$.

Further, we claim such pair of numbers is unique. For brevity, we introduce the notation

$$B(u, v) := \frac{1}{2} \int_{\mathbb{R}^N} (I_{\alpha} * |u|^2) |v|^2 dx, \quad \forall \ u, v \in E_\lambda.$$ 

Through direct calculation, we deduce that the Hessian matrix of $F_{\lambda,\mu}$ at $(s, t) \in (0, +\infty)^2$ is

$$H_{\lambda,\mu}(s, t) = \frac{2 - 2s^2}{(2s^2)^2} \begin{pmatrix} \frac{s^2}{2} - 2 \|u^+\|_{\lambda}^2 & 0 \\ 0 & \frac{t^2}{2} - 2 \|u^-\|_{\mu}^2 \end{pmatrix}$$

$$- \begin{pmatrix} B(u^+, u^+) & B(u^+, u^-) \\ B(u^+, u^-) & B(u^-, u^-) \end{pmatrix} - \frac{\mu (p - 2s^2)}{(2s^2)^2} \begin{pmatrix} \frac{s^2}{2} - 2 \|u^+\|_{\lambda}^2 & 0 \\ 0 & \frac{t^2}{2} - 2 \|u^-\|_{\mu}^2 \end{pmatrix}.$$ 

It follows from [17, Theorem 9.8] that $B(u^+, u^-)^2 < B(u^+, u^+)B(u^-, u^-)$. Then, noting $p \geq 2s^2$, we conclude that $H_{\lambda,\mu}(s, t)$ is negative defined for any $(s, t) \in (0, +\infty)^2$. Thereby, it is easy to know that $F_{\lambda,\mu}$ has at most one critical point on $(0, +\infty)^2$. Thus, $(s_{\lambda,\mu}, t_{\lambda,\mu})$ is the unique pair of positive numbers such that $\frac{1}{\lambda} s_{\lambda,\mu} u^+ + \frac{1}{\mu} t_{\lambda,\mu} u^- \in \mathcal{M}_{\lambda,\mu}$, and this lemma is proved. □

As a by-product, we may derive $\mathcal{M}_{\infty,\mu} \neq \emptyset$. Indeed, since $\mathcal{J}_{\lambda,\mu} = \mathcal{J}_{\infty,\mu}$ in $H_0^1(\Omega)$, we have

**Remark 2.2.** For any $\mu > 0$ and $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$, there exists a unique pair $(s_{\mu,\mu}, t_{\mu,\mu})$ of positive numbers such that $\frac{1}{\lambda} s_{\mu,\mu} u^+ + \frac{1}{\mu} t_{\mu,\mu} u^- \in \mathcal{M}_{\infty,\mu}$ and

$$\mathcal{J}_{\infty,\mu}(\frac{1}{\lambda} s_{\mu,\mu} u^+ + \frac{1}{\mu} t_{\mu,\mu} u^-) = \max_{s, t \geq 0} \mathcal{J}_{\infty,\mu}(\frac{1}{\lambda} s u^+ + \frac{1}{\mu} t u^-).$$
To facilitate the subsequent discussion, we show some properties of $M_{\lambda, \mu}$ in the following

**Lemma 2.3.** For any $\lambda > 0$ and $\mu > 0$, if $\{u_n\} \subset M_{\lambda, \mu}$ and $\lim _{n \to \infty} J_{\lambda, \mu}(u_n) = m_{\lambda, \mu}$, then $m_{\lambda, \mu} > 0$ and there exist some constants $C_{\lambda, \mu, 1}, C_{\lambda, \mu, 2} > 0$ such that $C_{\lambda, \mu, 2} \leq \|u_n\|_{\lambda} \leq C_{\lambda, \mu, 1}$ for all $n$.

**Proof.** From $M_{\lambda, \mu} \neq \emptyset$, we know $m_{\lambda, \mu} < +\infty$ for any $\lambda, \mu > 0$. Since $\{u_n\} \subset M_{\lambda, \mu}$, there holds

$$m_{\lambda, \mu} + o(1) = J_{\lambda, \mu}(u_n) - \frac{1}{p} \left( J_{\lambda, \mu}'(u_n), u_n \right) \geq \frac{p - 2}{2p} \|u_n\|_{\lambda}^2.$$  

(2.1)

Then there is constant $C_{\lambda, \mu, 1} > 0$ such that $\sup _n \|u_n\|_{\lambda} \leq C_{\lambda, \mu, 1}$. Thereby, (1.4) and (1.7) imply

$$\|u_n\|_{\lambda}^2 = \int _{\mathbb{R}^N} \left| \left( I_{\lambda} \ast |u_n| \right) |u_n| \right|^2 dx + \mu \int _{\mathbb{R}^N} |u_n| dx \leq A_\lambda C(N, \alpha) \nu_\lambda^{2+2\alpha} \|u_n\|_{\lambda}^2 + \mu \nu_\lambda^p \|u_n\|_{\lambda}^p \leq A_\lambda \nu_\lambda^{2+2\alpha} C_{\lambda, \mu, 1} \|u_n\|_{\lambda}^2 + \mu \nu_\lambda^p \|u_n\|_{\lambda}^p.$$  

As a consequence, there exists some constant $C_{\lambda, \mu, 2} > 0$ such that $\inf _\gamma \|u_n\|_{\lambda} \geq C_{\lambda, \mu, 2}$. Further, we deduce from (2.1) that $m_{\lambda, \mu} > 0$. Thus we complete the proof of this lemma. □

Next, following [5], we construct a sign-changing $(PS)_c$ sequence $\{u_n\}$ for $J_{\lambda, \mu}$, i.e. $u_n \neq 0$ for any $n$, $J_{\lambda, \mu}(u_n) \to c$ and $J_{\lambda, \mu}'(u_n) \to 0$ in $E_\lambda$ as $n \to \infty$. Let $P_\lambda$ be the cone of nonnegative functions in $E_\lambda$, $Q = [0, 1]^2$ and $\Gamma_{\lambda, \mu}$ be the set of continuous maps $\gamma : Q \to E_\lambda$ such that, for any $(s, t) \in Q$,

(a) $\gamma(s, 0) = 0$, $\gamma(0, t) \in P_\lambda$ and $\gamma(1, t) \in -P_\lambda$,

(b) $(J_{\lambda, \mu} \circ \gamma)(s, 1) \leq 0$ and

$$\int _{\mathbb{R}^N} \left( I_{\lambda} \ast |\gamma(s, 1)|^{2-} \right) |\gamma(s, 1)|^{2+} + \mu |\gamma(s, 1)|^p dx \|\gamma(s, 1)\|_{\lambda}^2 \geq 2.$$  

For any $u \in E_\lambda$ with $u \neq 0$, define $\gamma_{\sigma, \mu}(s, t) = \sigma t(1-s)u^+ + \sigma ts u^-$ for $\sigma > 0$ and $(s, t) \in Q$. It is easy to show $\gamma_{\sigma, \mu} \in \Gamma_{\lambda, \mu}$ for $\sigma > 0$ large enough. Therefore, $\Gamma_{\lambda, \mu} \neq \emptyset$. Define the functional

$$L_{\lambda, \mu}(u, v) = \left\{ \begin{array}{ll} \int _{\mathbb{R}^N} \left( I_{\lambda} \ast |u|^{2-} \right) \left( |u|^{2+} + |v|^{2+} \right) + \mu |u|^p \right\} dx \|u\|_{\lambda}^2, & u \neq 0, \\ 0, & u = 0. \end{array} \right.$$  

Clearly, $L_{\lambda, \mu} > 0$ if $u \neq 0$. Moreover, $u \in M_{\lambda, \mu}$ if and only if $L_{\lambda, \mu}(u^+, u^-) = L_{\lambda, \mu}(u^-, u^+) = 1$.

As a start point, we display a minimax characterization on $m_{\lambda, \mu}$ for any $\lambda > 0$ and $\mu > 0$.

**Lemma 2.4.** For any $\lambda > 0$ and $\mu > 0$, there holds

$$m_{\lambda, \mu} = \inf _{\gamma \in \Gamma_{\lambda, \mu}} \max _{(s, t) \in Q} J_{\lambda, \mu}(\gamma(s, t)).$$  

(2.2)

**Proof.** On the one hand, for every $u \in M_{\lambda, \mu}$, $\gamma_u(s, t) = \sigma t(1-s)u^+ + \sigma ts u^- \in \Gamma_{\lambda, \mu}$ for some $\sigma > 0$ large enough. Then it follows from Lemma 2.1 that

$$J_{\lambda, \mu}(u) = \max _{s,t \geq 0} J_{\lambda, \mu}(su^+ + tu^-) \geq \max _{(s, t) \in Q} J_{\lambda, \mu}(\gamma_u(s, t)) \geq \inf _{\gamma \in \Gamma_{\lambda, \mu}} \max _{(s, t) \in Q} J_{\lambda, \mu}(\gamma(s, t)).$$
Thereby, due to the arbitrariness of \( u \in \mathcal{M}_{\lambda, \mu} \) there results

\[
m_{\lambda, \mu} \geq \inf_{\gamma \in \Gamma_{\lambda, \mu}} \max_{(s, t) \in Q} J_{\lambda, \mu}(\gamma(s, t)).
\]

On the other hand, for each \( \gamma \in \Gamma_{\lambda, \mu} \) and \( t \in [0, 1] \), since \( \gamma(0, t) \in P_\lambda \) and \( \gamma(1, t) \in -P_\lambda \), we conclude

\[
\begin{align*}
\mathcal{L}_{\lambda, \mu}(\gamma(0, t)^+, \gamma(0, t)^-) - \mathcal{L}_{\lambda, \mu}(\gamma(0, t)^-, \gamma(0, t)^+) &= \mathcal{L}_{\lambda, \mu}(\gamma(0, t)^+, \gamma(0, t)^-) \geq 0, \quad (2.3) \\
\mathcal{L}_{\lambda, \mu}(\gamma(1, t)^+, \gamma(1, t)^-) - \mathcal{L}_{\lambda, \mu}(\gamma(1, t)^-, \gamma(1, t)^+) &= -\mathcal{L}_{\lambda, \mu}(\gamma(1, t)^-, \gamma(1, t)^+) \leq 0. \quad (2.4)
\end{align*}
\]

Meanwhile, due to \( \gamma(s, 0) = 0 \) for all \( s \in [0, 1] \), there holds

\[
\mathcal{L}_{\lambda, \mu}(\gamma(s, 0)^+, \gamma(s, 0)^-) + \mathcal{L}_{\lambda, \mu}(\gamma(s, 0)^-, \gamma(s, 0)^+) - 2 = -2, \quad \forall s \in [0, 1]. \quad (2.5)
\]

And, for each \( \gamma \in \Gamma_{\lambda, \mu} \), by the definition of \( \mathcal{L}_{\lambda, \mu} \) and the property (b) we have, for all \( s \in [0, 1] \),

\[
\begin{align*}
\mathcal{L}_{\lambda, \mu}(\gamma(s, 1)^+, \gamma(s, 1)^-) + \mathcal{L}_{\lambda, \mu}(\gamma(s, 1)^-, \gamma(s, 1)^+) - 2 &
\geq \int_{\mathbb{R}^N} \left[ (I_\ast |\gamma(s, 1)|^2_2)|\gamma(s, 1)|^2_2 + \mu |\gamma(s, 1)|^p \right] \, dx \\
\|\gamma(s, 1)\|_\lambda^2 &\geq 2. \quad (2.6)
\end{align*}
\]

Moreover, it is easy to verify that, for any \((s, t) \in \partial Q\),

\[
\left( \mathcal{L}_{\lambda, \mu}(\gamma(s, t)^+, \gamma(s, t)^-) - \mathcal{L}_{\lambda, \mu}(\gamma(s, t)^-, \gamma(s, t)^+) \right) \neq \left( 0, 0 \right). \quad (2.7)
\]

Then, by combining (2.3)–(2.7) with the Miranda theorem (see e.g. Lemma 2.4 in [13]), we derive that there exists some \((s_\gamma, t_\gamma) \in (0, 1)^2\) satisfying

\[
\begin{align*}
\mathcal{L}_{\lambda, \mu}(\gamma(s_\gamma, t_\gamma)^+, \gamma(s_\gamma, t_\gamma)^-) - \mathcal{L}_{\lambda, \mu}(\gamma(s_\gamma, t_\gamma)^-, \gamma(s_\gamma, t_\gamma)^+) &= 0, \\
\mathcal{L}_{\lambda, \mu}(\gamma(s_\gamma, t_\gamma)^+, \gamma(s_\gamma, t_\gamma)^-) + \mathcal{L}_{\lambda, \mu}(\gamma(s_\gamma, t_\gamma)^-, \gamma(s_\gamma, t_\gamma)^+) &= 2.
\end{align*}
\]

In view of this fact, we easily obtain

\[
\mathcal{L}_{\lambda, \mu}(\gamma(s_\gamma, t_\gamma)^+, \gamma(s_\gamma, t_\gamma)^-) = \mathcal{L}_{\lambda, \mu}(\gamma(s_\gamma, t_\gamma)^-, \gamma(s_\gamma, t_\gamma)^+) = 1,
\]

which implies \( \gamma(s_\gamma, t_\gamma) \in \mathcal{M}_{\lambda, \mu} \). Consequently, from the arbitrariness of \( \gamma \in \Gamma_{\lambda, \mu} \), we deduce

\[
\inf_{\gamma \in \Gamma_{\lambda, \mu}} \max_{(s, t) \in Q} J_{\lambda, \mu}(\gamma(s, t)) \geq m_{\lambda, \mu}.
\]

Now, by combining the above two sides, we know (2.2) holds. Thus this lemma is showed. \( \square \)

**Lemma 2.5.** For any \( \lambda > 0 \) and \( \mu > 0 \), \( J_{\lambda, \mu} \) possesses a sign-changing (PS)\(_{m_{\lambda, \mu}} \) sequence \( \{u_n\} \subset E_\lambda \).

**Proof.** We will end the proof in two steps. Firstly, we construct a (PS)\(_{m_{\lambda, \mu}} \) sequence for \( J_{\lambda, \mu} \).

Take a minimizing sequence \( \{w_n\} \subset \mathcal{M}_{\lambda, \mu} \) for \( m_{\lambda, \mu} \) and set \( \gamma_{\bar{c}, \bar{n}}(s, t) = \sigma t(1 - s)w_n^+ + \sigma tsw_n^- \).

By Lemma 2.3, it is easy to choose a sufficiently large constant \( \bar{c} > 0 \) such that \( \{\gamma_{\bar{c}, \bar{n}}\} \subset \Gamma_{\lambda, \mu} \).

Due to Lemmas 2.1 and 2.4, there holds

\[
\lim_{n \to \infty} \max_{(s, t) \in Q} J_{\lambda, \mu}(\gamma_{\bar{c}, \bar{n}}(s, t)) = \lim_{n \to \infty} J_{\lambda, \mu}(w_n) = m_{\lambda, \mu}. \quad (2.8)
\]
We assert that there exists some sequence \( \{ u_n \} \subset E_\lambda \) such that, as \( n \to \infty \),
\[
J_{\lambda, \mu}(u_n) \to m_{\lambda, \mu}, \quad J'_{\lambda, \mu}(u_n) \to 0, \quad \min_{(s, t) \in Q} \| u_n - \gamma_{\sigma, n}(s, t) \|_{\lambda} \to 0. \tag{2.9}
\]
If not, there exists some constant \( \delta_{\lambda, \mu} > 0 \) such that, for \( n \) suitably large, \( \gamma_{\sigma, n}(Q) \cap U_{\delta_{\lambda, \mu}} = \emptyset \), in which
\[
U_{\delta_{\lambda, \mu}} \triangleq \{ u \in E_\lambda : \exists v \in E_\lambda \text{ s.t. } \| v - u \|_{\lambda} \leq \delta_{\lambda, \mu}, \| \nabla J_{\lambda, \mu}(v) \| \leq \delta_{\lambda, \mu}, | J_{\lambda, \mu}(v) - m_{\lambda, \mu} | \leq \delta_{\lambda, \mu} \}.
\]
Then, by a variant of the classical deformation lemma due to Hofer (see [12, Lemma 1]), there exists a continuous map \( \eta_{\lambda, \mu} : [0, 1] \times E_\lambda \to E_\lambda \), which satisfies that, for some \( \varepsilon_{\lambda, \mu} \in (0, \frac{m_{\lambda, \mu}}{2}) \),
\[
\text{(i) } \eta_{\lambda, \mu}(0, u) = u, \ \eta_{\lambda, \mu}(\tau, u) = -\eta_{\lambda, \mu}(\tau, u), \ \forall \tau \in [0, 1], \ u \in E_\lambda,
\]
\[
\text{(ii) } \eta_{\lambda, \mu}(\tau, u) = u, \ \forall \tau \in J_{\lambda, \mu}^{m_{\lambda, \mu} - \varepsilon_{\lambda, \mu}} \cup \left( E_\lambda \setminus J_{\lambda, \mu}^{m_{\lambda, \mu} + \varepsilon_{\lambda, \mu}} \right), \ \forall \tau \in [0, 1],
\]
\[
\text{(iii) } \eta_{\lambda, \mu}
\begin{cases}
1, J_{\lambda, \mu}^{m_{\lambda, \mu} - \varepsilon_{\lambda, \mu}} \cup U_{\delta_{\lambda, \mu}} & \subset J_{\lambda, \mu}^{m_{\lambda, \mu} - \frac{\varepsilon_{\lambda, \mu}}{2}} \text{ for } \tau \in [0, 1],
\end{cases}
\]
\[
\text{(iv) } \eta_{\lambda, \mu}
\begin{cases}
1, (J_{\lambda, \mu}^{m_{\lambda, \mu} + \varepsilon_{\lambda, \mu}} \cap P_{\lambda}) \setminus U_{\delta_{\lambda, \mu}} & \subset J_{\lambda, \mu}^{m_{\lambda, \mu} - \frac{\varepsilon_{\lambda, \mu}}{2}} \cap P_{\lambda},
\end{cases}
\]
where the sublevel set \( J_{\lambda, \mu}^d := \{ u \in E_\lambda : J_{\lambda, \mu}(u) \leq d \} \) for \( d \in \mathbb{R} \). By (2.8), we choose large \( n \) such that
\[
\gamma_{\sigma, n}(Q) \subset J_{\lambda, \mu}^{m_{\lambda, \mu} + \varepsilon_{\lambda, \mu}} \text{ and } \gamma_{\sigma, n}(Q) \cap U_{\delta_{\lambda, \mu}} = \emptyset. \tag{2.10}
\]
Set the continuous map \( \tilde{\gamma}_{\lambda, \mu, n}(s, t) = \eta_{\lambda, \mu}(1, \gamma_{\sigma, n}(s, t)) \) for any \( (s, t) \in Q \). We claim \( \tilde{\gamma}_{\lambda, \mu, n} \in \Gamma_{\lambda, \mu} \).

Indeed, from \( \gamma_{\sigma, n}(s, 0) = 0 \) and (ii), it follows that \( \tilde{\gamma}_{\lambda, \mu, n}(s, 0) = \eta_{\lambda, \mu}(1, 0) = 0 \) for any \( s \in [0, 1] \). Since \( \gamma_{\sigma, n}(0, t), -\gamma_{\sigma, n}(1, t) \in P_{\lambda} \) and (2.10) implies \( \gamma_{\sigma, n}(0, t), -\gamma_{\sigma, n}(1, t) \in J_{\lambda, \mu}^{m_{\lambda, \mu} - \varepsilon_{\lambda, \mu}} \cup U_{\delta_{\lambda, \mu}} \), we deduce from (i), (iv) that \( \tilde{\gamma}_{\lambda, \mu, n}(0, t) \in P_{\lambda} \) and \( \tilde{\gamma}_{\lambda, \mu, n}(1, t) \in -P_{\lambda} \) for all \( t \in [0, 1] \). Also, \( J_{\lambda, \mu}(\gamma_{\sigma, n}(s, 1)) \leq 0 \) and (ii) imply \( \tilde{\gamma}_{\lambda, \mu, n}(s, 1) = \eta_{\lambda, \mu}(1, \gamma_{\sigma, n}(s, 1)) = \gamma_{\sigma, n}(s, 1) \) for any \( s \in [0, 1] \). Then, by \( \gamma_{\sigma, n} \in \Gamma_{\lambda, \mu} \), we know \( \tilde{\gamma}_{\lambda, \mu, n} \) satisfies the property (b). From the above arguments, we derive our claim \( \tilde{\gamma}_{\lambda, \mu, n} \in \Gamma_{\lambda, \mu} \).

Thereby, since (2.10) and (iii) imply \( \tilde{\gamma}_{\lambda, \mu, n}(Q) \subset J_{\lambda, \mu}^{m_{\lambda, \mu} - \frac{\varepsilon_{\lambda, \mu}}{2}} \), we conclude
\[
m_{\lambda, \mu} \leq \max_{(s, t) \in Q} J_{\lambda, \mu}(\tilde{\gamma}_{\lambda, \mu, n}(s, t)) \leq m_{\lambda, \mu} - \frac{\varepsilon_{\lambda, \mu}}{2},
\]
which is a contradiction. Thus there is a sequence \( \{ u_n \} \subset E_\lambda \) possessing the properties in (2.9).

Secondly, we prove \( u_n^+ \neq 0 \) for all large \( n \). By (2.9), there exists a sequence \( \{ v_n \} \) such that
\[
v_n = \alpha_n u_n^+ + \beta_n w_n^+ \in \gamma_{\sigma, n}(Q) \quad \text{and} \quad \| v_n - u_n \|_{\lambda} \xrightarrow{n \to \infty} 0. \tag{2.11}
\]
Due to \( \{ w_n \} \subset M_{\lambda, \mu} \) and \( p \in (2, 2^*) \), from (1.4), Lemma 2.3 and the Young inequality we have
\[
\| w_n^+ \|_{\lambda}^2 \leq A_\alpha C(N, \alpha)(v_2 \cdot C_{\lambda, \mu, 1})^2 \| w_n^+ \|_{2^*}^2 + \frac{2^* - p}{2} - \frac{2}{2^*} | w_n^+ \|_{2^*}^2 + \frac{2^* - 2}{2^* - 2} \| w_n^+ \|_{2^*}^2.
\]
Then, by (1.7), there holds
\[
\frac{p-2}{(2^*-2)\lambda^2} |w_n^+|^2_{2^*} \leq A_a C(N, \alpha) (\nu_2 C_{\lambda, \mu, 1})^{2^*} |w_n^+|_{2^*}^{2^*} + \frac{\mu p - 2}{2^* - 2} |w_n^+|_{2^*}^{2^*},
\]
which implies \( \inf_{\lambda} |w_n^+|^{2^*} > 0 \). In view of this fact, the second limiting formula in (2.11) and (1.7), to show \( u_n^+ \neq 0 \) for \( n \) large enough, it suffices to verify that \( \alpha_n \rightarrow 0 \) and \( \beta_n \rightarrow 0 \) up to subsequences. Suppose inversely \( \alpha_n \rightarrow 0 \) up to a subsequence. Then it follows from \( J_{\lambda, \mu} \in C(E, \mathbb{R}) \) and Lemma 2.3 that
\[
m_{\lambda, \mu} = \lim_{n \to \infty} J_{\lambda, \mu}(v_n) = \lim_{n \to \infty} J_{\lambda, \mu}(\alpha_n w_n^+ + \beta_n w_n^-) = \lim_{n \to \infty} J_{\lambda, \mu}(\beta_n w_n^-),
\]
which together with \( m_{\lambda, \mu} > 0 \) implies \( \beta := \sup_n \beta_n < +\infty \). Further, by Lemma 2.1, the Fubini theorem, Lemma 2.3, (1.4) and (1.7), we deduce
\[
m_{\lambda, \mu} \leq \lim_{n \to \infty} J_{\lambda, \mu}(w_n)
= \lim_{n \to \infty} \max_{s \geq 0} J_{\lambda, \mu}(sw_n^+ + tw_n^-)
\geq \lim_{n \to \infty} \max_{s \geq 0} J_{\lambda, \mu}(sw_n^* + \beta_n w_n^-)
= \lim_{n \to \infty} \max_{s \geq 0} \left[ \frac{s^2}{2} ||w_n^+||_{\lambda}^2 - \frac{s^2 \beta_n^2}{2 \cdot 2^*} \int_{\mathbb{R}^N} (I_\alpha * |w_n^+|^{2^*}) |w_n^+|^{2^*} dx - \frac{\mu sp}{p} \int_{\mathbb{R}^N} |w_n^+|^p dx \right]
+ \frac{\beta_n^2}{2} ||w_n^-||_{\lambda}^2 - \frac{\beta_n^2}{2 \cdot 2^*} \int_{\mathbb{R}^N} (I_\alpha * |w_n^-|^{2^*}) |w_n^-|^{2^*} dx + \frac{\mu \beta_n^2}{p} \int_{\mathbb{R}^N} |w_n^-|^p dx
- \frac{s^2 \beta_n^2}{2^*} \int_{\mathbb{R}^N} (I_\alpha * |w_n^-|^{2^*}) |w_n^-|^{2^*} dx
- \frac{s^2 \beta_n^2}{2} \int_{\mathbb{R}^N} (I_\alpha * |w_n^-|^{2^*}) |w_n^-|^{2^*} dx - \frac{\mu sp}{p} \int_{\mathbb{R}^N} |w_n^-|^p dx + J_{\lambda, \mu}(\beta_n w_n^-)
\geq \max_{s \geq 0} \left[ \frac{1}{2} C_{\lambda, \mu, 2} s^2 - \frac{1}{2} A_a C(N, \alpha) (\nu_2 C_{\lambda, \mu, 1})^{2^*} \beta_n^2 s^2 - \frac{\mu}{p} (v_\alpha C_{\lambda, \mu, 1})^p s^p \right]
- \frac{1}{2 \cdot 2^*} A_a C(N, \alpha) (\nu_2 C_{\lambda, \mu, 1})^{2^*} s^{2^*} + \lim_{n \to \infty} J_{\lambda, \mu}(\beta_n w_n^-)
> m_{\lambda, \mu},
\]
a contradiction. Naturally, \( \{\alpha_n\} \) has no subsequence tending to 0. Similarly, we can show \( \{\beta_n\} \) has no subsequence tending to 0. Thus \( u_n^+ \neq 0 \) for \( n \) large enough. This lemma is proved. \( \Box \)

Now, we estimate the least energy \( m_{\lambda, \mu} \) from above. By [9, Lemma 1.2], the best constant
\[
S_\alpha := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : u \in D^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*}) |u|^{2^*} dx = 1 \right\}
\]
is attained by the functions
\[
U_\varepsilon(\cdot) = \frac{\left[ N(N - 2) \varepsilon^2 \right]^{\frac{N-2}{2}}}{[C(N, \alpha) A_a S^2]^{\frac{N-2}{2}}} (\varepsilon^2 + | \cdot |^{2^*})^{\frac{N-2}{2}}, \quad \varepsilon > 0.
\]
Take $\delta > 0$ such that $\mathbb{B}_{5\delta} \subset \Omega$, and extract two cut-off functions $\varphi, \psi \in C_0^\infty(\Omega, [0, 1])$ satisfying
\[
\varphi(x) = \begin{cases} 
1, & x \in \mathbb{B}_{5\delta}, \\
0, & x \in \mathbb{B}_{2\delta}^c 
\end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} 
0, & x \in \mathbb{B}_{2\delta}, \\
1, & x \in \mathbb{B}_{4\delta} \setminus \mathbb{B}_{3\delta}, \\
0, & x \in \mathbb{B}_{5\delta}^c.
\end{cases}
\]

Define $u_\varepsilon = \varphi U_\varepsilon$ and $v_\varepsilon = \psi U_\varepsilon$. As in [3,4], through direct computation, we obtain, as $\varepsilon \to 0^+$,
\[
\int_{\Omega} |\nabla u_\varepsilon|^2 dx = S_\alpha^{N+\alpha} + O(\varepsilon^{N-2}), \quad (2.13)
\]
\[
\int_{\Omega} |u_\varepsilon|^2 dx = \begin{cases} 
O(\varepsilon), & N = 3, \\
O(\varepsilon^2 \ln \varepsilon), & N = 4, \\
O(\varepsilon^2), & N \geq 5
\end{cases} \quad (2.14)
\]
and
\[
\int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^2 |u_\varepsilon(y)|^2}{|x-y|^{N-\alpha}} dxdy = A_\alpha^{-1} S_\alpha^{N+\alpha} + O(\varepsilon^{N-2}). \quad (2.15)
\]

Additionally, as $\varepsilon \to 0^+$,
\[
\int_{\Omega} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 dx = O(\varepsilon^{N-2}) \quad \text{and} \quad \int_{\Omega} |v_\varepsilon(x)|^p dx \geq d_p \varepsilon^{(N-2)p} \text{ for some } d_p > 0. \quad (2.16)
\]

**Lemma 2.6.** There exists some $\mu_* > 0$ independent of $\lambda$ such that, for any $\lambda > 0$ and $\mu \geq \mu_*$,
\[
m_{\lambda, \mu} \leq m_{\infty, \mu} < m_* := \frac{2 + \alpha}{2(N + \alpha)} S_\alpha^{N+\alpha}.
\]

**Proof.** Since $M_{\infty, \mu} \subset M_{\lambda, \mu}$ and $\mathcal{J}_{\lambda, \mu} = \mathcal{J}_{\infty, \mu}$ on $M_{\infty, \mu}$, we easily derive $m_{\lambda, \mu} \leq m_{\infty, \mu}$. For any $\varepsilon > 0$ and $\mu > 0$, by Remark 2.2, there exist some constants $s_{\mu, \varepsilon} > 0, t_{\mu, \varepsilon} > 0$ such that $s_{\mu, \varepsilon} u_\varepsilon - t_{\mu, \varepsilon} v_\varepsilon \in M_{\infty, \mu}$ and $\mathcal{J}_{\infty, \mu}(s_{\mu, \varepsilon} u_\varepsilon - t_{\mu, \varepsilon} v_\varepsilon) = \max_{s,t>0} \mathcal{J}_{\infty, \mu}(su_\varepsilon - tv_\varepsilon)$. It suffices to show $\max_{s,t>0} \mathcal{J}_{\infty, \mu}(su_\varepsilon - tv_\varepsilon) < m_*$ for $\varepsilon > 0$ small enough. Noting $s_{\mu, \varepsilon} \cap \text{spt } v_\varepsilon = \emptyset$, we deduce
\[
\max_{s,t>0} \mathcal{J}_{\infty, \mu}(su_\varepsilon - tv_\varepsilon) \leq \max_{s>0} \mathcal{J}_{\infty, \mu}(su_\varepsilon) + \max_{t>0} \mathcal{J}_{\infty, \mu}(tv_\varepsilon). \quad (2.17)
\]

It easily follows from (2.13)–(2.15) that, for $\varepsilon > 0$ sufficiently small and all $\mu > 0, s > 0$,
\[
\mathcal{J}_{\infty, \mu}(su_\varepsilon) \leq S_\alpha^{N+\alpha} \left( s^2 - \frac{1}{4 \alpha^2} S_\alpha^{2.2s} \right).
\]

In view of this, there exist some sufficiently small $s_1 > 0$ and sufficiently large $s_2 > 0$ independent of $\varepsilon, \mu$ such that, for $\varepsilon > 0$ small enough and all $\mu > 0$,
\[
\max_{s \in (0, s_1)} \mathcal{J}_{\infty, \mu}(su_\varepsilon) < m_* \quad \text{and} \quad \max_{s \in (s_2, +\infty)} \mathcal{J}_{\infty, \mu}(su_\varepsilon) < 0.
\]
Moreover, from (2.13)–(2.15) again we conclude, for $\varepsilon > 0$ sufficiently small and any $\mu > 0$,

$$
\max_{s \in [s_1, s_2]} \mathcal{J}_{s, \mu}(su_c) \leq \max_{s > 0} \left( \frac{s^2}{2} \int_{\Omega} |\nabla u_c|^2 dx - \frac{s^{2-4\varepsilon}}{2} \mathcal{A}_n \int_{\Omega} \int_{\Omega} \frac{|u_c(x)|^2 |u_c(y)|^2}{|x - y|^{N-\alpha}} dxdy \right)
$$

$$
+ \frac{s^2}{2} \int_{\Omega} |u_c|^2 dx - \frac{\mu s^p}{p} \int_{\Omega} |u_c|^p dx
$$

$$
\leq \frac{2 + \alpha}{2(N + \alpha)} \sum_{n=1}^{N_0} \left[ 1 + O(\varepsilon^{N-2}) \right] \left[ 1 - O(\varepsilon^{N-2}) \right]
$$

$$
+ \frac{s^2}{2} \int_{\Omega} |u_c|^2 dx - \frac{\mu s^p \varepsilon^{N-(N-2)p}}{p} \int_{B_1} |U_1|^p dx
$$

$$
= \frac{2 + \alpha}{2(N + \alpha)} \sum_{n=1}^{N_0} \left[ O(\varepsilon^{N-2}) + \frac{s^2}{2} \int_{\Omega} |u_c|^2 dx - \frac{\mu s^p \varepsilon^{N-(N-2)p}}{p} \int_{B_1} |U_1|^p dx \right].
$$

If $N \geq 4$, or $N = 3$ and $\alpha \in (1, 3)$, by (2.14) and $p \geq 2\alpha$, we deduce, for $\varepsilon > 0$ small enough and $\mu > 0$,

$$
\eta_{N}(\varepsilon) := O(\varepsilon^{N-2}) + \frac{s^2}{2} \int_{\Omega} |u_c|^2 dx - \frac{\mu s^p \varepsilon^{N-(N-2)p}}{p} \int_{B_1} |U_1|^p dx < 0.
$$

If $N = 3$ and $\alpha \in (0, 1)$, take $\mu = \varepsilon^{2\alpha}$, by (2.14), there exists small $\varepsilon_1 > 0$ such that $\eta_3(\varepsilon) < 0$ for all $\varepsilon \in (0, \varepsilon_1]$. Based on the above discussion, for $\varepsilon > 0$ small enough and any $\mu \geq \varepsilon_1^{2\alpha}$ if $N = 3$ and $\alpha \in (0, 1)$, also, for $\varepsilon > 0$ small enough and any $\mu > 0$ if $N \geq 4$ or $N = 3$ and $\alpha \in (1, 3)$, we conclude

$$
\max_{s > 0} \mathcal{J}_{s, \mu}(su_c) < m_s.
$$

(2.18)

In addition, due to (2.16), there exists some $C_1 > 0$ such that, for $\varepsilon > 0$ small enough and any $\mu > 0$,

$$
\max_{t > 0} \mathcal{J}_{s, \mu}(tu_c) \leq \max_{t > 0} \left[ C_1 \varepsilon^{N-2}\alpha - \mu d_p (\varepsilon^{N-2}\alpha)^{\frac{p}{2}} \right] \leq \frac{(p - 2)(2C_1)^{\frac{1}{p}}}{2p(\mu d_p)^{\frac{1}{p}}}.
$$

(2.19)

Now, by combining (2.17), (2.18) and (2.19), there exists some large $\mu^* \in \left[ \frac{1}{\varepsilon_1}, +\infty \right)$ such that $\max_{s, t > 0} \mathcal{J}_{s, \mu}(tu_c - tv_c) < m_s$ for any $\mu \geq \mu^*$ and small $\varepsilon > 0$. Thus this lemma is proved.

In the forthcoming lemma, we show that $\mathcal{J}_{\lambda, \mu}$ satisfies the local $(PS)_c$ condition for $\lambda$ large.

**Lemma 2.7.** There exists some $\Lambda > 0$ independent of $\mu$ such that, for any $\lambda \geq \Lambda$ and $\mu \geq \mu^*$, each $(PS)_c$ sequence $\{u_n\} \subset E_\lambda$ for $\mathcal{J}_{\lambda, \mu}$, with level $c \in (0, m_s)$, has a convergent subsequence.

**Proof.** From the definition of $\{u_n\}$, there results

$$
m_s + o(1) + o(\|u_n\|_\lambda) \geq \mathcal{J}_{\lambda, \mu}(u_n) - \frac{1}{p} \left\langle \mathcal{J}_{\lambda, \mu}'(u_n), u_n \right\rangle \geq \frac{p - 2}{2p} \|u_n\|_\lambda^2.
$$

Then there exists some $C_2 > 0$ independent of $\lambda$ and $\mu$ such that $\limsup_n \|u_n\|_\lambda \leq C_2$. Naturally, $\{u_n\}$ is bounded in $E_\lambda$. Hence, there exists some $u \in E_\lambda$ such that, up to subsequences,

$$
\begin{align*}
\begin{cases}
    u_n \to u & \text{in } E_\lambda, \\
    u_n \to u & \text{in } L^p_{\text{loc}}(\mathbb{R}^N), \forall s \in [1, 2^*) \text{, as } n \to \infty, \\
    u_n(x) \to u(x) & \text{a.e. in } \mathbb{R}^N,
\end{cases}
\end{align*}
$$

(2.20)
Set $v_n = u_n - u$. Clearly, $\limsup_{n} \|v_n\|_\lambda \leq 2C_2$. We will show $\|v_n\|_\lambda \overset{n}{\to} 0$ up to a subsequence. Define

$$\beta = \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} v_n^2 \, dx.$$ 

We assert $\beta = 0$. Otherwise, $\beta > 0$. Due to $(V_5)$, there exists some large $R > 0$ such that

$$|\{x \in B^c_R(0) : V(x) \leq M\}| \leq \left(\frac{\beta S}{16C_2^2}\right)^\frac{3}{2}.$$ 

Then it follows from the Hölder and Sobolev inequalities that

$$\limsup_{n \to \infty} \int_{\{x \in B^c_R(0) : V(x) \leq M\}} v_n^2 \, dx \leq \limsup_{n \to \infty} \|v_n\|_\lambda^2 \leq \frac{\beta}{4}. \tag{2.21}$$

Moreover, if taking $\Lambda = \frac{1}{M} (16C_2^2\beta^{-1} - 1)$ and letting $\lambda \geq \Lambda$, we have

$$\limsup_{n \to \infty} \int_{\{x \in B^c_R(0) : V(x) > M\}} v_n^2 \, dx \leq \frac{1}{\lambda M + 1} \limsup_{n \to \infty} \|v_n\|_\lambda^2 \leq \frac{\beta}{4}. \tag{2.22}$$

Consequently, combining (2.20)–(2.22) leads to

$$\beta \leq \limsup_{n \to \infty} \int_{\mathbb{R}^N} v_n^2 \, dx = \limsup_{n \to \infty} \int_{B_1(0)} v_n^2 \, dx \leq \frac{\beta}{2},$$

which contradicts $\beta > 0$. That is, our claim $\beta = 0$ is true. Then, thanks to [29, Lemma 1.21],

$$v_n \to 0 \quad \text{in } L^2(\mathbb{R}^N), \quad \forall s \in (2, 2^*). \tag{2.23}$$

By (2.20), it is easy to show $\mathcal{J}_{\lambda, \mu}'(u) = 0$. Further, with $\langle \mathcal{J}_{\lambda, \mu}'(u_n), u_n \rangle = o(1)$ in hand, we deduce from (2.20), (2.23) and the nonlocal version of the Brézis–Lieb lemma (see e.g. [4, Lemma 2.2]) that

$$o(1) = \|v_n\|^2_\lambda - \int_{\mathbb{R}^N} (I_0 * |v_n|^2_\lambda) |v_n|^2 \, dx. \tag{2.24}$$

Set $\kappa = \limsup_{n \to \infty} \|v_n\|_\lambda$. Due to (2.24) and the definition of $S_\alpha$, there results $\kappa = 0$ or $\kappa \geq S_\alpha^{\frac{N+s}{N+\alpha}}$. We claim $\kappa = 0$. If not, because $\mathcal{J}_{\lambda, \mu}(u) \geq 0$, it follows from (2.20), (2.24) and Lemma 2.2 in [4] that

$$c = \lim_{n \to \infty} \mathcal{J}_{\lambda, \mu}(u_n) = \mathcal{J}_{\lambda, \mu}(u) + \frac{2 + \alpha}{2(N + \alpha)} \limsup_{n \to \infty} \|v_n\|^2_\lambda \geq \frac{2 + \alpha}{2(N + \alpha)} S_\alpha^{\frac{N+s}{N+\alpha}},$$

which contradicts $c < m_s$. Thus $u_n \to u$ in $E_\lambda$ up to a subsequence. This lemma is proved. \qed

Based on the above preliminary lemmas, we shall complete the proof of main results below.

**Proof of Theorem 1.2.** Let $\lambda \geq \Lambda$ and $\mu \geq \mu_s$. Thanks to Lemmas 2.5 and 2.6, $\mathcal{J}_{\lambda, \mu}$ has a sign-changing $(PS)_{m_{\lambda, \mu}}$ sequence $\{u_n\} \subset E_\lambda$, with $m_{\lambda, \mu} < m_s$. From Lemma 2.7, we derive that $u_n \to u_{\lambda, \mu}$ in $E_\lambda$ in the sense of subsequence. Then, there result $\mathcal{J}_{\lambda, \mu}'(u_{\lambda, \mu}) = 0$ in $E_{\lambda}^*$ and $\mathcal{J}_{\lambda, \mu}(u_{\lambda, \mu}) = m_{\lambda, \mu}$. Further, Lemma 2.3 implies $u_{\lambda, \mu}^\pm \not= 0$. That is, Eq. (1.6) has a ground state sign-changing solution $u_{\lambda, \mu}$.
Next, we show the concentration of ground state sign-changing solutions for Eq. (1.6) as $\lambda \to +\infty$. Given $\mu \geq \mu_*$ arbitrarily. For sequence $\{\lambda_n\} \subset [\Lambda, +\infty)$ with $\lambda_n \to +\infty$, let $u_{\lambda_n, \mu} \in E_{\lambda_n}$ be such that

$$u_{\lambda_n, \mu}^+ \neq 0, \quad J'_{\lambda_n, \mu}(u_{\lambda_n, \mu}) = 0 \quad \text{in } E_{\lambda_n}^*, \quad J_{\lambda_n, \mu}(u_{\lambda_n, \mu}) = m_{\lambda_n, \mu}.$$

By Lemma 2.6, it is easy to obtain

$$m_* > J_{\lambda_n, \mu}(u_{\lambda_n, \mu}) - \frac{1}{p} \left( J'_{\lambda_n, \mu}(u_{\lambda_n, \mu}), u_{\lambda_n, \mu} \right) > \frac{p-2}{2p} \|u_{\lambda_n, \mu}\|_{\lambda_n}^2.$$  \hfill (2.25)

Obviously, $\{u_{\lambda_n, \mu}\}$ is bounded in $H^1(\mathbb{R}^N)$. Then, there exists some $u_\mu \in H^1(\mathbb{R}^N)$ such that, up to subsequences,

$$\begin{align*}
&u_{\lambda_n, \mu} \xrightarrow{n} u_\mu \quad \text{in } H^1(\mathbb{R}^N), \\
&u_{\lambda_n, \mu} \xrightarrow{n} u_\mu \quad \text{in } L_{2s}^\infty(\mathbb{R}^N), \quad \forall \ s \in [1, 2^*), \\
&u_{\lambda_n, \mu}(x) \xrightarrow{n} u_\mu(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{2.26}
\end{align*}$$

It follows from the Fatou lemma, (2.25) and (2.26) that

$$0 \leq \int_{\mathbb{R}^N} V(x)u_\mu^2 dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} V(x)u_{\lambda_n, \mu}^2 dx \leq \liminf_{n \to \infty} \frac{\|u_{\lambda_n, \mu}\|_{\lambda_n}^2}{\lambda_n} = 0,$$

which together with (V6) implies $u_\mu|_{\mathbb{R}^N} = 0$. Then, $u_\mu \in H^1_0(\Omega)$, since $\partial \Omega$ is smooth. Thereby, for any $\omega \in H^1_0(\Omega)$, we derive from $\langle J'_{\lambda_n, \mu}(u_{\lambda_n, \mu}), \omega \rangle = 0$ and (2.26) that $J'_{\lambda_n, \mu}(u_\mu) = 0$.

Set $v_{\mu, n} = u_{\lambda_n, \mu} - u_\mu$. For any $\varepsilon > 0$, by (V5), there exists some large $R_\varepsilon > 0$ such that

$$\| \{ x \in B_{R_\varepsilon}^\infty : V(x) \leq M \} \| < \left[ \frac{(p-2)S\varepsilon}{4pm_*} \right]^\frac{2}{p-2}.$$

Then, due to the Hölder and Sobolev inequalities, the weakly lower semicontinuity of norm and (2.25), there holds

$$\int_{\{ x \in B_{R_\varepsilon}^\infty : V(x) \leq M \}} v_{\mu, n}^2 dx \leq \| \{ x \in B_{R_\varepsilon}^\infty : V(x) \leq M \} \|^\frac{2}{p-2} \| v_{n, \mu} \|_{\lambda_n}^2 \leq \varepsilon.$$

From the weakly lower semicontinuity of norm and (2.25), it follows that

$$\int_{\{ x \in B_{R_\varepsilon}^\infty : V(x) \geq M \}} v_{\mu, n}^2 dx \leq \| v_{n, \mu} \|_{\lambda_n}^2 \leq \frac{4pm_*}{(p-2)M\lambda_n} \to 0 \quad \text{as } n \to \infty.$$

Thereby, we deduce from (2.26) that $\| v_{\mu, n}|_{B_{R_\varepsilon}^\infty} \xrightarrow{n} 0$. Further, by (2.25), the Hölder and Sobolev inequalities, there holds

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N} |v_{\mu, n}|^p dx \leq \limsup_{n \to \infty} \left( \frac{2^{2^* - p}}{2^{2^* - 2}} |v_{\mu, n}|_{2^*}^{2^* - 2} + \frac{2^{2^* - p}}{2^{2^* - 2}} |v_{\mu, n}|_{2^*}^{2^* - 2} \right) \leq \left[ \frac{4pm_*}{(p-2)S} \right]^{\frac{2^*(p-2)}{2^*(p-2)}} \limsup_{n \to \infty} |v_{\mu, n}|_{2^*}^{2^*(p-2)} = 0. \tag{2.27}$$

By (2.26), (2.27), the nonlocal type of the Brézis–Lieb Lemma 2.2 in [4] and $J'_{\infty, \mu}(u_\mu) = 0$, we have

$$0 = \langle J'_{\lambda_n, \mu}(u_{\lambda_n, \mu}), u_{\lambda_n, \mu} \rangle = \| v_{\mu, n} \|_{\lambda_n}^2 - \int_{\mathbb{R}^N} \left( I_\mu * |v_{\mu, n}|^{2^*} \right) |v_{\mu, n}|^{2^*} dx + o(1). \tag{2.28}$$
Denote $\kappa_\mu = \limsup_{n \to \infty} \|v_{\nu,n}\|_{L^2}$. It follows from (2.28) and the definition of $S_n$ that $\kappa_\mu^2 \leq S_n^{-2} \kappa_\mu^2 S_n^{-2}$. Then, by (2.25), there results $\kappa_\mu = 0$ or $\kappa_\mu \geq S_n^{-2} \kappa_\mu^2 S_n^{-2}$. We assert $\kappa_\mu = 0$. If not, from Lemma 2.6, (2.25)–(2.28), the nonlocal type of the Brézis–Lieb lemma and $J_{\infty,\mu}'(u_\mu) = 0$, we have

$$m_\ast > \lim_{n \to \infty} J_{\lambda_n,\mu}(u_{\lambda_n,\mu})$$

$$= J_{\infty,\mu}(u_\mu) + \frac{2 + \alpha}{2(N + \alpha)} \limsup_{n \to \infty} \|v_{\nu,n}\|_{L^2}^2$$

$$= J_{\infty,\mu}(u_\mu) - \frac{1}{p} \left( J_{\infty,\mu}'(u_\mu), u_\mu \right) + \frac{2 + \alpha}{2(N + \alpha)} \kappa_\mu^2$$

$$\geq m_\ast,$$

a contradiction. Hence, $\|u_{\lambda_n,\mu} - u_\mu\|_{L^2} \not\to 0$. Then, it is easy to show $u_{\lambda_n,\mu} \to u_\mu$ in $H^1(\mathbb{R}^N)$. From $\left( J_{\lambda_n,\mu}'(u_{\lambda_n,\mu}), u_{\lambda_n,\mu}^\pm \right) = 0$, (1.4), the Young and Sobolev inequalities, we deduce that

$$\int_{\mathbb{R}^N} \left( I_k * |u_{\lambda_n,\mu}|^2 \right) |u_{\lambda_n,\mu}| \frac{2^*}{2} dx + \mu \|u_{\lambda_n,\mu}\|^p_p \leq A_k C(N, \alpha) \|u_{\lambda_n,\mu}\|^2_{L^2} \int_{\mathbb{R}^N} | u_{\lambda_n,\mu}^\pm |^2 dx + \frac{2^* - p}{2} \|u_{\lambda_n,\mu}\|^2 \frac{2^* - 2}{2} \|u_{\lambda_n,\mu}\|^2_{L^2'},$$

which together with (2.25) implies

$$\int_{\mathbb{R}^N} \left( I_k * |u_{\lambda_n,\mu}|^2 \right) |u_{\lambda_n,\mu}| \frac{2^*}{2} dx + \mu \|u_{\lambda_n,\mu}\|^p_p \leq A_k C(N, \alpha) \|u_{\lambda_n,\mu}\|^2_{L^2} \int_{\mathbb{R}^N} | u_{\lambda_n,\mu}^\pm |^2 dx + \frac{2^* - p}{2} \|u_{\lambda_n,\mu}\|^2 \frac{2^* - 2}{2} \|u_{\lambda_n,\mu}\|^2_{L^2'},$$

In view of this, there holds inf $\|u_{\lambda_n,\mu}\|^2_{L^2} > 0$. Therefore, $\|u_{\lambda_n,\mu} - u_\mu\| \not\to 0$ implies $\|u_\mu\|_{L^2}^2 > 0$. Naturally, $u_\mu^\pm \not= 0$ and then $u_\mu \in M_{\infty,\mu}$. Thus we derive from (2.26), the Fatou lemma and Lemma 2.6 that

$$m_{\infty,\mu} \leq J_{\infty,\mu}(u_\mu) - \frac{1}{p} \left( J_{\infty,\mu}'(u_\mu), u_\mu \right)$$

$$\leq \lim_{n \to \infty} \left[ \frac{p - 2}{2p} \|u_{\lambda_n,\mu}\|^2_{L^2} + \frac{2 \cdot 2^* - p}{2p} \left( \int_{\Omega} |\nabla u_{\lambda_n,\mu}|^2 + u_{\lambda_n,\mu}^2 \right) dx + A_k \int_{\Omega} \int_{\Omega} \frac{|u_{\lambda_n,\mu}(x)|^2 |u_{\lambda_n,\mu}(y)|^2}{|x - y|^{N+\alpha}} dx dy \right]$$

$$\leq \lim_{n \to \infty} \left[ \frac{p - 2}{2p} \|u_{\lambda_n,\mu}\|^2_{L^2} + \frac{2 \cdot 2^* - p}{2p} \left( \int_{\mathbb{R}^N} \left( I_k * |u_{\lambda_n,\mu}|^2 \right) |u_{\lambda_n,\mu}|^2 dx \right) \right]$$

$$= \lim_{n \to \infty} \left[ J_{\lambda_n,\mu}(u_{\lambda_n,\mu}) - \frac{1}{p} \left( J_{\lambda_n,\mu}'(u_{\lambda_n,\mu}), u_{\lambda_n,\mu} \right) \right]$$

$$\leq m_{\infty,\mu},$$

which leads to $J_{\infty,\mu}(u_\mu) = m_{\infty,\mu}$. Therefore, $u_\mu$ is a ground state sign-changing solution for Eq. (1.8).

Further, we certify the asymptotic behavior of ground state sign-changing solutions for Eq. (1.6) as $\mu \to +\infty$. Fix $\lambda \geq \Lambda$. For any sequence $\{\mu_n\} \subset [\mu_\ast, +\infty)$ with $\mu_n \to +\infty$, let $\{u_{\lambda,\mu_n}\} \subset E_\lambda$ satisfy

$$u_{\lambda,\mu_n}^\pm \not= 0, \quad J_{\lambda,\mu_n}'(u_{\lambda,\mu_n}) = 0 \text{ in } E_\lambda^+, \quad J_{\lambda,\mu_n}(u_{\lambda,\mu_n}) = m_{\lambda,\mu_n}.$$
It easily follows that
\[
m_{λ,μ_n} = J_{λ,μ_n}(u_{λ,μ_n}) - \frac{1}{p} \left( J'_{λ,μ_n}(u_{λ,μ_n}), u_{λ,μ_n} \right) \geq \frac{p-2}{2p} \|u_{λ,μ_n}\|_λ^2. \tag{2.29}
\]
We assert that \( \lim_{n \to \infty} m_{λ,μ_n} \to 0 \) in the sense of subsequence. Take \( ω \in H_0^1(Ω) \) such that \( ω^± ≠ 0 \). Due to Remark 2.2, there exist \( s_n > 0 \) and \( t_n > 0 \) such that \( s_nω^+ + t_nω^- \in M_{∞,μ_n} \). Then we have
\[
s_n^2 \int_Ω |∇ω^+|^2 + |ω^+|^2 \, dx
\[
= A_n s_n^2 \int_Ω \int_Ω \frac{|ω^+(x)|^2 |ω^+(y)|^2}{|x-y|^{N-a}} \, dx \, dy
\[
+ A_n (s_n t_n)^2 \int_Ω \int_Ω \frac{|ω^+(x)|^2 |ω^-(y)|^2}{|x-y|^{N-a}} \, dx \, dy + μ_n s_n^p \int_Ω |ω^+|^p \, dx, \tag{2.30}
\]
\[
t_n^2 \int_Ω |∇ω^-|^2 + |ω^-|^2 \, dx
\[
= A_n t_n^2 \int_Ω \int_Ω \frac{|ω^-(x)|^2 |ω^-(y)|^2}{|x-y|^{N-a}} \, dx \, dy
\[
+ A_n (t_n s_n)^2 \int_Ω \int_Ω \frac{|ω^+(x)|^2 |ω^-(y)|^2}{|x-y|^{N-a}} \, dx \, dy + μ_n t_n^p \int_Ω |ω^-|^p \, dx. \tag{2.31}
\]
From (2.30) and (2.31), we easily deduce that both \( \{s_n\} \) and \( \{t_n\} \) are bounded. Thereby, \( s_n \to s_0 \) and \( t_n \to t_0 \) up to subsequences. By using (2.30) and (2.31) again, we derive \( s_0 = t_0 = 0 \). Consequently, Lemmas 2.3 and 2.6 imply
\[
0 ≤ \limsup_{n \to ∞} m_{λ,μ_n} ≤ \limsup_{n \to ∞} m_{∞,μ_n} ≤ \limsup_{n \to ∞} J_{∞,μ_n}(s_nω^+ + t_nω^-)
\[
≤ \limsup_{n \to ∞} \left( s_n^2 \int_Ω |∇ω^+|^2 + |ω^+|^2 \, dx + t_n^2 \int_Ω |∇ω^-|^2 + |ω^-|^2 \, dx \right) = 0.
\]
Now, from (2.29) we conclude \( u_{λ,μ_n} \nrightarrow 0 \) in \( E_λ \). Naturally \( u_{λ,μ_n} \nrightarrow 0 \) in \( H^1(\mathbb{R}^N) \) in the sense of subsequence. Thus, based on the above arguments, we complete the proof of Theorem 1.2.

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