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The existence of solutions for the modified $(p(x), q(x))$-Kirchhoff equation

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Abstract. We consider the Dirichlet problem

$$-\Delta_{p(x)}^{K_p} u(x) - \Delta_{q(x)}^{K_q} u(x) = f(x, u(x), \nabla u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0,$$

driven by the sum of a $p(x)$-Laplacian operator and of a $q(x)$-Laplacian operator, both of them weighted by indefinite (sign-changing) Kirchhoff type terms. We establish the existence of weak solution and strong generalized solution, using topological tools (properties of Galerkin basis and of Nemitsky map). In the particular case of a positive Kirchhoff term, we obtain the existence of weak solution (= strong generalized solution), using the properties of pseudomonotone operators.

Keywords: Brouwer fixed point theorem, Galerkin basis, Kirchhoff term, Nemitsky map, pseudomonotone operator.

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1 Introduction

In this manuscript we consider equations driven by Kirchhoff type operators of the form $u \to -K(r, u)\Delta_{r(x)} u$ for functions u, defined on a bounded domain $\Omega \subseteq \mathbb{R}^N$ with smooth boundary $\partial\Omega$. The analysis is carried out in a suitable anisotropic Dirichlet Sobolev space $W_0^{1,r(x)}(\Omega)$, with variable exponent $r \in C(\overline{\Omega})$ satisfying certain regularity and bound conditions. The operator $\Delta_{r(x)}$ is the $r(x)$-Laplacian operator, which for every $u \in W_0^{1,r(x)}(\Omega)$ is defined by $\Delta_{r(x)} u = \operatorname{div}(|\nabla u|^{r(x)-2}\nabla u)$. Additionally, the nonlocal Kirchhoff type term $K(r, u)$ is assumed indefinite (sign changing) and given as

$$K(r, u) = a_r - b_r \int_\Omega \frac{1}{r(x)} |\nabla u|^{r(x)} dx, \quad \text{with } a_r, b_r > 0. \tag{K}$$

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Precisely, the Dirichlet problem we study is

$$-\Delta_{p(x)}^{K_p}u(x)-\Delta_{q(x)}^{K_q}u(x)=f(x,u(x),\nabla u(x))\quad\text{in }\Omega,\quad u\big|_{\partial\Omega}=0. \tag{P}$$

Here, we have the sum of two such Kirchhoff type operators $-\Delta_{p(x)}^{K_p}u:=-K(p,u)\Delta_{p(x)}u$ and $-\Delta_{q(x)}^{K_q}u:=-K(q,u)\Delta_{q(x)}u$, with variable exponents $p,q\in C(\overline{\Omega})$ such that

$$\begin{aligned}1<q^-&=\inf_{x\in\overline{\Omega}}q(x)\le q(x)\le q^+=\sup_{x\in\overline{\Omega}}q(x)\\&<p^-=\inf_{x\in\overline{\Omega}}p(x)\le p(x)\le p^+=\sup_{x\in\overline{\Omega}}p(x)<+\infty.\end{aligned}$$

The reaction (right hand side of (P)) is a Carathéodory function $f(x,z,y)$ (that is, for all $(z,y)\in\mathbb{R}\times\mathbb{R}^N$, $x\to f(x,z,y)$ is measurable and for almost all $x\in\Omega$, $(z,y)\to f(x,z,y)$ is continuous). The presence of the gradient ∇u is crucial to be considered when the convection in fluid dynamical processes cannot be neglected (that is, when an energy transfer is accomplished by moving particles). Turning to the Kirchhoff type term (K), it is related to physical modeling of the changes in length of a string subject to transverse vibrations. In [13], Kirchhoff generalized the classical D'Alembert wave equation

$$\rho\frac{\partial^2}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2}{\partial x^2}=0,$$

with ρ,P_0,h,E,L denoting physical parameters (respectively, mass density, initial tension, area of the cross-section, Young modulus of the material, length of the string) and describing the change of string's length during free vibration.

The existence results we establish use topological techniques (fixed-point arguments together with the theory of pseudomonotone operators) in order to overcome the loss of variational structure, due to the presence of gradient term in the reaction. The classical strategies are also adapted to deal with the nonlocal nature of the Kirchhoff term. Following the similar approach as in Vetro [26], we prove the existence of strong generalized solutions as well as weak solutions to (P). To have a more complete picture of the relevant literature, we mention that standard $-\Delta_p-\Delta_q$ operator was considered by Faria et al. [5] and Zeng & Papageorgiou [28], in the case of positive solutions. For single $-\Delta_p$ operator we mention Papageorgiou et al. [19], dealing also with positive solutions. Precisely, in [5] the authors adopt an approximating process involving a Schauder basis of $W_0^{1,p(x)}(\Omega)$, then apply a generalized strong maximum principle. In [28], the authors use the Leray–Schauder alternative principle in combination with the frozen variable method (to freeze the effects of the gradient term). In [19], the authors use also Leray–Schauder alternative principle, together with truncation and comparison techniques. Additionally, the case of double phase problems (that is, $-\Delta_p-\mu(x)\Delta_q$ operator, with suitable weight function $\mu(\cdot)$) was studied by Gasiński & Winkert [10], using surjectivity result of pseudomonotone operators. Finally, we mention the work of Motreanu [18] dealing with $-\Delta_p+\Delta_q$ operator. In that paper, the author uses a consequence of the Brouwer fixed point theorem, in respect of a Galerkin basis of $W_0^{1,p}(\Omega)$. A main feature of the present manuscript and of the works [18, 26] is the consideration of two different types of solutions of (P), that is, the authors employ both classical weak solutions and new concepts of strong generalized solutions. Additionally, [26] deals with the variable exponents Lebesgue and Sobolev spaces, in the case of a single $p(x)$-Kirchhoff type operator. The similar problem was previously studied by Wang et al. [27], in absence of Kirchhoff type term. Moreover, the Kirchhoff type term herein was considered by Hamdani et al. [11], whose reaction is not gradient dependent. Therefore, [11] employs a (classical) variational approach. It is worth mentioning that the Lions' work [16] originated a revival interest for equations involving a Kirchhoff term, but a large amount of manuscripts imposes a positive restriction to the values of the Kirchhoff term (that is, they consider a sign "+" instead of "−" in (K), deriving from the classical theory). The interested reader can also refer to Molica Bisci & Pizzimenti [17] (looking infinitely many solutions), Figueiredo & Nascimento [6] (looking for nodal (sign-changing) solutions), Santos Júnior & Siciliano [24] and Gasiński & Santos Júnior [8,9] (both of them introducing non positivity conditions on the Kirchhoff term). In the last three papers, the authors assume that Kirchhoff terms can vanish in many different points. Additionally, their strategy of proofs also involve fixed point results, and aims to establish both existence and nonexistence theorems. Before concluding this introduction, it is very important to say that recently in the literature, we find many papers where the authors study the existence and multiplicity of solutions to problems involving the Kirchhoff operator, Choquard-Pekar equations and functionals of double phase with variable exponents. As a partial list we mention the works by Albalawi [1], He et al. [12], Liang et al. [15], Qin et al. [21], Ragusa & Tachikawa [22], Hi et al. [25], and references therein.

2 Preliminaries

Referring to the books of Diening et al. [2] and of Rădulescu & Repovš [23], we provide the mathematical background of the present study. The natural setting where finding solutions to (P) is the anisotropic Dirichlet Sobolev space $W_0^{1,p(x)}(\Omega)$, which means the completion of $C_0^\infty(\Omega)$ with respect to the $W^{1,p(x)}$-norm defined below. Starting with

$$L^{p(x)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\},$$

which is the variable exponent Lebesgue space, we consider the norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_p \left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

Here $M(\Omega)$ means the space of all measurable functions $u : \Omega \to \mathbb{R}$, and

$$\rho_p(u) := \int_\Omega |u(x)|^{p(x)} dx \quad \text{for all } u \in L^{p(x)}(\Omega)$$

denotes the modular. As it is well known, $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$ is a separable, reflexive and uniformly convex Banach space. The norm $\|\cdot\|_{L^{p(x)}(\Omega)}$ and the modular $\rho_p(\cdot)$ are related each other by the following statements.

Theorem 2.1 ([4, Theorem 1.3]). *Let $u \in L^{p(x)}(\Omega)$, then we have:*

(i) $\|u\|_{L^{p(x)}(\Omega)} < 1 \ (= 1, \ > 1) \Leftrightarrow \rho_p(u) < 1 \ (= 1, \ > 1)$;

(ii) *if* $\|u\|_{L^{p(x)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^+}_{L^{p(x)}(\Omega)}$;

(iii) *if* $\|u\|_{L^{p(x)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)}$.

In view of Theorem 2.1, we obtain the relation:

$$\|u\|^{p^+}_{L^{p(x)}(\Omega)} + 1 \geq \rho_p(u) \geq \|u\|^{p^-}_{L^{p(x)}(\Omega)} - 1. \tag{2.1}$$

We are able to introduce the conjugate variable exponent to p, namely $p' \in C(\overline{\Omega})$ satisfying

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

As it is well known $L^{p(x)}(\Omega)^* = L^{p'(x)}(\Omega)$ and if $p^- > 1$ we have

$$\int_\Omega uw dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|w\|_{L^{p'(x)}(\Omega)} \leq 2\|u\|_{L^{p(x)}(\Omega)}\|w\|_{L^{p'(x)}(\Omega)},$$

for $u \in L^{p(x)}(\Omega)$, $w \in L^{p'(x)}(\Omega)$. This Hölder's inequality plays a crucial role in establishing suitable embedding results. We refer to [4, Theorem 1.11] for the continuity of the embedding $L^{p_1(x)}(\Omega) \hookrightarrow L^{p_2(x)}(\Omega)$, provided that $p_1, p_2 \in C(\overline{\Omega})$ with $p_1(x) \geq p_2(x) > 1$ for all $x \in \overline{\Omega}$. Using the variable exponent Lebesgue space, we can define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}, \quad p \in C(\overline{\Omega}).$$

Starting with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)} \quad (\text{where } \|\nabla u\|_{L^{p(x)}(\Omega)} = \||\nabla u|\|_{L^{p(x)}(\Omega)}),$$

we recall that

$$\|u\|_{L^{p(x)}(\Omega)} \leq c_1\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega), \text{ some } c_1 > 0, \tag{2.2}$$

see [2, Theorem 8.2.18]. Thus, as it is well known, $\|u\|_{W^{1,p(x)}(\Omega)}$ and $\|\nabla u\|_{L^{p(x)}(\Omega)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. This implies that we can use $\|\nabla u\|_{L^{p(x)}(\Omega)}$ instead of $\|u\|_{W^{1,p(x)}(\Omega)}$, and set

$$\|u\| = \|\nabla u\|_{L^{p(x)}(\Omega)} \quad \text{in } W_0^{1,p(x)}(\Omega) \quad (\text{by } (2.2)).$$

We mention that judicious choices of norms and norm inequalities are needed for establishing bounds and a priori estimates. Additionally, Fan & Zhao [4] established that with these norms the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, become Banach spaces which are separable and uniformly convex (hence reflexive). Now, for $p \in C(\overline{\Omega})$ we are able to define the critical Sobolev exponent p^* by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } N \leq p(x), \end{cases} \quad \text{for all } x \in \overline{\Omega}.$$

About the continuity and compactness of Sobolev embeddings, we recall the following well-known result.

Proposition 2.2. *Suppose $p \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$. If $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then $W^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ is continuous and compact.*

As already mentioned in the Introduction, our approach here makes use of properties of pseudomonotone operators. So, we collect some definitions and results, as follows.

Definition 2.3. For a reflexive Banach space X, let X^* the dual space of X and $\langle \cdot, \cdot \rangle$ the duality pairing. Let $A : X \to X^*$, then A is called

(i) to satisfy the (S_+)-property if $u_n \overset{w}{\to} u$ in X and $\limsup_{n\to+\infty} \langle A(u_n), u_n - u \rangle \le 0$ imply $u_n \to u$ in X;

(ii) pseudomonotone if $u_n \overset{w}{\to} u$ in X and $\limsup_{n\to+\infty} \langle A(u_n), u_n - u \rangle \le 0$ imply

$$\liminf_{n\to+\infty} \langle A(u_n), u_n - v \rangle \ge \langle A(u), u - v \rangle \text{ for all } v \in X;$$

(iii) coercive if

$$\lim_{\|u\|_X \to +\infty} \frac{\langle A(u), u \rangle}{\|u\|_X} = +\infty.$$

Remark 2.4. We point out that if the operator $A : X \to X^*$ is bounded, then pseudomonotonicity in Definition 2.3 (ii) is equivalent to $u_n \overset{w}{\to} u$ in X and $\limsup_{n\to+\infty} \langle A(u_n), u_n - u \rangle \le 0$ imply $A(u_n) \overset{w}{\to} A(u)$ and $\langle A(u_n), u_n \rangle \to \langle A(u), u \rangle$. In the following we are going to use this fact since our involved operators are bounded.

Pseudomonotone operators exhibit remarkable surjectivity properties. In particular, we have the following result, see, for example, Papageorgiou & Winkert [20, Theorem 6.1.57].

Theorem 2.5. *Let X be a real and reflexive Banach space. Let $A \colon X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and let $b \in X^*$. Then, the equation $A(u) = b$ admits a solution.*

From Gasiński & Papageorgiou [7, Lemma 2.2.27], we get the following result of continuous embedding and density.

Theorem 2.6. *Let X, Y be Banach spaces such that $X \subseteq Y$. If X is dense in Y and the embedding is continuous, then the embedding $Y^* \subseteq X^*$ is continuous too. Moreover, if X is reflexive then Y^* is dense in X^*.*

Our arguments of proofs are also based on Brouwer's fixed point theorem, which leads to the existence of solutions to certain operator equations as stated in the following proposition.

Proposition 2.7. *For a normed finite-dimensional space $(X, \|\cdot\|_X)$ and a continuous map $A : X \to X^*$, we have that:*

If there exists some $R > 0$ such that

$$\langle A(w), w \rangle \ge 0 \quad \text{for all } w \in X \text{ with } \|w\|_X = R,$$

then $A(w) = 0$ has a solution $\widehat{w} \in X$ such that $R \ge \|\widehat{w}\|_X$.

3 Hypotheses and results

In this section, we introduce the hypotheses on the data and collect the statements of our results. First, we put some restrictions on the exponent p, useful to give us the Rayleigh quotient

$$\widehat{\lambda} := \inf_{u \in W_0^{1,p(x)}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx} > 0. \tag{3.1}$$

$H(p)$: There exists $\xi_0 \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$ the function $p_x : \Omega_x \to \mathbb{R}$ defined by $p_x(z) = p(x + z\xi_0)$ is monotone, where $\Omega_x := \{z \in \mathbb{R} : x + z\xi_0 \in \Omega\}$.

We get (3.1) by [3, Theorem 3.3]. Alternatively, one can adopt a different condition, see for example [3, Theorem 3.4]. Here, we will also impose the condition:

$H'(p)$: $p \in C(\overline{\Omega})$ is finite with $p^+ < 2p^-$.

A similar condition was used in [11, 26]. Additionally, we impose growth conditions on the right hand side of (P). Precisely, our hypotheses will be the following:

$H(f)$: $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

(i) there exist $\sigma \in L^{\alpha'(x)}(\Omega)$, $\alpha \in C(\overline{\Omega})$ with $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $c > 0$ such that

$$|f(x,z,y)| \le c(\sigma(x) + |z|^{\alpha(x)-1} + |y|^{\frac{p(x)}{\alpha'(x)}}) \quad \text{for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}, \text{ all } y \in \mathbb{R}^N;$$

(ii) there exist $\sigma_0 \in L^1(\Omega)$ and $b_1, b_2 \ge 0$ such that

$$|f(x,z,y)z| \le \sigma_0(x) + b_1|z|^{p(x)} + b_2|y|^{p(x)} \quad \text{for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}, \text{ all } y \in \mathbb{R}^N.$$

Remark 3.1. Let $\lambda^* = b_1\widehat{\lambda}^{-1} + b_2$. By $H(f)$(ii) and $H(p)$, we get the following estimate:

$$\int_\Omega |f(x,u,\nabla u)u|dx \le \lambda^*\rho_p(\nabla u) + \|\sigma_0\|_{L^1(\Omega)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega). \tag{3.2}$$

Here we establish the existence of solutions both in the usual weak form and in a specific (for Dirichlet problem (P)) form. As it is well known, $u \in W_0^{1,p(x)}(\Omega)$ is weak solution whenever

$$\Big\langle -\Delta_{p(x)}^{K_p} u, w\Big\rangle + \Big\langle -\Delta_{q(x)}^{K_q} u, w\Big\rangle = \int_\Omega f(x, u(x), \nabla u(x))w(x)dx \tag{3.3}$$

for all $w \in W_0^{1,p(x)}(\Omega)$.

On the other hand, we introduce a new definition of strong generalized solution to (P), as follows (see, the corresponding notion of [26]).

Definition 3.2. $u \in W_0^{1,p(x)}(\Omega)$ is a strong generalized solution to (P), if we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$ verifying the convergences:

(i) $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$, as $n \to +\infty$;

(ii) $-\Delta_{p(x)}^{K_p} u_n - \Delta_{q(x)}^{K_q} u_n - f(\cdot, u_n(\cdot), \nabla u_n(\cdot)) \xrightarrow{w} 0$ in $W^{-1,p'(x)}(\Omega)$, as $n \to +\infty$;

(iii) $\lim_{n\to+\infty} \Big\langle -\Delta_{p(x)}^{K_p} u_n - \Delta_{q(x)}^{K_q} u_n, u_n - u\Big\rangle = 0$.

Remark 3.3. We point out that every weak solution to (P) satisfies the conditions in Definition 3.2. It is sufficient to use as test sequence, $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$ defined by $u_n = u$ for all $n \in \mathbb{N}$.

In view of the above remark, we provide an answer to the question:

When does a strong generalized solution to (P) lead to a weak solution?

Note that the source of difficulty in answering this question, is related to the indefinite behavior of (K). Thus we assume the following non-negative bound conditions:

$$\liminf_{n\to+\infty} |K(p,u_n)| > 0 \quad \text{and} \quad K(p,u_n)\,K(q,u_n) \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{K_+}$$

Proposition 3.4. *Consider a strong generalized solution of (P), namely $u \in W_0^{1,p(x)}(\Omega)$, in respect to the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$. Then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (P), provided that hypotheses $H(f)$ and (K_+) hold.*

We shall prove two auxiliary propositions needed for the proof of the main result. These propositions require both the hypotheses $H(p)$ and $H(f)$, as given in the statements. Since $W_0^{1,p(x)}(\Omega)$ is a separable Banach space, then we consider a Galerkin basis of $W_0^{1,p(x)}(\Omega)$, which means that there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ of vector subspaces of $W_0^{1,p(x)}(\Omega)$ satisfying

(j) $\dim(X_n) < +\infty$ for all $n \in \mathbb{N}$;

(jj) $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$;

(jjj) $\overline{\cup_{n=1}^{\infty} X_n} = W_0^{1,p(x)}(\Omega)$.

Proposition 3.5. *Consider a Galerkin basis of $W_0^{1,p(x)}(\Omega)$, namely $\{X_n\}_{n\in\mathbb{N}}$. Then for all $n \in \mathbb{N}$ we can find $u_n \in X_n$ with*

$$\Big\langle -\Delta_{p(x)}^{K_p} u_n, w \Big\rangle + \Big\langle -\Delta_{q(x)}^{K_q} u_n, w \Big\rangle = \int_\Omega f(x, u_n(x), \nabla u_n(x)) w(x)\,dx \tag{3.4}$$

for all $w \in X_n$, provided that hypotheses $H(p)$ and $H(f)$ hold.

Remark 3.6. From Theorem 2.1 we deduce that $S \subseteq W_0^{1,p(x)}(\Omega)$ is bounded in its norm if the set $\{\rho_p(\nabla u) : u \in S\}$ is bounded.

Focusing on the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} X_n$ mentioned in Proposition 3.5 (see also the corresponding proof, in next section), we will show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Proposition 3.7. *Consider the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} X_n$ generated in Proposition 3.5. Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$, provided that hypotheses $H(p)$ and $H(f)$ hold.*

Consequently, we prove our first existence result.

Theorem 3.8. *Problem (P) admits a strong generalized solution $u \in W_0^{1,p(x)}(\Omega)$, provided that hypotheses $H(p)$ and $H(f)$ hold.*

The analogous of Propositions 3.5 and 3.7, and Theorem 3.8 can be obtained imposing $H'(p)$ instead of $H(p)$ (see also Remark 4.1 at the end of Section 4).

4 Proofs of results

In this section we collect the technical proofs of the results stated previously.

Proof of Proposition 3.4. We only prove the case $\liminf_{n\to+\infty} K(p,u_n) > 0$, the other cases can be proved by a similar argument. The previous assumption ensures that we can suppose that

$$K(p,u_n) \geq \beta > 0 \quad \text{and} \quad K(q,u_n) \geq 0 \quad \text{for all } n \in \mathbb{N}, \tag{4.1}$$

are true at least for a relabeled subsequence of $\{u_n\}_{n\in\mathbb{N}}$. Next, we recall that the $-\Delta_{q(x)}$ operator is monotone and hence

$$\langle -\Delta_{q(x)} u_n, u_n - u\rangle \geq \langle -\Delta_{q(x)} u, u_n - u\rangle \quad \text{for all } n \in \mathbb{N}.$$

Multiplying both sides of last inequality by $K(q,u_n)$, then we get

$$K(q,u_n)\langle -\Delta_{q(x)} u_n, u_n - u\rangle \geq K(q,u_n)\langle -\Delta_{q(x)} u, u_n - u\rangle \quad \text{for all } n \in \mathbb{N} \text{ (by (4.1))},$$

that is, adopting the notation introduced at the beginning of this manuscript,

$$\Big\langle -\Delta^{K_q}_{q(x)} u_n, u_n - u\Big\rangle \geq K(q,u_n)\langle -\Delta_{q(x)} u, u_n - u\rangle \quad \text{for all } n \in \mathbb{N}.$$

It is clear that using condition (iii) of Definition 3.2, we get

$$\begin{aligned}\limsup_{n\to+\infty} \Big\langle -\Delta^{K_p}_{p(x)} u_n, u_n - u\Big\rangle &= \limsup_{n\to+\infty} \Big[\Big\langle -\Delta^{K_p}_{p(x)} u_n, u_n - u\Big\rangle - K(q,u_n)\langle \Delta_{q(x)} u, u_n - u\rangle\Big] \\ &\leq \lim_{n\to+\infty} \Big\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n, u_n - u\Big\rangle = 0.\end{aligned}$$

From the previous inequality and (4.1), we deduce

$$\limsup_{n\to+\infty} \langle -\Delta_{p(x)} u_n, u_n - u\rangle \leq 0,$$

and hence we retrieve the $(S)_+$-property of the $p(x)$-Laplacian operator, provided that $u_n \to u$ in $W^{1,p(x)}_0(\Omega)$, as $n \to +\infty$. Using condition (ii) of Definition 3.2, we deduce that

$$-\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - f(\cdot, u_n(\cdot), \nabla u_n(\cdot)) \xrightarrow{w} 0 \quad \text{in } W^{-1,p'(x)}(\Omega),$$

which implies

$$-\Delta^{K_p}_{p(x)} u - \Delta^{K_q}_{q(x)} u - f(\cdot, u(\cdot), \nabla u(\cdot)) = 0,$$

and hence we conclude that $u \in W^{1,p(x)}_0(\Omega)$ is a weak solution to problem (P) (recall the definition of weak solution in (3.3)). □

Proof of Proposition 3.5. Fixed $n \in \mathbb{N}$, let $A_n : X_n \to X^*_n$ be the operator defined by

$$\langle A_n(u), w\rangle = \Big\langle -\Delta^{K_p}_{p(x)} u, w\Big\rangle + \Big\langle -\Delta^{K_q}_{q(x)} u, w\Big\rangle - \int_\Omega f(x, u(x), \nabla u(x)) w(x) dx$$

for all $u, w \in X_n$.

Now, by the estimate (3.2), we get

$$\begin{aligned}\langle -A_n(w), w\rangle &= \left(b_p \int_\Omega \frac{1}{p(x)}|\nabla w|^{p(x)}dx - a_p\right)\int_\Omega |\nabla w|^{p(x)}dx \\ &\quad + \left(b_q \int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}dx - a_q\right)\int_\Omega |\nabla w|^{q(x)}dx - \int_\Omega f(x,w,\nabla w)w dx \\ &\geq \left(b_p \int_\Omega \frac{1}{p(x)}|\nabla w|^{p(x)}dx - a_p\right)\int_\Omega |\nabla w|^{p(x)}dx \\ &\quad - a_q \int_\Omega |\nabla w|^{q(x)}dx - \int_\Omega |f(x,w,\nabla w)w|dx \\ &\geq \frac{b_p}{p^+}\rho_p^2(\nabla w) - a_p\rho_p(\nabla w) - a_q\int_\Omega (1+|\nabla w|^{p(x)})dx \\ &\quad - \lambda^*\rho_p(\nabla w) - \|\sigma_0\|_{L^1(\Omega)} \quad \text{(by (3.2))} \\ &\geq \frac{b_p}{p^+}\rho_p^2(\nabla w) - (a_p + a_q + \lambda^*)\rho_p(\nabla w) - a_q|\Omega| - \|\sigma_0\|_{L^1(\Omega)},\end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of the set Ω. So, we have

$$\langle -A_n(w), w\rangle \geq \frac{b_p}{p^+}\rho_p^2(\nabla w) - (a_p + a_q + \lambda^*)\rho_p(\nabla w) - C \quad \text{for all } w \in X_n,$$

where $C = a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}$. Now, if $\rho_p(\nabla w) > 1$ we get

$$\begin{aligned}\langle -A_n(w), w\rangle &\geq \frac{b_p}{p^+}\rho_p^2(\nabla w) - (a_p + a_q + \lambda^* + C)\rho_p(\nabla w) \\ &= \left[\frac{b_p}{p^+}\rho_p(\nabla w) - \left(a_p + a_q + \lambda^* + C\right)\right]\rho_p(\nabla w),\end{aligned}$$

which gives us the condition

$$\langle -A_n(w), w\rangle \geq 0 \quad \text{if } \rho_p(\nabla w) \geq \frac{p^+}{b_p}(a_p + a_q + \lambda^* + C).$$

Let

$$R > \max\left\{\left[\frac{p^+}{b_p}(a_p + a_q + \lambda^* + C)\right]^{1/p^-}, 1\right\}$$

be fixed. For each $w \in X_n$ with $\|w\| = R$ we obtain

$$\langle -A_n(w), w\rangle \geq 0 \quad \text{(recall we have } \|w\| = \|\nabla w\|_{L^{p(x)}(\Omega)} \leq \rho_p^{\frac{1}{p^-}}(\nabla w)).$$

A simple application of Proposition 2.7 ensures that $-A_n(w) = 0$ (and hence, $A_n(w) = 0$) possesses a solution $u_n \in X_n$. This is sufficient to conclude that the equation (3.4) is proved. □

Proof of Proposition 3.7. The crucial point of the proof consists in showing that

$$\rho_p(\nabla u_n) \leq \max\left\{\frac{p^+}{b_p}\left(a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}\right), 1\right\} \quad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

Hence, we start obtaining the inequality

$$\rho_p(\nabla u_n) \leq \frac{p^+}{b_p}\left(a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}\right),$$

provided that $\rho_p(\nabla u_n) > 1$. By (3.4), putting $w = u_n$ we get

$$\begin{aligned}\frac{b_p}{p^+}\rho_p^2(\nabla u_n) &\leq a_p\rho_p(\nabla u_n) + a_q\rho_q(\nabla u_n) - \frac{b_q}{q^+}\rho_q^2(\nabla u_n) - \int_\Omega f(x,u_n,\nabla u_n)u_n dx\\ &\leq (a_p+a_q)\rho_p(\nabla u_n) + a_q|\Omega| + \int_\Omega |f(x,u_n,\nabla u_n)u_n|dx\\ &\leq (a_p+a_q)\rho_p(\nabla u_n) + a_q|\Omega| + \lambda^*\rho_p(\nabla u_n) + \|\sigma_0\|_{L^1(\Omega)} \quad \text{(by (3.2))}.\end{aligned}$$

Keeping in mind that $\rho_p(\nabla u_n) > 1$, it follows that

$$\begin{aligned}&\frac{b_p}{p^+}\rho_p^2(\nabla u_n) \leq \left(a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}\right)\rho_p(\nabla u_n),\\ \Rightarrow\quad &\frac{b_p}{p^+}\rho_p(\nabla u_n) \leq a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)},\end{aligned}$$

and multiplying both sides by $\frac{p^+}{b_p}$, we get

$$\rho_p(\nabla u_n) \leq \frac{p^+}{b_p}\left(a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}\right).$$

This concludes the proof of inequality (4.2). Consequently, we get that $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^\infty X_n$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$. □

Proof of Theorem 3.8. First, we introduce the Nemitsky map corresponding to the Carathéodory function f. Namely, $N_f^*: W_0^{1,p(x)}(\Omega) \subset L^{\alpha(x)}(\Omega) \to L^{\alpha'(x)}(\Omega)$ defined by

$$N_f^*(u)(\cdot) = f(\cdot, u(\cdot), \nabla u(\cdot)) \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Hypothesis $H(f)$(i) implies that $N_f^*(\cdot)$ is well-defined, bounded and continuous, see Fan & Zhao [4] and Kováčik & Rákosník [14]. By Theorem 2.6, the embedding $i^*: L^{\alpha'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is continuous and hence the operator $N_f: W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by $N_f = i^* \circ N_f^*$ is bounded and continuous.

Now, we have established in Proposition 3.7, that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^\infty X_n$ (generated in Proposition 3.5) is bounded in the anisotropic Dirichlet Sobolev space $W_0^{1,p(x)}(\Omega)$. Additionally, this Sobolev space is reflexive, and hence for some $u \in W_0^{1,p(x)}(\Omega)$, we suppose that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p(x)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{\alpha(x)}(\Omega). \tag{4.3}$$

Since the Nemitsky map is bounded, then we deduce that

$$\{N_f(u_n)\}_{n\in\mathbb{N}} \quad \text{is bounded in } W^{-1,p'(x)}(\Omega).$$

We already know that $-\Delta_{p(x)}^{K_p}, -\Delta_{q(x)}^{K_q}: W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ are bounded, and hence

$$\left\{-\Delta_{p(x)}^{K_p}u_n - \Delta_{q(x)}^{K_q}u_n - N_f(u_n)\right\}_{n\in\mathbb{N}} \quad \text{is bounded in } W^{-1,p'(x)}(\Omega). \tag{4.4}$$

Consequently, for a relabeled subsequence of (4.4) we get

$$-\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n) \xrightarrow{w} g \quad \text{in } W^{-1,p'(x)}(\Omega), \text{ for some } g \in W^{-1,p'(x)}(\Omega), \tag{4.5}$$

as the dual space $W^{-1,p'(x)}(\Omega)$ is reflexive too.

Choosing w in $\cup_{n=1}^{\infty} X_n$, there will be $n(w) \in \mathbb{N}$ such that w belongs to $X_{n(w)}$. By Proposition 3.5, we deduce that (3.4) holds true for every $n \geq n(w)$. Letting n to infinity in (3.4), we obtain

$$\langle g, w \rangle = 0 \quad \text{for all } w \in \cup_{n=1}^{\infty} X_n.$$

The density of $\cup_{n=1}^{\infty} X_n$ in $W_0^{1,p(x)}(\Omega)$ (as $\{X_n\}_{n\in\mathbb{N}}$ is a Galerkin basis), leads to the conclusion $g = 0$, and using (4.5) we get

$$-\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n) \xrightarrow{w} 0 \quad \text{in } W^{-1,p'(x)}(\Omega). \tag{4.6}$$

Turning to equation (3.4), we consider $w = u_n$ and obtain

$$\Big\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n), u_n \Big\rangle = 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

By (4.6) we have

$$\Big\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n), u \Big\rangle \to 0, \quad \text{as } n \to +\infty,$$

and using (4.7) we get

$$\lim_{n\to+\infty} \Big\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n), u_n - u \Big\rangle = 0. \tag{4.8}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $W_0^{1,p(x)}(\Omega)$, it is bounded and so $\{N_f^*(u_n)\}_{n\in\mathbb{N}}$ is bounded. Using this fact along with Hölder's inequality and the compact embedding $W_0^{1,p(x)} \hookrightarrow L^{\alpha(x)}(\Omega)$ (see Proposition 2.2), we get

$$\begin{aligned}\left| \int_\Omega f(x, u_n, \nabla u_n)(u_n - u)\, dx \right| &\leq 2 \|N_f^*(u_n)\|_{L^{\alpha'(x)}(\Omega)} \|u - u_n\|_{L^{\alpha(x)}(\Omega)} \\ &\leq 2 \left(\sup_{n\in\mathbb{N}} \|N_f^*(u_n)\|_{L^{\alpha'(x)}(\Omega)} \right) \|u - u_n\|_{L^{\alpha(x)}(\Omega)} \to 0,\end{aligned}$$

as $n \to +\infty$. It follows that

$$\lim_{n\to+\infty} \Big\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n, u_n - u \Big\rangle = 0 \quad \text{(recall (4.8))}. \tag{4.9}$$

Combining (4.3), (4.6) and (4.9) we conclude that $u \in W_0^{1,p(x)}(\Omega)$ is a strong generalized solution to (P). This completes the proof. □

Remark 4.1. Changing $H(p)$ by $H'(p)$, the proofs of Propositions 3.5 and 3.7 above need minor adaptations. Thus, to avoid repetitions, we omit the details. We leave to the reader the easy computations, see also the similar lines in Section 4, pp. 12–13, of [26].

5 Case of positive Kirchhoff term

In this section, we briefly discuss the existence of weak solutions to (P), in the case the Kirchhoff type term (K) is substituted by the classical positive Kirchhoff term in the literature, that is

$$\widetilde{K}(r,u) = a_r + b_r \int_\Omega \frac{1}{r(x)} |\nabla u|^{r(x)} dx, \quad \text{with } a_r, b_r > 0. \tag{5.1}$$

This means that our hypothesis (K_+) this time is trivially satisfied as from (5.1) we have

$$\widetilde{K}(r,u) \geq a_r > 0 \quad \text{for all } u \in W_0^{1,r(x)}(\Omega),$$

and consequently we focus only on the notion of weak solution. Indeed, every weak solution obtained in this case, is a strong generalized solution too (recall (K_+)).

The main problem (P) becomes as follows

$$-\Delta_{p(x)}^{\widetilde{K}_p} u(x) - \Delta_{q(x)}^{\widetilde{K}_q} u(x) = f(x,u(x),\nabla u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{P_+}$$

This time, $-\Delta_{p(x)}^{\widetilde{K}_p}, -\Delta_{q(x)}^{\widetilde{K}_p} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ are the operators defined by

$$\begin{aligned}\Big\langle -\Delta_{p(x)}^{\widetilde{K}_p} u, w \Big\rangle &= \widetilde{K}(p,u)\langle -\Delta_{p(x)} u, w\rangle \\ &= \widetilde{K}(p,u) \int_\Omega |\nabla u|^{p(x)-2} (\nabla u, \nabla w)_{\mathbb{R}^N} dx \quad \text{for all } u,w \in W_0^{1,p(x)}(\Omega),\end{aligned}$$

$$\begin{aligned}\Big\langle -\Delta_{q(x)}^{\widetilde{K}_q} u, w \Big\rangle &= \widetilde{K}(q,u)\langle -\Delta_{q(x)} u, w\rangle \\ &= \widetilde{K}(q,u) \int_\Omega |\nabla u|^{q(x)-2} (\nabla u, \nabla w)_{\mathbb{R}^N} dx \quad \text{for all } u,w \in W_0^{1,p(x)}(\Omega).\end{aligned}$$

Simplifying, $-\Delta_{r(x)}^{\widetilde{K}_r} : W_0^{1,r(x)}(\Omega) \to W^{-1,r'(x)}(\Omega)$ can be seen as positive-weight version of the operator $-\Delta_{r(x)} : W_0^{1,r(x)}(\Omega) \to W^{-1,r'(x)}(\Omega)$, in respect to the theory of pseudomonotone operators. Since $-\Delta_{r(x)}$ is continuous, bounded, strictly monotone convex and of type $(S)_+$, we deduce trivially that $-\Delta_{r(x)}^{\widetilde{K}_r}$ is continuous, bounded and of type $(S)_+$.

Our approach remains purely topological (because of the presence of convection), so we involve the Nemitsky map $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, and introduce the operator $A : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$A(u) = -\Delta_{p(x)}^{\widetilde{K}_p} u - \Delta_{q(x)}^{\widetilde{K}_q} u - N_f(u) \quad \text{for all } u \in W_0^{1,p(x)}(\Omega). \tag{5.2}$$

Clearly, this operator is bounded and continuous. We establish the following existence theorem.

Theorem 5.1. *If hypotheses $H(p)$ and $H(f)$ hold, then problem (P_+) admits at least a weak solution.*

A similar theorem can be established using hypothesis $H'(p)$ instead of $H(p)$. In both the cases, the new strategy develops through two steps: the proof of pseudo-monotonicity of $A(\cdot)$ and the proof of coercivity of $A(\cdot)$.

Proof of Theorem 5.1. In the first step of the proof, we establish the pseudo-monotonicity of $A(\cdot)$ defined by (5.2), in the sense of Remark 2.4. To this end, let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}$ be a sequence such that

$$u_n \overset{w}{\to} u \quad \text{in } W_0^{1,p(x)} \quad \text{and} \quad \limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0. \tag{5.3}$$

Using (5.3) we deduce that

$$\limsup_{n\to+\infty}\Big[\Big\langle -\Delta_{p(x)}^{\widetilde{K}_p}u_n - \Delta_{q(x)}^{\widetilde{K}_q}u_n, u_n - u\Big\rangle - \int_\Omega f(x,u_n,\nabla u_n)(u_n-u)dx\Big] \leq 0. \tag{5.4}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $W_0^{1,p(x)}(\Omega)$, it is bounded and so $\{N_f^*(u_n)\}_{n\in\mathbb{N}}$ is bounded. Using this fact along with Hölder's inequality and the compact embedding $W_0^{1,p(x)} \hookrightarrow L^{\alpha(x)}(\Omega)$ (see Proposition 2.2), we get

$$\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)dx \to 0, \quad \text{as } n\to+\infty. \tag{5.5}$$

Therefore (5.4) leads to the following chain of implications

$$\begin{aligned}
&\limsup_{n\to+\infty}\Big\langle -\Delta_{p(x)}^{\widetilde{K}_p}u_n - \Delta_{q(x)}^{\widetilde{K}_q}u_n, u_n - u\Big\rangle \leq 0,\\
\Rightarrow\ &\limsup_{n\to+\infty}\Big[\Big\langle -\Delta_{p(x)}^{\widetilde{K}_p}u_n, u_n - u\Big\rangle + \widetilde{K}(q,u_n)\langle -\Delta_{q(x)}u, u_n-u\rangle\Big] \leq 0,\\
\Rightarrow\ &\limsup_{n\to+\infty}\Big\langle -\Delta_{p(x)}^{\widetilde{K}_p}u_n, u_n - u\Big\rangle \leq 0\\
\Rightarrow\ &u_n \to u \text{ in } W_0^{1,p(x)}(\Omega) \quad (\text{since } -\Delta_{p(x)}^{\widetilde{K}_p} \text{ has the } (S)_+\text{-property}).
\end{aligned} \tag{5.6}$$

Since $A(\cdot)$ is continuous, using (5.6) we get the convergences $A(u_n)\to A(u)$ and $\langle A(u_n),u_n\rangle \to \langle A(u),u\rangle$. So, we conclude that $A(\cdot)$ is a pseudomonotone operator.

It remains to prove the coercivity of $A(\cdot)$. Using hypothesis $H(f)$(ii), we deduce that

$$\begin{aligned}
\langle A(u),u\rangle &= \left(a_p + b_p\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)}dx\\
&\quad + \left(a_q + b_q\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)}dx - \int_\Omega f(x,u,\nabla u)u dx\\
&\geq \frac{b_p}{p^+}\rho_p^2(\nabla u) + a_p\rho_p(\nabla u) + \frac{b_q}{q^+}\rho_q^2(\nabla u) + a_q\rho_q(\nabla u) - \int_\Omega |f(x,u,\nabla u)u|dx\\
&\geq \left[\frac{b_p}{p^+}\rho_p(\nabla u) + a_p - \lambda^*\right]\rho_p(\nabla u) - \|\sigma_0\|_{L^1(\Omega)} \quad (\text{by } (3.2)),
\end{aligned}$$

and hence we get

$$\langle A(u),u\rangle \geq \left[\frac{b_p}{p^+}(\|u\|^{p^-}-1) + a_p - \lambda^*\right](\|u\|^{p^-}-1) - \|\sigma_0\|_{L^1(\Omega)} \quad (\text{by } (2.1)).$$

Therefore the coercivity of $A(\cdot)$ follows immediately since $1 < p^-$. Now, we can apply Theorem 2.5 to the operator $A(\cdot)$, and hence we deduce that there exists $\widehat{u} \in W_0^{1,p(x)}(\Omega)$ such that $A(\widehat{u}) = 0$. Obviously, such $\widehat{u} \in W_0^{1,p(x)}(\Omega)$ is a weak solution to (P_+). □

Remark 5.2. When we use hypothesis $H(f)$(ii) and $H'(p)$ to prove the coercivity of $A(\cdot)$, the precise calculations are as follows

$$\begin{aligned}\langle A(u),u\rangle &= \left(a_p + b_p\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)}dx \\ &\quad + \left(a_q + b_q\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)}dx - \int_\Omega f(x,u,\nabla u)udx \\ &\geq \frac{b_p}{p^+}\rho_p^2(\nabla u) - |a_p - b_2|\rho_p(\nabla u) - b_1\int_\Omega |u|^{p(x)}dx - \|\sigma_0\|_{L^1(\Omega)} \\ &\geq \frac{b_p}{p^+}\|u\|^{2p^-} - C\|u\|^{p^+} \quad \text{for some } C>0 \text{ if } \|u\|>1,\end{aligned}$$

and hence the coercivity of $A(\cdot)$ is proved.

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