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Half-linear differential equations: Regular variation, principal solutions, and asymptotic classes

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Abstract. We are interested in the structure of the solution space of second-order half-linear differential equations taking into account various classifications regarding asymptotics of solutions. We focus on an exhaustive analysis of the relations among several types of classes which include the classes constructed with respect to the values of the limits of solutions and their quasiderivatives, the classes of regularly varying solutions, the classes of principal and nonprincipal solutions, and the classes of the solutions that obey certain asymptotic formulae. Many of our observations are new even in the case of linear differential equations, and we provide also the revision of existing results.

Keywords: half-linear differential equation, regularly varying function, principal solution, asymptotic formula.

2020 Mathematics Subject Classification: 26A12, 34D05, 34E10.

1 Introduction

We consider the half-linear differential equation

$$(r(t)\Phi(u'))' + p(t)\Phi(u) = 0, \tag{1.1}$$

$t \in [a, \infty)$, $a > 0$, where $r(t) > 0$, $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$, $\alpha > 1$. By Φ^{-1} we mean the inverse of Φ. Note that $\Phi^{-1}(u) = |u|^{\beta-1} \operatorname{sgn} u$, where β is the conjugate number to α, i.e.,

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

We study asymptotic properties of equation (1.1) from several points of view. We deal with the sets of solutions classified according to the values of their limits and the limits of their quasiderivatives, the classes of regularly varying solutions (with prescribed indices), the classes of principal and nonprincipal solutions, and the classes of solutions satisfying quite

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precise asymptotic formulae. We provide an exhaustive discussion concerning the relations among these classes and, in fact, in each setting we describe the entire solution space of (1.1). A big part of our results is new even in the linear case (where such a comprehensive treatment has not been known previously). In addition, we offer a revision and completion of existing results and place them into a broader context. To be more precise, all the results where $p > 0$ (and $L > 0$) are new, with the exception of some of the inclusions involving the formulae in terms of $\mathcal{L}$, which are established in [22, Section 5]. We utilize in the proofs also another results from [22], namely Theorem 3.3 and Lemma 3.5 on regular variation of the elements of the solution space. As for the case $p < 0$, all the results where $\eta < 0$ (in Theorem 2.1 and Theorem 2.2) or $\delta + \alpha < \gamma$ (the entire Theorem 2.3) or $\eta_i < 0$ (in Theorem 2.4) are new. Moreover, the results in the case $p < 0$ are newly supplemented by the formulae in terms of $\mathcal{B}_k$, and some of the known inclusions involving $\mathcal{G}_k, \mathcal{H}_k$ are completed in sense of equalities. The known results which are included in Theorem 2.1 and Theorem 2.2 (except of those involving $\mathcal{L}$) are taken from [19, Section 6] and [23, Section 4], see also Lemma 3.19. The relations with the formulae involving $\mathcal{L}$ in Theorems 2.1, 2.2, 2.4 for the case $p < 0$ and $L < 0$ are taken from [20, Theorem 2, Theorem 4]. Thanks to the parallel analysis of the cases $p < 0$ and $p > 0$, we can see similarities and differences between these two cases. This concerns not only the statements, but also the proofs, some of them can be unified, some other require A different approach. Further relations and comparisons with existing results are spread throughout the text.

Some phenomena which can occur only in the purely half-linear case (i.e., $\alpha \neq 2$) are revealed. Recall that (1.1) arises out when studying radially symmetric solutions of certain partial differential equations with p-Laplacian, thus the results can be useful in theory of PDEs. Our observations are important also from stability point of view and can find applications in a description of Poincaré–Perron solutions which are associated to perturbations of some autonomous nonlinear differential equations.

An important role in our theory is played by the condition

$$\lim_{t\to\infty} \frac{t^{\alpha} p(t)}{r(t)} = C_{\gamma}. \tag{1.2}$$

This condition guarantees that the set of all positive solutions of (1.1) consists of regularly varying solutions of known indices which are related to the value of the limit $C_\gamma \in (-\infty, (|\alpha - 1 - \gamma|/\alpha)^{\alpha}]$, γ being the index of regular variation of r, see Theorem 3.3. As for the existence of a regularly varying solution of (1.1), note that there are known conditions in certain integral (more general) forms that are not only sufficient but also necessary (1.1), see [9,10]. Since we assume regular variation of p and r (as we wish to include precise asymptotic formulae into our relations among the classes), the integral conditions reduce to (1.2), and thereby (1.2) actually becomes also necessary, see Lemma 3.5. We however emphasize that thanks to Theorem 3.3 we work with the entire solution space, and there is no sign condition on p a-priori needed.

A deeper approach to asymptotic formulae (including the critical – double root cases, see below) and related problems in the framework not only of Karamata theory, but also of de Haan theory (the classes Gamma and Pi) can be found in [19,20,22,23]. Relations of regularly varying solutions of (1.1) to Poincaré–Perron solutions are examined in [21,22]. For further results concerning asymptotics of half-linear differential equations in the framework of regular variation see [6,9–11,14–17]. A very important work which shows how the Karamata theory can be applied to study qualitative properties of various differential equations is the monograph [12] by Marić, see also [18], where the progress after the year 2000 is summarized.

Recall that by the Sturm type separation theorem which extends to half-linear equations, see [6, Chapter 1], a solution of (1.1) is oscillatory (i.e., it is not of eventually one sign) if and only if all solutions of (1.1) are oscillatory. Hence, we can classify equation (1.1) as oscillatory or nonoscillatory as in the linear case. We are interested in behavior of nonoscillatory solutions of (1.1). Since the solution space (1.1) is homogeneous, without loss of generality we may consider only the set

$$\mathcal{S} = \{y : y(t) \text{ is a positive solution of (1.1) for large } t\}.$$

Assuming that p is eventually of one sign we get that all solutions in $\mathcal{S}$ are eventually monotone, thus any such a solution belongs to one of the classes

$$\mathcal{IS} = \{y \in \mathcal{S} : y'(t) > 0 \text{ for large } t\}, \qquad \mathcal{DS} = \{y \in \mathcal{S} : y'(t) < 0 \text{ for large } t\}.$$

The classes $\mathcal{IS}, \mathcal{DS}$ can further be divided into four mutually disjoint subclasses

$$\mathcal{IS}_B = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = M_y \in (0,\infty)\right\}, \qquad \mathcal{IS}_\infty = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = \infty\right\},$$
$$\mathcal{DS}_B = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = M_y \in (0,\infty)\right\}, \qquad \mathcal{DS}_0 = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = 0\right\}.$$

The so-called quasiderivative $y^{[1]}$ of $y \in \mathcal{S}$ is defined by $y^{[1]} = r\Phi(y')$. We introduce the following convention that is pertinent to the limits of solutions and their quasiderivatives:

$$\mathcal{IS}_{uv} = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = u,\ \lim_{t\to\infty} y^{[1]}(t) = v\right\},$$
$$\mathcal{DS}_{uv} = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = u,\ \lim_{t\to\infty} |y^{[1]}(t)| = v\right\};$$

for the subscripts of $\mathcal{IS}$ and $\mathcal{DS}$, by $u = B$ and $v = B$ we mean that the value of u and v, respectively, is a positive number. Denote

$$J_p = \int_a^\infty |p(s)|\,\mathrm{d}s, \qquad J_r = \int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s, \tag{1.3}$$

Let $p < 0$. Then

$$\mathcal{S} = \mathcal{IS} \cup \mathcal{DS}, \quad \text{where } \mathcal{IS} \neq \emptyset \neq \mathcal{DS}, \tag{1.4}$$

see [5], [6, Chapter 4]. It is almost immediate (thanks to monotonicity) that

$$\mathcal{IS} = \mathcal{IS}_{\infty\infty} \cup \mathcal{IS}_{\infty B} \cup \mathcal{IS}_{B\infty} \cup \mathcal{IS}_{BB}$$

and

$$\mathcal{DS} = \mathcal{DS}_{00} \cup \mathcal{DS}_{0B} \cup \mathcal{DS}_{B0} \cup \mathcal{DS}_{BB},$$

see also [5], [6, Chapter 4]. The solutions in $\mathcal{IS}_{\infty\infty}$ are called *strongly increasing* and the solutions in $\mathcal{DS}_{00}$ are called *strongly decreasing*, together they form *extremal* solutions. The solutions in $\mathcal{IS}_{\infty B}$ are called *regularly increasing* and the solutions in $\mathcal{DS}_{0B}$ are called *regularly decreasing*.

Let $p > 0$. If $J_r = \infty$, then $\mathcal{DS} = \emptyset$ while if $J_p = \infty$, then $\mathcal{IS} = \emptyset$, see [6, Chapter 4]. Note that if $J_r = \infty = J_p$, then $\mathcal{S} = \emptyset$ since (1.1) is oscillatory by the Leighton–Wintner type criterion, see [6, Theorem 1.2.9]. Moreover, it is easy to show that if $J_r = \infty$ (and $J_p < \infty$), then

$$\mathcal{S} = \mathcal{IS} = \mathcal{IS}_{\infty B} \cup \mathcal{IS}_{\infty 0} \cup \mathcal{IS}_{B0}, \tag{1.5}$$

while if $J_p = \infty$ (and $J_r < \infty$), then

$$\mathcal{S} = \mathcal{DS} = \mathcal{DS}_{B\infty} \cup \mathcal{DS}_{0\infty} \cup \mathcal{DS}_{0B}, \tag{1.6}$$

see [4]. The solutions in $\mathcal{IS}_{\infty B}$ and $\mathcal{DS}_{B\infty}$ are called *dominant*, the solutions in $\mathcal{IS}_{\infty 0}$ and $\mathcal{DS}_{0\infty}$ are called *intermediate*, the solutions in $\mathcal{IS}_{B0}$ and $\mathcal{DS}_{0B}$ are called *subdominant*. An important role in studying (non)emptiness of the subclasses $\mathcal{IS}_{uv}$ and $\mathcal{DS}_{uv}$ and related problems is played by the integral conditions (3.1). Some of these relations will be used in our proofs. For more information in this direction, see [2–6].

If (1.1) is nonoscillatory, then there exists a nontrivial solution y of (1.1) such that for every nontrivial solution u of (1.1) with $u \neq \lambda y$, $\lambda \neq 0$, we have

$$\frac{y'(t)}{y(t)} < \frac{u'(t)}{u(t)} \quad \text{for large } t,$$

see, e.g., [6, Section 4.2]. Such a solution is said to be *principal solution*. Solutions of (1.1) which are not principal are called *nonprincipal solutions*. Principal solutions are unique up to a constant multiple. We denote

$$\mathcal{P} = \{y \in \mathcal{S} : y \text{ is principal}\}.$$

Some characterizations of principal solutions are presented in Theorems 3.20–3.26 for the purposes of our later use, see also [2,3,13]. Note that the situation concerning a description of principal solutions is substantially more complicated in the case $p > 0$ than in the case $p < 0$ for half-linear equations.

A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying (at infinity) of index* ϑ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \quad \text{for every } \lambda \in (0, \infty); \tag{1.7}$$

we write $f \in \mathcal{RV}(\vartheta)$. If $\vartheta = 0$, we speak about *slowly varying* functions; we write $f \in \mathcal{SV}$, thus $\mathcal{SV} = \mathcal{RV}(0)$. If $f \in \mathcal{RV}(\vartheta)$, then relation (1.7) holds uniformly on each compact λ-set in $(0, \infty)$ (the so-called *Uniform Convergence Theorem*, see, e.g., [1]). It follows that $f \in \mathcal{RV}(\vartheta)$ if and only if there exists a function $L \in \mathcal{SV}$ such that $f(t) = t^{\vartheta} L(t)$ for every t. The slowly varying component of $f \in \mathcal{RV}(\vartheta)$ will be denoted by L_f, i.e.,

$$L_f(t) := \frac{f(t)}{t^{\vartheta}}, \tag{1.8}$$

unless stated otherwise. We adopt notation (1.8) also for negative functions f such that $|f| \in \mathcal{RV}(\vartheta)$. The so-called *Representation Theorem* (see, e.g., [1]) says the following: $f \in \mathcal{RV}(\vartheta)$ if and only if

$$f(t) = \varphi(t) t^{\vartheta} \exp\left\{\int_a^t \frac{\psi(s)}{s}\, \mathrm{d}s\right\}, \tag{1.9}$$

$t \geq a$, for some $a > 0$, where φ, ψ are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$ and $\lim_{t\to\infty} \psi(t) = 0$. A function $f \in \mathcal{RV}(\vartheta)$ can alternatively be represented as

$$f(t) = \varphi(t) \exp\left\{\int_a^t \frac{\omega(s)}{s}\, \mathrm{d}s\right\}, \tag{1.10}$$

$t \geq a$, for some $a > 0$, where φ, ω are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$ and $\lim_{t\to\infty} \omega(t) = \vartheta$. A regularly varying function f is said to be *normalized regularly varying*, we write $f \in \mathcal{NRV}(\vartheta)$, if $\varphi(t) \equiv C$ in (1.9) or in (1.10). If (1.9) holds with $\vartheta = 0$ and $\varphi(t) \equiv C$, we say that f is *normalized slowly varying*, we write $f \in \mathcal{NSV}$. We denote

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{S} \cap \mathcal{SV}, \qquad \mathcal{S}_{\mathcal{RV}}(\vartheta)c = \mathcal{S} \cap \mathcal{RV}(\vartheta),$$
$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{S} \cap \mathcal{NSV}, \qquad \mathcal{S}_{\mathcal{NRV}}(\vartheta) = \mathcal{S} \cap \mathcal{NRV}(\vartheta);$$

a similar convention is used when $\mathcal{S}$ is replaced by $\mathcal{DS}$ or $\mathcal{IS}$. Some properties of regularly varying functions are gathered in Proposition 3.1 and Theorem 3.2; for more information see [1,8].

The condition

$$|p| \in \mathcal{RV}(\delta), \qquad r \in \mathcal{RV}(\gamma), \tag{1.11}$$

which plays an important role in our theory, in fact is not needed for showing regular variation of solutions to (1.1), but it enables us to provide a precise asymptotic description. We will assume that $\delta \neq -1$ and $\gamma \neq \alpha - 1$ which leads to avoiding the critical (double-root – see (2.3)) setting. The critical setting (which is considered in connection with searching precise asymptotic formulae in [20,22] and requires a more refined approach) could be treated also in the framework of our topic – a finer classification would however be needed. Denote

$$G(t) = \Phi^{-1}\left(\frac{tp(t)}{r(t)}\right), \quad J = \int_a^\infty |G(t)|\,\mathrm{d}t, \quad H(t) = \frac{t^{\alpha-1}p(t)}{r(t)}, \quad R = \int_a^\infty |H(t)|\,\mathrm{d}t. \tag{1.12}$$

If (1.11) holds and $\delta + \alpha = \gamma$, then

$$G(t) = \frac{1}{t}\Phi^{-1}\left(\frac{L_p(t)}{L_r(t)}\right) \quad \text{and} \quad H(t) = \frac{L_p(t)}{tL_r(t)} \tag{1.13}$$

by Proposition 3.1. Observe that if $\alpha \neq 2$, then the situation where $J = \infty$ and $R < \infty$ (or vice versa) can occur under the conditions (1.11) and $\delta + \alpha = \gamma$. An example can easily be constructed via the relations in (1.13). This fact substantially affects the structure of the solution space of (1.1) which turns out to be more complex than in the linear case. Lemma 3.7 describes a connection of J, R with the integrals in (3.1) which play a central role in studying the existence problems in the classes $\mathcal{IS}_{uv}, \mathcal{DS}_{uv}$. To simplify writing asymptotic formulae, we adopt the notation

$$\mathfrak{E}(\sigma, \tau, K, f) = \exp\left\{\int_\sigma^\tau (1 + o(1))Kf(s)\,\mathrm{d}s\right\},$$

where $o(1)$ is meant either as $\tau \to \infty$ when $\tau < \infty$ or as $\sigma \to \infty$ when $\tau = \infty$. As usually, for f, g which are either both positive or both negative, the relation $f(t) \sim g(t)$ as $t \to \infty$ means $\lim_{t\to\infty} f(t)/g(t) = 1$, while $f(t) = o(g(t))$ as $t \to \infty$ means $\lim_{t\to\infty} f(t)/g(t) = 0$. The sets presented below are introduced for purposes of an easy and synoptic incorporation of asymptotic formulae to other classifications; the constants M_y, N_y are defined by

$$M_y = \lim_{t\to\infty} y(t), \qquad N_y = \lim_{t\to\infty} y^{[1]}(t).$$

The sets $\mathcal{G}_1, \mathcal{G}_2, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ are pertinent to the solutions in the classes $\mathcal{SV}$ and $\mathcal{RV}(\varrho)$, respectively, where

$$\varrho = \frac{\alpha - 1 - \gamma}{\alpha - 1}, \tag{1.14}$$

under the condition $C_\gamma = 0$, and are defined by:

$$\mathcal{G}_1 = \left\{ y \in \mathcal{S} : y(t) = \mathfrak{E}(a, t, -1/\Phi^{-1}(\delta+1), G) \right\},$$
$$\mathcal{G}_2 = \left\{ y \in \mathcal{S} : y(t) = M_y \mathfrak{E}(t, \infty, 1/\Phi^{-1}(\delta+1), G) \right\},$$

and

$$\mathcal{H}_1 = \left\{ y \in \mathcal{S} : y(t) = y(t_0) + \int_{t_0}^{t} r^{1-\beta}(s) \mathfrak{E}(a, s, -(\beta-1)/\Phi(\varrho), H) \, \mathrm{d}s \right\},$$
$$\mathcal{H}_2 = \left\{ y \in \mathcal{S} : y(t) = \int_{t}^{\infty} r^{1-\beta}(s) \mathfrak{E}(a, s, -(\beta-1)/\Phi(\varrho), H) \, \mathrm{d}s \right\},$$
$$\mathcal{H}_3 = \left\{ y \in \mathcal{S} : y(t) = y(t_0) + \int_{t_0}^{t} r^{1-\beta}(s) \Phi^{-1}(N_y) \mathfrak{E}(s, \infty, (\beta-1)/\Phi(\varrho), H) \, \mathrm{d}s \right\},$$
$$\mathcal{H}_4 = \left\{ y \in \mathcal{S} : y(t) = \int_{t}^{\infty} r^{1-\beta}(s) \Phi^{-1}(-N_y) \mathfrak{E}(s, \infty, (\beta-1)/\Phi(\varrho), H) \, \mathrm{d}s \right\}.$$

If $\int_a^\infty |H(s)| \, \mathrm{d}s = \infty$, then $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_0$ (see Lemma 3.14), where

$$\mathcal{H}_0 = \left\{ y \in \mathcal{S} : y(t) = t r^{1-\beta}(s) \mathfrak{E}(a, t, -(\beta-1)/\Phi(\varrho), H) \right\}.$$

The sets $\mathcal{L}_1, \mathcal{L}_2$ which are designed for the case $C_\gamma \neq 0$ and for an alternative description in the case $C_\gamma = 0$ with $\mathcal{RV}(\varrho)$ solutions, are given by:

$$\mathcal{L}_1(\vartheta, \eta) = \left\{ y \in \mathcal{S} : y(t) = t^\vartheta \mathfrak{E}\left(a, t, \frac{1-\beta}{\Phi(\vartheta) - C_\gamma/\vartheta}, L(\vartheta, \eta, \cdot) \right) \right\},$$
$$\mathcal{L}_2(\vartheta, \eta) = \left\{ y \in \mathcal{S} : y(t) = D t^\vartheta \mathfrak{E}\left(t, \infty, \frac{\beta-1}{\Phi(\vartheta) - C_\gamma/\vartheta + \eta|\vartheta|^{\alpha-2}}, L(\vartheta, \eta, \cdot) \right) \right\},$$

where

$$L(\vartheta, \eta, t) = \frac{1}{t} \left[\frac{t^\alpha p(t)}{r(t)} - C_\gamma + \Phi(\vartheta) \left(\frac{t r'(t)}{r(t)} - \gamma \right) \right], \quad \text{with } |L(\vartheta, \eta, \cdot)| \in \mathcal{RV}(\eta - 1),$$

and $D = \lim_{t\to\infty} y(t)/t^\vartheta$. If A is a set, then by the equality $A = \mathcal{L}(\vartheta, \eta)$ we mean that

$$A = \begin{cases} \mathcal{L}_1(\vartheta, \eta) & \text{if } \int_a^\infty |L(\vartheta, \eta, s)| \, \mathrm{d}s = \infty, \\ \mathcal{L}_2(\vartheta, \eta) & \text{if } \int_a^\infty |L(\vartheta, \eta, s)| \, \mathrm{d}s < \infty. \end{cases} \tag{1.15}$$

In view of Proposition 3.1, if $\eta < 0$ and $A = \mathcal{L}(\vartheta, \eta)$, then $A = \mathcal{L}_2(\vartheta, \eta)$. Note that in our results we actually have $\lim_{t\to\infty} L(\vartheta, \eta, t) = 0$, thus by the Representation Theorem (1.9), we get $\mathcal{L}(\vartheta, \eta) \subset \mathcal{RV}(\vartheta)$, $\vartheta \in \mathbb{R}, \eta \leq 0$. If $C_\gamma = 0$, then

$$L(0, \eta, t) = H(t) \quad \text{and} \quad L(\varrho, \eta, t) = \frac{L_p(t)}{t L_r(t)} - \Phi(\varrho) \frac{L_r'(t)}{L_r(t)}.$$

The sets $\mathcal{B}_1, \ldots, \mathcal{B}_6$ are pertinent to the situations where y and/or $y^{[1]}$ have a real nonzero limit and are defined as follows:

$$\mathcal{B}_1 = \left\{ y \in \mathcal{S} : M_y - y(t) \sim \frac{\Phi^{-1}(N_y)}{\varrho} t r^{1-\beta}(t) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_2 = \left\{ y \in \mathcal{S} : N_y - y^{[1]}(t) \sim \frac{\Phi(M_y)}{\delta+1} t p(t) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_3 = \left\{ y \in \mathcal{S} : M_y - y(t) \sim \frac{M_y(\alpha-1)}{\Phi^{-1}(\delta+1)(\delta+\alpha-\gamma)} t G(t) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_4 = \left\{ y \in \mathcal{S} : N_y - y^{[1]}(t) \sim \frac{-N_y}{\Phi(\varrho)(\delta+\alpha-\gamma)} t H(t) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_5 = \left\{ y \in \mathcal{S} : t|G(t)| = o(|M_y - y(t)|) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_6 = \left\{ y \in \mathcal{S} : t|H(t)| = o(|N_y - y^{[1]}(t)|) \text{ as } t \to \infty \right\}.$$

2 Main results

In this section we present the main results that are formulated as four theorems; we distinguish, in particular, whether C_γ is zero or not and whether γ is equal to $\delta + \alpha$ or not.

First note that under the assumptions of Theorems 2.1–2.4, we have, for a given $\vartheta \in \mathbb{R}$,

$$\mathcal{S}_{\mathcal{RV}}(\vartheta) = \mathcal{S}_{\mathcal{NRV}}(\vartheta), \tag{2.1}$$

see Remark 3.4. Therefore we omit writing this relation in formulations of the theorems since it holds in each case. It is worthy of noting that because of the properties of principal solutions, in the sets that are equal to $\mathcal{P}$, we have uniqueness up to a constant multiple. This, in particular, means that, for example, in the case (i-a) of Theorem 2.1, there is only one slowly varying solution provided we fix its value at a point.

In Theorems 2.1-2.3, we need to take $\delta \neq -1$, $\gamma \neq \alpha - 1$; Theorem 2.4 does not require an inequality. In fact, the equality in the settings of Theorems 2.1-2.3 would lead to somehow critical cases (which correspond with double roots in (2.3) and/or border-line version of the Karamata integration theorem). Actually, the critical cases can be treated, but a more sophisticated approach is needed and introducing new special asymptotic subclasses is necessary. The main ingredients in analyzing these cases are suitable transformations to non-critical cases and applications of existing results (including the new ones in this paper). We will not go further in this direction. For some considerations concerning the critical case see [20,22].

The first two theorems deal with $\mathcal{SV}$ and $\mathcal{RV}(\varrho)$ solutions under the condition $\gamma = \delta + \alpha$. Recall that ϱ is defined in (1.14).

Theorem 2.1. *Let $C_\gamma = 0$ and (1.11) hold, where $\gamma = \delta + \alpha$. For the relations involving the class $\mathcal{L}(\varrho, \eta)$ assume, in addition, $|L(\varrho, \eta, \cdot)| \in \mathcal{RV}(\eta - 1)$, $\eta \leq 0$, and if the condition $\delta < -1$ is supposed, let, in addition, $\delta < -1 + \eta(\alpha - 1)$. Then $\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho)$, $\mathcal{S}_{\mathcal{NSV}} \neq \emptyset$, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \neq \emptyset$, and the following hold:*

(i) Assume that $J = \infty$ and $R = \infty$.

(i-a) If $p < 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{G}_1 = \mathcal{P}, \quad \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{H}_1 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta).$$

(i-b) If $p < 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{G}_1, \quad \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{H}_2 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(i-c) If $p > 0$ and $\delta < -1$, then

$$\mathcal{S} = \mathcal{IS} = \mathcal{IS}_{\infty 0} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho), \text{ with } \mathcal{S}_{\mathcal{NSV}} = \mathcal{G}_1 = \mathcal{P},\ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{H}_1 = \mathcal{H}_0 = \mathcal{L}(\varrho,\eta).$$

(i-d) If $p > 0$ and $\delta > -1$, then

$$\mathcal{S} = \mathcal{DS} = \mathcal{DS}_{0\infty} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho), \text{ with } \mathcal{S}_{\mathcal{NSV}} = \mathcal{G}_1,\ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{H}_2 = \mathcal{H}_0 = \mathcal{L}(\varrho,\eta) = \mathcal{P}.$$

(ii) Assume that $J < \infty$ and $R < \infty$.

(ii-a) If $p < 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5 = \mathcal{P},\ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty B} = \mathcal{H}_3 = \mathcal{B}_6 = \mathcal{L}(\varrho,\eta).$$

(ii-b) If $p < 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5,\ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho,\eta) = \mathcal{P}.$$

(ii-c) If $p > 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5 = \mathcal{P},\ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty B} = \mathcal{H}_3 = \mathcal{B}_6 = \mathcal{L}(\varrho,\eta).$$

(ii-d) If $p > 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5,\ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho,\eta) = \mathcal{P}.$$

Observe that Theorem 2.1 and Theorem 2.2 have the same general assumptions. They differ in the conditions regarding mutual behavior of J and R. We emphasize that the combinations $J = \infty \wedge R < \infty$ and $J < \infty \wedge R = \infty$, which are assumed in Theorem 2.2, can occur only in the purely half-linear case (i.e., $\alpha \neq 2$), and that is why we separate them into a particular theorem. In view of equalities in (1.13), it is easy to find a suitable example illustrating this setting. Indeed, take $L_r(t) = 1$ and $L_p(t) = 1/\ln^{\omega} t$, where $1 < \omega < \alpha - 1$ or $\alpha - 1 < \omega < 1$. It so arises out that the structure of the solution space in the half-linear case is generally more complex than in the linear one under our setting. In particular, under the conditions of Theorem 2.2, there can coexist strongly monotone solutions with non-extremal ones or intermediate solutions with dominant or subdominant ones. See also [4,5] where the problem of coexistence and non-linear setting is discussed in a more general context.

Theorem 2.2. *Let (1.11) hold, where $\gamma = \delta + \alpha$, and $C_\gamma = 0$. For the relations involving the class $\mathcal{L}(\varrho,\eta)$ assume, in addition, $|L(\varrho,\eta,\cdot)| \in \mathcal{RV}(\eta - 1)$, $\eta \leq 0$, and if the condition $\delta < -1$ is supposed, let, in addition, $\delta < -1 + \eta(\alpha - 1)$. Then $\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho)$, $\mathcal{S}_{\mathcal{NSV}} \neq \emptyset$, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \neq \emptyset$, and the following hold:*

(i) Assume that $J = \infty$ and $R < \infty$.

(i-a) If $p < 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{G}_1 = \mathcal{P},\ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho,\eta).$$

(i-b) If $p < 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{G}_1,\ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho,\eta) = \mathcal{P}.$$

(i-c) If $p > 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{\infty 0} = \mathcal{G}_1 = \mathcal{P},\ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho,\eta).$$

(i-d) If $p > 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{0\infty} = \mathcal{G}_1, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(ii) Assume that $J < \infty$ and $R = \infty$.

(ii-a) If $p < 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{H}_1 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta).$$

(ii-b) If $p < 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{H}_2 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(ii-c) If $p > 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty\infty} = \mathcal{H}_1 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta).$$

(ii-d) If $p > 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_2 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

The next theorem can be seen as a complement of Theorems 2.1 and 2.2 in the sense that the condition $\delta + \alpha = \gamma$ will not be satisfied. We assume $\delta + \alpha < \gamma$ which implies $C_\gamma = 0, J < \infty, R < \infty$; this can be seen from Proposition 3.1 (see the proof of Theorem 2.3). On the other hand, in contrast to the case of equality $\delta + \alpha = \gamma$, the strict inequality allows us to consider a richer variety of combinations of conditions $\delta < -1, \delta > -1, \gamma < \alpha - 1, \gamma > \alpha - 1$. Observe that under the setting of Theorem 2.3, there are no extremal or intermediate solutions. The case $\delta + \alpha > \gamma$ is not considered since then there are no regularly varying solutions. Indeed, by Proposition 3.1, we then have $|C_\gamma| = \infty$. If $p < 0$, then by [23], the set $\mathcal{S}$ is nonempty and consists entirely of the solutions in the de Haan classes Γ and Γ_-, which are subsets of rapidly varying functions. If $p > 0$, then equation (1.1) is oscillatory by Hille–Nehari type criteria, see [6, Chapter 3], and so $\mathcal{S}$ is empty. In fact, to show that there are no $\mathcal{RV}$ solutions, we can argue in a alternative way, namely that the necessary condition is not fulfilled, see Lemma 3.5.

Theorem 2.3. *Let (1.11) hold, where $\gamma > \delta + \alpha$. For the relations involving the class $\mathcal{L}(\varrho, \eta)$ assume, in addition, $|L(\varrho, \eta, \cdot)| \in \mathcal{RV}(\eta - 1)$, $\eta \leq 0$, and if the condition $\gamma < \alpha - 1$ is supposed, let, in addition, $\gamma < (\alpha - 1)(1 + \eta)$. Then $\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho)$, $\mathcal{S}_{\mathcal{NSV}} \neq \emptyset$, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \neq \emptyset$, and the following hold:*

(i) Assume that $\delta < -1$ and $\gamma < \alpha - 1$.

(i-a) If $p < 0$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty B} = \mathcal{H}_3 = \mathcal{B}_4 = \mathcal{L}(\varrho, \eta).$$

(i-b) If $p > 0$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty B} = \mathcal{H}_3 = \mathcal{B}_4 = \mathcal{L}(\varrho, \eta).$$

(ii) Assume that $\delta > -1$ and $\gamma > \alpha - 1$.

(ii-a) If $p < 0$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_4 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(ii-b) If $p > 0$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_3, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_4 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(iii) Assume that $\delta < -1$ and $\gamma > \alpha - 1$.

(iii-a) If $p < 0$, then

$$\begin{aligned} &\mathcal{IS}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{BB} = \mathcal{B}_1 = \mathcal{B}_2 \neq \emptyset, \\ &\mathcal{DS}_{\mathcal{NSV}} = \mathcal{DS}_{B0} \cup \mathcal{DS}_{BB}, \ \ \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3 \neq \emptyset, \ \ \mathcal{DS}_{BB} = \mathcal{B}_1 = \mathcal{B}_2 \neq \emptyset, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_4 = \mathcal{L}(\varrho, \eta) = \mathcal{P}. \end{aligned}$$

(iii-b) If $p > 0$, then

$$\begin{aligned} &\mathcal{IS}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B0} \cup \mathcal{IS}_{BB}, \ \ \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3 \neq \emptyset, \ \ \mathcal{IS}_{BB} = \mathcal{B}_1 = \mathcal{B}_2 \neq \emptyset, \\ &\mathcal{DS}_{\mathcal{NSV}} = \mathcal{DS}_{BB} = \mathcal{B}_1 = \mathcal{B}_2 \neq \emptyset, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_4 = \mathcal{L}(\varrho, \eta) = \mathcal{P}. \end{aligned}$$

One can see that the case $\delta > -1$ and $\gamma < \alpha - 1$ is not considered in the previous theorem. This is quite natural because there are no regularly varying solutions; the reasons are almost the same as in the case $\alpha + \delta > \gamma$ (discussed before Theorem 2.3). Indeed, if $p < 0$, there are solutions only in the de Haan classes Γ and Γ_-, see [23]. If $p > 0$, then (1.1) is oscillatory by Hille–Wintner type criterion, see [6]. Alternatively, we can again argue by Lemma 3.5 since $\delta + \alpha > -1 + \gamma + 1 = \gamma$.

The next theorem can be seen as a complement to the previous ones in the sense that previously was assumed (or was guaranteed) $C_\gamma = 0$ and now we take $C_\gamma \neq 0$. Note that $C_\gamma \neq 0$ and $r \in \mathcal{RV}(\gamma)$ imply $|p| \in \mathcal{RV}(\gamma - \alpha)$. Indeed, from (1.2) and Proposition 3.1, we have $|p(t)| \sim |C_\gamma| t^{-\alpha} r(t) \in \mathcal{RV}(-\alpha + \gamma)$ as $t \to \infty$. In general, we do not need to exclude the critical case $\gamma = \alpha - 1$. However, if we take $C_\gamma > 0$, then necessarily $\gamma \neq \alpha - 1$ since we assume $C_\gamma \leq K_\gamma$, where

$$K_\gamma = \left(\frac{|\alpha - 1 - \gamma|}{\alpha} \right)^\alpha . \tag{2.2}$$

We denote

$$\vartheta_i = \Phi(\lambda_i), \qquad \vartheta_1 \leq \vartheta_2,$$

where $\lambda_1 \leq \lambda_2$ are the (real) roots of

$$F_\gamma(\lambda) := |\lambda|^\beta + \frac{\gamma + 1 - \alpha}{\alpha - 1} \lambda + \frac{C_\gamma}{\alpha - 1} = 0. \tag{2.3}$$

If $\eta_2 = 0$ in Theorem 2.4, then we do not need to assume $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$, since this inequality is satisfied automatically thanks to the properties of the roots, see Lemma 3.6. Observe that under the setting of Theorem 2.4, there are only extremal solutions (when $p < 0$) or intermediate solutions (when $p > 0$). In the case $C_\gamma = K_\gamma$, generally oscillation or nonoscillation of (1.1) can occur. Nonoscillation is guaranteed e.g. by $t^\alpha p(t)/r(t) \leq C_\gamma$ (this follows from the Sturm type theorem, see [6]), or by the conditions of [9, Theorem 2.2, Theorem 3.2], or by some suitable nonoscillation criterion, see, e.g., [6, Chapter 3].

Theorem 2.4. *Let $C_\gamma \in (-\infty, K_\gamma] \setminus \{0\}$ and $r \in \mathcal{NRV}(\gamma) \cap C^1$, $\gamma \in \mathbb{R}$. For the relations involving the classes $\mathcal{L}(\vartheta_i, \eta_i)$, $i = 1, 2$, assume, in addition, $|L(\vartheta_i, \eta_i, \cdot)| \in \mathcal{RV}(\eta_i - 1)$, where $\eta_1, \eta_2 \le 0$, and $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$. Then $\mathcal{S} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$, $\mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \neq \emptyset$, $i = 1, 2$, and the following hold:*

(i) Assume that $C_\gamma < 0$. Then

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{L}(\vartheta_1, \eta_1) = \mathcal{P}, \quad \vartheta_1 < 0,$$
$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{L}(\vartheta_2, \eta_2), \quad \vartheta_2 > 0.$$

(ii) Assume that $0 < C_\gamma \le K_\gamma$; the strict inequality $C_\gamma < K_\gamma$ is required only when the relations involving the classes $\mathcal{L}(\vartheta_i, \eta_i)$, $i = 1, 2$, are considered. If $C_\gamma = K_\gamma$, we assume, in addition, nonoscillation of (1.1).

(ii-a) If $\gamma < \alpha - 1$, then

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{S} = \mathcal{IS} = \mathcal{IS}_{\infty 0}, \quad \vartheta_1, \vartheta_2 > 0,$$
$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) = \mathcal{L}(\vartheta_1, \eta_1) = \mathcal{P}, \quad \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{L}(\vartheta_2, \eta_2).$$

(ii-b) If $\gamma > \alpha - 1$, then

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{S} = \mathcal{DS} = \mathcal{DS}_{0\infty}, \quad \vartheta_1, \vartheta_2 < 0$$
$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) = \mathcal{L}(\vartheta_1, \eta_1) = \mathcal{P}, \quad \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{L}(\vartheta_2, \eta_2).$$

For various examples that illustrate, in particular, the asymptotic formulae in particular settings, see [20,22]. Among others it is shown that the situation where $\int_a^\infty |L(\vartheta_1, \eta_1, s)| \, ds = \infty$ and $\int_a^\infty |L(\vartheta_2, \eta_2, s)| \, ds < \infty$ (or vice versa) can occur even when $\eta_1 = \eta_2 = 0$.

In [21, 22] we explore how some of the above results can be applied to the half-linear equation of the form

$$(\Phi(y'))' + a(t)\Phi(y') + b(t)\Phi(y) = 0$$

to analyze its Poincaré–Perron solutions (that is the solutions y such that $\lim_{t\to\infty} y'(t)/y(t)$ exists as a finite number). The equation can be viewed as a perturbation of the equation with constant coefficients. A key role is played by a suitable transformation, and we believe that the new results of this paper could be extended in this sense. Another direction is an extension to the critical (double-root) case which is roughly explained at the beginning of this section. Since theory of regularly varying sequences is at disposal and difference equations often show their particularities (when compared with their continuous counterparts), a discrete version of our results is also of interest.

3 Auxiliary statements and proofs

We start with selected properties of regularly varying functions.

Proposition 3.1.

(i) If $f \in \mathcal{RV}(\vartheta)$, then $\ln f(t)/\ln t \to \vartheta$ as $t \to \infty$. It then clearly implies that $\lim_{t\to\infty} f(t) = 0$ provided $\vartheta < 0$, and $\lim_{t\to\infty} f(t) = \infty$ provided $\vartheta > 0$.

(ii) If $f \in \mathcal{RV}(\vartheta)$, then $f^\alpha \in \mathcal{RV}(\alpha\vartheta)$ for every $\alpha \in \mathbb{R}$.

(iii) If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, $f_2(t) \to \infty$ as $t \to \infty$, then $f_1 \circ f_2 \in \mathcal{RV}(\vartheta_1\vartheta_2)$.

(iv) If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1,2$, then $f_1 + f_2 \in \mathcal{RV}(\max\{\vartheta_1, \vartheta_2\})$.

(v) If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1,2$, then $f_1 f_2 \in \mathcal{RV}(\vartheta_1 + \vartheta_2)$.

(vi) If $f_1, \ldots, f_n \in \mathcal{RV}$, $n \in \mathbb{N}$, and $R(x_1, \ldots, x_n)$ is a rational function with nonnegative coefficients, then $R(f_1, \ldots, f_n) \in \mathcal{RV}$.

(vii) If $L \in \mathcal{SV}$ and $\vartheta > 0$, then $t^{\vartheta} L(t) \to \infty$, $t^{-\vartheta} L(t) \to 0$ as $t \to \infty$.

(viii) If $f \in \mathcal{RV}(\vartheta)$ and a measurable g is such that $g(t) \sim f(t)$ as $t \to \infty$. Then $g \in \mathcal{RV}(\vartheta)$.

(ix) If $f \in \mathcal{RV}(\vartheta)$, $\vartheta \neq 0$, then there exists $g \in C^1$ with $g(t) \sim f(t)$ as $t \to \infty$ and such that $tg'(t)/g(t) \to \vartheta$, whence $g \in \mathcal{NRV}(\vartheta)$. Moreover, g can be taken such that $|g'| \in \mathcal{NRV}(\vartheta - 1)$.

(x) Let f be eventually positive and differentiable, and let $\lim_{t\to\infty} tf'(t)/f(t) = \vartheta$. Then $f \in \mathcal{NRV}(\vartheta)$.

(xi) If $|f'| \in \mathcal{RV}(\vartheta)$, $\vartheta \neq -1$, with f' being eventually of one sign, then $f \in \mathcal{NRV}(\vartheta + 1)$.

Proof. The proofs of (i)–(x) are either easy or can be found in [1,8]. For (xi) see [19]. □

The following statement (the so-called Karamata integration theorem) is of great importance in our theory.

Theorem 3.2 ([1]). *Let $L \in \mathcal{SV}$.*

(i) If $\vartheta < -1$, then $\int_t^\infty s^\vartheta L(s)\,\mathrm{d}s \sim t^{\vartheta+1} L(t)/(-\vartheta - 1)$ as $t \to \infty$.

(ii) If $\vartheta > -1$, then $\int_a^t s^\vartheta L(s)\,\mathrm{d}s \sim t^{\vartheta+1} L(t)/(\vartheta + 1)$ as $t \to \infty$.

(iii) If $\int_a^\infty L(s)/s\,\mathrm{d}s$ converges, then $\widetilde{L}(t) = \int_t^\infty L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; if $\int_a^\infty L(s)/s\,\mathrm{d}s$ diverges, then $\widetilde{L}(t) = \int_a^t L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; in both cases, $L(t)/\widetilde{L}(t) \to 0$ as $t \to \infty$.

Finiteness of the limit in (1.2) guarantees (in nonoscillatory case) regular variation of all positive solutions.

Theorem 3.3 ([22]). *Let $r \in \mathcal{RV}(\gamma)$, $\gamma \in \mathbb{R}$, and $C_\gamma \in (-\infty, K_\gamma]$ be defined by (1.2), $K_\gamma = (|\alpha - 1 - \gamma|/\alpha)^\alpha$. We assume, in addition, nonoscillation of (1.1) when $C = K_\gamma$ with $t^\alpha p(t)/r(t) \not\leq K_\gamma$ (in all other cases, nonoscillation is automatically guaranteed). Then $\mathcal{S} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$ with $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \neq \emptyset \neq \mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$, where $\lambda_i = \Phi(\vartheta_i)$, $i = 1,2$, are the roots of (2.3).*

Remark 3.4. In the proof of Theorem 3.3 it is actually shown that for any $y \in \mathcal{S}$, we have $\lim_{t\to\infty} ty'(t)/y(t) \in \{\vartheta_1, \vartheta_2\}$. That is why any regularly varying solution is automatically normalized; in other words, (2.1) holds. But even without a-priori assuming (1.2), it can be proved that $\mathcal{S}_{\mathcal{RV}}(\vartheta) \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta)$ under the assumption of regular variation of r, by means of Lemma 3.5 and Proposition 3.1. Normality follows also from the asymptotic formulae or from monotonicity of solutions and quasiderivatives with the help of the properties of regularly varying functions.

Under our setting, condition (1.2) is necessary for the existence of a regularly varying solution.

Lemma 3.5 ([22]). *Let (1.11) hold with $\delta \neq -1$ and $\gamma \neq \alpha - 1$. If $\mathcal{S}_{\mathcal{RV}}(\vartheta) \neq \emptyset$, where $\lambda = \Phi(\vartheta)$ is a real root of (2.3), then $\lim_{t\to\infty} t^\alpha p(t)/r(t) = C_\gamma$ and $\delta + \alpha \leq \gamma$.*

Lemma 3.6 ([22]). *Let $\lambda_1^\pm \leq \lambda_2^\pm$ denote the (real) roots of (2.3) when $\operatorname{sgn}(\alpha - 1 - \gamma) = \pm 1$ and let $\lambda_1 \leq \lambda_2$ denote the (real) roots of (2.3) when $\gamma = \alpha - 1$. Set $\vartheta_i^\pm = \Phi(\lambda_i^\pm)$ and $\vartheta_i = \Phi(\lambda_i)$, $i = 1,2$.*

(i) Let $\gamma \neq \alpha - 1$. *If* $C_\gamma < 0$, *then* $\vartheta_1^\pm \vartheta_2^\pm < 0$, $|\vartheta_1^+| = \vartheta_2^-$, *and* $|\vartheta_1^-| = \vartheta_2^+ > |\varrho|$. *If* $C_\gamma = 0$, *then* $\vartheta_1^+ = \vartheta_2^- = 0$ *and* $-\vartheta_1^- = \vartheta_2^+ = |\varrho|$. *If* $C_\gamma \in (0, K_\gamma)$, *then* $\vartheta_1^\pm \vartheta_2^\pm > 0$ *and* $\vartheta_1^+ = |\vartheta_2^-| < |\gamma + 1 - \alpha|/\alpha < \vartheta_2^+ = |\vartheta_1^-| < |\varrho|$. *If* $C_\gamma = K_\gamma$, *then* $-\vartheta_1^- = -\vartheta_2^- = \vartheta_1^+ = \vartheta_2^+ = |\gamma + 1 - \alpha|/\alpha$.
(ii) Let $\gamma = \alpha - 1$. *Then* $C_\gamma \leq 0$ *with* $\vartheta_1 = \vartheta_2 = 0$ *when* $C_\gamma = 0$ *while* $\vartheta_{1,2} = \pm(|C_\gamma|/(\alpha-1))^{1/\alpha}$ *when* $C_\gamma < 0$.

Denote

$$J_1 = \int_a^\infty V_1(t)\,\mathrm{d}t, \quad J_2 = \int_a^\infty V_2(t)\,\mathrm{d}t, \quad R_1 = \int_a^\infty W_1(t)\,\mathrm{d}t, \quad R_2 = \int_a^\infty W_2(t)\,\mathrm{d}t, \tag{3.1}$$

where

$$V_1(t) = r^{1-\beta}(t)\left(\int_a^t |p(s)|\,\mathrm{d}s\right)^{\beta-1}, \qquad V_2(t) = r^{1-\beta}(t)\left(\int_t^\infty |p(s)|\,\mathrm{d}s\right)^{\beta-1},$$
$$W_1(t) = |p(t)|\left(\int_a^t r^{1-\beta}(s)\,\mathrm{d}s\right)^{\alpha-1}, \qquad W_2(t) = |p(t)|\left(\int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s\right)^{\alpha-1}.$$

These integrals naturally occur when studying (non)emptiness of the classes $\mathcal{IS}_{uv}, \mathcal{DS}_{uv}$ and play an important role also in characterization of principal solutions, see [2–6]. Later, in the proofs we use some of these results.

Since we work in the framework of regular variation, some specific and useful properties of V_1, V_2, W_1, W_2 can be derived.

Lemma 3.7. *Let* (1.11) *hold. Then*

(i) $V_i(t) \sim |G(t)|/|\delta + 1|^{\beta-1}$ *as* $t \to \infty$, *where* $i = 1$ *when* $\delta > -1$ *while* $i = 2$ *when* $\delta < -1$.

(ii) $W_i(t) \sim |H(t)|/|\gamma(1-\beta) + 1|^{\alpha-1}$ *as* $t \to \infty$, *where* $i = 1$ *when* $\gamma < \alpha - 1$ *while* $i = 2$ *when* $\gamma > \alpha - 1$.

(iii) If $\delta < -1$, *then* $V_1(t) \sim J_p^{\beta-1} r^{1-\beta}(t)$ *as* $t \to \infty$, *where* J_p *is defined in* (1.3).

(iv) If $\gamma > \alpha - 1$, *then* $W_1(t) \sim J_r^{\alpha-1}|p(t)|$ *as* $t \to \infty$, *where* J_r *is defined in* (1.3).

Proof. The asymptotic formulae in (i) and (ii) follow from the Karamata Integration Theorem (Theorem 3.2). The relations in (iii) and (iv) are obvious; convergence of the integrals J_p and J_r, respectively, is a consequence of Theorem 3.2. □

Remark 3.8. Let (1.11) hold. If $\delta > -1$, then $\int_a^\infty |p(s)|\,\mathrm{d}s = \infty$, thus $\int_a^\infty V_2(s)\,\mathrm{d}s$ cannot converge. If $\gamma < \alpha - 1$, then $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$, thus $\int_a^\infty W_2(s)\,\mathrm{d}s$ cannot converge. Now from Lemma 3.7 it easily follows that:

(i) Let $\delta > -1$. Then a) $J_1 = \infty \Leftrightarrow J = \infty$, b) $J_2 = \infty$.

(ii) Let $\delta < -1$. Then a) $J_1 = \infty \Leftrightarrow J_r = \infty$, b) $J_2 = \infty \Leftrightarrow J = \infty$.

(iii) Let $\gamma < \alpha - 1$. Then a) $R_1 = \infty \Leftrightarrow R = \infty$, b) $R_2 = \infty$.

(iv) Let $\gamma > \alpha - 1$. Then a) $R_1 = \infty \Leftrightarrow J_p = \infty$, b) $R_2 = \infty \Leftrightarrow R = \infty$.

The first statement in the following lemma is sometimes called the *reciprocity principle* and equation (3.2) is called the *reciprocal equation* (to equation (1.1)).

Lemma 3.9. *Let y be a solution of (1.1) with $p \neq 0$. If $u = |y^{[1]}|$, then u is a solution of*

$$(\widehat{r}(t)\Phi^{-1}(u'))' + \widehat{p}(t)\Phi^{-1}(u) = 0, \tag{3.2}$$

where $\widehat{r} = |p|^{1-\beta}$ and $\widehat{p} = r^{1-\beta}\operatorname{sgn} p$. In particular, if $y \in \mathcal{S}$, then

$$u \in \widehat{\mathcal{S}} = \{u : u \text{ is an eventually positive solution of } (3.2)\}.$$

If $\widehat{G}(t) = \Phi(t\widehat{p}(t)/\widehat{r}(t))$ and $\widehat{H}(t) = t^{\beta-1}\widehat{p}(t)/\widehat{r}(t)$, then

$$\widehat{G} = H \quad \text{and} \quad \widehat{H} = G. \tag{3.3}$$

If (1.11) holds, then

$$|\widehat{p}| \in \mathcal{RV}(\widehat{\delta}) \text{ and } \widehat{r} \in \mathcal{RV}(\widehat{\gamma}), \text{ where } \widehat{\delta} = \gamma(1-\beta) \text{ and } \widehat{\gamma} = \delta(1-\beta). \tag{3.4}$$

Proof. Since $u' = -p\Phi(y)$, we get $y = -|p|^{1-\beta}\Phi^{-1}(u')\operatorname{sgn} p$. From $u = r\Phi(y')$, we have $y' = r^{1-\beta}\Phi^{-1}(u)$. Thus we find that u satisfies (3.2). The relations in (3.3) are obvious. The relations in (3.4) follow easily by Proposition 3.1. □

Remark 3.10. For the notation of subclasses of $\widehat{\mathcal{S}}$ we use the "circumflex analog" of the notation of subclasses of $\mathcal{S}$. For instance, $\widehat{\mathcal{DS}}$ and $\widehat{\mathcal{DS}}_{B0}$ mean the set of eventually decreasing solutions of (3.2) and the subset of $\widehat{\mathcal{DS}}$ where $u \in \widehat{\mathcal{DS}}_{B0}$ tends to a positive constant with $\lim_{t\to\infty} \widehat{r}(t)\Phi^{-1}(u(t)) = 0$, respectively. Similarly we approach to the notation of the classes for the solutions satisfying prescribed asymptotic formulae. For example, $\widehat{\mathcal{G}}_2$ is defined as $\widehat{\mathcal{G}}_2 = \{u \in \widehat{\mathcal{S}} : u(t) = M_u\mathfrak{E}(t, \infty, 1/\Phi(\widehat{\delta}+1), \widehat{G})\}$, where $M_u = \lim_{t\to\infty} u(t)$.

Lemma 3.11. *Let (1.11) be satisfied with $\delta \neq -1$ and $\gamma \neq \alpha - 1$. Then the following hold:*

(i) $\mathcal{DS}_{B0} \cup \mathcal{DS}_{B\infty} \cup \mathcal{IS}_{B0} \cup \mathcal{IS}_{B\infty} \subseteq X$, where $X = \mathcal{B}_5$ when $\delta + \alpha = \gamma$, while $X = \mathcal{B}_3$ when $\delta + \alpha < \gamma$.

(ii) $\mathcal{DS}_{0B} \cup \mathcal{IS}_{\infty B} \subseteq X$, where $X = \mathcal{B}_6$ when $\delta + \alpha = \gamma$, while $X = \mathcal{B}_4$ when $\delta + \alpha < \gamma$.

(iii) $\mathcal{IS}_{BB} \cup \mathcal{DS}_{BB} \subseteq \mathcal{B}_i$, $i = 1, 2$.

Proof. (i) Let $y \in \mathcal{DS}_{B0} \cup \mathcal{DS}_{B\infty} \cup \mathcal{IS}_{B0} \cup \mathcal{IS}_{B\infty}$. Then $y \in \mathcal{S}_{\mathcal{SV}}$, and so $y \in \mathcal{S}_{\mathcal{NSV}}$, see Remark 3.4. Integrating (1.1) we get $y^{[1]}(t) \sim P_y(t)$ as $t \to \infty$, where $P_y(t) = \int_t^\infty p(s)\Phi(y(s))\,\mathrm{d}s$ or $P_y(t) = -\int_{t_0}^t p(s)\Phi(y(s))\,\mathrm{d}s$ according to whether $\delta < -1$ or $\delta > -1$, respectively. Applying Theorem 3.2 and using $y(t) \sim M_y$, where $M_y = \lim_{t\to\infty} y(t)$, in both cases we get

$$y^{[1]}(t) \sim \frac{-1}{\delta+1} tp(t)\Phi(y(t)) \sim \frac{-1}{\delta+1} tp(t)\Phi(M_y)$$

as $t \to \infty$, thus $y'(t) \sim -M_y G(t)/\Phi^{-1}(\delta+1)$ as $t \to \infty$. Integrating the last relation from t to ∞, we obtain

$$M_y - y(t) \sim \frac{-M_y}{\Phi^{-1}(\delta+1)} \int_t^\infty G(s)\,\mathrm{d}s \tag{3.5}$$

as $t \to \infty$. Assume that $\delta + \alpha = \gamma$. Since $|G| \in \mathcal{RV}(-1)$, from Theorem 3.2 we get $t|G(t)| = o\left(\int_t^\infty |G(s)|\,\mathrm{d}s\right)$ as $t \to \infty$. Combining the last relation with (3.5), we find that $y \in \mathcal{B}_5$. Assume that $\delta + \alpha < \gamma$. Then, in view of Proposition 3.1, $|G| \in \mathcal{RV}(\zeta - 1)$, where $\zeta =$

$(\beta-1)(\delta+1-\gamma)+1$. From $\delta+\alpha<\gamma$ we have $\zeta<0$. Hence, Theorem 3.2 yields $\int_t^\infty G(s)\,\mathrm{d}s \sim -tG(t)/\zeta$ as $t\to\infty$, thus (3.5) implies $y\in\mathcal{B}_3$.

(ii) Let $y\in\mathcal{DS}_{0B}\cup\mathcal{IS}_{\infty B}\subseteq X$. Set $u=|y^{[1]}|$. We have $u=\pm y^{[1]}$ according to whether $y\in\mathcal{IS}$ or $y\in\mathcal{DS}$, respectively. Then u satisfies (3.2), and $\lim_{t\to\infty}u(t)=M_u$ where $M_u=|N_y|$, $N_y=\lim_{t\to\infty}y^{[1]}(t)$. Since, in addition,

$$u^{[1]}=\widehat{r}\Phi^{-1}(u')=|p|^{1-\beta}\Phi^{-1}(\pm(r\Phi(y'))')=\mp|p|^{1-\beta}\Phi^{-1}(p\Phi(y))=\mp y\operatorname{sgn}p, \tag{3.6}$$

we get

$$u\in\widehat{\mathcal{DS}}_{B0}\cup\widehat{\mathcal{DS}}_{B\infty}\cup\widehat{\mathcal{IS}}_{B0}\cup\widehat{\mathcal{IS}}_{B\infty}. \tag{3.7}$$

We use the convention introduced in Lemma 3.9 and Remark 3.10. The reciprocal version of $\delta+\alpha\le\gamma$ is $\widehat{\delta}+\beta\le\widehat{\gamma}$; it is easy to see that the inequalities are in fact the same. In view of (3.7), we can apply part (i) of Lemma 3.11 to the reciprocal equation. If $\delta+\alpha<\gamma$, then

$$\begin{aligned}|N_y|-|y^{[1]}(t)|=M_u-u(t)&\sim\frac{M_u(\beta-1)}{\Phi(\widehat{\delta}+1)(\widehat{\delta}+\beta-\widehat{\gamma})}t\Phi\left(\frac{t\widehat{p}(t)}{\widehat{r}(t)}\right)\\&=\frac{|N_y|}{\Phi(\gamma(1-\beta)+1)(-\gamma+\beta/(\beta-1)+\delta)}t^\alpha\frac{p(t)}{r(t)}=\frac{-|N_y|}{\Phi(\varrho)(\delta+\alpha-\gamma)}tH(t)\end{aligned}$$

as $t\to\infty$. Consequently, $y\in\mathcal{B}_4$. Similarly we find that $\mathcal{B}_6$ is reciprocal version of $\mathcal{B}_5$.

(iii) Let $y\in\mathcal{IS}_{BB}\cup\mathcal{DS}_{BB}$. From (1.1), $(y^{[1]}(t))'\sim -M_y^{\alpha-1}p(t)$ as $t\to\infty$, where $M_y=\lim_{t\to\infty}y(t)$. Theorem 3.2 yields

$$N_y-y^{[1]}(t)\sim -M_y^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s\sim\frac{-M_y^{\alpha-1}}{-(\delta+1)}tp(t)$$

as $t\to\infty$, where $N_y=\lim_{t\to\infty}y^{[1]}(t)$. This implies $\mathcal{IS}_{BB}\cup\mathcal{DS}_{BB}\subseteq\mathcal{B}_2$. From the relation $y^{[1]}(t)\sim N_y$ as $t\to\infty$, which is equivalent to $y'(t)\sim\Phi^{-1}(N_y/r(t))$, by Theorem 3.2, we obtain

$$M_y-y(t)\sim\Phi^{-1}(N_y)\int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s\sim\frac{\Phi^{-1}(N_y)}{-((1-\beta)\gamma+1)}tr^{1-\beta}(t)=-\frac{\Phi^{-1}(N_y)}{\varrho}tr^{1-\beta}(t)$$

as $t\to\infty$. This implies $\mathcal{IS}_{BB}\cup\mathcal{DS}_{BB}\subseteq\mathcal{B}_1$. □

Lemma 3.12. *Let (1.11) be satisfied with $\delta\ne-1$ and $\gamma\ne\alpha-1$. Then the following hold:*

(i) If $J=\infty$, then $(\mathcal{DS}_{00}\cup\mathcal{IS}_{\infty\infty}\cup\mathcal{IS}_{\infty0}\cup\mathcal{DS}_{0\infty})\cap\mathcal{SV}\subseteq\mathcal{G}_1$.

(ii) If $J<\infty$, then $\mathcal{IS}_{B0}\cup\mathcal{DS}_{B0}\cup\mathcal{IS}_{B\infty}\cup\mathcal{DS}_{B\infty}\subseteq\mathcal{G}_2$.

Proof. Take $y\in\mathcal{S}_{\mathcal{SV}}$. Note that in (i) slow variation is assumed, in (ii) it clearly holds, and $\mathcal{S}_{\mathcal{SV}}=\mathcal{S}_{\mathcal{NSV}}$, see Remark 3.4. We have $|p|\Phi(y)\in\mathcal{RV}(\delta)$ by Proposition 3.1. Let $\delta<-1$. Then $\int_a^\infty|p(s)|\Phi(y(s))\,\mathrm{d}s<\infty$ by Theorem 3.2. Observe that the classes considered in the lemma, which correspond to this setting, are $\mathcal{DS}_{x0},\mathcal{IS}_{x0}$. Indeed, from (1.1) we have

$$\left|y^{[1]}(t)-y^{[1]}(t_0)\right|=\int_{t_0}^t|p(s)|\Phi(y(s))\,\mathrm{d}s \tag{3.8}$$

and because of the convergence of the integral we cannot have $\lim_{t\to\infty}|y^{[1]}(t)|=\infty$. Assume that y belongs to such classes. Integrating (1.1) from t to ∞, Theorem 3.2 yields

$$-y^{[1]}(t)=-\int_t^\infty p(s)\Phi(y(s))\,\mathrm{d}s\sim\frac{1}{\delta+1}tp(t)\Phi(y(t)) \tag{3.9}$$

as $t \to \infty$. Similarly, under the condition $\delta > -1$, which corresponds to the classes $\mathcal{DS}_{x\infty}, \mathcal{IS}_{x\infty}$ (this follows from (3.8) and the divergence of the integral), integration of (1.1) from t_0 to t and Theorem 3.2 lead to

$$y^{[1]}(t) = y^{[1]}(t_0) - \int_{t_0}^{t} p(s)\Phi(y(s))\,\mathrm{d}s \sim -\int_{t_0}^{t} p(s)\Phi(y(s))\,\mathrm{d}s \sim -\frac{1}{\delta+1} tp(t)\Phi(y(t)) \tag{3.10}$$

as $t \to \infty$. Consequently, no matter what $\delta \neq -1$ is, both (3.9) and (3.10) lead to

$$\frac{y'(t)}{y(t)} \sim \Phi^{-1}\left(\frac{-1}{\delta+1}\right)\Phi^{-1}\left(\frac{tp(t)}{r(t)}\right) = \Phi^{-1}\left(\frac{-1}{\delta+1}\right)G(t) \tag{3.11}$$

as $t \to \infty$. The following observation which was established in [22] will be useful in the sequel. Let $A \in \mathbb{R}$, $\varepsilon_1(t) \to 0$ as $t \to \infty$, and f be a positive function such that $\int_a^\infty f(t)\,\mathrm{d}t = \infty$. Then there exists $\varepsilon_2(t) \to 0$ as $t \to \infty$ such that

$$A + \int_a^t (1+\varepsilon_1(s))f(s)\,\mathrm{d}s = \int_a^t (1+\varepsilon_2(s))f(s)\,\mathrm{d}s. \tag{3.12}$$

If $J = \infty$, then integration of (3.11) from t_0 to t yields

$$\begin{aligned} \ln y(t) &= \ln y(t_0) + \int_{t_0}^t (1+o(1))\Phi^{-1}\left(\frac{-1}{\delta+1}\right)G(s)\,\mathrm{d}s \\ &= \int_{t_0}^t (1+o(1))\Phi^{-1}\left(\frac{-1}{\delta+1}\right)G(s)\,\mathrm{d}s \\ &= \int_{a}^t (1+o(1))\Phi^{-1}\left(\frac{-1}{\delta+1}\right)G(s)\,\mathrm{d}s \end{aligned}$$

as $t \to \infty$, where we applied (3.12) twice. Taking exponential, we find that $y \in \mathcal{G}_1$. If $J < \infty$, then integration of (3.11) from t to ∞ yields

$$-\ln\frac{y(t)}{M_y} = \int_t^\infty \Phi^{-1}\left(\frac{-(1+o(1))}{\delta+1}\right)G(s)\,\mathrm{d}s$$

as $t \to \infty$, where $M_y = \lim_{t\to\infty} y(t)$, which leads to $y \in \mathcal{G}_2$. □

Remark 3.13. Let (1.11) hold with $\delta \neq -1$ and $\gamma \neq \alpha - 1$. Let $\mathcal{S}_{\mathcal{NSV}} \neq \emptyset$ and recall that it implies (1.2) with $C_\gamma = 0$ by Lemma 3.5. Assume that $J = \infty$ and note that then necessarily $\delta + \alpha = \gamma$. Indeed, $\delta + \alpha < \gamma$ would imply $J < \infty$ while $\delta + \alpha > \gamma$ would imply $\mathcal{S}_{\mathcal{NSV}} = \emptyset$. From [19, Section 6] and [23, Section 4] it follows that if $p < 0$, then

$$\begin{aligned} \mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{DS}_{00} \quad \text{provided} \quad \delta < -1, \\ \mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{IS}_{\infty\infty} \quad \text{provided} \quad \delta > -1. \end{aligned}$$

From [22, Section 5] we have, if $p > 0$, then

$$\begin{aligned} \mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{IS}_{\infty 0} \quad \text{provided} \quad \delta < -1, \\ \mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{DS}_{0\infty} \quad \text{provided} \quad \delta > -1. \end{aligned}$$

Assume that $J < \infty$. From [19, Section 6], [22, Section 5], and [23, Section 4] we have, if $p < 0$, then

$$\begin{aligned} \mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{DS}_{B0} \quad \text{provided} \quad \delta < -1, \gamma < \alpha - 1, \\ \mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{IS}_{B\infty} \quad \text{provided} \quad \delta > -1, \gamma > \alpha - 1. \end{aligned}$$

From [22, Section 5] we have, if $p > 0$, then

$$\begin{aligned} \mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{IS}_{B0} \quad \text{provided} \quad \delta < -1, \gamma < \alpha - 1, \\ \mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{DS}_{B\infty} \quad \text{provided} \quad \delta > -1, \gamma > \alpha - 1. \end{aligned}$$

Lemma 3.14. *Let (1.11) be satisfied with $\delta \neq -1$ and $\gamma \neq \alpha - 1$. Then the following hold:*

(i) If $R = \infty$, then $(\mathcal{DS}_{00} \cup \mathcal{DS}_{0\infty}) \cap \mathcal{RV}(\varrho) \subseteq \mathcal{H}_2 = \mathcal{H}_0$ and $(\mathcal{IS}_{\infty\infty} \cup \mathcal{IS}_{\infty 0}) \cap \mathcal{RV}(\varrho) \subseteq \mathcal{H}_1 = \mathcal{H}_0$.

(ii) If $R < \infty$, then $\mathcal{DS}_{0B} \subseteq \mathcal{H}_4$ and $\mathcal{IS}_{\infty B} \subseteq \mathcal{H}_3$.

(iii) If $R = \infty$, then $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_0$.

Proof. We will prove the case when $R < \infty$ for the class $\mathcal{DS}_{0B}$ with details. The other cases in (i) and (ii) can be proved similarly. Let $y \in \mathcal{DS}_{0B}$. Set $u = -y^{[1]}$. Then u satisfies reciprocal equation (3.2) and $u \in \widehat{S}$ by Lemma 3.9. Since $y \in \mathcal{DS}_{0B}$, we get $u(t) \sim M_u$ as $t \to \infty$, where $M_u = -N_y = -\lim_{t\to\infty} y^{[1]}(t)$. As in (3.6), we get $u^{[1]} = y \operatorname{sgn} p$, and therefore $u^{[1]}(t) \to 0$ as $t \to \infty$. Consequently, $u \in \widehat{\mathcal{DS}}_{B0}$ or $u \in \widehat{\mathcal{IS}}_{B0}$ according to whether $p < 0$ or $p > 0$, respectively. In view of Lemma 3.12-(ii), we get $u \in \widehat{\mathcal{G}}_2$, that is

$$u(t) = M_u \exp\left\{\int_t^\infty \frac{1+o(1)}{\Phi(\widehat{\delta}+1)} \Phi\left(\frac{s\widehat{p}(s)}{\widehat{r}(s)}\right) \mathrm{d}s\right\}$$

as $t \to \infty$. We use the convention from Lemma 3.9 and Remark 3.10. Thus we find that

$$-r(t)\Phi(y'(t)) = u(t) = -N_y \exp\left\{\int_t^\infty (1+o(1)) \frac{1}{\Phi(\varrho)} H(s)\,\mathrm{d}s\right\},$$

which yields

$$y'(t) = \Phi^{-1}(N_y) r^{1-\beta}(t) \exp\left\{\int_t^\infty (1+o(1)) \frac{\beta-1}{\Phi(\varrho)} H(s)\,\mathrm{d}s\right\},$$

as $t \to \infty$. Since $y \in \mathcal{DS}_0$, integration from t to ∞ leads to $y \in \mathcal{H}_4$.

It remains to prove $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_0$ when $R = \infty$. Take $y \in \mathcal{H}_1$. In view of (1.13) and representation (1.9), we have $\mathfrak{E}(a, \cdot, -(\beta-1)\Phi(\varrho), H) \in \mathcal{SV}$. Therefore, $r^{1-\beta}\mathfrak{E}(a, \cdot, -(\beta - 1)\Phi(\varrho), H) \in \mathcal{RV}(\gamma(1-\beta))$ by Proposition 3.1. Hence, from Theorem 3.2 and thanks to divergence of $\int_a^\infty |H(t)|\,\mathrm{d}t$, utilizing (3.12), we obtain

$$\begin{aligned} y(t) &= (1+o(1)) \frac{t r^{1-\beta}(t)}{|\varrho|} \mathfrak{E}\left(a, t, -\frac{\beta-1}{\Phi(\varrho)}, H\right) \\ &= t r^{1-\beta}(t)\, \mathrm{e}^{\ln \frac{1+o(1)}{|\varrho|}}\, \mathfrak{E}\left(a, t, -\frac{\beta-1}{\Phi(\varrho)}, H\right) = t r^{1-\beta}(t) \mathfrak{E}\left(a, t, -\frac{\beta-1}{\Phi(\varrho)}, H\right) \end{aligned}$$

as $t \to \infty$. Thus $\mathcal{H}_1 \subseteq \mathcal{H}_0$. Using similar ideas, we obtain the opposite inclusion. The equality $\mathcal{H}_2 = \mathcal{H}_0$ can be proved analogously. □

Remark 3.15. Let (1.11) hold with $\delta \neq -1$ and $\gamma \neq \alpha - 1$. From the reciprocity principle (see Lemma 3.9) combined with the ideas of Remark 3.13, recalling the relations $u = \pm y^{[1]}$, $u^{[1]} = \mp y \operatorname{sgn} p$ (see (3.6)) and $\widehat{G} = H$ (see (3.3)), we obtain the following claims. Assume $R = \infty$ (which implies $\delta + \alpha = \gamma$). Then

$$\begin{aligned} &\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty\infty} && \text{provided } \delta < -1, p < 0, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{00} && \text{provided } \delta > -1, p < 0, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty 0} && \text{provided } \delta < -1, p > 0, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0\infty} && \text{provided } \delta > -1, p > 0. \end{aligned}$$

Assume $R < \infty$. Then

$$\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty B} \quad \text{provided} \quad \delta < -1, \gamma < \alpha - 1,$$
$$\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0B} \quad \text{provided} \quad \delta > -1, \gamma > \alpha - 1.$$

Lemma 3.16 ([22]). *Let $r \in \mathcal{NRV}(\gamma) \cap C^1$, $\gamma \in \mathbb{R}$, and (1.2) hold with $C_\gamma < K_\gamma$, K_γ being defined by (2.2). Assume that $|L(\vartheta_i, \eta_i, \cdot)| \in \mathcal{RV}(\eta_i - 1)$, $i = 1, 2$, where $\Phi(\vartheta_1) < \Phi(\vartheta_2)$ are the roots of (2.3), $\eta_1, \eta_2 \le 0$, and $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$. Then*

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \subseteq \mathcal{L}_k(\vartheta_i, \eta_i), \qquad i = 1, 2, \tag{3.13}$$

where $k = 1$ when $\int_a^\infty |L(\vartheta_i, \eta_i, s)|\,\mathrm{d}s = \infty$, while $k = 2$ when $\int_a^\infty |L(\vartheta_i, \eta_i, s)|\,\mathrm{d}s < \infty$; if $C_\gamma = 0$, we consider only the nonzero root in (3.13).

Lemma 3.17. *Let (1.11) be satisfied with $p > 0$, $\delta \neq -1$, and $\gamma \neq \alpha - 1$. Then the following hold:*

(i) If $y \in \mathcal{S}_1 \cap \mathcal{RV}(\vartheta)$, $\vartheta \in \mathbb{R}$, where $\mathcal{S}_1 = \mathcal{IS}_{\infty 0} \cup \mathcal{DS}_{0\infty} \cup \mathcal{IS}_{B0} \cup \mathcal{DS}_{B\infty}$, then $|y^{[1]}| \in \mathcal{RV}(\delta + 1 + (\alpha - 1)\vartheta)$ and $|y'| \in \mathcal{RV}((\beta - 1)(\delta + 1 - \gamma) + \vartheta)$. If $y \in \mathcal{S}_1 \cap \mathcal{RV}(\vartheta)$ and $\delta + \alpha = \gamma$, then $|y'| \in \mathcal{RV}(\vartheta - 1)$. If, in addition $\vartheta = \varrho$, then $|y^{[1]}| \in \mathcal{SV}$.

(ii) If $y \in \mathcal{S}_2 \cap \mathcal{RV}(\vartheta)$, where $\mathcal{S}_2 = \mathcal{IS}_{\infty B} \cup \mathcal{DS}_{0B}$, then $\vartheta = \varrho$, $|y^{[1]}| \in \mathcal{SV}$, and $|y'| \in \mathcal{RV}(\varrho - 1)$.

Proof. (i) Let $y \in \mathcal{S}_1 \cap \mathcal{RV}(\vartheta)$. Then $|y^{[1]}|$ tends to 0 or ∞ and $p\Phi(y) \in \mathcal{RV}(\delta + \vartheta(\alpha - 1))$ by Proposition 3.1. Hence, integrating (1.1) from t_0 to t or from t to ∞ (according to whether $\delta + \vartheta(\alpha - 1)$ is positive or negative, respectively), realizing that $y^{[1]}(t) - y^{[1]}(t_0) \sim y^{[1]}(t)$ in the former case, and using Theorem 3.2, we get

$$|y^{[1]}(t)| \sim \frac{1}{|\delta + 1 + \vartheta(\alpha - 1)|} t p(t) \Phi(y(t))$$

as $t \to \infty$, which implies $|y^{[1]}| \in \mathcal{RV}(\delta + 1 + \vartheta(\alpha - 1))$. In view of Proposition 3.1, we get $|y'| \in \mathcal{RV}((\beta - 1)[\delta + 1 + (\alpha - 1)\vartheta - \gamma]) = \mathcal{RV}((\beta - 1)(\delta + 1 - \gamma) + \vartheta)$. If $\delta + \alpha = \gamma$, then the last index reduces to $\vartheta - 1$. If $\vartheta = \varrho$, then for the index associated to $|y^{[1]}|$ we have $\delta + 1 + \vartheta(\alpha - 1) = \delta + \alpha - \gamma = 0$.

(ii) Let $y \in \mathcal{S}_2 \cap \mathcal{RV}(\vartheta)$. Then $y^{[1]}(t) \sim N_y$ as $t \to \infty$, i.e.

$$y'(t) \sim \Phi^{-1}(N_y) r^{1-\beta}(t) \tag{3.14}$$

as $t \to \infty$. Integrating this relation from t_0 to t or from t to ∞ (according to whether $\gamma < \alpha - 1$ or $\gamma > \alpha - 1$, respectively), realizing that $y(t) - y(t_0) \sim y(t)$ in the former case, and using Theorem 3.2, we get

$$y(t) \sim \frac{|\Phi^{-1}(N_y)|}{|(1 - \beta)\gamma + 1|} t r^{1-\beta} \in \mathcal{RV}((1 - \beta)\gamma + 1) = \mathcal{RV}(\varrho),$$

thus $\vartheta = \varrho$. In view of (3.14), we get $|y'| \in \mathcal{RV}(-\gamma(1 - \beta)) = \mathcal{RV}(\varrho - 1)$. □

Lemma 3.18. *Let (1.11) hold with $\gamma \neq \alpha - 1$. If $\mathcal{NSV} \cap (\mathcal{DS}_0 \cup \mathcal{IS}_\infty) \neq \emptyset$, then $\gamma = \delta + \alpha$, $\mathcal{NSV} \cap \mathcal{DS}_0 = \mathcal{DS}_{00} \cup \mathcal{DS}_{0\infty}$, and $\mathcal{NSV} \cap \mathcal{IS}_\infty = \mathcal{IS}_{\infty\infty} \cup \mathcal{IS}_{\infty 0}$.*

Proof. Take $y \in \mathcal{NSV} \cap (\mathcal{DS}_0 \cup \mathcal{IS}_\infty)$. Then, in view of Proposition 3.1, $|(r\Phi(y'))'| = |p|y^{\alpha-1} \in \mathcal{RV}(\delta)$. If $\int_a^\infty |p(s)|y^{\alpha-1}(s)\,\mathrm{d}s$ diverges, then $\lim_{t\to\infty} |y^{[1]}(t)| = \infty$, and the Karamata Integration Theorem (Theorem 3.2) applied to equation (1.1) after integration yields

$$r(t)|y'(t)|^{\alpha-1} \sim |r(t)\Phi(y'(t)) - r(t)\Phi(y'(t_0))| \sim \int_{t_0}^t |p(s)|y^{\alpha-1}(s)\,\mathrm{d}s \in \mathcal{RV}(\delta+1)$$

as $t \to \infty$. Similarly, if $\int_a^\infty |p(s)|y^{\alpha-1}(s)\,\mathrm{d}s$ converges, then

$$r(t)|y'(t)|^{\alpha-1} = \int_t^\infty |p(s)|y^{\alpha-1}(s)\,\mathrm{d}s \in \mathcal{RV}(\delta+1)$$

by Theorem 3.2. Indeed, $\lim_{t\to\infty} y^{[1]}(t) = N_y$ would lead to $y'(t) \sim \Phi^{-1}(N_y)r^{1-\beta}(t)$, so $y \in \mathcal{RV}(\varrho)$, $\varrho \neq 0$, contradiction. Thus in any case, $|y'|^{\alpha-1} \in \mathcal{RV}(\delta+1-\gamma)$, and therefore $|y'| \in \mathcal{RV}((\delta+1-\gamma)/(\alpha-1))$ by Proposition 3.1. Since $y \in \mathcal{DS}_0$ or $y \in \mathcal{IS}_\infty$, in view of the Karamata Theorem, $y \in \mathcal{RV}((\delta+1-\gamma)/(\alpha-1)+1) = \mathcal{RV}((\delta+\alpha-\gamma)(\beta-1))$. But $y \in \mathcal{SV}$, and so it must hold that $\gamma = \delta+\alpha$. □

In spite of the fact that many of the claims which are included in the next statement were already proved above (as it was within the more general setting), for completeness and easier reference we prefer to present some conclusions from [19] in the form of individual lemma.

Lemma 3.19 ([19]). *Let $p < 0$, $C_\gamma = 0$, and (1.11) hold, where $\gamma = \delta + \alpha$.*

(i) Let $\delta < -1$. If $J = \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{DS} = \mathcal{DS}_{00} \subseteq \mathcal{G}_1$. If $J < \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{DS} = \mathcal{DS}_{B0} \subseteq \mathcal{G}_2$. If $R = \infty$, then $\mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} \subseteq \mathcal{H}_1$. If $R < \infty$, then $\mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty B} \subseteq \mathcal{H}_3$.

(ii) Let $\delta > -1$. If $J = \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{IS} = \mathcal{IS}_{\infty\infty} \subseteq \mathcal{G}_1$. If $J < \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{IS} = \mathcal{IS}_{B\infty} \subseteq \mathcal{G}_2$. If $R = \infty$, then $\mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{00} \subseteq \mathcal{H}_2$. If $R < \infty$, then $\mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0B} \subseteq \mathcal{H}_4$.

Theorem 3.20 ([5]). *Let $p < 0$. Then*

$$\mathcal{P} = \begin{cases} \mathcal{DS}_B & \text{if } J_1 = \infty \text{ and } J_2 < \infty, \\ \mathcal{DS}_0 & \text{otherwise.} \end{cases}$$

The lower limit a in the integrals in Theorems 3.21, 3.24, 3.26 is taken such that $y(t) > 0$ and $y'(t) \neq 0$ for $t \geq a$. In the paper [3], an example is given showing that condition (3.15) cannot be omitted. As we will see, in our proofs, the cases where (3.15) fails to hold can fortunately be treated by Theorem 3.24.

Theorem 3.21 ([3,6]). *Let $p > 0$ and (1.1) be nonoscillatory. Assume that*

$$J_r = \infty \text{ and } \alpha \geq 2 \quad \text{or} \quad J_p = \infty \text{ and } 1 < \alpha \leq 2. \tag{3.15}$$

Then, for $y \in \mathcal{S}$,

$$y \in \mathcal{P} \quad \text{if and only if} \quad \int_a^\infty \mathcal{F}[y](t)\,\mathrm{d}t = \infty,$$

where $\mathcal{F}[y] = y'/(y^2 y^{[1]})$.

Theorem 3.22 ([2]). *Let $p > 0$ and (1.1) be nonoscillatory. Assume that $J_r + J_p = \infty$. Then*

$$y \in \mathcal{P} \quad \text{if and only if} \quad |y^{[1]}| \in \widehat{\mathcal{P}},$$

where $\widehat{\mathcal{P}} = \{u \in \widehat{\mathcal{S}} : u \text{ is principal}\}$.

For $\xi \in (1,\infty)$, define the function $\varphi_\xi : [0,1] \to \mathbb{R}$ by

$$\varphi_\xi(t) = \begin{cases} \frac{1-t^\xi}{1-t} + (1-t)^{\xi-1} & \text{if } t \in [0,1), \\ \xi & \text{if } t = 1. \end{cases} \tag{3.16}$$

Denote $m := \min\{\varphi_\beta(t) : t \in [0,1]\}$, $M := \max\{\varphi_\beta(t) : t \in [0,1]\}$, where β is the conjugate number of α.

Lemma 3.23. *It holds that $\varphi_\beta(0) = 2$, $\varphi_\beta(1) = \beta$, $\varphi_\beta(1/2) = 2$, and $m > 1$. If $1 < \alpha < 2$ (i.e., $\beta > 2$), then φ_β is strictly convex on $[0,1]$ and, in particular, $M = \beta$.*

Proof. The equalities $\varphi_\beta(0) = 2$, $\varphi_\beta(1) = \beta$, $\varphi_\beta(1/2) = 2$ are obvious. The convexity of φ_β on $[0,1]$ when $\alpha \in (1,2)$ can be demonstrated via standard calculus tools. The equality $M = \beta$ follows from the convexity of φ_β. □

Theorem 3.24 ([13]). *Let $J_r = \infty$ and $y \in \mathcal{S}$. Denote $\mathcal{T}_K[y] = r^{1-\beta}y^{-K}$, $K \in \mathbb{R}$.*

(i) If $y \in \mathcal{P}$, then $\int_a^\infty \mathcal{T}_m[y](s)\,\mathrm{d}s = \infty$.

(ii) If $\int_a^\infty \mathcal{T}_M[y](s)\,\mathrm{d}s = \infty$, then $y \in \mathcal{P}$.

Remark 3.25. By means of the reciprocity principle (see Lemma 3.9), with help of Theorem 3.22, the condition $J_r = \infty$ in Theorem 3.24 can actually be relaxed to $J_r + J_p = \infty$; $\mathcal{T}_m, \mathcal{T}_M$ are then appropriately modified. For details see the proofs of Theorems 2.1, 2.2, 2.3, and 2.4, where this trick is used.

Theorem 3.26 ([2]). *Let $p > 0$ and $J_r + J_p < \infty$. Then*

$$y \in \mathcal{P} \quad \text{if and only} \quad \int_a^\infty \frac{1}{r^{\beta-1}(t)y^2(t)}\,\mathrm{d}t = \infty.$$

In view of their common setting, it is senseful to prove Theorems 2.1 and 2.2 simultaneously.

Proof of Theorems 2.1 and 2.2. Let $p < 0$. If $J = \infty$ and $\delta < -1$, then $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{DS}_{00} \subseteq \mathcal{G}_1$ by Lemma 3.12-(i) and Remark 3.13. Since $G(t) = \Phi^{-1}(L_p(t)/L_r(t))/t$ and $\lim_{t\to\infty} L_p(t)/L_r(t) = 0$, we have $\mathcal{G}_1 \subseteq \mathcal{S}_{\mathcal{SV}} = \mathcal{S}_{\mathcal{NSV}}$ by the Representation Theorem (see (1.9)) and Remark 3.4. From [23] we know that $\mathcal{DS} \subseteq \mathcal{NSV}$, thus $\mathcal{DS}_{00} \subseteq \mathcal{NSV}$. In view of (1.4)

$$\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho), \qquad \mathcal{S}_{\mathcal{NSV}} \neq \emptyset, \qquad \mathcal{S}_{\mathcal{NRV}}(\varrho) \neq \emptyset \tag{3.17}$$

(which follows from Theorem 3.3), we get $\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{G}_1$. Analogously we obtain $\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{G}_1$ when $J = \infty$ and $\delta > -1$. If $J < \infty$, then in a similar manner as above we use Lemma 3.12-(ii), the obvious fact $\mathcal{G}_2 \subseteq \mathcal{SV}$, (3.17), (1.4), Lemma 3.19, and, in addition, Lemma 3.11-(i), to get $\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5$ when $\delta < -1$ and $\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5$ when $\delta > -1$. Note that Lemma 3.11-(i) yields $\mathcal{DS}_{B0} \subseteq \mathcal{B}_5$ and $\mathcal{IS}_{B\infty} \subseteq \mathcal{B}_5$, respectively. The opposite inclusions are obvious, since y belonging to

$\mathcal{B}_5$ is slowly varying and there are no other slowly varying solutions than $\mathcal{DS}_{B0}$ and $\mathcal{IS}_{B\infty}$, respectively, see Remark 3.13. Let $R = \infty$ and $\delta < -1$. Observe that $\mathcal{H}_1 \subseteq \mathcal{RV}(\varrho)$. Indeed, if $y \in \mathcal{H}_1$, then $y(t) \sim \int_{t_0}^{t} r^{1-\beta}(s)\mathfrak{E}(a, s, -(\beta-1)/\Phi(\varrho), H)\,\mathrm{d}s \in ([\gamma(1-\beta)+0]+1) = \mathcal{RV}(\varrho)$, where we use (1.9), Proposition 3.1, and Theorem 3.2. From Lemma 3.14-(i) and Remark 3.15, taking into account Lemma 3.19, (3.17), and (1.4), we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{H}_1$. Lemma 3.14-(iii) gives $\mathcal{H}_1 = \mathcal{H}_0$. Because $\delta + \alpha = \gamma$ and ϱ is the bigger root of (2.3) when $\delta < -1$, the condition $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$ from Lemma 3.16 reads as $\delta < -1 + \eta(\alpha - 1)$ which is assumed in Theorems 2.1, 2.2. Since also all other assumptions of Lemma 3.16 are satisfied, we may apply it to obtain $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{L}(\varrho, \eta)$; we use convention (1.15). Since $\lim_{t\to\infty} tL(\vartheta, \eta, t) = 0$, from the Representation Theorem (see (1.9)) it follows that $\mathcal{L}(\varrho, \eta) \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Analogously we proceed when $R = \infty$ and $\delta > -1$. Let us only note that in this case, ϱ is the lesser root of (2.3) (since $\varrho < 0$) and therefore we do not need to verify the condition $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$ from Lemma 3.16. The case $R < \infty$ can also be treated similarly; we use, in addition, Lemma 3.14-(ii) and Lemma 3.11. Next we derive the relations with $\mathcal{P}$. If $\delta > -1$, then $J_p = \infty$ by Theorem 3.2, thus $J_2 = \infty$. Hence, $\mathcal{P} = \mathcal{DS}_0$ by Theorem 3.20. If $\delta < -1$, then, in view of $\delta + \alpha = \gamma$, we have $\gamma < \alpha - 1$, thus $r^{1-\beta} \in \mathcal{RV}((1-\beta)\gamma)$ with the index greater than -1, and so $J_r = \infty$ (see Theorem 3.2), which implies $J_1 = \infty$. Further, by Lemma 3.7, $V_2(t) \sim |G(t)|/|\delta+1|^{\beta-1} \in \mathcal{RV}(-1)$ as $t \to \infty$. Hence, in general, J_2 can converge or diverge. But we see that $J_2 = \infty$ if and only if $J = \infty$. According to Theorem 3.20, if $J = \infty$, then $\mathcal{P} = \mathcal{DS}_0$, while if $J < \infty$, then $\mathcal{P} = \mathcal{DS}_B$. Adding the relations between $\mathcal{P}$ and $\mathcal{DS}_0$ resp. $\mathcal{P}$ and $\mathcal{DS}_B$ to the other relations we obtain the complete picture in the case $p < 0$.

Let $p > 0$. First of all note that by Theorem 3.3, (3.17) holds. Assume that $\delta < -1$. Then $\gamma < \alpha - 1$, $r^{1-\beta}$ thus has the index of regular variation greater than -1, and so $J_r = \infty$ by Theorem 3.2. Hence, (1.5) holds. Note that ϱ in (3.17) is now positive. If $J = \infty$, then by Lemma 3.12 and Remark 3.13, we get $\mathcal{S}_{\mathcal{NSV}} \cap \mathcal{IS}_{\infty 0} \subseteq \mathcal{G}_1$ and $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{IS}_{\infty 0}$. In view of $\mathcal{G}_1 \subseteq \mathcal{S}_{\mathcal{NSV}}$ (which follows from (1.9)) and (2.1), we have $\mathcal{S}_{\mathcal{NSV}} = \mathcal{G}_1 = \mathcal{IS}_{\infty 0}$. If $R = \infty$, then by Lemma 3.14 and Remark 3.15, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \cap \mathcal{IS}_{\infty 0} \subseteq \mathcal{H}_1$ and $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty 0}$. In view of $\mathcal{H}_1 \subseteq \mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{S}_{\mathcal{NRV}}(\varrho)$ (which follows from (1.9)), we have $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{H}_1 = \mathcal{IS}_{\infty 0}$. By Lemma 3.14, $\mathcal{H}_1 = \mathcal{H}_0$. Assume that $J = \infty$ and $R = \infty$. Because of (3.17), (1.5), and the observations from the previous parts, we have $\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty 0} \subseteq \mathcal{IS} = \mathcal{S}$. If $J < \infty$, then by Lemma 3.12 and Remark 3.13, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{IS}_{B0}$, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{G}_2$. If $y \in \mathcal{IS}_B$, then it is clearly slowly varying and we get $\mathcal{IS}_{B0} = \mathcal{S}_{\mathcal{NSV}}$. Since $\mathcal{G}_2 \subseteq \mathcal{SV}$ and (2.1) holds, we have $\mathcal{S}_{\mathcal{NSV}} = \mathcal{G}_2$. In view of Lemma 3.11, we obtain $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{B}_5$; the opposite inclusion obviously holds as well. If $R < \infty$, then by Lemma 3.14 and Remark 3.15 it follows that $\mathcal{IS}_{\infty B} \subseteq \mathcal{H}_3$ and $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty B}$. From (1.9), Proposition 3.1, and Theorem 3.2, we have $\mathcal{H}_3 \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. If $y \in \mathcal{IS}_{\infty B}$, then $y^{[1]}(t) \sim N_y \in (0, \infty)$ as $t \to \infty$. Expressing y' and integrating, Theorem 3.2 and Proposition 3.1 yield

$$y(t) \sim \left| \frac{\Phi(N_y)}{\gamma(1-\beta)+1} \right| t r^{1-\beta}(t) \in \mathcal{RV}(\gamma(1-\beta)+1) = \mathcal{RV}(\varrho) \tag{3.18}$$

as $t \to \infty$. Hence, $\mathcal{IS}_{\infty B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Consequently, in view of the fact that regular variation of solutions is normalized, we have $\mathcal{IS}_{\infty B} = \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{H}_3$. From Lemma 3.11 we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{B}_6$. The opposite inclusion is obvious. The settings $J < \infty, R < \infty$, or $J = \infty, R < \infty$, or $J < \infty, R = \infty$, can be treated by suitable combinations of the above presented observations. Similarly as in the case $p < 0$, with the help Lemma 3.16, we show $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{L}(\varrho, \eta)$; we use convention (1.15).

The case $p > 0$ and $\delta > -1$ can be proved analogously to the case $p > 0$ and $\delta < -1$ (applying again (3.17), (1.9), Lemma 3.11, Lemma 3.12, Remark 3.13, Lemma 3.14, Lemma 3.16), and therefore it is omitted.

In the last part of the proof we will show how $\mathcal{P}$ is related to the other classes when $p > 0$. From the above established classification we see that any $y \in \mathcal{S}$ must belong either to $\mathcal{S}_1$ or $\mathcal{S}_2$ under the assumptions of Theorem 2.1 and Theorem 2.2. By Lemma 3.17 we have that $\mathcal{F}[y]$ is regularly varying. Let Ω denote the index of regular variation of $\mathcal{F}[y]$.

Assume first that (3.15) holds. If $y \in \mathcal{S}_{\mathcal{NSV}}$, then $\Omega = -\delta - 2$. If $\delta < -1$, then $\Omega > -1$, and so $\int_a^\infty \mathcal{F}[y](s)\,\mathrm{d}s = \infty$ by Theorem 3.2. This yields $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{P}$ by Theorem 3.21. Similarly we obtain $\mathcal{S}_{\mathcal{NSV}} \cap \mathcal{P} = \emptyset$ when $\delta > -1$. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Then $\Omega = (\beta-1)(\delta+1-\gamma) + \varrho - 2\varrho = -\varrho - 1$, see Lemma 3.17. If $\delta < -1$, then $\varrho > 0$, i.e., $-\varrho - 1 < -1$, which implies $\int_a^\infty \mathcal{F}[y](s)\,\mathrm{d}s < \infty$, and we obtain $\mathcal{S}_{\mathcal{NRV}}(\varrho) \cap \mathcal{P} = \emptyset$ by Theorem 3.21. Similarly we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{P}$ when $\delta > -1$. Altogether, in view of (3.17), $\mathcal{P} = \mathcal{S}_{\mathcal{NSV}}$ when $\delta < -1$, while $\mathcal{P} = \mathcal{S}_{\mathcal{NRV}}(\varrho)$ when $\delta > -1$.

Assume now that (3.15) fails to hold. The constants m, M will have the same meaning as in Theorem 3.24. Let $J_r = \infty$ (this means $\gamma < \alpha - 1$, thus, $\delta < -1$ since we assume $\gamma \neq \alpha - 1$ and $\delta + \alpha = \gamma$) and $\alpha < 2$. If $y \in \mathcal{S}_{\mathcal{NSV}}$, then $r^{1-\beta}y^{-M} \in \mathcal{RV}(-\gamma/(\alpha-1))$ by Proposition 3.1. In view of $\gamma < \alpha - 1$, the index is greater -1, and so $\int_a^\infty r^{1-\beta}(s)y^{-M}(s)\,\mathrm{d}s = \infty$ by Theorem 3.2. Hence, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{P}$ by Theorem 3.24. If $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$, then $r^{1-\beta}y^{-m} \in \mathcal{RV}(-\gamma/(\alpha-1) - \varrho m)$. It clearly holds $-\gamma/(\alpha-1) - \varrho m < -1$ if and only if $(\alpha - 1 - \gamma)(1 - m) < 0$. The latter inequality holds since $m > 1$, see Lemma 3.23, and $\alpha - 1 > \gamma$. Consequently, $\int_a^\infty r^{1-\beta}(s)y^{-M}(s)\,\mathrm{d}s < \infty$ by Theorem 3.2, and so Theorem 3.24 yields $\mathcal{S}_{\mathcal{NRV}}(\varrho) \cap \mathcal{P} = \emptyset$. In view of (3.17), we have $\mathcal{S}_{\mathcal{NSV}} = \mathcal{P}$.

Let $J_p = \infty$ (i.e., $\delta > -1$, i.e., $\gamma > \alpha - 1$) and $\alpha > 2$. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$ and note that $\varrho < 0$ and $\mathcal{S} = \mathcal{DS}$. Set $u = -y^{[1]}$. Then u is positive and satisfies (3.2), thus $u \in \widehat{\mathcal{S}}$. By Lemma 3.17, $u \in \widehat{\mathcal{S}}_{\mathcal{NSV}}$. Because of our assumptions we have $\widehat{\delta} < -1$ and $\widehat{\gamma} < \beta - 1$, where $\widehat{\delta}$ and $\widehat{\gamma}$ are defined in (3.4). Thus we can apply Theorem 3.24 to reciprocal equation (3.2). Denote $\widehat{M} = \max\{\varphi_\alpha(t) : t \in [0,1]\}$ and note that φ_α can be understood as a reciprocal counterpart to φ_β. Since $\widehat{r}^{1-\alpha}u^{-\widehat{M}} \in \mathcal{RV}(-\widehat{\gamma}(\alpha-1))$, where $\widehat{\gamma}(\alpha-1) > -1$, we have $\int_a^\infty \widehat{r}^{1-\alpha}(s)u^{-\widehat{M}}(s)\,\mathrm{d}s = \infty$, which implies $u \in \widehat{\mathcal{P}}$. In view of Theorem 3.22, we get $y \in \mathcal{P}$, thus $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{P}$. Now take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$ and $x \in \mathcal{S}_{\mathcal{NSV}}$. Then, since we have $ty'(t)/y(t) \to \varrho < 0$ and $tx'(t)/x(t) \to 0$ with $t \to \infty$, we get $y'(t)/y(t) < x'(t)/x(t)$ for large t, hence $x \notin \mathcal{P}$ by definition. Consequently, $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{P}$. $\square$

Proof of Theorem 2.3. Since $t^\alpha p(t)/r(t) \in \mathcal{RV}(\delta + \alpha - \gamma)$ (by Proposition 3.1) and $\delta + \alpha < \gamma$, we have $C_\gamma = 0$. Consequently (3.17) holds. The following observation will be repeatedly utilized in the sequel. Thanks to (1.11), $|G| \in \mathcal{RV}((\delta + 1 - \gamma)(\beta - 1))$ and $|H| \in \mathcal{RV}(\alpha - 1 + \delta - \gamma)$ by Proposition 3.1. It is easy to see that $\delta + \alpha < \gamma$ is equivalent to $(\delta + 1 - \gamma)(\beta - 1) < -1$. Hence, both the indices of $|G|$ and $|H|$ are less than -1, and so

$$J < \infty \quad \text{and} \quad R < \infty. \tag{3.19}$$

(i-a) Let $\delta < -1$, $\gamma < \alpha - 1$, and $p < 0$. Take $y \in \mathcal{S}_{\mathcal{NSV}}$. Then $y \in \mathcal{DS}$. Indeed, if $y \in \mathcal{IS}$, then $y^{[1]}$ is positive increasing, hence there is $A > 0$ such that $y^{[1]}(t) \geq A$ for large t, say $t \geq t_0$. Consequently, by Theorem 3.2 and Proposition 3.1,

$$y(t) \geq y(t_0) + A^{\beta-1}\int_{t_0}^t r^{1-\beta}(s)\,\mathrm{d}s \in \mathcal{RV}(\gamma(1-\beta)) = \mathcal{RV}(\varrho).$$

Hence, y is greater than or equal to a regularly varying function with a positive index, thus cannot be slowly varying. Using similar arguments we find that for $y \in \mathcal{DS}$, the quasiderivative $y^{[1]}$ (which is negative increasing) must tend to zero. Moreover, $y \in \mathcal{DS}_{B0}$. Indeed, if $y \in \mathcal{DS}_{00}$, then $\gamma = \delta + \alpha$ (see Lemma 3.18), which contradicts to $\delta + \alpha < \gamma$. Hence, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{DS}_{B0} \subseteq \mathcal{DS}$. On the other hand, if $y \in \mathcal{DS}$, then it cannot be in $\mathcal{NRV}(\varrho)$ since $\varrho > 0$ (and the functions with a positive index always tend to infinity, see Proposition 3.1), consequently, in view of (3.17), we get $\mathcal{DS} \subseteq \mathcal{S}_{\mathcal{NSV}}$. Therefore, $\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{B0} = \mathcal{DS}$. Consider the class $\mathcal{S}_{\mathcal{NRV}}(\varrho)$. From the previous part we know that slowly varying solutions cannot be increasing. Recalling (3.17), we get $\mathcal{IS} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Applying Remark 3.8 and (3.19) we obtain $J_2 < \infty$ and $R_1 < \infty$. Condition $\gamma < \alpha - 1$ implies $J_r = \infty$ and that is why $J_1 = \infty$ and $R_2 = \infty$, see Remark 3.8. According to [5, Theorem 1], see also [6, Chapter 4], we get $\mathcal{IS} = \mathcal{IS}_{\infty B}$. Moreover $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$ cannot be decreasing since $\varrho > 0$, thus $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}$. We obtain $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty B} = \mathcal{IS}$.

It is not difficult to see that the relations of $\mathcal{S}_{\mathcal{NSV}}$ with $\mathcal{G}_2, \mathcal{B}_3$ and of $\mathcal{S}_{\mathcal{NRV}}(\varrho)$ with $\mathcal{H}_3, \mathcal{B}_4, \mathcal{L}(\varrho, \eta)$ follow similarly as they were established in the proof of Theorems 2.1 and 2.2, with the help of Lemma 3.12, Remark 3.13, Lemma 3.14, Remark 3.15, Lemma 3.11, Lemma 3.16, formula (1.9), and [22, Section 5].

(i-b) Let $\delta < -1$, $\gamma < \alpha - 1$, and $p > 0$. Thanks to $\gamma < \alpha - 1$ and $r^{1-\beta} \in \mathcal{RV}(\gamma(1-\beta))$, we have $J_r = \infty$ (see Theorem 3.2), which implies (1.5). Take $y \in \mathcal{S}_{\mathcal{NSV}}$. Then $\lim_{t\to\infty} y^{[1]}(t) = 0$. Indeed, $y^{[1]}$ is positive decreasing and if $y^{[1]}(t) \sim N_y > 0$ as $t \to \infty$, then as in (3.18), we get $y \in \mathcal{RV}(\varrho)$, contradiction with $y \in \mathcal{S}_{\mathcal{NSV}}$. Moreover, y cannot be in $\mathcal{IS}_{\infty 0}$ otherwise we would get $\gamma = \delta + \alpha$ (see Lemma 3.18), which contradicts to $\delta + \alpha < \gamma$. Consequently, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{IS}_{B0}$. The opposite inclusion is obvious, in view of (2.1). Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$. From the previous part we get that $y \in \mathcal{IS}_{\infty 0} \cup \mathcal{IS}_{\infty B}$. We claim that $\mathcal{IS}_{\infty 0} = \emptyset$. Indeed, if $y \in \mathcal{IS}_{\infty 0}$, then as in (3.9) we obtain

$$y^{[1]}(t) \sim \frac{-tp(t)\Phi(y(t))}{\delta + 1} \tag{3.20}$$

as $t \to \infty$, which leads to (3.11). Integration of this relation from t to ∞, in view $\lim_{t\to\infty} y(t) = \infty$, would give $J = \infty$. This however contradicts to (3.19). Hence, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty B}$. In fact, we have the equality here because of $\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{B0}$ and (3.17). The relations of $\mathcal{S}_{\mathcal{NSV}}$ and $\mathcal{S}_{\mathcal{NRV}}(\varrho)$ with $\mathcal{G}, \mathcal{H}, \mathcal{L}, \mathcal{B}$ type classes can be treated as in the part (i-a).

(ii-a) Let $\delta > -1$, $\gamma > \alpha - 1$, and $p < 0$. Take $y \in \mathcal{S}_{\mathcal{NSV}}$. Then $y \in \mathcal{IS}$. Indeed, if $y \in \mathcal{DS}_{\mathcal{NSV}}$, then $y^{[1]}$ is negative increasing, thus $\lim_{t\to\infty} y^{[1]}(t) \in (-\infty, 0]$. But at the same time, as in (3.10) we get (3.20), where $|tp\Phi(y(t))| \in \mathcal{RV}(\delta + 1)$. Hence, $y^{[1]}(t) \in \mathcal{RV}(\delta + 1)$, which yields $\lim_{t\to\infty} y^{[1]}(t) = \infty$, contradiction with $y \in \mathcal{DS}$. We have $\mathcal{IS} = \mathcal{IS}_{B\infty} \cup \mathcal{IS}_{\infty\infty}$. But if $y \in \mathcal{IS}_{\infty\infty}$, then $\gamma = \delta + \alpha$ by Lemma 3.18, contradiction with $\gamma > \delta + \alpha$. Thus $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{IS}_{B\infty}$. The opposite inclusion clearly holds as well, in view of (2.1). Consider the class $\mathcal{S}_{\mathcal{NRV}}(\varrho)$. First note that $\mathcal{DS} = \mathcal{DS}_{0B}$. Indeed, similarly as in the proof of the part (i-a), from (3.19), Lemma 3.7, and Remark 3.8, we find that $J_1 < \infty, J_2 = \infty, R_1 = \infty$, and $R_2 < \infty$, and the claim follows by [5, Theorem 1], see also [6, Chapter 4]. Since $\varrho < 0$, $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$ cannot be in $\mathcal{IS}$ (see Proposition 3.1), therefore $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0B}$. On the other hand, if $y \in \mathcal{DS}_{0B}$, then $y^{[1]}(t) \sim N_y < 0$ as $t \to \infty$ which yields (3.18), and so $\mathcal{DS}_{0B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$.

(ii-b) Let $\delta > -1$, $\gamma > \alpha - 1$, and $p > 0$. Since $p \in \mathcal{RV}(\delta)$, we have $J_p = \infty$, and so (1.6) holds. Take $y \in \mathcal{S}_{\mathcal{NSV}}$. Then $y^{[1]}$ is negative decreasing and from (3.20), we get $\lim_{t\to\infty} y^{[1]}(t) = -\infty$. Moreover, y cannot be in $\mathcal{DS}_{0\infty}$, otherwise we would get $\gamma = \delta + \alpha$, see Lemma 3.18. Consequently, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{DS}_{B\infty}$. The opposite inclusion is obvious. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$. We know that $y \in \mathcal{DS}_{0\infty} \cup \mathcal{DS}_{0B}$. We claim that $y \notin \mathcal{DS}_{0\infty}$. Indeed, if

$y \in \mathcal{DS}_{0\infty}$, then from (3.10) we get (3.11). Since $y(t) \to 0$ as $t \to \infty$, integration of (3.11) yields $J = \infty$, contradiction with (3.19). Thus, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0B}$ and in view of $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{DS}_{B\infty}$ and (3.17), we get $\mathcal{DS}_{0B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. The relations of $\mathcal{S}_{\mathcal{NSV}}$ and $\mathcal{S}_{\mathcal{NRV}}(\varrho)$ with $\mathcal{G}, \mathcal{H}, \mathcal{L}, \mathcal{B}$ type classes in the setting of (ii-a) and (ii-b) can be treated as in the part (i).

(iii) Let $\delta < -1$ and $\gamma > \alpha - 1$. Then $J_p < \infty$ and $J_r < \infty$. Hence, clearly $J_i < \infty, R_i < \infty$, $i = 1,2$. Assume that $p < 0$. By [5, Theorem 1], see also [6, Chapter 4], we get $\mathcal{IS} = \mathcal{IS}_B$. Hence, $\mathcal{IS} \subseteq \mathcal{S}_{\mathcal{SV}} = \mathcal{S}_{\mathcal{NSV}}$, in view of (2.1). If $y \in \mathcal{IS}$, then from (1.1), $(y^{[1]}(t))' \sim -M_y^{\alpha-1} p(t)$ as $t \to \infty$, where $M_y = \lim_{t\to\infty} y(t)$, and because of the convergence of J_p, we get $\mathcal{IS} = \mathcal{IS}_{BB}$. Indeed, $y^{[1]}$ is positive increasing and if $\lim_{t\to\infty} y^{[1]}(t) = \infty$, then $J_p = \infty$, contradiction. By [5, Theorem 1], see also [6, Chapter 4], we get $\mathcal{DS} = \mathcal{DS}_{0B} \cup \mathcal{DS}_B$, where both subclasses are nonempty. As in (3.18), we obtain $y \in \mathcal{RV}(\varrho)$ provided $y \in \mathcal{DS}_{0B}$, thus $\mathcal{DS}_{0B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Since $\varrho < 0$ and except of $\mathcal{DS}_{0B}$ all other possible subclasses $(\mathcal{IS}_B, \mathcal{DS}_B)$ are subsets of $\mathcal{SV}$, we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0B}$. Further, in view of [5, Theorem 1], $\mathcal{DS}_B = \mathcal{DS}_{B0} \cup \mathcal{DS}_{BB}$, where both subclasses are nonempty. Altogether we get $\mathcal{DS}_{B0} \cup \mathcal{DS}_{BB} \cup \mathcal{IS}_{BB} = \mathcal{S}_{\mathcal{NSV}}$.

From Lemma 3.11 we get $\mathcal{DS}_{0B} \subseteq \mathcal{B}_4$, $\mathcal{DS}_{B0} \subseteq \mathcal{B}_3$, and $\mathcal{DS}_{BB} \cup \mathcal{IS}_{BB} \subseteq \mathcal{B}_j$, $j = 1,2$. Lemma 3.12 yields $\mathcal{DS}_{B0} \subseteq \mathcal{G}_2$. From Lemma 3.14 and Lemma 3.16, we obtain $\mathcal{DS}_{0B} \subseteq \mathcal{H}_4$ and $\mathcal{DS}_{0B} \subseteq \mathcal{L}(\varrho, \eta)$, respectively. By definition, if $y \in \mathcal{B}_4 \cap \mathcal{DS}$, then $y \in \mathcal{DS}_{0B} \cup \mathcal{DS}_{BB}$. Suppose by a contradiction that $y \in \mathcal{DS}_{BB}$. We know that $\mathcal{DS}_{BB} \subseteq \mathcal{B}_2$. Thus, $|N_y - y^{[1]}| \in \mathcal{RV}(\delta + 1)$ by Proposition 3.1. But at the same time we have $y \in \mathcal{B}_4$, which yields $|N_y - y^{[1]}| \in \mathcal{RV}(\alpha + \delta - \gamma)$ by Proposition 3.1. This implies – because of necessary equality of indices of regular variation – that $\gamma = \alpha - 1$, contradiction. Thus $\mathcal{B}_4 \cap \mathcal{DS} \subseteq \mathcal{DS}_{0B}$. By definition and because of the above established classification, if $y \in \mathcal{B}_3 \cap \mathcal{DS}$, then $y \in \mathcal{DS}_{BB} \cup \mathcal{DS}_{B0}$. Let $y \in \mathcal{DS}_{BB}$. We know that $\mathcal{DS}_{BB} \subseteq \mathcal{B}_1$ by Lemma 3.11. Consequently, by Proposition 3.1, $|M_y - y| \in \mathcal{RV}(1 + \gamma(1 - \beta))$. But at the same time we have $y \in \mathcal{B}_3$, and so $|M_y - y| \in \mathcal{RV}((\beta - 1)(\delta + 1 - \gamma) + 1)$. For the indices we then get $(\beta - 1)(\alpha - 1 - \gamma) = (\beta - 1)(\delta + 1 - \gamma + \alpha - 1)$, which gives $\delta = -1$, contradiction. Thus $\mathcal{B}_3 \cap \mathcal{DS} \subseteq \mathcal{DS}_{B0}$. By definition, $\mathcal{B}_j \cap \mathcal{IS} \subseteq \mathcal{IS}_{BB}$ and $\mathcal{B}_j \cap \mathcal{DS} \subseteq \mathcal{DS}_{BB}$, $j = 1,2$. If $y \in \mathcal{G}_2 \cap \mathcal{DS}$, then $y \in \mathcal{DS}_B$. Differentiating the relation which defines $\mathcal{G}_2$, applying Φ to the both sides and multiplying by r, we obtain, as $t \to \infty$, $|y^{[1]}| \sim Kt|p(t)| \in \mathcal{RV}(\delta + 1)$, where K is a positive constant. Consequently, in view of Proposition 3.1, $y \in \mathcal{DS}_{B0}$. If $y \in \mathcal{H}_4$ or $y \in \mathcal{L}(\varrho, \eta)$, then clearly the only class for y among the ones that are allowed in the setting $\delta < -1$, $\gamma > \alpha - 1$, $p < 0$ is $\mathcal{DS}_{0B}$.

Assume that $p > 0$. By [4, Theorems 2 and 4 and their proofs], we have $\mathcal{S} = \mathcal{IS}_{B0} \cup \mathcal{IS}_{BB} \cup \mathcal{DS}_{0B} \cup \mathcal{DS}_{BB}$ with all these subclasses to be nonempty. Hence, $\mathcal{IS} \cup \mathcal{DS}_{BB} \subseteq \mathcal{S}_{\mathcal{NSV}}$. In view of (3.18), $\mathcal{DS}_{0B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Taking into account (3.17), we get $\mathcal{DS}_{BB} \cup \mathcal{IS}_{B0} \cup \mathcal{IS}_{BB} = \mathcal{S}_{\mathcal{NSV}}$ and $\mathcal{DS}_{0B} = \mathcal{S}_{\mathcal{NRV}}(\varrho)$. The relations with the classes $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{G}_2, \mathcal{H}_3$, and $\mathcal{L}(\varrho, \eta)$ can be shown similarly as in the case $p < 0$

In the last part of this proof we establish the relations with the class $\mathcal{P}$ under the condition $\delta + \alpha < \gamma$. First consider the case $p < 0$. Let $\gamma < \alpha - 1$ and $\delta < -1$. Then, as it was established in the previous parts, $J_1 = \infty$ and $J_2 < \infty$. Theorem 3.20 now yields $\mathcal{P} = \mathcal{DS}_B$. From the previous computations we know that $\mathcal{DS}_B = \mathcal{DS}$. Let $\gamma > \alpha - 1$ and $\delta > -1$. Then, as was established already earlier, we have $J_1 < \infty$. Theorem 3.20 and the equality $\mathcal{DS}_0 = \mathcal{DS}$ (which holds to be true) in this case yield $\mathcal{P} = \mathcal{DS}$. If $\delta < -1$ and $\gamma > \alpha - 1$, then $J_p < \infty$ and $J_r < \infty$. Consequently, $J_1 < \infty$ and thus Theorem 3.20 yields $\mathcal{P} = \mathcal{DS}_0$. The above established classification implies $\mathcal{DS}_0 = \mathcal{DS}_{0B}$, hence $\mathcal{P} = \mathcal{DS}_{0B}$.

Let $p > 0$. If $y \in \mathcal{S}_{\mathcal{NSV}}$, then $r^{1-\beta} y^{-M} \in \mathcal{RV}(-\gamma/(\alpha - 1))$ by Proposition 3.1. If $\gamma < \alpha - 1$ and $\delta < -1$, then $\int_a^\infty \mathcal{T}_m[y](s)\,\mathrm{d}s = \infty$, and hence $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{P}$, in view of Theorem 3.24. Since $\varrho > 0$, $ty'(t)/y(t) \to 0$ and $tx'(t)/x(t) \to \varrho$ as $t \to \infty$ for $x \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$, we get

$\mathcal{S}_{\mathcal{NRV}}(\varrho) \cap \mathcal{P} = \emptyset$ by definition. Consequently, $\mathcal{S}_{\mathcal{NSV}} = \mathcal{P}$. Assume that $\gamma > \alpha - 1$ and $\delta > -1$. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Then by the classification made in the previous parts, we obtain $y \in \mathcal{S}_2$, $\mathcal{S}_2$ being defined in Lemma 3.17, and $\mathcal{F}[y] \in \mathcal{RV}(-\varrho - 1)$ (see Lemma 3.17), $\mathcal{F}$ being defined in Theorem 3.21. Since $\varrho < 0$, we have $\int_a^\infty \mathcal{F}[y]\,\mathrm{d}s = \infty$. Assuming (3.15), we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{P}$ by Theorem 3.21. Further, $\mathcal{S}_{\mathcal{NSV}} \cap \mathcal{P} = \emptyset$ by definition, since for $x \in \mathcal{S}_{\mathcal{NSV}}$, $tx'(t)/t \to 0$ as $t \to \infty$ and $\varrho < 0$. Thus $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{P}$. If (3.15) fails to hold, then we can proceed similarly as at the end of the proof of Theorems 2.1 and 2.2, since the discussion made there is valid no matter whether $\delta + \alpha = \gamma$ or $\delta + \alpha < \gamma$. We again obtain $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{P}$. It remains to examine principal solutions when $\delta < -1$ and $\gamma > \alpha - 1$, i.e., $J_p + J_r < \infty$ under the condition $p > 0$. We will use Theorem 3.26. If $y \in \mathcal{S}_{\mathcal{NSV}}$, then $r^{1-\beta}y^{-2} \in \mathcal{RV}(\gamma(1-\beta))$. The index is less than -1, thus $\int_a^\infty r^{1-\beta}(s)y^{-2}(s)\,\mathrm{d}s < \infty$ and $\mathcal{S}_{\mathcal{NSV}} \cap \mathcal{P} = \emptyset$ by Theorem 3.26. If $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$, then $r^{1-\beta}y^{-2} \in \mathcal{RV}(\gamma(1-\beta) - 2\varrho) = \mathcal{RV}(-1-\varrho)$. In view of $\varrho < 0$, the index is greater than -1, thus $\int_a^\infty r^{1-\beta}(s)y^{-2}(s)\,\mathrm{d}s = \infty$, and $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{P}$ by Theorem 3.26. Hence, in view of (3.17), $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{P}$. □

Proof of Theorem 2.4. Let $p < 0$. Since

$$\mathcal{S} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2), \qquad \mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \neq \emptyset, \qquad i = 1, 2, \tag{3.21}$$

$\mathcal{S} = \mathcal{IS} \cup \mathcal{DS}$, and $\vartheta_1 < 0 < \vartheta_2$ (see Lemma 3.6), in view of Proposition 3.1, we get $\mathcal{IS} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$ and $\mathcal{DS} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1)$. Thanks to the positivity of ϑ_2, we have $\mathcal{IS} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) \subseteq \mathcal{IS}_\infty \subseteq \mathcal{IS}$ by Proposition 3.1. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{IS} = \mathcal{IS}_\infty$. Since $y^{[1]}$ is positive increasing, we have $\mathcal{IS}_\infty = \mathcal{IS}_{\infty\infty} \cup \mathcal{IS}_{\infty B}$. But if $y \in \mathcal{IS}_{\infty B}$, we get $y \in \mathcal{RV}(\varrho)$ by Lemma 3.17-(ii), contradiction because of $\vartheta_2 \neq \varrho$ (see Lemma 3.6). Therefore $\mathcal{IS} = \mathcal{IS}_{\infty\infty}$. Similarly we find that $\mathcal{DS} \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \subseteq \mathcal{DS}_0 = \mathcal{DS}_{00} \cup \mathcal{DS}_{0B} = \mathcal{DS}_{00} \subseteq \mathcal{DS}$, and the equalities follow. From Lemma 3.16, $\mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \subseteq \mathcal{L}(\vartheta_i, \eta_i)$, $i = 1, 2$. Condition (1.2) and $r \in \mathcal{NRV}(\gamma) \cap C^1$ imply $\lim_{t\to\infty} tL(\vartheta_i, \eta_i, t) = 0$. Hence, by the Representation Theorem (see (1.9)), $\mathcal{L}(\vartheta_i, \eta_i) \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta_i)$, $i = 1, 2$. In view of Theorem 3.20, $\mathcal{P} = \mathcal{DS}_B$ or $\mathcal{P} = \mathcal{DS}_0$. But $\mathcal{DS}_B = \emptyset$, thus only the latter possibility occurs. Note that $J_2 = \infty$ by (1.2).

Let $p > 0$. Since we assume that $C_\gamma \in (0, K_\alpha]$, we have $\gamma \neq \alpha - 1$, otherwise K_α would be zero. Let $\gamma < \alpha - 1$. Then $J_r = \infty$ by Theorem 3.2, and so (1.5) holds. The class $\mathcal{IS}_{B0}$ is empty because of (3.21), where ϑ_1, ϑ_2 are positive by Lemma 3.6. The class $\mathcal{IS}_{\infty B}$ is also empty. Indeed, if $y \in \mathcal{IS}_{\infty B}$, then $y \in \mathcal{RV}(\varrho)$ by Lemma 3.17. But according to Lemma 3.6, $0 < \vartheta_1 \leq \vartheta_2 < \varrho$, contradiction. Thus $\mathcal{IS} \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) \subseteq \mathcal{IS}_{\infty 0} \subseteq \mathcal{IS}$. Let $\gamma > \alpha - 1$. Then $J_p = \infty$ since $p(t) \sim C_\gamma t^{-\alpha} r(t) \in \mathcal{RV}(\gamma - \alpha)$. Thus (1.6) holds. Similarly as before (using Lemma 3.6 and Lemma 3.17), we get $\mathcal{DS}_{B\infty} = \emptyset = \mathcal{DS}_{0B}$. Consequently, $\mathcal{DS} \subseteq \mathcal{NRV}(\vartheta_1) \cup \mathcal{NRV}(\vartheta_2) \subseteq \mathcal{DS}_{0\infty} \subseteq \mathcal{DS}$. The inclusions $\mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \subseteq \mathcal{L}(\vartheta_i, \eta_i) \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta_i)$, $i = 1, 2$, can be proved analogously as in the case $p < 0$.

Finally we show the relations with the class $\mathcal{P}$ when $p > 0$. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\vartheta)$, where $\vartheta = \vartheta_1$ or $\vartheta = \vartheta_2$. From the previous part we know that $y \in \mathcal{IS}_{\infty 0} \cup \mathcal{DS}_{0\infty} \subseteq \mathcal{S}_1$. Recall that $\delta = \gamma - \alpha$ and $\gamma \neq \alpha - 1$. Assume that (3.15) holds. From Lemma 3.17 and Proposition 3.1, we get $\mathcal{F}[y] \in \mathcal{RV}(\Omega)$, $\mathcal{F}$ being defined in Theorem 3.21, where $\Omega = \vartheta - 1 - 2\vartheta - \delta - 1 - (\alpha - 1)\vartheta = \alpha - \gamma - 2 - \alpha\vartheta$. Clearly, $\Omega \gtrless -1$ if and only if $\vartheta \gtrless (\alpha - 1 - \gamma)/\alpha$. Since $C_\gamma \in (0, K_\alpha]$, from Lemma 3.6 we have $\vartheta_1 < (\alpha - 1 - \gamma)/\alpha < \vartheta_2$. Thus $\int_a^\infty \mathcal{F}[y](s)\,\mathrm{d}s = \infty$ when $\vartheta = \vartheta_1$ while $\int_a^\infty \mathcal{F}[y](s)\,\mathrm{d}s < \infty$ when $\vartheta = \vartheta_2$ by Theorem 3.2. Theorem 3.21 yields $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \subseteq \mathcal{P}$ and $\mathcal{S}_{\mathcal{NRV}}(\vartheta_2) \cap \mathcal{P} = \emptyset$. In view of (3.21), we get $\mathcal{P} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1)$. Now assume that (3.15) fails to hold and let $J_r = \infty$ (i.e., in our setting, $\gamma < \alpha - 1$) and $\alpha < 2$. The constant M is defined in Theorem 3.24. If $y \in \mathcal{S}_{\mathcal{NRV}}(\vartheta_1)$, then $r^{1-\beta}y^{-M} \in \mathcal{RV}(-\gamma(\beta - 1) - M\vartheta_1)$ by

Proposition 3.1. For the index we have $-\gamma(\beta-1)-M\vartheta_1>-1$ if and only if $M\vartheta_1(\alpha-1)<\alpha-1-\gamma$. From Lemma 3.23 we know that $M=\beta$; recall we assume $\alpha<2$. Thus the inequality $M\vartheta_1(\alpha-1)<\alpha-1-\gamma$ reads as $\vartheta_1<(\alpha-1-\gamma)/\alpha$ which is true by Lemma 3.6. Consequently, $\int_a^\infty \mathcal{T}_M[y](s)\,\mathrm{d}s=\infty$, and so Theorem 3.24 yields $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)\subseteq\mathcal{P}$. The class $\mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$ will be treated later. Now assume that (3.15) fails to hold in the sense that $J_p=\infty$ and $\alpha>2$. Note that then $\delta>-1$ and $\gamma>\alpha-1$, and so $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)\cup\mathcal{S}_{\mathcal{NRV}}(\vartheta_2)=\mathcal{DS}=\mathcal{DS}_{0\infty}\subseteq\mathcal{S}_1$, where ϑ_1,ϑ_2 are negative. Take $y\in\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)$ and set $u=-y^{[1]}$. Then $u\in\widehat{\mathcal{S}}$, see Lemma 3.9. We want to show that $u\in\widehat{\mathcal{P}}$; $\widehat{\mathcal{P}}$ is the set of principal solutions in $\widehat{\mathcal{S}}$. Denote $\widehat{\vartheta}_1=\gamma-\alpha+\vartheta_1(\alpha-1)+1$. Then, owing to Lemma 3.17, $u\in\mathcal{RV}(\widehat{\vartheta}_1)$. Recall that $\widehat{\gamma}$ is the index of regular variation of $\widehat{r}$ and let $\widehat{M}=\max\{\varphi_\alpha(t):t\in[0,1]\}$. Thanks to Proposition 3.1, we have $\widehat{r}^{1-\alpha}u^{-\widehat{M}}\in\mathcal{RV}(\widehat{\Psi})$, where $\widehat{\Psi}=-\widehat{\gamma}(\alpha-1)-\widehat{\vartheta}_1\widehat{M}$. Since we assume $\alpha>2$, we have $\beta<2$, and Lemma 3.23 yields $\widehat{M}=\alpha$. Recalling $\widehat{\gamma}=(\alpha-\gamma)/(\alpha-1)$, for the index $\widehat{\Psi}$ we get $\widehat{\Psi}=-\alpha+\gamma-\alpha(\gamma-\alpha+\vartheta_1(\alpha-1)+1)$. It is now easy to see that $\widehat{\Psi}>-1$ if and only if $\vartheta_1<(\alpha-1-\gamma)/\alpha$ where the last inequality is true by Lemma 3.6. Hence, $\int_a^\infty \widehat{r}^{1-\alpha}(s)u^{-\widehat{M}}(s)\,\mathrm{d}s=\infty$, and noting that $\int_a^\infty \widehat{r}^{1-\alpha}(s)\,\mathrm{d}s=\infty$ (since $\widehat{\gamma}(1-\alpha)>-1$), applying Theorem 3.24 to reciprocal equation (3.2), we get $u\in\widehat{\mathcal{P}}$. According to Theorem 3.22 we have $y\in\mathcal{P}$, and so again $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)\subseteq\mathcal{P}$. The rest of the observations is made under the general assumption $\gamma\neq\alpha-1$. Take $y_i\in\mathcal{S}_{\mathcal{NRV}}(\vartheta_i)$, $i=1,2$. Then, no matter whether (3.15) holds or does not hold, $\lim_{t\to\infty} ty_i'(t)/y_i(t)=\vartheta_i$, $i=1,2$, and since $\vartheta_1<\vartheta_2$, we get $y_1'(t)/y_1(t)<y_2'(t)/y_2(t)$ for large t, which implies $\mathcal{S}_{\mathcal{NRV}}(\vartheta_2)\cap\mathcal{P}=\emptyset$. Altogether we get $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)=\mathcal{P}$. □

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