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Stability of delay equations

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Abstract. For a large class of nonautonomous linear delay equations with distributed delay, we obtain the equivalence of hyperbolicity, with the existence of an exponential dichotomy, and Ulam–Hyers stability. In particular, for linear equations with constant or periodic coefficients and with a simple spectrum these two properties are equivalent. We also show that any linear delay equation with an exponential dichotomy and its sufficiently small Lipschitz perturbations are Ulam–Hyers stable.

Keywords: Ulam–Hyers stability, delay equations, exponential dichotomies.

2020 Mathematics Subject Classification: 37D99.

1 Introduction

We consider delay equations with distributed delay with the objective of relating hyperbolicity and Ulam–Hyers stability. More precisely, the aim of our work is twofold. In a first part, we show that any linear delay equation with an exponential dichotomy and its sufficiently small Lipschitz perturbations are Ulam–Hyers stable. We emphasize that we consider arbitrary nonautonomous delay equations with distributed delay. In a second part, we obtain a converse for a large class of linear equations by showing that hyperbolicity and Ulam–Hyers stability are equivalent properties. This includes in particular linear delay equations with constant coefficients, always with distributed delay, provided for example that the generator has a simple spectrum. We also consider delay equations with periodic coefficients.

Before proceeding, we recall the notion of Ulam–Hyers stability for an autonomous delay equation (the general nonautonomous case is analogous but is left for the main text). Let $|\cdot|$ be a norm on $\mathbb{C}^n$. Given $r > 0$, we denote by $C = C([-r,0],\mathbb{C}^n)$ the Banach space of all continuous functions $\phi\colon [-r,0] \to \mathbb{C}^n$ equipped with the supremum norm $\|\cdot\|$. Now let $L\colon C \to \mathbb{C}^n$ be a bounded linear operator and let $f\colon C \to \mathbb{C}^n$ be a continuous function. We say that the equation

$$v' = Lv_t + f(v_t), \tag{1.1}$$

where $v_t(\theta) = v(t+\theta)$ for $\theta \in [-r,0]$, is *Ulam–Hyers stable* if there exists $\kappa > 0$ such that for each $\varepsilon > 0$ and each continuous function $v\colon [-r,+\infty) \to \mathbb{C}^n$ of class C^1 on $[0,+\infty)$ (taking the

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right-hand derivative at 0) satisfying

$$\sup_{t\geq 0}\left|v'(t) - Lv_t - f(v_t)\right| < \varepsilon,$$

there exists a solution $w\colon [-r, +\infty) \to \mathbb{C}^n$ of equation (1.1) satisfying

$$\sup_{t\geq 0}\|v_t - w_t\| < \kappa\varepsilon$$

We detail briefly the origins and developments of Ulam–Hyers stability (sometimes the names are reordered in the literature), particularly in the context of differential equations and dynamical systems. Often it is also called Ulam–Hyers–Rassias stability. The concept goes back to a question of Ulam [37] for functional equations. Hyers [17] soon gave a solution for a particular functional equation and much later Rassias [34] made a considerable generalization for a notion of stability that includes the one studied by Hyers as a particular case (we refer the reader to the book [18] for details and for many additional references). The notion essentially requires that if there exists an approximate solution of a differential equation, in the sense that it satisfies the differential equation up to a certain error, then there exists an actual solution that is sufficiently close to the approximate solution. For many developments of the theory we refer the reader to the books [10, 20, 36] and the references therein.

The developments described above include in particular many works giving conditions leading to Ulam–Hyers stability, both for linear and nonlinear differential equations, or even that are equivalent to Ulam–Hyers stability for some classes of equations. The first to consider Ulam–Hyers stability in the context of differential equations seem to have been Alsina and Ger [1] (see [35] for a generalization). Further developments include for example the works [6, 13, 14, 19, 32, 33] as well as related work for difference equations, such as [5, 11, 31]. There are various other variants, including for integral equations, differential integral equations, impulsive differential equations, and partial differential equations. There are also some works for delay equations, such as [16, 21, 26, 27, 38], although to our best knowledge never for distributed delays and never considering the problem of whether hyperbolicity is equivalent to Ulam–Hyers stability. These two aspects are precisely the main novelties of our work.

We are mainly interested in the relation between Ulam–Hyers stability and hyperbolicity. The equivalence between these two properties, under some additional assumptions, has been established in a few cases. Namely, this was established in [25] for differential equations with constant coefficients and in [6] for differential equations with periodic coefficients. Related results for discrete time were obtained, respectively, in [5] for constant coefficients and in [12] for periodic coefficients. To the possible extent, and similarly also under some additional assumptions, we want to obtain related results for delay equations. To our best knowledge, no similar problem was considered before for delays equations.

Certainly, our work is related to all these works since we study similar properties, but the techniques (either for delays equations or others) cannot be used in our work. This is due to the fact that we consider distributed delays, for which in particular the variation of constants formula requires extending some operators to a space of discontinuous functions. Moreover, unlike in all former works concerning Ulam–Hyers stability for delay equations, which put their emphasis on Lipschitz properties and then deduce the stability of the equation, our emphasis is instead on the hyperbolicity of the linear part, which allows us in particular to give a complete characterization of Ulam–Hyers stability for linear delay equations.

Incidentally, Ulam–Hyers stability can be described, equivalently, as the shadowing of approximate orbits and specifically as what is called Lipschitz shadowing (we refer the reader to the books [28, 29] for details and references). Nonetheless, the two theories first emerged independently. Shadowing theory was mainly motivated by hyperbolic dynamics. In particular, Anosov's closing lemma [2] shows how to shadow pseudo-orbits by periodic orbits. The general shadowing theorem of Anosov [3] and Bowen [9] leads to the structural stability of hyperbolic sets. On the other hand, it was shown by Pilyugin and Tikhomirov [30] that the Lipschitz shadowing property of a diffeomorphism is equivalent to its structural stability. These closing and shadowing results have important generalizations to nonuniformly hyperbolic systems. In particular, a closing lemma was proved by Katok in [22]. It is also a Lipschitz shadowing result, although some applications require its sharper bounds. We refer the reader to the book [7] for a detailed presentation of these results, but we refrain from giving further references since our work concerns only delay equations (with continuous time).

In the remainder of the introduction we recall briefly the notion of hyperbolicity and we formulate our main results in the particular case of autonomous delay equations. This allows us to avoid some technicalities that are present in the general nonautonomous case and for which we refer to the main text.

We consider an autonomous delay equation

$$v' = Lv_t, \tag{1.2}$$

where $L\colon C \to \mathbb{C}^n$ is a bounded linear operator. For each initial condition $v_0 = \phi \in C$, equation (1.2) has a unique solution v on $[-r, +\infty)$. These solutions determine a semigroup $S(t)\colon C \to C$, for $t \geq 0$, defined by

$$S(t)\phi = v_t(\cdot, 0, \phi) \quad \text{for } \phi \in C.$$

It is a strongly continuous semigroup with generator $A\colon D(A) \to C$ given by

$$A\phi := \lim_{t \searrow 0} \frac{S(t)\phi - \phi}{t} = \phi'$$

in the domain $D(A)$ formed by all $\phi \in C$ such that $\phi' \in C$ and $\phi'(0) = L\phi$. It turns out that the spectrum $\sigma(A)$ is composed entirely of eigenvalues.

Now we can formulate prototypes of our results in the particular case of autonomous equations (we refer to the main text for general results).

Theorem 1.1. *If the spectrum $\sigma(A)$ does not intersect the imaginary axis and the function $f\colon C \to \mathbb{C}^n$ satisfies*

$$|f(\phi) - f(\psi)| \leq K\|\phi - \psi\| \quad \textit{for all } \phi, \psi \in C,$$

then provided that K is sufficiently small the equation $v' = Lv_t + f(v_t)$ is Ulam–Hyers stable.

One can take $f = 0$ to obtain a result for the linear equation $v' = Lv_t$.

Theorem 1.2. *For a linear equation $v' = Lv_t$, if the spectrum $\sigma(A)$ does not intersect the imaginary axis, then the equation is Ulam–Hyers stable.*

We also consider the converse problem for a linear delay equation, among other results in the main text. Again we consider here only autonomous delay equations.

Theorem 1.3. *Assume that any $\lambda \in \sigma(A)$ on the imaginary axis is a simple eigenvalue. Then the equation $v' = Lv_t$ is Ulam–Hyers stable if and only if the spectrum $\sigma(A)$ does not intersect the imaginary axis.*

In addition, we show that for differential difference equations of the form

$$v' = A_0 v + \sum_{i=1}^{k} A_i v(t-\tau_i),$$

for some positive numbers $\tau_1 < \tau_2 < \cdots < \tau_k$ and some $n \times n$ matrices A_i for $i = 0, \ldots, k$, the simplicity condition in Theorem 1.3 is an open condition.

A more general condition than the simplicity of the spectrum is considered in the main text. It corresponds to assume that the Jordan form of each eigenvalue on the imaginary axis is diagonal. We also consider equations with periodic coefficients and, using the version of Floquet theory for delay equations, we obtain an appropriate version of the former theorem.

2 Preliminaries

In this section we recall a few notions and results from the theory of delay equations. This includes the notions of an exponential dichotomy and of an exponential trichotomy. We refer the reader to the books [8, 15] for details as well as proofs of all the results recalled in this section.

2.1 Basic notions

Let $|\cdot|$ be a norm on $\mathbb{C}^n$. Given $r > 0$ (the delay), we denote by $C = C([-r,0],\mathbb{C}^n)$ the Banach space of all continuous functions $\phi\colon [-r,0] \to \mathbb{C}^n$ equipped with the supremum norm

$$\|\phi\| = \sup_{-r\le\theta\le0} |\phi(\theta)|. \tag{2.1}$$

We consider perturbations of a linear delay equation of the form

$$v' = L(t)v_t + g(t), \tag{2.2}$$

writing $v_t(\theta) = v(t+\theta)$ for $\theta \in [-r,0]$ and where:

1. $L(t)\colon C \to \mathbb{C}^n$, for $t \ge 0$, are bounded linear operators such that the map $t \mapsto L(t)$ is strongly continuous on $[0,+\infty)$ and
$$\sup_{t\ge0} \int_t^{t+1} \|L(\tau)\|\, d\tau < +\infty; \tag{2.3}$$
2. $g\colon [0,+\infty) \to \mathbb{C}^n$ is a bounded continuous function, that is,
$$\sup_{t\ge0} |g(t)| < +\infty.$$

We recall that a map $t \mapsto L(t)$ is said to be *strongly continuous* on $[0,+\infty)$ if $t \mapsto L(t)\phi$ is continuous on $[0,+\infty)$ for each $\phi \in C$. It follows from the uniform boundedness principle and the strong continuity of the map $t \mapsto L(t)$ that

$$\sup_{\tau\in[s,t]} \|L(\tau)\| < +\infty \quad \text{for all } t > s. \tag{2.4}$$

Note that condition (2.3) holds for example when the map $t \mapsto L(t)$ is bounded and so in particular when the operators $L(t)$ are independent of t or are periodic in t.

A continuous function $v\colon [s-r,a) \to \mathbb{C}^n$ with $a \le +\infty$ is called a *solution* of equation (2.2) if

$$v(t) = v(s) + \int_s^t (L(\tau)v_\tau + g(\tau))\, d\tau \quad \text{for } t \in [s,a). \tag{2.5}$$

Since v is uniformly continuous on bounded intervals, the map $\tau \mapsto v_\tau$ is continuous. This implies that the function

$$h(\tau) = L(\tau)v_\tau + g(\tau)$$

is also continuous. Indeed,

$$\begin{aligned} |h(\tau) - h(\overline{\tau})| &\le |L(\tau)v_\tau - L(\overline{\tau})v_{\overline{\tau}}| + |g(\tau) - g(\overline{\tau})| \\ &\le |L(\tau)(v_\tau - v_{\overline{\tau}})| + |L(\tau)v_{\overline{\tau}} - L(\overline{\tau})v_{\overline{\tau}}| + |g(\tau) - g(\overline{\tau})| \end{aligned}$$

and the right-hand side converges to 0 when $\tau \to \overline{\tau}$, in view of (2.4) and the continuity of the maps $\tau \mapsto v_\tau$, $\tau \mapsto L(\tau)v_{\overline{\tau}}$ and g. Therefore, by (2.5), any solution of equation (2.2) is of class C^1 on $[s,a)$ and satisfies

$$v'(t) = L(t)v_t + g(t) \quad \text{for } t \in [s,a),$$

taking the right-hand derivative at s. Moreover, it follows from standard results on the existence and uniqueness of solutions of a delay equation that equation (2.2) has a unique solution on $[s-r,+\infty)$ for each initial condition $v_s = \phi \in C$. These solutions can be expressed in terms of the variation of constants formula (which we recall in the following section).

2.2 Linear equations

Now we consider the particular case of a linear equation

$$v' = L(t)v_t, \tag{2.6}$$

with the same hypotheses on the operators $L(t)$ as before. Equation (2.6) determines an evolution family $T(t,s)\colon C \to C$, for $t \ge s \ge 0$, defined by

$$T(t,s)\phi = v_t(\cdot, s, \phi) \quad \text{for } \phi \in C, \tag{2.7}$$

where v is the unique solution of equation (2.6) on $[s-r,+\infty)$ with $v_s = \phi$. One can easily verify that indeed

$$T(t,s) = \mathrm{Id} \quad \text{and} \quad T(t,\tau)T(\tau,s) = T(t,s)$$

for any $t \ge \tau \ge s \ge 0$. Moreover, one can show that

$$\|T(t,s)\| \le \exp\left(\int_s^t \|L(\tau)\|\, d\tau\right)$$

for any $t \ge s \ge 0$ and so it follows from any of the properties (2.3) and (2.4) that each $T(t,s)$ is bounded.

It turns out that the linear operators $T(t,s)$ can be extended to a certain space of discontinuous functions. Let C_0 be the set of all functions $\phi\colon [-r,0] \to \mathbb{C}^n$ that are continuous on $[-r,0)$ and for which the limit

$$\phi(0^-) = \lim_{\theta \to 0^-} \phi(\theta)$$

exists. This is a Banach space when equipped with the supremum norm $\|\cdot\|$ in (2.1). We write the linear operator $L(t)\colon C \to \mathbb{C}^n$ as a Riemann–Stieltjes integral

$$L(t)\phi = \int_{-r}^{0} d\eta(t,\theta)\phi(\theta) \tag{2.8}$$

for some measurable map $\eta\colon [0,+\infty) \times [-r,0] \to M_n$, where M_n is the set of all $n \times n$ matrices with complex entries, such that $\theta \mapsto \eta(t,\theta)$ has bounded variation and is left-continuous for each $t \geq 0$. We extend the linear operator $L(t)$ to C_0 using the integral in (2.8) (we continue to denote the extension by $L(t)$ since there is no danger of confusion). Finally, given $t \geq 0$, we define a linear operator $T_0(t,s)$ on the space C_0 by

$$T_0(t,s)\phi = v_t(\cdot,s,\phi) \quad \text{for } \phi \in C_0,$$

where v is the unique solution of equation (2.6) on $[s-r,+\infty)$ with $v_s = \phi$.

By the variation of constants formula for delay equations, the unique solution v of equation (2.2) on $[s-r,+\infty)$ with $v_s = \phi \in C$ satisfies

$$v_t = T(t,s)\phi + \int_s^t T_0(t,\tau)X_0 g(\tau)\, d\tau \tag{2.9}$$

for all $t \geq s$, where $X_0\colon \mathbb{C}^n \to C_0$ is the linear operator defined by

$$(X_0 p)(\theta) = \begin{cases} 0 & \text{if } -r \leq \theta < 0, \\ p & \text{if } \theta = 0 \end{cases}$$

for each $p \in \mathbb{C}^n$. Identity (2.9) means that

$$v(t+\theta) = (T(t,s)\phi)(\theta) + \int_s^{t+\theta} \big(T_0(t,\tau)X_0 g(\tau)\big)(\theta)\, d\tau \tag{2.10}$$

for all $t \geq s$ and $\theta \in [-r,0]$ with $t+\theta \geq s$. In particular, this formula gives the solution $v(t)$ taking $\theta = 0$.

2.3 Partial hyperbolicity

We say that the linear equation (2.6) has an *exponential trichotomy* if:

1. there exist projections $P(t), Q(t), R(t)\colon C \to C$ for $t \geq 0$ satisfying

   $$P(t) + Q(t) + R(t) = \mathrm{Id}$$

   such that for any $t \geq s \geq 0$ we have

   $$P(t)T(t,s) = T(t,s)P(s), \quad Q(t)T(t,s) = T(t,s)Q(s)$$

   and

   $$R(t)T(t,s) = T(t,s)R(s);$$

2. the linear operator

   $$\overline{T}(t,s) := T(t,s)|_{\ker P(s)}\colon \ker P(s) \to \ker P(t) \tag{2.11}$$

   is onto and invertible for each $t \geq s \geq 0$;

3. there exist $\mu, \nu, D > 0$ with $\mu < \nu$ such that for any $t \geq s \geq 0$ we have
$$\|T(t,s)P(s)\| \leq De^{-\nu(t-s)}, \quad \|T(t,s)Q(s)\| \geq D^{-1}e^{\nu(t-s)}$$
and
$$D^{-1}e^{-\mu(t-s)} \leq \|T(t,s)R(s)\| \leq De^{\mu(t-s)}.$$

An *exponential dichotomy* is an exponential trichotomy with $R(s) = 0$ for some $s \geq 0$ (and so with $R(s) = 0$ for all $s \geq 0$). For each $t \geq s \geq 0$ we denote the inverse of the operator $\overline{T}(t,s)$ in (2.11) by
$$\overline{T}(s,t) := \overline{T}(t,s)^{-1} \colon \ker P(t) \to \ker P(s).$$

The *stable, unstable and center spaces* of an exponential trichotomy (or of an exponential dichotomy) at time t are defined, respectively, by
$$E(t) = P(t)(C), \quad F(t) = Q(t)(C) \quad \text{and} \quad G(t) = R(t)(C).$$
Clearly,
$$C = E(t) \oplus F(t) \oplus G(t).$$
The unstable and center spaces are always finite-dimensional, with dimensions independent of t (see for example [8, Chapter 10]). For each $t \geq 0$ we define linear operators
$$P_0(t), Q_0(t), R_0(t) \colon \mathbb{C}^n \to C_0$$
by
$$Q_0(t) = \overline{T}(t, t+r)Q(t+r)T_0(t+r,t)X_0,$$
$$R_0(t) = \overline{T}(t, t+r)R(t+r)T_0(t+r,t)X_0$$
and
$$P_0(t) = X_0 - Q_0(t) - R_0(t).$$
Then
$$P_0(t)p \in C_0 \setminus C, \quad Q_0(t)p \in F \subset C \quad \text{and} \quad R_0(t)p \in G \subset C$$
for each $p \in \mathbb{C}^n$. The following result extends the exponential bounds of an exponential trichotomy to the space C_0.

Proposition 2.1. *If condition (2.3) holds and equation (2.6) has an exponential trichotomy, then there exist $\mu, \nu, N > 0$ such that for any $t \geq s \geq 0$ we have*
$$\|T_0(t,s)P_0(s)\| \leq Ne^{-\nu(t-s)}, \quad \|T(t,s)Q_0(s)\| \geq N^{-1}e^{\nu(t-s)}$$
and
$$N^{-1}e^{-\mu(t-s)} \leq \|T(t,s)R_0(s)\| \leq Ne^{\mu(t-s)}.$$

Proposition 2.1 also holds for an exponential dichotomy, in which case we have $R(s) = 0$ for all $s \geq 0$ and so also $R_0(s) = 0$ for all $s \geq 0$.

3 From hyperbolicity to Ulam–Hyers stability

In this section we establish the Ulam–Hyers stability of an arbitrary nonautonomous linear delay equation with an exponential dichotomy and of its sufficiently small Lipschitz perturbations.

3.1 Basic notions

We first introduce the notion of Ulam–Hyers stability for a delay equation. We consider general perturbations of a nonautonomous linear delay equation. Namely, we assume that:

1. $L(t)\colon C \to \mathbb{C}^n$, for $t \geq 0$, are bounded linear operators such that the map $t \mapsto L(t)$ is strongly continuous on $[0, +\infty)$ and (2.3) holds;

2. $f\colon [0, +\infty) \times C \to \mathbb{C}^n$ is a continuous function.

We say that the equation

$$v' = L(t)v_t + f(t, v_t) \tag{3.1}$$

is *Ulam–Hyers stable* if there exists $\kappa > 0$ such that for each $\varepsilon > 0$ and each continuous function $v\colon [-r, +\infty) \to \mathbb{C}^n$ of class C^1 on $[0, +\infty)$ (taking the right-hand derivative at 0) satisfying

$$\sup_{t\geq 0} \left| v'(t) - L(t)v_t - f(t, v_t) \right| < \varepsilon, \tag{3.2}$$

there exists a solution $w\colon [-r, +\infty) \to \mathbb{C}^n$ of equation (3.1) satisfying

$$\sup_{t\geq 0} \| v_t - w_t \| < \kappa\varepsilon \tag{3.3}$$

Before proceeding, we make a few comments on this notion of stability. We must assume that each function v has derivative on $[0, +\infty)$ so that the supremum in (3.2) is well defined. But in fact one can show that any solution of equation (3.1) is of class C^1 on the interval $[0, +\infty)$ (taking the right-hand derivative at 0). Indeed, let w be any solution of the equation and consider the continuous function $g(t) = f(t, w_t)$. Then, as detailed in Section 2.1, any solution of equation (2.2) is of class C^1 on the interval $[0, +\infty)$ (taking the right-hand derivative at 0). But the function w is a solution of this equation, which thus gives the desired result. On the other hand, this also motivates assuming that the function v in (3.2) is of class C^1 on $[0, +\infty)$.

3.2 Linear case

The following theorem is our first result relating Ulam–Hyers stability and hyperbolicity. It considers the particular case of a nonautonomous *linear* equation (2.6) and shows that the existence of an exponential dichotomy yields the Ulam–Hyers stability of the equation. The proof has the advantage of being more direct than in the general nonlinear case since we construct explicitly the function w in (3.3).

Theorem 3.1. *If the equation $v' = L(t)v_t$ has an exponential dichotomy, then it is Ulam–Hyers stable.*

Proof. Take $\varepsilon > 0$ and a continuous function $v\colon [-r, +\infty) \to \mathbb{C}^n$ of class C^1 on the interval $[0, +\infty)$ satisfying

$$\sup_{t\geq 0} |v'(t) - L(t)v_t| < \varepsilon.$$

Consider the continuous function $g\colon [0, +\infty) \to \mathbb{C}^n$ given by

$$g(t) = v'(t) - L(t)v_t.$$

Note that $\sup_{t\geq 0}|g(t)| < \varepsilon$. For each $t \geq 0$ let

$$w(t) = v(t) - \int_0^t \big(T_0(t, \tau)P_0 g(\tau)\big)(0)\, d\tau + \int_t^{+\infty} \big(\overline{T}(t, \tau)Q_0 g(\tau)\big)(0)\, d\tau.$$

Then for any $t \geq 0$ and $\theta \in [-r, 0]$ with $t + \theta \geq 0$ we have

$$w_t(\theta) = v_t(\theta) - \int_0^{t+\theta} \big(T_0(t+\theta, \tau)P_0 g(\tau)\big)(0)\, d\tau + \int_{t+\theta}^{+\infty} \big(\overline{T}(t+\theta, \tau)Q_0 g(\tau)\big)(0)\, d\tau$$
$$= v_t(\theta) - \int_0^{t+\theta} \big(T_0(t, \tau)P_0 g(\tau)\big)(\theta)\, d\tau + \int_{t+\theta}^{+\infty} \big(\overline{T}(t, \tau)Q_0 g(\tau)\big)(\theta)\, d\tau.$$

This can be written in the form

$$w_t = v_t - \int_0^t T_0(t, \tau)P_0 g(\tau)\, d\tau + \int_t^{+\infty} \overline{T}(t, \tau)Q_0 g(\tau)\, d\tau, \tag{3.4}$$

in a similar manner to that in (2.10). It follows from Proposition 2.1 that

$$\int_0^t \|T_0(t, \tau)P_0 g(\tau)\|\, d\tau \leq \sup_{s\geq 0}|g(s)| \int_0^t N e^{-\nu(t-\tau)}\, d\tau$$
$$= \sup_{s\geq 0}|g(s)| \frac{N(1-e^{-\nu t})}{\nu} < \frac{N\varepsilon}{\nu}$$

and, similarly,

$$\int_t^{+\infty} \|\overline{T}(t, \tau)Q_0 g(\tau)\|\, d\tau \leq \sup_{s\geq 0}|g(s)| \frac{N}{\nu} < \frac{N\varepsilon}{\nu},$$

for all $t \geq 0$. Therefore, the function $w \colon [-r, +\infty) \to \mathbb{C}^n$ is well defined. Moreover, for any $t \geq s \geq 0$ we have

$$v_t - w_t = \int_s^t T_0(t, \tau)X_0 g(\tau)\, d\tau - \int_s^t T_0(t, \tau)P_0 g(\tau)\, d\tau - \int_s^t T(t, \tau)Q_0 g(\tau)\, d\tau$$
$$+ \int_0^t T_0(t, \tau)P_0 g(\tau)\, d\tau - \int_t^{+\infty} \overline{T}(t, \tau)Q_0 g(\tau)\, d\tau$$
$$= \int_s^t T_0(t, \tau)X_0 g(\tau)\, d\tau + \int_0^s T_0(t, \tau)P_0 g(\tau)\, d\tau - \int_s^{+\infty} \overline{T}(t, \tau)Q_0 g(\tau)\, d\tau.$$

On the other hand, it also follows from (3.4) that

$$T(t, s)(v_s - w_s) = \int_0^s T_0(t, \tau)P_0 g(\tau)\, d\tau - \int_s^{+\infty} \overline{T}(t, \tau)Q_0 g(\tau)\, d\tau$$

(see for example [8, Section 3.4]). Therefore,

$$v_t - w_t = T(t, s)(v_s - w_s) + \int_s^t T_0(t, \tau)X_0 g(\tau)\, d\tau$$

for all $t \geq s \geq 0$. It follows from the variation of constants formula that

$$(v - w)' = L(t)(v_t - w_t) + g(t).$$

Since v satisfies the equation

$$v' = L(t)v_t + g(t),$$

we conclude that $w' = L(t)w_t$. Moreover,

$$\|v_t - w_t\| \leq \left\| \int_0^t T_0(t, \tau)P_0 g(\tau)\, d\tau - \int_t^{+\infty} \overline{T}(t, \tau)Q_0 g(\tau)\, d\tau \right\|$$
$$\leq \int_0^t \|T_0(t, \tau)P_0 g(\tau)\|\, d\tau + \int_t^{+\infty} \|\overline{T}(t, \tau)Q_0 g(\tau)\|\, d\tau$$

and so

$$\sup_{t\ge 0}\|v_t - w_t\| < \frac{2N\varepsilon}{\nu}.$$

This shows that equation (2.6) is Ulam–Hyers stable with $\kappa = 2N/\nu$. □

A consequence of Theorem 3.1 is the following result. Let $M(t)\colon C \to \mathbb{C}^n$, for $t \ge 0$, be bounded linear operators such that the map $t \mapsto M(t)$ is strongly continuous on $[0, +\infty)$.

Corollary 3.2. *If the equation $v' = L(t)v_t$ has an exponential dichotomy, then there exists $\delta > 0$ such that if*

$$\sup_{t\ge 0}\int_t^{t+1} \|L(\tau) - M(\tau)\|\, d\tau < \delta, \tag{3.5}$$

then the equation $v' = M(t)v_t$ is Ulam–Hyers stable.

Proof. When $v' = L(t)v_t$ has an exponential dichotomy and $\delta > 0$ is sufficiently small, condition (3.5) implies that the equation $v' = M(t)v_t$ also has an exponential dichotomy (see Theorem 6.1 in [8]). Hence, the desired statement follows readily from Theorem 3.1. □

3.3 Nonlinear case

The following theorem is our main result relating Ulam–Hyers stability and hyperbolicity for a *nonlinear* delay equation obtained from perturbing a linear equation with an exponential dichotomy by a continuous map that is Lipschitz on the space variable.

Theorem 3.3. *Assume that the equation $v' = L(t)v_t$ has an exponential dichotomy and that there exists $K > 0$ such that*

$$|f(t,\phi) - f(t,\psi)| \le K\|\phi - \psi\| \quad \text{for all } t \ge 0 \text{ and } \phi, \psi \in C. \tag{3.6}$$

If K is sufficiently small, then equation (3.1) is Ulam–Hyers stable.

Proof. Take $\varepsilon > 0$ and a continuous function $v\colon [-r, +\infty) \to \mathbb{C}^n$ of class C^1 on $[0, +\infty)$ satisfying (3.2). We consider also the continuous function $g\colon [0, +\infty) \to \mathbb{C}^n$ defined by

$$g(t) = v'(t) - L(t)v_t - f(t, v_t),$$

which satisfies $\sup_{t\ge 0}|g(t)| < \varepsilon$. We want to show that there exists a continuous function $w\colon [-r, +\infty) \to \mathbb{C}^n$ satisfying

$$\begin{aligned} w_t = v_t &- \int_0^t T_0(t,\tau)P_0(\tau)\big[f(\tau, v_\tau) - f(\tau, w_\tau) + g(\tau)\big]\, d\tau \\ &+ \int_t^{+\infty} \overline{T}(t,\tau)Q_0(\tau)\big[f(\tau, v_\tau) - f(\tau, w_\tau) + g(\tau)\big]\, d\tau \end{aligned} \tag{3.7}$$

for all $t \ge 0$ such that the map $t \mapsto v_t - w_t$ is bounded. Let

$$u(t) = w(t) - v(t) \quad \text{and so} \quad u_t = w_t - v_t. \tag{3.8}$$

Moreover, let

$$h(t,\phi) = f(t, v_t) - f(t, \phi + v_t) + g(t). \tag{3.9}$$

Then identity (3.7) becomes

$$u_t = -\int_0^t T_0(t,\tau)P_0(\tau)h(\tau,u_\tau)\,d\tau + \int_t^{+\infty} \overline{T}(t,\tau)Q_0(\tau)h(\tau,u_\tau)\,d\tau.$$

Now we consider the map $G\colon C_b \to C_b$ defined by

$$G(u)_t = -\int_0^t T_0(t,\tau)P_0(\tau)h(\tau,u_\tau)\,d\tau + \int_t^{+\infty} \overline{T}(t,\tau)Q_0(\tau)h(\tau,u_\tau)\,d\tau,$$

where C_b denotes the Banach space of all bounded continuous functions $u\colon [-r,+\infty) \to \mathbb{C}^n$ equipped with the supremum norm

$$\|u\|_\infty = \sup_{t\geq -r}|u(t)|.$$

For each $t \geq 0$ and $u, \overline{u} \in C_b$ we have

$$\begin{aligned}
\|G(u)_t - G(\overline{u})_t\| \leq & \int_0^t \big\|T_0(t,\tau)P_0(\tau)(f(\tau,\overline{u}_\tau + v_\tau) - f(\tau,u_\tau + v_\tau))\big\|\,d\tau \\
&+ \int_t^{+\infty} \big\|\overline{T}(t,\tau)Q_0(\tau)(f(\tau,\overline{u}_\tau + v_\tau) - f(\tau,u_\tau + v_\tau))\big\|\,d\tau \\
&\leq K\|u-\overline{u}\|_\infty \left(\int_0^t Ne^{-\nu(t-\tau)}\,d\tau + \int_t^{+\infty} Ne^{-\nu(\tau-t)}\,d\tau\right) \\
&\leq \frac{2KN}{\nu}\|u-\overline{u}\|_\infty.
\end{aligned}$$

Therefore,

$$\|G(u) - G(\overline{u})\|_\infty = \sup_{t\geq 0}\|G(u)_t - G(\overline{u})_t\| \leq \frac{2KN}{\nu}\|u-\overline{u}\|_\infty$$

and so the map G is a contraction provided that K is sufficiently small. Moreover, taking $\overline{u} = 0$ we obtain

$$\|G(u)\|_\infty = \sup_{t\geq 0}\|G(u)_t\| \leq \frac{2KN}{\nu}\|u\|_\infty + \sup_{t\geq 0}\|G(0)_t\|.$$

Since $h(\tau,0) = g(\tau)$, proceeding as before we get

$$\sup_{t\geq 0}\|G(0)_t\| \leq \frac{2N}{\nu}\sup_{t\geq 0}|g(t)| < \varepsilon\frac{2N}{\nu}$$

and so

$$\|G(u)\|_\infty \leq \frac{2KN}{\nu}\|u\|_\infty + \varepsilon\frac{2N}{\nu}.$$

This shows that the map G is well-defined. Hence, by the contraction mapping principle, there exists $u \in C_b$ satisfying (3.7). Moreover, u satisfies

$$\|u\|_\infty \leq \frac{2KN}{\nu}\|u\|_\infty + \varepsilon\frac{2N}{\nu}$$

and for $K < \nu/(2N)$ we obtain

$$\|u\|_\infty \leq \varepsilon\frac{2N}{\nu - 2KN}. \tag{3.10}$$

Finally, proceeding as in the proof of Theorem 3.1 with $g(t)$ replaced by $h(t,u_t)$, we find that

$$u_t = T(t,s)u_s - \int_s^t T_0(t,\tau)X_0 h(\tau,u_\tau)\,d\tau$$

for all $t \geq s \geq 0$. It follows from the variation of constants formula that

$$u' = L(t)u_t - h(t, u_t).$$

Since v satisfies the equation

$$v' = L(t)v_t + f(t, v_t) + g(t),$$

it follows from (3.8) and (3.9) that

$$w' = L(t)w_t + f(t, w_t).$$

By (3.10), we finally obtain

$$\sup_{t \geq 0} \|v_t - w_t\| < \varepsilon \frac{2N}{\nu - 2KN},$$

and so equation (3.1) is Ulam–Hyers stable with $\kappa = 2N/(\nu - 2KN)$. □

We were informed by the referee that Theorem 3.3 was obtained independently in [4] and extended to the case of weighted Ulam–Hyers stability.

3.4 Measurable right-hand side

While Theorem 3.3 considers equations with continuous right-hand side, one can consider a class of equations with measurable right-hand side. Similarly, one can also consider a notion of Ulam–Hyers stability for which the approximate solution need not be of class C^1 on $[0, +\infty)$.

Theorem 3.4. *Let $L(t) \colon C \to \mathbb{C}^n$ be linear operators for $t \geq 0$, bounded for almost all t, with $t \mapsto L(t)\phi$ measurable for each $\phi \in C$, and satisfying (2.3), and let $f \colon [0, +\infty) \times C \to \mathbb{C}^n$ be a measurable function satisfying (3.6). Then there exists $\kappa > 0$ such that for each $\varepsilon > 0$ and each continuous function $v \colon [-r, +\infty) \to \mathbb{C}^n$ with measurable derivative on $[0, +\infty)$ satisfying (3.2), there exists a solution $w \colon [-r, +\infty) \to \mathbb{C}^n$ of equation (3.1) satisfying (3.3).*

One can in fact replace condition (2.3) by the more general requirement that there exist constants $C, \omega > 0$ such that

$$\|T_0(t, s)\| \leq Ce^{\omega(t-s)} \quad \text{for } t \geq s.$$

The proof of Theorem 3.4 follows almost verbatim the proof of Theorem 3.3, although now the functions g and h may be only measurable in t.

4 From Ulam–Hyers stability to hyperbolicity

In this section we establish the converse of Theorem 3.1 for a large class of autonomous linear equations $v' = Lv_t$ and, more generally, linear equations $v' = L(t)v_t$ for which the map $t \mapsto L(t)$ is periodic. This class includes for example all equations for which the generator of the semigroup induced by the equation has a simple spectrum on the imaginary axis. We first recall a few basic notions, including the spectral properties of the generator, since these are necessary for the proofs. Again we refer the reader to the books [8, 15] for details as well as proofs of these basic notions.

4.1 Basic notions

In this section we consider perturbations of a linear delay equation of the form

$$v' = Lv_t + g(t), \tag{4.1}$$

where $L\colon C \to \mathbb{C}^n$ is a bounded linear operator and $g\colon \mathbb{R} \to \mathbb{C}^n$ is a bounded continuous function. Note that conditions 1 and 2 in Section 2.1 are automatically satisfied. Therefore, for each initial condition $v_s = \phi \in C$ equation (4.1) has a unique solution on $[s-r, +\infty)$. This solution is of class C^1 on the interval $(s, +\infty)$ and satisfies

$$v'(t) = Lv_t + g(t) \quad \text{for } t \in [s, +\infty),$$

taking the right-hand derivative at s.

Now we consider the particular case of a linear equation

$$v' = Lv_t. \tag{4.2}$$

This includes for example the differential difference equations of the form

$$v' = A_0 v + \sum_{i=1}^{k} A_i v(t - \tau_i) \tag{4.3}$$

for some positive numbers $\tau_1 < \tau_2 < \cdots < \tau_k$ and some $n \times n$ matrices A_i for $i = 0, \ldots, k$. Equation (4.2) determines a semigroup $S(t)\colon C \to C$, for $t \geq 0$, defined by

$$S(t)\phi = v_t(\cdot, 0, \phi) \quad \text{for } \phi \in C,$$

where v is the unique solution of equation (4.2) on $[-r, +\infty)$ with $v_0 = \phi$. In fact,

$$S(t-s) = T(t,s) \quad \text{for any } t \geq s \geq 0,$$

where $T(t,s)$ is the evolution family in (2.7). One can also extend the linear operators $S(t)$ to the space C_0. Namely, we first write the linear operator $L\colon C \to \mathbb{C}^n$ in the form

$$L\phi = \int_{-r}^{0} d\eta(\theta)\phi(\theta)$$

for some left-continuous measurable map $\eta\colon [-r, 0] \to M_n$ of bounded variation and then we use it to extend L to C_0. Given $t \geq 0$, we define a linear operator $S_0(t)$ on the space C_0 by

$$S_0(t)\phi = v_t(\cdot, 0, \phi) \quad \text{for } \phi \in C_0,$$

where v is the unique solution of equation (4.2) on $[-r, +\infty)$ with $v_0 = \phi$.

We note that equation (4.2) has an *exponential trichotomy* if:

1. there exist projections $P, Q, R\colon C \to C$ satisfying $P + Q + R = \mathrm{Id}$ such that for all $t \geq 0$ we have
$$PS(t) = S(t)P, \quad QS(t) = S(t)Q \quad \text{and} \quad RS(t) = S(t)R;$$
2. the linear operator
$$\overline{S}(t) := S(t)|_{\ker P}\colon \ker P \to \ker P$$
is onto and invertible for each $t \geq 0$;

3. there exist $\mu, \nu, D > 0$ with $\mu < \nu$ such that for any $t \geq 0$ we have
$$\|S(t)P\| \leq De^{-\nu t}, \quad \|S(t)Q\| \geq D^{-1}e^{\nu t}$$
and
$$D^{-1}e^{-\mu t} \leq \|S(t)R\| \leq De^{\mu t}. \tag{4.4}$$

An *exponential dichotomy* is an exponential trichotomy with $R = 0$. It turns out that any autonomous linear equation (4.2) has an exponential trichotomy (possibly with $R = 0$), as a consequence of the spectral properties of the generator of $S(t)$ (see Proposition 4.1).

We define linear operators $P_0, Q_0, R_0 \colon \mathbb{C}^n \to C_0$ by
$$Q_0 = \overline{S}(-r)QS_0(r)X_0, \quad R_0 = \overline{S}(-r)RS_0(r)X_0$$
and
$$P_0 = X_0 - Q_0 - R_0.$$
For each $p \in \mathbb{C}^n$ we have
$$P_0 p \in C_0 \setminus C, \quad Q_0 p \in F \subset C \quad \text{and} \quad R_0 p \in G \subset C.$$
In a similar manner to that in Proposition 2.1, one can extend the exponential bounds of an exponential trichotomy to the space C_0 (notice that condition (2.3) is now automatically satisfied).

Finally, we recall some properties of the semigroup $S(t)$ and its generator that will be used later on.

Proposition 4.1. *The following properties hold:*

1. *$S(t)$ is a strongly continuous semigroup with generator $A \colon D(A) \to C$ given by*
$$A\phi := \lim_{t \searrow 0} \frac{S(t)\phi - \phi}{t} = \phi'$$
in the domain
$$D(A) = \{\phi \in C : \phi' \in C, \phi'(0) = L\phi\};$$
2. *the spectrum $\sigma(A)$ is composed of eigenvalues, for each $\gamma \in \mathbb{R}$ there are finitely many numbers $\lambda \in \sigma(A)$ satisfying $\operatorname{Re}\lambda > \gamma$, and*
$$\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < +\infty;$$
3. *the generalized eigenspace space M_λ of each $\lambda \in \sigma(A)$ is finite-dimensional, there exists $k \in \mathbb{N}$ such that $M_\lambda = \ker(A - \lambda \mathrm{Id})^k$ and*
$$C = M_\lambda \oplus N_\lambda \quad \text{with } N_\lambda = \operatorname{Im}(A - \lambda \mathrm{Id})^k;$$
4. *if $\Phi_\lambda = \{\phi_1, \ldots, \phi_d\}$ is a basis for M_λ, then there exists a $d \times d$ matrix B_λ with a single eigenvalue λ such that $A\Phi_\lambda = \Phi_\lambda B_\lambda$,*
$$S(t)\Phi_\lambda = \Phi_\lambda e^{B_\lambda t} \quad \text{for } t \in \mathbb{R}$$
and so also
$$(S(t)\Phi_\lambda)(\theta) = \Phi_\lambda(0)e^{B_\lambda(t+\theta)} \quad \text{for } t \in \mathbb{R} \text{ and } \theta \in [-r, 0]. \tag{4.5}$$

The identity $A\Phi_\lambda = \Phi_\lambda B_\lambda$ in property 4 means that $A\Phi_\lambda a = \Phi_\lambda B_\lambda a$ for any $a \in \mathbb{C}^d$ and a similar observation applies to the remaining identities. A consequence of the former proposition is that any autonomous linear equation (4.2) has an exponential trichotomy (possibly with $R = 0$), in fact with an arbitrarily small constant μ in (4.4).

The stable, unstable and center spaces of an exponential trichotomy are now independent of time and are given, respectively, by

$$E = P(C), \quad F = Q(C) \quad \text{and} \quad G = R(C).$$

In fact, we have

$$E = \bigcap_{\operatorname{Re}\lambda \geq 0} N_\lambda, \quad F = \bigoplus_{\operatorname{Re}\lambda > 0} M_\lambda \quad \text{and} \quad G = \bigoplus_{\operatorname{Re}\lambda = 0} M_\lambda.$$

Clearly, $C = E \oplus F \oplus G$ and the spaces F and G are finite-dimensional.

Moreover, we have the following result.

Proposition 4.2. *Equation (4.2) has an exponential dichotomy if and only if the spectrum $\sigma(A)$ does not intersect the imaginary axis.*

4.2 Autonomous case

Now we consider the converse of Theorem 3.1 for an autonomous linear delay equation assuming that for any eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ the corresponding matrix B_λ (see Proposition 4.1) has a diagonal Jordan form. Under this assumption, we present our main result for a *linear* delay equation: the Ulam–Hyers stability of the equation implies that there exists an exponential dichotomy.

Theorem 4.3. *If equation (4.2) is Ulam–Hyers stable, then either it has an exponential dichotomy or for some eigenvalue $\lambda \in \sigma(A)$ with* $\operatorname{Re}\lambda = 0$ *the matrix B_λ has a nondiagonal Jordan form.*

Proof. Assume that equation (4.2) does not have an exponential dichotomy and that for any eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ the matrix B_λ has a diagonal Jordan form. Since the equation does not have an exponential dichotomy, by Proposition 4.2 indeed there exists an eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ (for which the matrix B_λ has thus a diagonal Jordan form). Take $\phi_\lambda \in M_\lambda$ normalized so that $|\phi_\lambda(0)| < 1$. Then necessarily $\phi_\lambda(0) \neq 0$ for some $\phi_\lambda \in M_\lambda$ since otherwise it would follow from (4.5) that the generalized eigenspace was $M_\lambda = \{0\}$. More precisely, by property (4.5), the solutions $w(t)$ of equation (4.2) with $w_0 \in M_\lambda$ are

$$w(t) = c(S(t)\phi_\lambda)(0) = ce^{\lambda t}\phi_\lambda(0) \tag{4.6}$$

with $c \in \mathbb{C}$. Given $\varepsilon > 0$, let

$$\psi(t) = \varepsilon e^{\lambda t}\phi_\lambda(0)/(1 + r\|L\|)$$

and consider the function

$$g(t) = \psi(t) - \int_{-r}^{0} d\eta(\theta)\theta\psi(t+\theta). \tag{4.7}$$

Note that g is continuous and that

$$\sup_{t\geq 0}|g(t)| \leq \|\psi\|(1 + r\|L\|) < \varepsilon.$$

The unbounded function $v(t) = t\psi(t)$ satisfies

$$v'(t) = \psi(t) + t\psi'(t) = \psi(t) + tL\psi_t$$

since ψ is a solution of equation (4.2). Moreover,

$$Lv_t = \int_{-r}^{0} d\eta(\theta)(t+\theta)\psi(t+\theta) = tL\psi_t + \int_{-r}^{0} d\eta(\theta)\theta\psi(t+\theta)$$

and so

$$v'(t) - Lv_t = \psi(t) - \int_{-r}^{0} d\eta(\theta)\theta\psi(t+\theta) = g(t).$$

This shows that v is an unbounded function satisfying

$$\sup_{t\geq 0}|v'(t) - Lv_t| < \varepsilon.$$

In order to obtain a contradiction, we consider an arbitrary solution $w(t)$ of the linear equation (4.2). Note that

$$\|Pw_t\| = \|S(t)Pw_0\| \leq De^{-\nu t}\|w_0\| \leq D\|w_0\|$$

and

$$\|Qw_t\| = \|S(t)Qw_0\| \geq D^{-1}e^{\nu t}\|w_0\|$$

for all $t \geq 0$. Now we observe that there are finitely many eigenvalues λ of A with $\operatorname{Re}\lambda = 0$. Moreover, the matrix B_λ of each of them has a diagonal Jordan form. Finally, denoting by $\Pi_{\lambda'}$ the projection onto $M_{\lambda'}$ it follows readily from (4.6) that

$$\|Rw_t\| = \|S(t)Rw_0\| \leq \sum_{\operatorname{Re}\lambda'=0} \|S(t)\Pi_{\lambda'}w_0\| = \sum_{\operatorname{Re}\lambda'=0} \|\Pi_{\lambda'}w_0\|$$

for all $t \geq 0$. Therefore, there exists a constant $N > 0$ such that

$$\|(P+R)w_t\| = \|S(t)(P+R)w_0\| \leq N$$

for all $t \geq 0$. This implies that if $Qw_0 \neq 0$, then

$$\begin{aligned}\|v_t - w_t\| &\geq \|Qw_t\| - \|v_t - (P+R)w_t\| \\ &\geq D^{-1}e^{\nu t}\|w_0\| - \sup_{\theta\in[-r,0]} |t+\theta|\varepsilon|\phi_\lambda(0)|/(1+r\|L\|) - N \to +\infty\end{aligned}$$

when $t \to +\infty$, which shows that

$$\sup_{t\geq 0}\|v_t - w_t\| = +\infty \tag{4.8}$$

when $Qw_0 \neq 0$. Now we assume that $Qw_0 = 0$. In this case we have

$$\begin{aligned}\|v_t - w_t\| &\geq |v(t)| - \|(P+R)w_t\| \\ &\geq t\varepsilon|\phi_\lambda(0)|/(1+r\|L\|) - N \to +\infty.\end{aligned}$$

when $t \to +\infty$. This shows that (4.8) also holds when $Qw_0 = 0$, which contradicts the hypothesis that equation (4.2) is Ulam–Hyers stable. Therefore, either the equation has an exponential dichotomy or for some eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ the matrix B_λ has a nondiagonal Jordan form. □

The more general case of a linear delay equation with periodic coefficients is considered later on in Section 4.4. However, the proof requires substantial additional material that also needs to be introduced. For this reason we have preferred to give first the former streamlined proof for autonomous equations. In its turn, it is this proof that motivates the approach for delay equations with periodic coefficients.

4.3 Diagonal Jordan forms

The main difficulty in considering nondiagonal normal forms for the eigenvalues of the generator A is that one may not be able to obtain a bounded function g as in (4.7). Indeed, if the function ψ is obtained from a generalized eigenvector a in the form

$$\psi(t) = \varepsilon(S(t)\Phi_\lambda a)(0),$$

then g may not be bounded, simply because it may involve nonconstant polynomials. This means that this approach need not work for an arbitrary autonomous linear equation $v' = Lv_t$.

A corollary of the former Theorems 3.1 and 4.3 is a complete characterization of the Ulam–Hyers stability of a linear delay equation when the eigenvalues of the generator A on the imaginary axis have diagonal Jordan forms.

Corollary 4.4. *Assume that for any $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ the matrix B_λ has a diagonal Jordan form. Then equation (4.2) is Ulam–Hyers stable if and only if it has an exponential dichotomy.*

It remains an open problem whether a similar characterization holds without the hypothesis on diagonal Jordan forms. We are not aware of counterexamples, even though explicit computations are always somewhat involved.

On the other hand, at least for differential difference equations as in (4.3) we can show that if the spectrum of the generator A is simple on the imaginary axis (for which thus the hypothesis on diagonal Jordan forms holds), then any sufficiently close equation is Ulam–Hyers stable if and only if it has an exponential dichotomy. More precisely, we have the following result.

Corollary 4.5. *For equation (4.3) assume that any eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ is simple. Then there exists $\delta > 0$ such that any equation*

$$v' = A_0'v + \sum_{i=1}^{k} A_i'v(t-\tau_i) \tag{4.9}$$

with

$$\|A_i' - A_i\| < \delta \quad \text{for } i = 0, \dots, k \tag{4.10}$$

is Ulam–Hyers stable if and only if it has an exponential dichotomy.

Proof. We recall that the eigenvalues of the generator A are the roots of the characteristic equation $\det\Delta(\lambda) = 0$, where

$$\Delta(\lambda) = \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta) - \lambda \mathrm{Id}.$$

For equation (4.3) this becomes

$$h(\lambda) := \det\left(A_0 + \sum_{i=1}^{k} A_i e^{-\lambda\tau_i} - \lambda\mathrm{Id}\right) = 0.$$

Since the function h is holomorphic, one can use Rouché's theorem to deduce the continuity of the eigenvalues of A on the matrices A_i. In particular, given an eigenvalue λ of multiplicity m, there exists $\delta > 0$ such that for any equation (4.9) satisfying (4.10) there are exactly m eigenvalues, counted with multiplicities, of the corresponding generator of the induced semigroup (see [23, 24] for details and related discussions).

This has the following consequence. If for equation (4.3) any eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re} \lambda = 0$ is simple, then any equation (4.9) satisfying (4.10) for some sufficiently small $\delta > 0$ has the property that any eigenvalue of the generator of the corresponding semigroup on the imaginary axis is also simple. It follows from Corollary 4.4 that any such equation is Ulam–Hyers stable if and only if it has an exponential dichotomy. □

4.4 Periodic case

In this section we consider the more general case of a linear equation

$$v' = L(t)v_t, \tag{4.11}$$

where $L(t)\colon C \to \mathbb{C}^n$, for $t \geq 0$, are bounded linear operators such that:

1. the map $t \mapsto L(t)$ is strongly continuous on $[0, +\infty)$;
2. there exists $\omega > 0$ such that $L(t + \omega) = L(t)$ for all $t \geq 0$.

Note that condition (2.3) is automatically satisfied.

We recall a few properties of the solutions of the linear equation (4.11) that are necessary for the arguments. We refer the reader to the book [15] for details. Consider the operator $U\colon C \to C$ defined by

$$U\phi = T(\omega, 0)\phi$$

with the evolution family $T(t,s)$ as in (2.7). The spectrum $\sigma(U)$ is a countable compact subset of $\mathbb{C}$ accumulating at most at 0 and any number $\mu \in \sigma(U) \setminus \{0\}$ is an eigenvalue of U, called a *characteristic multiplier* of equation (4.11). Moreover, any number $\lambda \in \mathbb{C}$ satisfying $\mu = e^{\lambda\omega}$ for some eigenvalue $\mu \neq 0$ is called a *characteristic exponent* of the equation.

Proposition 4.6. *Given a characteristic multiplier μ, the following properties hold:*

1. *for each $s \in \mathbb{R}$ there exists a splitting $C = M_\mu(s) \oplus N_\mu(s)$ into closed subspaces with $M_\mu(s)$ is finite-dimensional such that*
$$U(s)M_\mu(s) \subset M_\mu(s) \quad \text{and} \quad U(s)N_\mu(s) \subset N_\mu(s),$$
where $U(s) = T(s + \omega, s)$;
2. *we have*
$$\sigma(U(s)|_{M_\mu(s)}) = \{\mu\} \quad \text{and} \quad \sigma(U(s))|_{N_\mu(s)}) = \sigma(U) \setminus \{\mu\};$$
3. *if $\Phi_{\mu,s}$ is a basis for $M_\mu(s)$, then $T(t,s)\Phi_{\mu,s}$ is a basis for the space $M_\mu(t)$ for each $t \in \mathbb{R}$;*
4. *if $\dim M_\mu(s) = d$, then there exist a $d \times d$ matrix C_μ and vectors $P(t) \in C^d$ for $t \in \mathbb{R}$ such that $\sigma(e^{C_\mu\omega}) = \{\mu\}$,*
$$P(t + \omega) = P(t) \quad \text{for } t \in \mathbb{R}$$
and
$$T(t,0)\Phi_{\mu,0} = P(t)e^{C_\mu t} \quad \text{for } t \in \mathbb{R}. \tag{4.12}$$

Identity (4.12) means that

$$(T(t,0)\Phi_{\mu,0})(\theta) = P(t)(\theta)e^{C_\mu t}$$

for all $t \in \mathbb{R}$ and $\theta \in [-r, 0]$. In particular, taking $\theta = 0$ we find that any solution $v(t)$ of equation (4.11) with initial condition in the space $M_\mu(0)$ for some characteristic multiplier $\mu = e^{\lambda\omega}$ is obtained multiplying the exponential $e^{\lambda t}$ by a polynomial in t whose coefficients are ω-periodic in t.

A consequence of the former properties is that any linear equation (4.11) with periodic coefficients has an exponential trichotomy (possibly with projections $R(s) = 0$ for each $s \in \mathbb{R}$), whose stable, unstable and center spaces at time s are given, respectively, by

$$E(s) = \bigcap_{|\mu|\geq 1} N_\mu(s), \quad F(s) = \bigoplus_{|\mu|>1} M_\mu(s) \quad \text{and} \quad G(s) = \bigoplus_{|\mu|=1} M_\mu(s).$$

Note that $F(s)$ and $G(s)$ are always finite-dimensional (for example since each space $M_\mu(s)$ is finite-dimensional and since $\sigma(U)$ accumulates at most at 0, although it is always the case that the unstable and center spaces of an exponential trichotomy are finite-dimensional). Moreover, equation (4.11) has an exponential dichotomy if and only if $\sigma(U)$ does not intersect the unit circle S^1.

The following result is a generalization of Theorem 4.3 for linear equations with periodic coefficients.

Theorem 4.7. *If equation (4.11) is Ulam–Hyers stable, then either it has an exponential dichotomy or for some characteristic multiplier $\mu \in \sigma(U)$ with $|\mu| = 1$ the matrix C_μ has a nondiagonal Jordan form.*

Proof. The proof is analogous to that of Theorem 4.7 and so we only give a sketch. Assume that equation (4.11) does not have an exponential dichotomy and that for any characteristic multiplier $\mu \in \sigma(U)$ with $|\mu| = 1$ the matrix C_μ has a diagonal Jordan form. Since by hypothesis the equation does not have an exponential dichotomy, there exists such a characteristic multiplier. Writing $\mu = e^{\lambda\omega}$ with $\operatorname{Re}\lambda = 0$, we consider the solutions $w(t)$ of equation (4.11) with $w_0 \in M_\mu$ of the form $w(t) = ce^{\lambda t}p(t)$ with $c \in \mathbb{C}$ and where $p\colon \mathbb{R} \to \mathbb{C}^n$ is a continuous function such that $p(t+\omega) = p(t)$ for all $t \in \mathbb{R}$. Given $\varepsilon > 0$, let $\psi(t) = \varepsilon e^{\lambda t}p(t)$ and

$$g(t) = \psi(t) - \int_{-r}^0 d\eta(t,\theta)\theta\psi(t+\theta).$$

Note that

$$\begin{aligned} |g(t)| &\leq |\psi(t)| + \|L(t)\| \sup_{\theta\in[-r,0]} |\theta\psi(t+\theta)| \\ &\leq |\varepsilon e^{\lambda t}p(t)| + \|L(t)\| \sup_{\theta\in[-r,0]} r|\varepsilon e^{\lambda(t+\theta)}p(t+\theta)| \\ &\leq \varepsilon|p(t)| + \|L(t)\| \sup_{\theta\in[-r,0]} r\varepsilon|p(t+\theta)| \leq \varepsilon\alpha, \end{aligned}$$

where

$$\alpha := \sup_{t\in\mathbb{R}} \left(|p(t)| + \|L(t)\| \sup_{\theta\in[-r,0]} r|p(t+\theta)| \right)$$

is finite because it is the supremum of a periodic function. One can easily verify that $v(t) = t\psi(t)$ satisfies $v'(t) - L(t)v_t = g(t)$ and so

$$\sup_{t\geq 0}|v'(t) - L(t)v_t| < \varepsilon\alpha.$$

In a similar manner to that in the proof of Theorem 4.7 we obtain

$$\|P(t)w_t\| = \|T(t,0)P(0)w_0\| \leq De^{-\nu t}\|w_0\| \leq D\|w_0\| \tag{4.13}$$

and

$$\|Q(t)w_t\| = \|T(t,0)Q(0)Qw_0\| \geq D^{-1}e^{\nu t}\|w_0\|$$

for all $t \geq 0$. Moreover, denoting by $\Pi_{\mu'}$ the projection onto $M_{\mu'}(0)$ we have

$$\|R(t)w_t\| = \|T(t,0)R(0)w_0\| \leq \sum_{|\mu'|=1} \|T(t,0)\Pi_{\mu'}w_0\|$$

for all $t \geq 0$. On the other hand, writing $\mu' = e^{\lambda'\omega}$ with $\operatorname{Re}\lambda' = 0$, the solutions $w(t)$ of equation (4.11) with $w_0 \in M_{\mu'}$ are given by $w(t) = ce^{\lambda' t}q(t)$ with $c \in \mathbb{C}$ and where $q\colon \mathbb{R} \to \mathbb{C}^n$ is a continuous function such that $q(t+\omega) = q(t)$ for all $t \in \mathbb{R}$. Therefore,

$$\begin{aligned}
\|T(t,0)\Pi_{\mu'}w_0\| &\leq \sup_{\theta\in[-r,0]} |ce^{\lambda'(t+\theta)}q(t+\theta)| \\
&\leq \sup_{t\in\mathbb{R}} \sup_{\theta\in[-r,0]} |cq(t+\theta)| \\
&= \sup_{t\in\mathbb{R}} |cq(t)| < +\infty,
\end{aligned}$$

for some $c \in \mathbb{R}$. Since there are finitely many characteristic multipliers on S^1, this implies that there exists a constant $K > 0$ such that

$$\|R(t)w_t\| \leq \sum_{|\mu'|=1} \|T(t,0)\Pi_{\mu'}w_0\| \leq K. \tag{4.14}$$

Finally, by (4.13) and (4.14) there exists $N > 0$ such that

$$\|(\mathrm{Id} - Q(t))w_t\| \leq N$$

for all $t \geq 0$. If $Q(0)w_0 \neq 0$, then

$$\begin{aligned}
\|v_t - w_t\| &\geq \|Q(t)w_t\| - \|v_t - (\mathrm{Id} - Q(t))w_t\| \\
&\geq D^{-1}e^{\nu t}\|w_0\| - \sup_{\theta\in[-r,0]} \varepsilon|(t+\theta)e^{\lambda(t+\theta)}p(t+\theta)| - N \to +\infty
\end{aligned}$$

when $t \to +\infty$, because the function $e^{\lambda t}p(t)$ is bounded. On the other hand, if $Q(0)w_0 = 0$, then

$$\|v_t - w_t\| \geq |v(t)| - \|(\mathrm{Id} - Q(t))w_t\| \geq t\varepsilon|e^{\lambda t}p(t)| - N.$$

Note that the function $|e^{\lambda t}p(t)| = |p(t)|$ is ω-periodic and so its maximum is attained at some times $t_k = t_0 + k\omega$ with $k \in \mathbb{N}$. This implies that

$$\|v_{t_k} - w_{t_k}\| \geq t_k\varepsilon \max_{t\in\mathbb{R}} |p(t)| - N \to +\infty$$

when $k \to +\infty$. In both cases property (4.8) holds and so we obtain a contradiction to the hypothesis that equation (4.11) is Ulam–Hyers stable. This yields the desired statement. □

Building on the proof of the former theorem we formulate a result for arbitrary nonautonomous linear delay equations $v' = L(t)v_t$ under certain additional assumptions. We refrain from including the proof since it corresponds to make slight changes in the former argument.

Theorem 4.8. *Assume that:*

1. *$L(t) \colon C \to \mathbb{C}^n$, for $t \geq 0$, are bounded linear operators such that the map $t \mapsto L(t)$ is bounded and strongly continuous on $[0, +\infty)$;*
2. *equation (4.11) has an exponential trichotomy such that all solutions with initial condition in the center space $G(0)$ are bounded;*
3. *there exist $\delta > 0$ and a solution ψ of equation (4.11) with initial condition in $G(0)$ such that $|\psi(t_n)| > \delta$ for some sequence $t_n \to +\infty$.*

If equation (4.11) is Ulam–Hyers stable, then it has an exponential dichotomy.

Note that by property 1 condition (2.3) is automatically satisfied.

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