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Dynamical behavior of a parametrized family of one-dimensional maps

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Abstract. The connection of these maps to homoclinic loops acts like an amplifier of the map behavior, and makes it interesting also in the case where all map orbits approach zero (but in many possible ways). We introduce so-called 'flat' intervals containing exactly one maximum or minimum, and so-called 'steep' intervals containing exactly one zero point of $f_{\mu,\omega}$ and no zero of $f'_{\mu,\omega}$. For specific parameters μ and ω, we construct an open set of points with orbits staying entirely in the 'flat' intervals in section three. In section four, we describe orbits staying in the 'steep' intervals (for open parameter sets), and in section five (for specific parameters) orbits regularly changing between 'steep' and 'flat' intervals. Both orbit types are described by symbol sequences, and it is shown that their Lebesgue measure is zero.

Keywords: homoclinic behavior, one-dimensional maps, symbolic dynamics, measure.

2020 Mathematics Subject Classification: Primary 37E05, Secondary 34C37, 37B10, 37D45.

1 Introduction

Our aim in this paper is to analyze the dynamics of certain parametrized families $f_{\mu,\omega}$ of one-dimensional maps. These arise in the dynamics of flows in three dimensions of saddle-focus homoclinic connections which were studied by Šil'nikov [6] and Holmes [2]. Holmes considered maps f similar to

$$f_{\mu,\omega} : x \to x^{\mu} \sin(\omega \ln(x))$$

for $\mu > 1, \omega > 0$ (and odd continuation). The property $\mu > 1$ implies that all points $x \in (-1, 1)$ approach 0 under f^n as $n \to \infty$. The connection of the map f to a doubly homoclinic loop (as explained below) implies that the small difference between $f^n(x)$ being positive or negative corresponds to the 'macroscopic' difference that the $n + 1$st return will take place along the upper or lower branch of the homoclinic loop, and is therefore of interest. Holmes claimed that the set Z of points x for which there exists an $n_x \in \mathbb{N}$ such that $f^{n_x}(x) = 0$ can be a dense subset of $[0, 1]$, but it seems that this proof is not conclusive. (We briefly write f for $f_{\mu,\omega}$ now.) In section four, we are interested in the orbit $x, f(x), f(f(x)), \ldots$ We first assign

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to x a symbolic trajectory $s_0, s_1, s_2, \ldots$ where $s_n = \text{sign}(f^n(x))$. Then we construct sets Ω_n^c (depending on a parameter c and $n \in \mathbb{N}$) of points with the first n iterates contained in certain 'steep' intervals and following arbitrary symbol sequences. We show that Ω_n^c is contained in the closure of the set Z, but $\Omega_\infty^c = \bigcap_{n\in\mathbb{N}} \Omega_n^c$ has measure zero. The remark on the bottom of the page 395 of [2] conjectures that open sets of points with orbit only in the 'flat' intervals can exist for certain parameters. (These 'flat' intervals are disjoint to Z.) We prove this in Section 3.

In the last section, we focus on constructing another type of orbit whose points travel regularly from a 'flat' interval to a 'steep' interval, then again from the 'steep' interval to a 'flat' interval. These points form a Cantor type set and are described by sequences of the type $(L, R, R, L, \ldots)$, indicating whether iterates of the initial points are to the left or to the right of corresponding maxima of f. Taking counter images $f^{-1}(J)$ of intervals J with $f^{-1}(J)$ close to a quadratic maximum of f involves inversion of the second order Taylor expansion and thus taking square roots. We also show that, despite the expanding effect of the square root, the measure of the points with such orbits (and thus the measure of the Cantor set) is also zero.

1.1 Motivation of the map

We consider the differential equation

$$\begin{aligned} \dot{x} &= sx - \nu y + F_1(x,y,z) \\ \dot{y} &= \nu x + sy + F_2(x,y,z) \quad \text{or} \quad \dot{X} = F(X), \\ \dot{z} &= \lambda z + F_3(x,y,z) \end{aligned} \tag{1.1}$$

where $X = (x, y, z)$, with smooth functions F_1, F_2, F_3 which vanish at the origin together with their derivatives and assume that there exists a doubly homoclinic connection associated to a saddle-focus singularity at the origin $(0,0,0)$ with eigenvalues $s \pm i\nu$, $s < 0$, $\nu \neq 0$, $\lambda > 0$. We also assume that the saddle value satisfies $s + \lambda < 0$ and F possesses symmetry under the change of sign, $F(X) = -F(-X)$. Here, note that while the stable manifold $W^s(0)$ is two-dimensional, the unstable manifold $W^u(0)$ is one-dimensional. The global unstable manifold $W^u(0)$ consists of the homoclinic loops and is contained in $W^s(0)$ (see Figure 1.1). Note also that in case $s + \lambda < 0$ stable periodic orbits bifurcate from the homoclinic loop as described by L. P. Šil'nikov in reference [5], even in case of only one homoclinic loop.

Furthermore, to obtain expressions for a Poincaré first return map defined by the trajectories close to the homoclinic loop Λ, we assume that the vector field is linear (i.e. $F_1 = F_2 = F_3 = 0$) in a neighborhood of $(0,0,0)$. First, in a neighborhood of $(0,0,0)$ we introduce a cross section Σ_0 that is transversal to Λ and has a nonzero projection to the unstable direction. The second property is an automatic consequence of the first in three dimensions. The stable manifold W^s_{loc} splits Σ_0 into the upper and lower components Σ_0^+ and Σ_0^- respectively, and the homoclinic loop intersects Σ_0 at some point $p = (\xi, 0, 0) \in \Lambda \cap \Sigma_0$ on W^s_{loc}. We next introduce two cross-sections $\Sigma_1^\mp$ transversal to W^u_{loc}. Using the trajectories which travel from Σ_0^+ to Σ_1^+ we aim at computing local maps $G_0^+ : \Sigma_0^+ \to \Sigma_1^+$ and $G_0^- : \Sigma_0^- \to \Sigma_1^-$. These local maps associate to each point $p \in \Sigma_0$ the first intersection with Σ_1 of the trajectory which starts at p. Thus, a local map G_0 is defined by the flow on subsets $\Sigma_0^\mp$ of Σ_0. Note that since the upper and lower homoclinic orbit of the system have analogous behavior, we shall continue with one (the upper homoclinic loop) of them. For simplification we assume that there exist $\xi > 0$, $\zeta > 0$ such that $\Sigma_0^+ \subset \{(\xi, y, z) : (y, z) \in \mathbb{R}^2\}$ and $\Sigma_1^+ \subset \{(x, y, \zeta) : (x, y) \in \mathbb{R}^2\}$.

The solution $(x(t), y(t), z(t))$ of (1.1), which starts from a point $(x_0 = \xi, y_0, z_0) \in \Sigma_0^+$ close to the origin at the time $t = 0$ and ends up at the point $(x_1, y_1, z_1 = \zeta) \in \Sigma_1^+$ at the time $t = \tau$, is written (taking into account only the linear terms in (1.1)) as follows:

$$\begin{aligned} x(t) + iy(t) &= e^{(s+iv)t}(x_0 + iy_0) = e^{(s+iv)t}(\xi + iy_0) \\ z(t) &= z_0 e^{\lambda t}. \end{aligned} \tag{1.2}$$

The flight time τ that the trajectory takes from Σ_0^+ to Σ_1^+ is given by $\tau = \frac{1}{\lambda}\ln(\frac{\zeta}{z_0})$. Substituting τ and ξ into formula (1.2), we get the following expression for the local map G_0^+, in complex notation:

$$x_1 + iy_1 = e^{(s+iv)\tau(z_0)}(x_0 + iy_0) = e^{(s+iv)\tau(z_0)}(\xi + iy_0). \tag{1.3}$$

On the other hand, due to the existence of the homoclinic connection and its transversal intersection with Σ_0 and Σ_1^+, we also have a Poincaré type map

$$G_1^+ : \Sigma_1^+ \to \Sigma_0$$

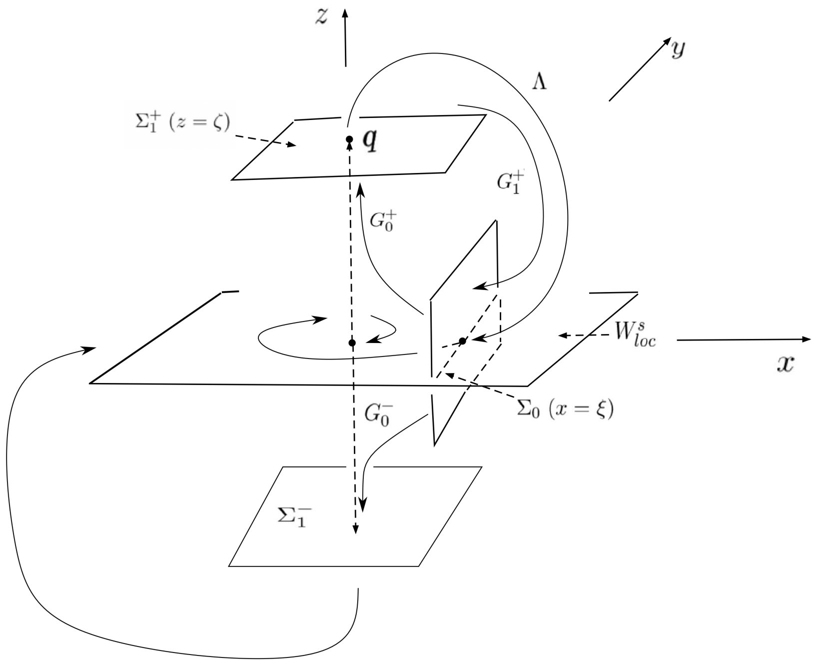

Figure 1.1: Cross sections Σ_0, Σ_1 and homoclinic orbit Λ.

Hence, for $(x_1, y_1, z_1 = \zeta) \in \Sigma_1^+$ we have $G_1^+(x_1, y_1, z_1 = \zeta) = (\xi, y_2, z_2) \in \Sigma_0$. With $DG_1^+(0, 0, \zeta)$ represented by the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial y_1} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial y_1} \end{pmatrix}(0,0),$$

we have for the composite map

$$(G_1^+ \circ G_0^+) : (\xi, y_0, z_0) \to (\xi, y_2, z_2),$$

$$\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \gamma x_1 + \delta y_1 \end{pmatrix} + h.o.t. \text{ (higher order terms)},$$

and finally we get

$$z_2 = \gamma x_1 + \delta y_1. \tag{1.4}$$

Substituting the value of $\tau(z_0)$, x_1 and y_1 in (1.4), in particular for $y_0 = 0$, one obtains

$$z_2 = \xi e^{s\tau(z_0)} \left[\gamma \cos + \delta \sin\right] (\nu\tau(z_0)).$$

Hence, with $c := \xi\sqrt{\gamma^2+\delta^2}$ and choosing φ with $e^{i\varphi}\sqrt{\gamma^2+\delta^2} = \delta + i\gamma$, the z-component after one return is approximately given by

$$z_0 \to z_2 = c\left(\frac{\zeta}{z_0}\right)^{\frac{s}{\lambda}} \left[\sin\left(\frac{\nu}{\lambda}\left(\ln\frac{\zeta}{z_0}\right) + \varphi\right)\right].$$

Note that $\frac{s}{\lambda} < -1$ so $\mu := -\frac{s}{\lambda} > 1$, with $x := \frac{z_0}{\zeta}$, and $\omega = \frac{\nu}{\lambda}$ we can rewrite the last equation as

$$z_2 = cx^{\mu}\left[\sin\left(-\omega\left(\ln x\right) + \varphi\right)\right].$$

This motivates the study of the one-dimensional map $f_{\omega,\mu} : [-1,1] \to [-1,1]$ given by the following simpler expression

$$f_{\mu,\omega}(x) = \begin{cases} x^{\mu}\sin(\omega\ln(x)), & x > 0, \\ 0, & x = 0, \\ -f_{\mu,\omega}(-x), & x < 0, \end{cases}$$

where we use x instead of z from now on. Here, note that odd continuation in the definition of $f_{\mu,\omega}$ is motivated by the corresponding symmetry of vector field. The above process shows how to arrive at this map starting from homoclinic orbits; similar considerations are given in Šil'nikov, L. P. [6], P. J. Holmes [2], or J. Guckenheimer/P. Holmes [1, pp. 320–321]. Analogous infinite-dimensional examples with attracting homoclinic behavior (not necessarily with a double loop) were studied by Walther in [7] and by Ignatenko in [3]. The maps of this kind (see Figure 1.2) were also studied by M. J. Pacifico, A. Rovella and M. Viana [4], but for $\mu < 1$, which has expansion properties of $f_{\mu,\omega}$ as a consequence. Briefly, they proved that a family of one dimensional maps with infinitely many critical points exhibit global chaotic behavior in a persistent way: For a positive Lebesgue measure set of values μ, the map f has positive Lyapunov exponent at every critical value and at Lebesgue almost all points in its domain; moreover, f is topologically transitive, i.e. has dense orbits [4].

After giving some preparatory calculations for the following chapters, we are going to study the orbit $f^n_{\omega,\mu}(x) = f^n(x)$; $n = 1,2,3,\ldots$ of a typical point $x \in (0,1)$. If $f^n(x) = 0$ for some $n < \infty$, then it is clear that all $\left(f^j(x)\right)_{j\geq n}$ will equal to 0. To orbits of f we can associate symbol sequences $(s_j) = \left(\operatorname{sign} f^j(x)\right)_{j\geq 0} = (+1,+1,-1,\ldots)$. $f^n(x) = 0$ implies that $s_n = 0$, then $s_k = 0$ for all $k \geq n$. Here $+1, -1$ and 0 correspond to the upper, to the lower homoclinic branch or to the stable manifold $W^s(0)$ in terms of the original motivation. Consequently, the following questions arise:

(i) Are all symbol sequences possible or not?

(ii) Does the symbol sequence change in every interval? (Is there chaotic motion?)

(iii) Is it possible to construct open intervals where the symbol sequence does not change?

In the fifth chapter, we shall also consider symbol sequences different from $(\operatorname{sign} f^j(x))$, describing whether $f^n(x)$ is to the left or to the right hand side of maximum points of f.

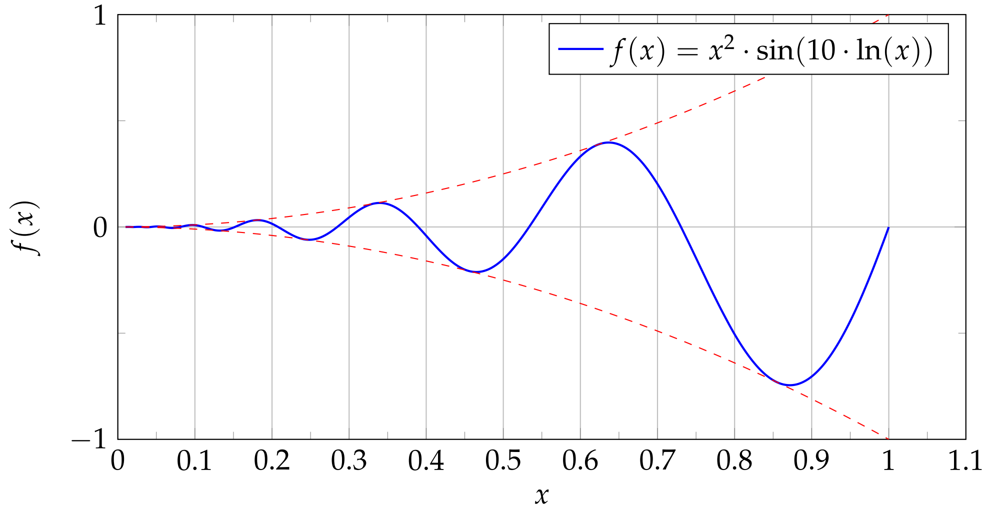

Figure 1.2: Graph of f for $\mu = 2, \omega = 10$.

2 Formulas for the derivatives of $f_{\mu,\omega}$

Lemma 2.1. *Define for $\mu, \omega > 0$ the map*

$$f_{\mu,\omega}(x) = \begin{cases} x^\mu \sin(\omega \ln(x)), & x > 0, \\ 0, & x = 0, \\ -f_{\mu,\omega}(-x), & x < 0. \end{cases}$$

Assume now $\mu \in (2, \infty)$, $\omega > 0$. Set $\varphi_j := \arctan\left(\frac{\omega}{\mu+1-j}\right) \in \left(0, \frac{\pi}{2}\right)$ and

$$g_{\omega,\mu+1-j} := \sqrt{(\mu + 1 - j)^2 + \omega^2}$$

for $j \in \{1, 2, 3\}$. It is convenient to also define the more general class of functions

$$f_{\mu,\omega,\varphi}(x) := x^\mu \sin\left(\omega \ln(x) + \varphi\right).$$

Then, the following formulas hold for $x \in \mathbb{R}$, if $\mu > 3$:

(i)

$$f'_{\mu,\omega}(x) = g_{\omega,\mu} \cdot f_{\mu-1,\omega,\varphi_1}(x), \tag{2.1}$$

$$\cos(\varphi_1) = \frac{\mu}{\sqrt{\mu^2 + \omega^2}} = \frac{\mu}{g_{\omega,\mu}}, \tag{2.2}$$

$$\sin(\varphi_1) = \frac{\omega}{\sqrt{\mu^2 + \omega^2}} = \frac{\omega}{g_{\omega,\mu}}. \tag{2.3}$$

(ii)

$$f''_{\mu,\omega}(x) = g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot f_{\mu-2,\omega,\varphi_1+\varphi_2}(x). \tag{2.4}$$

(iii)

$$f'''_{\mu,\omega}(x) = g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot f_{\mu-3,\omega,\varphi_1+\varphi_2+\varphi_3}(x). \tag{2.5}$$

Proof. (i) From the definition of φ_j, we have $\varphi_1 = \arctan(\frac{\omega}{\mu})$, and also from the definition of $g_{\omega,\mu+1-j}$, we have $g_{\omega,\mu} = \sqrt{\mu^2+\omega^2}$. It follows that

$$\cos(\varphi_1) = \frac{\mu}{g_{\omega,\mu}} \text{ and } \sin(\varphi_1) = \frac{\omega}{g_{\omega,\mu}}.$$

This proves (2.2) and (2.3). For $x > 0$, we have

$$\begin{aligned} f'_{\mu,\omega,\varphi}(x) &= x^{\mu}\cos(\omega\ln(x)+\varphi)\left(\frac{1}{x}\omega\right) + x^{\mu-1}\mu\sin(\omega\ln(x)+\varphi) \\ &= x^{\mu-1}\left(\mu\sin(\omega\ln(x)+\varphi)+\omega\cos(\omega\ln(x)+\varphi)\right). \end{aligned}$$

By multiplying and dividing the last equation with $g_{\omega,\mu}$, we have

$$f'_{\mu,\omega,\varphi}(x) = g_{\omega,\mu}\cdot x^{\mu-1}\left(\frac{\mu}{g_{\omega,\mu}}\sin(\omega\ln(x)+\varphi)+\frac{\omega}{g_{\omega,\mu}}\cos(\omega\ln(x)+\varphi)\right). \tag{2.6}$$

Putting (2.2) and (2.3) in (2.6), we finally obtain

$$\begin{aligned} f'_{\mu,\omega,\varphi}(x) &= g_{\omega,\mu}\cdot x^{\mu-1}\left(\cos(\varphi_1)\cdot\sin(\omega\ln(x)+\varphi)+\sin(\varphi_1)\cdot\cos(\omega\ln(x)+\varphi)\right) \\ &= g_{\omega,\mu}\cdot x^{\mu-1}\sin(\omega\ln(x)+\varphi+\varphi_1) \\ &= g_{\omega,\mu}\cdot f_{\mu-1,\omega,\varphi+\varphi_1}(x). \end{aligned} \tag{2.7}$$

(ii) Further, using (2.7) with $\varphi+\varphi_1$ instead of φ, and $\mu-1$ instead of μ, we see that

$$\begin{aligned} f''_{\mu,\omega}(x) = f''_{\mu,\omega,0}(x) &= \left(g_{\omega,\mu}\cdot f_{\mu-1,\omega,\varphi_1}\right)'(x) \\ &= g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot f_{\mu-2,\omega,\varphi_1+\varphi_2}(x), \end{aligned}$$

which proves (2.4).

(iii) Using (2.7) we obtain (2.5) analogously. □

Lemma 2.2. *Let $\mu > 3$ and $\omega > 0$ be given. Define $q := e^{-\frac{\pi}{\omega}}$ and φ_j as in Lemma 2.1. Then, the following properties are satisfied in $(0,1]$:*

(i) $f_{\mu,\omega}$ has the zero points

$$q^k = e^{-\frac{k\pi}{\omega}}, \tag{2.8}$$

($k \in \mathbb{N}$) and

$$f'_{\mu,\omega}(q^k) = (-1)^k\,\omega q^{k(\mu-1)}. \tag{2.9}$$

(ii) $f_{\mu,\omega}$ has the extremal points

$$m_k = q^k e^{\frac{-\varphi_1}{\omega}} \tag{2.10}$$

and

$$f_{\mu,\omega}(m_k) = (-1)^{k+1}\cdot\exp\left(-\frac{k\pi\mu+\varphi_1\mu}{\omega}\right)\cdot\sin(\varphi_1). \tag{2.11}$$

(iii) If μ is an even integer, and $\beta \in \mathbb{N}$ is odd and $l(k) := k\mu + \beta$, then $f_{\mu,\omega}$ has a maximum at

$$m_{l(k)} = q^{l(k)} e^{\frac{-\varphi_1}{\omega}}. \tag{2.12}$$

Proof. (i) We first find the zeros of $f_{\mu,\omega}$. For $x \in (0,1)$ one has

$$\sin(\omega \ln(x)) = 0 \quad \Leftrightarrow \quad \exists k \in \mathbb{N} : \omega \ln(x) = -k\pi \quad \Leftrightarrow \quad \exists k \in \mathbb{N} \;\; \ln(x) = \frac{-k\pi}{\omega},$$

and hence $x = e^{\frac{-k\pi}{\omega}}$. With $q = e^{-\frac{\pi}{\omega}}$, the zeros of $f_{\mu,\omega}$ in $(0, q]$ are given by $x = e^{\frac{-k\pi}{\omega}} = q^k$. Therefore, by inserting q^k in (2.1), we have

$$\begin{aligned} f'_{\mu,\omega}(q^k) &= q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\omega \left(\ln q^k\right) + \varphi_1) \\ &= q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\omega \left(\ln e^{\frac{-k\pi}{\omega}}\right) + \varphi_1) \\ &= (-1)^k q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\varphi_1). \end{aligned}$$

Using (2.3) we obtain

$$\begin{aligned} f'_{\mu,\omega}(q^k) &= (-1)^k q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \frac{\omega}{g_{\omega,\mu}} \\ &= (-1)^k \omega q^{k(\mu-1)}. \end{aligned}$$

Hence, assertion **(i)** is proved.

(ii) Let $k \in \mathbb{N}$. We find the extremum points of $f_{\mu,\omega}$ in the interval $I_k = \left[q^{k+1}, q^k\right]$ by solving $f'_{\mu,\omega}(x) = 0$ for $x \in I_k$. Since $x > 0$, $x^{\mu-1} \neq 0$. So, we have

$$\sin(\omega (\ln x) + \varphi_1) = 0,$$

and hence $x = e^{\frac{-k\pi-\varphi_1}{\omega}}$. The last expression equals to $q^k e^{\frac{-\varphi_1}{\omega}} = m_k$, which proves (2.10). Furthermore, for the extremum point m_k of $f_{\mu,\omega}$ in the interval $\left(q^{k+1}, q^k\right)$ we have

$$\begin{aligned} f_{\mu,\omega}(m_k) &= m_k^{\mu} \sin(\omega \ln(m_k)) \\ &= \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu} \sin(\omega \ln \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)) \\ &= \left(e^{-\frac{k\pi}{\omega}} e^{-\frac{\varphi_1}{\omega}}\right)^{\mu} \sin(\omega \ln \left(e^{-\frac{k\pi}{\omega}} e^{-\frac{\varphi_1}{\omega}}\right)) \\ &= \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\omega \frac{-k\pi - \varphi_1}{\omega}) \\ &= (-1)^{k+1} \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\varphi_1). \end{aligned}$$

(iii) Substituting $l(k)$ instead of k in (2.11), we have

$$\begin{aligned} f_{\mu,\omega}\left(m_{l(k)}\right) &= (-1)^{l(k)+1} \exp\left(-\frac{l(k)\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\varphi_1) \\ &= \exp\left(-\frac{l(k)\pi\mu + \varphi_1\mu}{\omega}\right) (-1)^{k\mu+\beta+1} \sin(\varphi_1) \end{aligned}$$

Therefore, it is clear that $f_{\mu,\omega}\left(m_{l(k)}\right) > 0$ (and hence $f_{\mu,\omega}$ has a maximum at $m_{l(k)}$), if μ is even and β is odd. $\square$

We shall frequently use the simple lemma below.

Lemma 2.3. *Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $|f'| \geq c$, or $|f'| \leq d$ (c and d are constant), then we have*

$$c\,|b-a| \leq |f(b) - f(a)| \leq d\,|b-a|\,. \tag{2.13}$$

Proof. (Follows from the mean value theorem.) □

3 The behavior of orbits remaining in some 'flat' intervals

In this part we find some parameters μ and ω such that $f_{\mu,\omega}$ maps some extremal points m_k to some other extremal points $m_{\ell(k)}$ (see Figure 3.1). Then, we construct some open intervals U_k around m_k and orbits of $f_{\mu,\omega} = f$ which are entirely contained in $\bigcup_{k\in\mathbb{N}} U_k$.

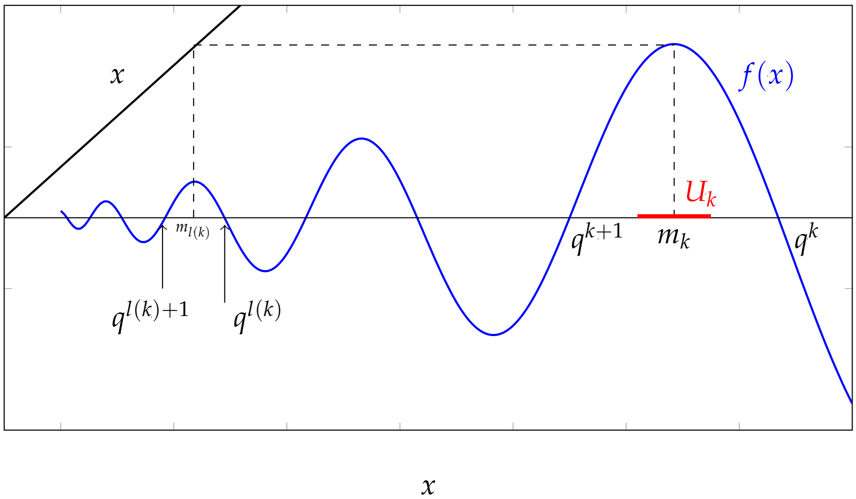

Figure 3.1: $f(m_k) = m_{\ell(k)}$ for special parameters (picture not produced with realistic parameters, for better visibility).

Theorem 3.1. *For $k \in \mathbb{N}$, $\omega > 0$, and even integer $\mu > 5$, define*

$$\eta := \min\left\{\frac{q}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}, \frac{e^{-\frac{\varphi_1}{\omega}} - q}{2}, \frac{1-e^{-\frac{\varphi_1}{\omega}}}{2}\right\}, \tag{3.1}$$

and set $\ell(k) := k\mu + 1$ (which corresponds to $\beta = 1$ in assertion (iii) of Lemma 2.2), $\delta_k := \eta q^k$, $\delta_{\ell(k)} := \eta q^{\ell(k)}$. Then, for every large enough even integer μ there exists a corresponding ω such that the following properties are satisfied:

(i) With the intervals $U_k = (m_k - \delta_k, m_k + \delta_k)$ one has $f(U_k) \subset U_{\ell(k)}$ and

$$\forall k \in \mathbb{N} : f^{-1}(\{0\}) \cap U_k = \emptyset.$$

(ii) If k is odd, then for $x \in U_k$, the orbits $\left(f^j(x)\right)_{j\in\mathbb{N}}$ all have the symbol sequence

$$(s_j) = \left(\operatorname{sign} f^j(x)\right)_{j\in\mathbb{N}} = (+1, +1, +1, \dots)\,.$$

(iii) The set

$$Z = \{x \mid \exists n \in \mathbb{N} : f^n(x) = 0\} \tag{3.2}$$

is disjoint to $\bigcup_k U_k$ *and, in particular, is not dense in* $[-1, 1]$.

The proof is divided into several lemmas.

Lemma 3.2. *Let* $k \in \mathbb{N}$ *and define* φ_1 *as in Lemma 2.1. Define* η *and* δ_k *as in Theorem 3.1, and*

$$\overline{\eta} := \min\left\{\frac{e^{-\frac{\varphi_1}{\omega}} - q}{2}, \frac{1 - e^{-\frac{\varphi_1}{\omega}}}{2}\right\}.$$

Then we have

$$(m_k - \delta_k, m_k + \delta_k) \subset [m_k - \delta_k, m_k + \delta_k] \subset \left[m_k - \overline{\eta}q^k, m_k + \overline{\eta}q^k\right] \subset \left(q^{k+1}, q^k\right).$$

Proof. From (3.1) we have $\eta \leq \overline{\eta}$. Multiplying both sides with q^k, and using (2.10) we have

$$\delta_k \leq \overline{\eta}q^k = \min\left\{\frac{q^k e^{-\frac{\varphi_1}{\omega}} - q^{k+1}}{2}, \frac{q^k - q^k e^{-\frac{\varphi_1}{\omega}}}{2}\right\} = \min\left\{\frac{m_k - q^{k+1}}{2}, \frac{q^k - m_k}{2}\right\},$$

it follows that $(m_k - \delta_k, m_k + \delta_k) \subset [m_k - \delta_k, m_k + \delta_k] \subset \left[m_k - \overline{\eta}q^k, m_k + \overline{\eta}q^k\right] \subset \left(q^{k+1}, q^k\right)$. □

Lemma 3.3. *Define* φ_1 *as in Lemma 2.1, and define* $\ell(k)$ *as in Theorem 3.1. Then the following statements are true.*

(i) For every even integer $\mu \geq 32$*, there exists* $\omega \in (0, 1)$ *such that for all* $k \in \mathbb{N}$ f *has the property*

$$f(m_k) = m_{\ell(k)}.$$

(ii) For any choice of ω *as in assertion* ***(i)****, one has* $\omega \to 0$ *as* $\mu \to \infty$.

Proof. **(i)** From (2.11) we have for all $k \in \mathbb{N}$

$$|f(m_k)| = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right)\sin(\varphi_1). \tag{3.3}$$

On the other hand, from the third assertion of Lemma 2.2 we know that for even μ, f has a maximum at the point

$$m_{\ell(k)} = \exp\left(-\frac{\pi\ell(k) + \varphi_1}{\omega}\right). \tag{3.4}$$

Using (2.3), (3.3) and (3.4), we obtain the following equivalences:

$$\begin{aligned} m_{\ell(k)} = f(m_k) \quad &\Leftrightarrow \quad \exp\left(-\frac{\pi\ell(k) + \varphi_1}{\omega}\right) = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \cdot \sin(\varphi_1) \\ &\Leftrightarrow \quad \exp\left(-\frac{\pi\ell(k) + \varphi_1}{\omega}\right) = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \cdot \frac{1}{\sqrt{1 + \frac{\mu^2}{\omega^2}}} \\ &\Leftrightarrow \quad \exp\left(\frac{-\pi}{\omega}\left[k\mu - \ell(k)\right] + \frac{\varphi_1(1 - \mu)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}. \end{aligned} \tag{3.5}$$

Substituting $\ell(k) = k\mu + 1$ in (3.5), we have

$$\exp\left(\frac{\pi + \varphi_1(1-\mu)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}$$

or, using the definition of φ_1,

$$\exp\left(\frac{\pi - (\mu - 1)\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}\,. \tag{3.6}$$

In view of (3.6), we define

$$F(\omega, \mu) = \exp\left(\frac{\pi - (\mu - 1)\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) - \sqrt{1 + \frac{\mu^2}{\omega^2}} \tag{3.7}$$

for all $\omega > 0$ and $\mu > 1$. We try to find (ω, μ) with $F(\omega, \mu) = 0$ (see Figure 3.2). Noting that for fixed μ, $\lim_{\omega \to 0} F(\omega, \mu) = +\infty$, it is enough to find at least one pair (ω, μ) with $F(\omega, \mu) < 0$. For $\omega = 1$, we have

$$\begin{aligned} F(1, \mu) &= \exp\left(\pi - (\mu - 1)\arctan\left(\frac{1}{\mu}\right)\right) - \sqrt{1 + \mu^2} \\ &= \exp\left(\pi - \mu\arctan\left(\frac{1}{\mu}\right) + \arctan\left(\frac{1}{\mu}\right)\right) - \sqrt{1 + \mu^2}. \end{aligned} \tag{3.8}$$

Since $\arctan'(x) = \frac{1}{1+x^2}$, we have $\arctan'(x) \geq \frac{1}{2}$ for $|x| \leq 1$. Hence, (2.13) shows $\arctan(x) \geq \frac{1}{2}x$ for $x \in [0, 1]$. It follows that for $\mu > 1$,

$$\mu \arctan\left(\frac{1}{\mu}\right) \geq \frac{1}{2}. \tag{3.9}$$

Using (3.9) and $\arctan\left(\frac{1}{\mu}\right) < \frac{\pi}{4}$ for $\mu > 1$ in (3.8), we have

$$F(1, \mu) \leq \exp\left(\pi - \frac{1}{2} + \frac{\pi}{4}\right) - \sqrt{1 + \mu^2} = \exp\left(\frac{5\pi}{4} - \frac{1}{2}\right) - \sqrt{1 + \mu^2}.$$

From the fact that $\exp\left(\frac{5\pi}{4} - \frac{1}{2}\right) < 32$, we have $F(\omega, \mu) < 0$, if we set $\omega = 1$ and $\mu \geq 32$. With the intermediate value theorem, it is trivial that $F(\omega, \mu)$ has at least one zero point $\omega \in (0, 1)$. It follows that (3.7) is satisfied with this ω depending on the even integer $\mu \geq 32$. Hence, the proof of assertion (i) is completed.

(ii) Consider a sequence μ_k, $\mu_k \to \infty$ with corresponding $\omega_k \in (0, 1)$ such that $F(\mu_k, \omega_k) = 0$. Then $\sqrt{1 + \frac{\mu_k^2}{\omega_k^2}} \to \infty$. Further, $(\mu_k - 1)\arctan\left(\frac{\omega_k}{\mu_k}\right)$ is bounded. The exponential term in (3.7) must go to $+\infty$, so $\omega_k \to 0$ necessarily. This completes the proof of (ii) and the proof of Lemma 3.3. □

Remark 3.4. Consider the equation (3.5). Because $\mu > 1$, so $\frac{\varphi_1(1-\mu)}{\omega} < 0$, and $\sqrt{1 + \frac{\mu^2}{\omega^2}} > 1$, the term $\frac{-\pi}{\omega}[k\mu - \ell(k)]$ must be positive, if we have a solution. Accordingly, $\ell(k) > k\mu$ must be satisfied. It means (3.6) has no solution for $\ell(k) \leq k\mu$. Thus $\ell(k) \geq k\mu + 1$ necessarily; we made the choice $\ell(k) = k\mu + 1$.

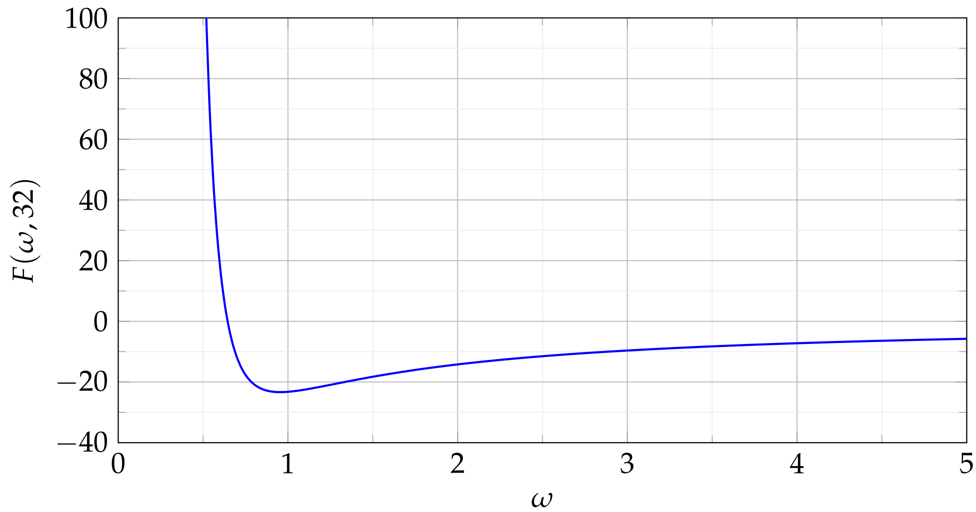

Figure 3.2: Graph of $F(\cdot, \mu)$ for $\mu = 32$.

Numerical observations. In order to find a numerical solution we use two starting points where $F(\cdot, \mu)$ has opposite signs and at the 9 th step of a bisection method we obtained $\omega = 0.69895$ and $\mu = 24$ as an appropriate $F(\omega, \mu) = 0$. Although one can obtain some other solution points ω, for some other the parameters μ, we numerically found out that there is no solution for $\mu < 3.1$.

Lemma 3.5. *Choose an even integer $\mu \geq 32$ and $\omega \in (0, 1)$ with the properties as in Lemma 3.3. Define $\ell(k)$, η, δ_k and $\delta_{\ell(k)}$ as in Theorem 3.1. Then with the intervals $U_k = (m_k - \delta_k, m_k + \delta_k)$, we have $f(U_k) \subset U_{\ell(k)}$.*

Proof. Let μ and ω be as in the assumption of the lemma, and $x \in U_k$. With $\ell(k) = k\mu + 1$ we claim that

$$\left| f(x) - m_{\ell(k)} \right| < \delta_{\ell(k)} = \eta q^{\ell(k)} . \tag{3.10}$$

From the second order Taylor expansion, we have

$$f(x) = f(m_k) + f'(m_k)(x - m_k) + \frac{f''(\xi)}{2}(x - m_k)^2 \tag{3.11}$$

with $\xi \in (m_k - \delta_k, m_k + \delta_k)$. Since $\mu > 2$, note that we also have

$$q^{(k+1)(\mu-2)} \leq |\xi|^{\mu-2} \leq q^{k(\mu-2)}. \tag{3.12}$$

Substituting the equality (3.11) in the left hand side of (3.10), we get

$$\left| f(x) - m_{\ell(k)} \right| = \left| f(m_k) + f'(m_k)(x - m_k) + f''(\xi)\frac{(x - m_k)^2}{2} - m_{\ell(k)} \right| .$$

From the fact that we now have fixed parameters μ, ω with the property $f(m_k) = m_{\ell(k)}$ as in Lemma 3.3 and using $f'(m_k) = 0$ and $(x - m_k) < \delta_k$, the last equality gives

$$\left| f(x) - m_{\ell(k)} \right| \leq \left| f''(\xi)\frac{\delta_k^2}{2} \right| .$$

Using (2.4) in the last equality, we obtain

$$\left|f(x)-m_{\ell(k)}\right|=\left|g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot\sin\left(\omega\ln(\xi)+\varphi_1+\varphi_2\right)|\xi|^{\mu-2}\frac{\delta_k^2}{2}\right|. \tag{3.13}$$

Using the upper estimate of (3.12) and substituting the value of δ_k in (3.13), we get

$$\begin{aligned}\left|f(x)-m_{\ell(k)}\right| &\leq \left|g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot\left|q^k\right|^{\mu-2}\frac{\eta^2q^{2k}}{2}\right| \\ &=\left|g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k\mu}\frac{\eta\eta}{2}\right|. \end{aligned}\tag{3.14}$$

Finally, using the definition of η from (3.1) in (3.14), we have

$$\begin{aligned}\left|f(x)-m_{\ell(k)}\right| &\leq \left|g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k\mu}\frac{\eta}{2}\frac{q}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}\right| \\ &=q^{k\mu+1}\frac{\eta}{2}<\eta q^{k\mu+1}=\eta q^{\ell(k)}=\delta_{\ell(k)}.\end{aligned}$$

□

***Proof of Theorem 3.1**.* Choose μ, ω as in Lemma 3.3, and let $\ell(k)$ be as in Theorem 3.1.

(i) Lemma 3.5 shows $f(U_k)\subset U_{\ell(k)}$ and the definition of $U_{\ell(k)}$ implies $0\notin U_{\ell(k)}$, so

$$f^{-1}(\{0\})\cap U_k=\emptyset.$$

(ii) If k is odd and μ is as above (therefore even), then all $\ell^j(k)$ $(j\geq 0)$ are odd and all $U_{\ell^j(k)}$ are intervals around maxima of f, where f is positive. Hence the assertion is proved.

(iii) For $k_0\in\mathbb{N}$, $x\in U_{k_0}$ and $n\in\mathbb{N}_0$, $f^n(x)\in\bigcup_{k\in\mathbb{N}}U_k$, in particular $f^n(x)\neq 0$, which proves assertion (iii). □

Note that the possible existence of the orbits which remain close to critical points, i.e implying non-density has been mentioned as a remark by P. J. Holmes in the bottom of the page 395 of [2] with only a vague indication of proof. With this section we gave a rigorous proof of that idea.

4 Behavior of the map $f_{\mu,\omega}$ in some 'steep' intervals

In this section we first construct some orbits whose points stay entirely in so-called 'steep' intervals, and then analyze the measure of the set of points which have such orbits. In contrast to Sections 3 and 5, where the parameters μ and ω are connected by the conditions given in assertion (i) of Lemma 3.3 and in (5.1), in this section both of them can be varied independently.

Consider the interval $(-m_k,-m_{k+1})$ or (m_{k+1},m_k). From Lemma 2.2 we have

$$\left|f'_{\mu,\omega}(q^{k+1})\right|=\omega\left(q^{k+1}\right)^{(\mu-1)}.$$

Since $f'_{\mu,\omega}(\mp m_k)=f'_{\mu,\omega}(\mp m_{k+1})=0$, continuity of $f'_{\mu,\omega}$ implies that we can choose a 'steep' interval S_k, either as a subset of (m_{k+1},m_k) or as a subset of $(-m_k,-m_{k+1})$, on which $|f'_{\mu,\omega}|$ satisfies a lower estimate. We begin by specifying the boundaries of the 'steep' interval S_k and by giving some new notations.

We use the notation $|I|$ for the length of an interval I.

Definition 4.1. Let $k \in \mathbb{N}$ and $c \in (0,1)$. Define

$$a_k := \min \left\{ x \in \left(m_{k+1}, q^{k+1} \right] : \left| f'_{\mu,\omega}(x) \right| \geq c\omega \left(q^{k+1} \right)^{(\mu-1)} \text{ on } \left[x, q^{k+1} \right] \right\}$$

and

$$b_k := \max \left\{ x \in \left[q^{k+1}, m_k \right) : \left| f'_{\mu,\omega}(x) \right| \geq c\omega \left(q^{k+1} \right)^{(\mu-1)} \text{ on } \left[q^{k+1}, x \right] \right\}.$$

Note that $q^{k+2} < a_k < q^{k+1} < b_k < q^k$ (see Figure 4.1). Given a symbol sequence of the form

$$\mathbf{s} = (s_0, s_1, s_2, \dots) \in \{+1, -1\}^{\mathbb{N}_0},$$

where symbols represent the signs of $f^j_{\mu,\omega}(x)$ for some starting value x, we construct corresponding orbits of $f_{\mu,\omega}$. Note that in terms of the motivation by the three dimensional vector field, such orbits correspond to solutions converging to the doubly homoclinic loop, and taking turns along the upper and lower homoclinic orbit according to the symbol sequence. For $0 \leq a \leq b$, define

$$[a,b]_{+1} := [a,b],$$
$$[a,b]_{-1} := [-b,-a],$$

and define 'steep' intervals by

$$S^c_{k,s} := [a_k, b_k]_s = \begin{cases} [a_k, b_k], & \text{if } s = +1, \\ [-b_k, -a_k], & \text{if } s = -1. \end{cases}$$

So, we have

$$\left| f'_{\mu,\omega}(x) \right| \geq c\omega \left(q^{k+1} \right)^{(\mu-1)} \quad \text{for } x \in S^c_{k,s},\ s \in \{\pm 1\},\ k \in \mathbb{N}. \tag{4.1}$$

We also define $S_{k,\pm 1} := S^c_{k,+1} \cup S^c_{k,-1}$ and define the union of all 'steep' intervals by

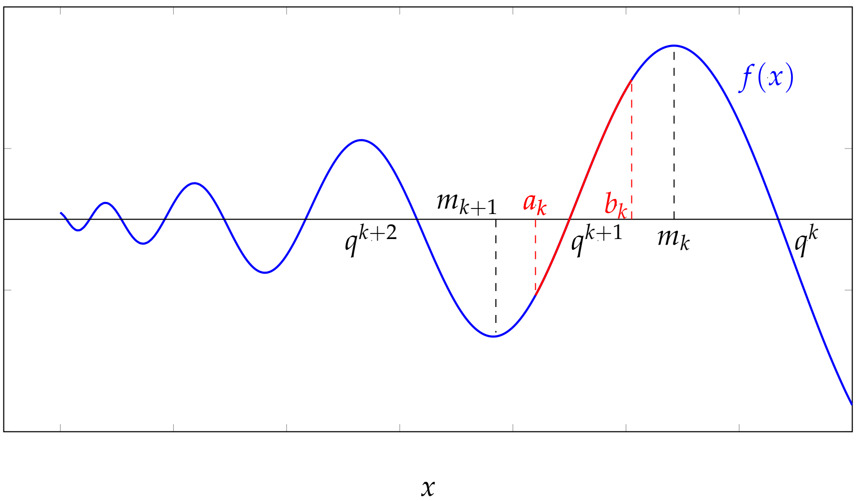

Figure 4.1: One interval (q^{k+2}, q^k), with corresponding 'steep' interval $S^c_{k,+1} = [a_k, b_k]$.

$$\Psi^c = \bigcup_{k\in\mathbb{N}} S_{k,\pm1}.$$

Note that for $s \in \{\pm1\}$, $S^c_{k,s} \subset (m_{k+1}, m_k)_s$, and hence

$$\left|S^c_{k,s}\right| \leq m_k - m_{k+1} = q^k e^{-\frac{\varphi_1}{\omega}} (1-q) \tag{4.2}$$

Setting $f := f_{\mu,\omega}$, we define sets of points with forward orbits which are contained in these 'steep' intervals (see Figure 4.2). Namely,

$$\Omega^c_n = \bigcap_{j=0}^{n} f^{-j}(\Psi^c)\,;\ \Omega^c_\infty = \bigcap_{j=0}^{\infty} f^{-j}(\Psi^c)\,.$$

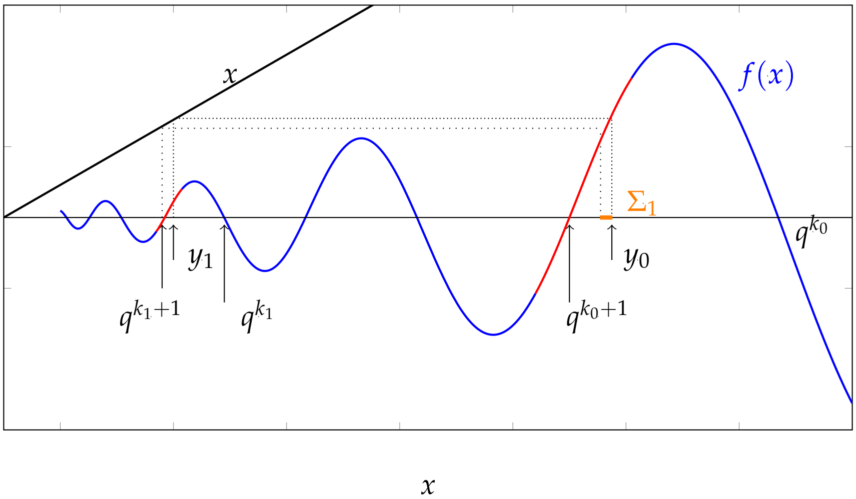

Figure 4.2: Graphical construction of Σ_1 from $\Sigma_0 = [q^{k_0+1}, y_0]$ (in case $s_0 = s_1 = +1$).

Theorem 4.2. *Let $c \in (0,1)$. Assume $\mu > 1$ and define $S^c_{k,\pm1}$ and Ω^c_∞ as above. Then for $k_0 \in \mathbb{N}$ the following statements are true:*

(i) For every symbol sequence $\mathbf{s} = (s_0, s_1, s_2, \dots)$ there exists a point $y_0 \in \left(S^c_{k_0,s_0} \cap \Omega^c_\infty\right)$ with the property that sign $f^j(y_0)$ *is given by $s_j \in \{\pm1\}$, where $j \in \mathbb{N}_0$.*

(ii) Let $\omega > \frac{1}{c} + \pi(\mu+1)$. Then with the set Z from (3.2) *we have $\left(S^c_{k_0,s_0} \cap \Omega^c_\infty\right) \subset \overline{Z}$.*

(iii) Let $c \in \left(\frac{2}{\pi}, 1\right)$ and $\omega > \frac{c\pi^2(2\mu+3)}{2(c\pi-2)}$. Then $\Omega^c_\infty \subset \overline{Z}$, and Ω^c_∞ has Lebesgue measure zero.

Remark 4.3. A similar argument is sketched in the page 395 of [2], with the purpose to show that Z can be dense, but it seems that the method gives density only in a set of measure zero (see part **(iii)** of the above theorem).

The proof starts with the following lemma.

Lemma 4.4. *Let $k_0 \in \mathbb{N}$, $c \in (0,1)$ and $\mathbf{s} = (s_0, s_1, s_2, \dots) \in \{+1, -1\}^{\mathbb{N}_0}$ be given. Define $S^c_{k_0,s_j}$ as in the passage given before Theorem 4.2. Then the following statements are true:*

(i) There exists a point $y_0 \in S^c_{k_0,s_0}$ and a sequence $k_0 < k_1 < k_2 < \cdots$ such that $\forall j \in \mathbb{N}_0$ $f^j(y_0) \in S^c_{k_j,s_j}$, in particular, $y_0 \in \Omega^c_\infty$.

(ii) Let $y_0 \in (S^c_{k_0,s_0} \cap \Omega^c_\infty)$ be given and define the sequence $k_0 < k_1 < k_2 < \cdots$ by $f^j(y_0) = y_j \in S^c_{k_j,s_j}$ $(j \in \mathbb{N}_0)$. Then there exists a sequence (Σ_j) of intervals in $S^c_{k_0,s_0}$ with $\Sigma_j \supset \Sigma_{j+1} \ni y_0$, $(f^j)' \neq 0$ on Σ_j and

$$f^j(\Sigma_j) = \left[q^{k_j+1}, y_j\right]_{s_j} = \begin{cases} \left[q^{k_j+1}, y_j\right], & \text{if } s_j = +1, \\ \left[y_j, -q^{k_j+1}\right], & \text{if } s_j = -1 \end{cases} \subset S^c_{k_j,s_j} \quad \text{for } j \in \mathbb{N}_0, \tag{4.3}$$

in particular, $Z \cap \Sigma_j \neq \emptyset$ for all $j \in \mathbb{N}_0$.

(iii) For $y_0 \in \left(S^c_{k_0,s_0} \cap \Omega^c_\infty\right)$ and $k_0, k_1, k_2, \dots$ as in assertion (ii) and all $j \in \mathbb{N}$ we have

$$\left|\left(f^j\right)'(y_0)\right| \geq (c\omega)^j \left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1}. \tag{4.4}$$

(iv) Let y_0 and the sequence $k_0 < k_1 < k_2 < \cdots$ be as in (ii). Then

$$\forall j \in \mathbb{N}: \; q^{k_j\mu} \geq q^{k_{j+1}+2}. \tag{4.5}$$

(v) Let $\omega > \frac{1}{c} + \pi(\mu+1)$. Let y_0 and the associated Σ_j be as in assertion (ii) and φ_1 be as in Lemma 2.1. Then $|\Sigma_j| \leq \frac{q^{k_0} e^{-\frac{\varphi_1}{\omega}}(1-q)}{(c\omega q^{\mu+1})^j}$ and $c\omega q^{\mu+1} > 1$; in particular, $|\Sigma_j| \to 0$, as $j \to \infty$.

Proof. **(i)** Let $k_0 \in \mathbb{N}$ and $\mathbf{s} = (s_0, s_1, s_2, \dots)$ be given. For $S^c_{k_0,s_0} = [a_{k_0}, b_{k_0}]_{s_0}$ it is clear that $f(S^c_{k_0,s_0})$ is an interval which contains 0 in its interior, and since $a_k \to 0$, $b_k \to 0$ as $k \to \infty$, there exists $k_1 > k_0$ with $S^c_{k_1,s_1} \subset f(S^c_{k_0,s_0})$. Further $f|_{S^c_{k_0,s_0}}$ is injective, and we set

$$J_1 := \left(f|_{S^c_{k_0,s_0}}\right)^{-1}\left(S^c_{k_1,s_1}\right).$$

(f maps J_1 bijectively onto $S^c_{k_1,s_1}$.) Similarly, there exists $k_2 > k_1$ with $S^c_{k_2,s_2} \subset f(S^c_{k_1,s_1})$, and a closed subinterval $J_2 \subset J_1$ such that $f^2|_{J_2} : J_2 \to S^c_{k_2,s_2}$ is bijective. Thus, we obtain a nested sequence

$$J_1 \supset J_2 \supset J_3 \supset \cdots$$

of closed intervals and sequence of numbers

$$k_0 < k_1 < k_2 < \cdots$$

with the property that $f^j(J_j) = S^c_{k_j,s_j}$, $j = 1, 2, 3, \dots$ Furthermore, the intersection of nested closed intervals $\bigcap_{j\in\mathbb{N}} J_j$ is not empty. It means that there exists a point $y_0 \in \bigcap_{j\in\mathbb{N}} J_j$ which follows the symbol sequence $\mathbf{s}$, and this result completes the proof of assertion (i).

(ii) For the proof of this assertion we use a recursive construction. Define

$$\Sigma_0 := \left[q^{k_0+1}, y_0\right]_{s_0} = \begin{cases} \left[q^{k_0+1}, y_0\right], & \text{if } s_0 = +1 \\ \left[y_0, -q^{k_0+1}\right], & \text{if } s_0 = -1 \end{cases} \subset S^c_{k_0,s_0}.$$

Then $y_0 \in \Sigma_0$, and the definition of $S^c_{k_0,s_0}$ implies $f' \neq 0$ on Σ_0, so (4.3) holds for $j = 0$. Assume Σ_j with the properties in (4.3) is constructed and we want to construct $\Sigma_{j+1} \subset \Sigma_j$ such that (4.3) is also satisfied for $j+1$. We have, observing that sign $(y_j) = s_j$,

$$f\left(\left[q^{k_j+1}, y_j\right]_{s_j}\right) = \left[0, f\left(y_j\right)\right]_{s_{j+1}} = \left[0, y_{j+1}\right]_{s_{j+1}},$$

and $f^j|_{\Sigma_j}$ as well as $f|_{\left[q^{k_j+1}, y_j\right]_{s_j}}$ are invertible. Hence, we can define

$$\Sigma_{j+1} = \left(f^{-j}|_{\Sigma_j}\right)^{-1}\left(f|_{\left[q^{k_j+1}, y_j\right]_{s_j}}\right)^{-1}\left(\left[q^{k_{j+1}+1}, y_{j+1}\right]_{s_{j+1}}\right).$$

Then $y_0 \in \Sigma_{j+1} \subset \Sigma_j$, the chain rule shows $\left(f^{j+1}\right)' \neq 0$ on Σ_{j+1}, and

$$\left(f^{j+1}\right)\left(\Sigma_{j+1}\right) = \left[q^{k_{j+1}+1}, y_{j+1}\right]_{s_{j+1}} \subset S^c_{k_{j+1},s_{j+1}}.$$

Hence, the recursive construction is completed. Note also that for $j \in \mathbb{N}$, Σ_j contains a point x_j with $f^j\left(x_j\right) = q^{k_j+1}$, so $f^{j+1}(x_j) = f\left(q^{k_j+1}\right) = 0$, hence $x_j \in \Sigma_j \cap Z$.

(iii) By the chain rule the derivative $\left(f^j\right)'$ at $y_0 \in \bigcap_{j\in\mathbb{N}} \Sigma_j$ can be calculated as the product of the derivatives of f along the orbit

$$\left|\left(f^j\right)'(y_0)\right| = \left|f'(y_0) \cdot f'(y_1) \cdot \ldots \cdot f'\left(y_{j-2}\right) \cdot f'\left(y_{j-1}\right)\right| = \prod_{n=0}^{j-1}\left|f'(y_n)\right|.$$

Using (4.1) for each derivative in the last equality, we have

$$\begin{aligned}\left|\left(f^j\right)'(y_0)\right| &= \prod_{n=0}^{j-1}\left|f'(y_n)\right| \geq (c\omega)^j \prod_{n=0}^{j-1}\left(q^{k_n+1}\right)^{\mu-1} \\ &= (c\omega)^j\left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1}.\end{aligned}$$

This gives the proof of (4.4).

(iv) Let now $y_0 \in S^c_{k_0,s_0}$ and sequence $k_0 < k_1 < k_2 < \cdots$ as in (ii) be given. With Σ_j from (4.3) we have $f^j\left(\Sigma_j\right) \subset S^c_{k_j,s_j}$, and so

$$f^{j+1}(y_0) \in S^c_{k_{j+1},s_{j+1}} \cap f^{j+1}\left(\Sigma_j\right) \subset f\left(f^j\left(\Sigma_j\right)\right) \subset f\left(S^c_{k_j,s_j}\right), \quad \text{for } j \in \mathbb{N}_0$$

which implies $f\left(S^c_{k_j,s_j}\right) \cap S^c_{k_{j+1},s_{j+1}} \neq \emptyset$. Moreover, since $|f| \leq q^{k_j\mu}$ on $S^c_{k_j,s_j}$, we obviously have $q^{k_j\mu} \geq \max\{|f(x)| : x \in S^c_{k_j,s_j}\}$. Together with

$$\max\left\{|f(x)| : x \in S^c_{k_j,s_j}\right\} \geq \min\left\{|y| : y \in S^c_{k_{j+1},s_{j+1}}\right\},$$

we conclude

$$q^{k_j\mu} \geq \max\left\{|f(x)| : x \in S^c_{k_j,s_j}\right\} \geq \min\left\{|y| : y \in S^c_{k_{j+1},s_{j+1}}\right\} = a_{k_{j+1}} \geq q^{k_{j+1}+2}.$$

Hence, the proof of (iv) is also completed.

(v) Finally, from (2.13) we know that on Σ_j we have

$$\left|\Sigma_j\right| \leq \frac{\left|\left(f^j\right)\left(\Sigma_j\right)\right|}{\min_{\Sigma_j}\left|\left(f^j\right)'\right|}. \tag{4.6}$$

From (4.3) we have $\left|\left(f^j\right)\left(\Sigma_j\right)\right| \leq \left|S^c_{k_j,s_j}\right|$, and from (4.2) we have $\left|S^c_{k_j,s_j}\right| \leq q^{k_j}e^{-\frac{\varphi_1}{\omega}}(1-q)$. Combining both inequalities, we get

$$\left|\left(f^j\right)\left(\Sigma_j\right)\right| \leq \left|S^c_{k_j,s_j}\right| \leq q^{k_j}e^{-\frac{\varphi_1}{\omega}}(1-q). \tag{4.7}$$

Using (4.7) and (4.4) in (4.6), we obtain

$$\left|\Sigma_j\right| \leq \frac{q^{k_j}e^{-\frac{\varphi_1}{\omega}}(1-q)}{(c\omega)^j\left(\prod\limits_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1}}. \tag{4.8}$$

By using (4.5) we can estimate the denominator of (4.8) as follows:

$$\begin{aligned}
(c\omega)^j\left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1} &= (c\omega)^j\cdot\left(\prod_{n=0}^{j-1} q\right)^{\mu-1}\left(\prod_{n=0}^{j-1} q^{k_n}\right)^{\mu-1} \\
&= (c\omega)^j\cdot\frac{q^{j(\mu-1)}\prod\limits_{n=0}^{j-1} q^{k_n\mu}}{\prod\limits_{n=0}^{j-1} q^{k_n}} \geq (c\omega)^j\cdot q^{j(\mu-1)}\cdot\frac{\prod\limits_{n=0}^{j-1} q^{k_{n+1}+2}}{\prod\limits_{n=0}^{j-1} q^{k_n}} \\
&= (c\omega)^j\cdot q^{(\mu-1)j}\cdot\frac{q^{2j}\prod\limits_{n=0}^{j-1} q^{k_{n+1}}}{\prod\limits_{n=0}^{j-1} q^{k_n}} = \left(c\omega q^{\mu+1}\right)^j\cdot\frac{q^{k_j}}{q^{k_0}}.
\end{aligned}$$

Substituting this estimate in (4.8), we finally have

$$\left|\Sigma_j\right| \leq \frac{q^{k_j}e^{-\frac{\varphi_1}{\omega}}(1-q)\,q^{k_0}}{\left(c\omega q^{\mu+1}\right)^j q^{k_j}} = \frac{q^{k_0}e^{-\frac{\varphi_1}{\omega}}(1-q)}{\left(c\omega q^{\mu+1}\right)^j}.$$

To show that $\left|\Sigma_j\right| \to 0$ as $j\to\infty$, it is enough to show $\left(c\omega q^{\mu+1}\right) > 1$. Note that the first order Taylor expansion of $q^{\mu+1} = \exp\left(-\frac{\pi}{\omega}(\mu+1)\right)$ is

$$\exp\left(-\frac{\pi}{\omega}(\mu+1)\right) = 1 - \frac{\pi(\mu+1)}{\omega} + R_1(\xi),$$

where $R_1(\xi) = \frac{\exp''(\xi)}{2}\left(\frac{\pi(\mu+1)}{\omega}\right)^2 > 0$, and $\xi \in \left(-\frac{\pi(\mu+1)}{\omega}, 0\right)$. The assumption of (e) gives us $\frac{1}{c} + \pi(\mu+1) < \omega$, and hence

$$1 < c\omega - c\pi(\mu+1) = c\omega\left(1 - \frac{\pi(\mu+1)}{\omega}\right).$$

Since $R_1(\xi) > 0$, we obtain

$$1 < c\omega\left(1 - \frac{\pi(\mu+1)}{\omega}\right) < c\omega\left(1 - \frac{\pi(\mu+1)}{\omega} + R_1(\xi)\right) = c\omega \exp\left(-\frac{\pi}{\omega}(\mu+1)\right) = c\omega q^{\mu+1},$$

and this completes the proof of (v). □

The next lemma estimates the measure of the points in the 'steep' interval $S^c_{k_0,+1}$ which have the first n iterates in the union of all 'steep' intervals.

Lemma 4.5. *Let $k_0 \in \mathbb{N}$, $c \in \left(\frac{1}{\pi}, 1\right)$. Let Ψ^c and $S^c_{k_0,\pm1}$ be as in the passage before Theorem 4.2. Define φ_1 as in Lemma 2.1. Then for $k_0 \in \mathbb{N}$ we have*

$$\left| S^c_{k_0,+1} \cap \bigcap_{i=1}^{n} f^{-i}(\Psi^c) \right| \le \frac{2^n q^{k_0} e^{-\frac{\varphi_1}{\omega}}(1-q)}{\left(c\omega q^{\mu+1}(1-q)\right)^n}. \tag{4.9}$$

(The same estimate holds for $S^c_{k_0,-1}$.)

Proof. Let $k_0 \in \mathbb{N}$ and $c \in \left(\frac{1}{\pi}, 1\right)$ be given. It is clear that $f\left(S^c_{k_0,+1}\right)$ contains infinitely many 'steep' intervals, because it is a neighborhood of zero. Assume $\ell, i \in \mathbb{N}$ are such that $S^c_{k_0,+1} \cap f^{-i}\left(S^c_{\ell,\pm1}\right) \ne \emptyset$. Since $\left|f^i(x)\right| \le |x|^{\mu^i}$ on $S^c_{k_0,\pm1}$, one must have $q^{k_0\mu^i} \ge \min\{|y| : y \in S^c_{\ell,\pm1}\} \ge q^{\ell+2}$. It follows that $\ell \ge k_0\mu^i - 2 \ge k_0\mu - 2$. Thus

$$f^{-i}(\Psi^c) = f^{-i}\left(\bigcup_{\ell\in\mathbb{N}} S^c_{\ell,\pm1}\right) = f^{-i}\left(\bigcup_{\substack{\ell\in\mathbb{N} \\ f^{-i}\left(S^c_{\ell,\pm1}\right)\ne\emptyset}} S^c_{\ell,\pm1}\right) = f^{-i}\left(\bigcup_{\ell\ge k_0\mu-2} S^c_{\ell,\pm1}\right).$$

Hence, the intersection in (4.9) equals $S^c_{k_0,+1} \cap \bigcap_{i=1}^n f^{-i}\left(\bigcup_{\ell\ge k_0\mu-2} S^c_{\ell,\pm1}\right)$. We now prove (4.9) by induction over n. For $n = 1$,

$$\begin{aligned}\left|S^c_{k_0,+1} \cap f^{-1}(\Psi^c)\right| &= \left|S^c_{k_0,+1} \cap f^{-1}\left(\bigcup_{\ell\ge k_0\mu-2} S^c_{\ell,\pm1}\right)\right| \\ &= \sum_{\ell\ge k_0\mu-2} \left|S^c_{k_0,+1} \cap f^{-1}\left(S^c_{\ell,\pm1}\right)\right|.\end{aligned} \tag{4.10}$$

From (4.2) we have

$$\left|S^c_{\ell,\pm1}\right| \le 2q^\ell e^{-\frac{\varphi_1}{\omega}}(1-q). \tag{4.11}$$

Using (2.13), (4.1) and (4.11) in (4.10), we have

$$\begin{aligned}\left|S^c_{k_0,+1} \cap f^{-1}(\Psi^c)\right| &= \sum_{\ell\ge k_0\mu-2} \left|S^c_{k_0,+1} \cap f^{-1}\left(S^c_{\ell,\pm1}\right)\right| \le \sum_{\ell\ge k_0\mu-2} \frac{1}{c\omega q^{(k_0+1)(\mu-1)}} \left|S^c_{\ell,\pm1}\right| \\ &\le \frac{2e^{-\frac{\varphi_1}{\omega}}(1-q)}{c\omega q^{(k_0+1)(\mu-1)}} \sum_{\ell\ge k_0\mu-2} q^\ell.\end{aligned} \tag{4.12}$$

Here, note that

$$\sum_{\ell\ge k_0\mu-2} q^\ell = \sum_{\ell\ge\lceil k_0\mu-2\rceil} q^\ell = q^{\lceil k_0\mu-2\rceil}\frac{1}{1-q}, \tag{4.13}$$

where $\lceil \cdot \rceil$ denotes the ceiling function. Setting $\varepsilon(k_0) := \lceil k_0\mu - 2\rceil - (k_0\mu - 1) \in [-1, 0)$ and using (4.13) in (4.12), we obtain

$$\begin{aligned}\left|S^c_{k_0,+1} \cap f^{-1}(\Psi^c)\right| &\leq \frac{2e^{-\frac{\varphi_1}{\omega}}}{c\omega q^{(k_0+1)(\mu-1)}} q^{\lceil k_0\mu-2\rceil} = \frac{2q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{c\omega} \cdot \frac{q^{\lceil k_0\mu-2\rceil}}{q^{k_0\mu-1}} \cdot \frac{1}{q^{\mu}} \\ &= 2q^{\varepsilon(k_0)} \cdot \frac{q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{c\omega q^{\mu}} \leq \frac{2q^{-1}q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{c\omega q^{\mu}} = \frac{2q^{k_0}e^{-\frac{\varphi_1}{\omega}}(1-q)}{c\omega q^{\mu+1}(1-q)}\end{aligned}$$

which proves the case $n = 1$.

Assume the assertion is true for n, i.e, for all $k_0 \in \mathbb{N}$ we have

$$\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n} f^{-i}(\Psi^c)\right| \leq q^{k_0}e^{-\frac{\varphi_1}{\omega}}(1-q)\left(\frac{2}{c\omega q^{\mu+1}(1-q)}\right)^n, \tag{4.14}$$

then the same estimate is true for $S^c_{k_0,-1}$. Now we show that it is true for $n+1$. Using (4.10) for the third equality we obtain

$$\begin{aligned}\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c)\right| &= \left|S^c_{k_0,+1} \cap f^{-1}(\Psi^c) \cap \cdots \cap f^{-n-1}(\Psi^c)\right| \\ &= \left|S^c_{k_0,+1} \cap f^{-1}\left(\bigcap_{i=0}^{n} f^{-i}(\Psi^c)\right)\right| \\ &= \left|S^c_{k_0,+1} \cap f^{-1}\left(\left(\bigcup_{\ell \geq k_0\mu-2} S^c_{\ell,\pm1}\right) \cap \bigcap_{i=0}^{n} f^{-i}(\Psi^c)\right)\right|.\end{aligned}$$

Note that $S^c_{\ell,\pm1} \subset \Psi^c$ implies

$$S^c_{\ell,\pm1} \cap \bigcap_{i=0}^{n} f^{-i}(\Psi^c) = S^c_{\ell,\pm1} \cap \bigcap_{i=1}^{n} f^{-i}(\Psi^c).$$

So, we obtain

$$\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c)\right| = \left|S^c_{k_0,+1} \cap f^{-1}\left(\bigcup_{\ell \geq k_0\mu-2}\left(S^c_{\ell,\pm1} \cap \bigcap_{i=1}^{n} f^{-i}(\Psi^c)\right)\right)\right| \tag{4.15}$$

Using (2.13), (4.1), (4.11), (4.13) and (4.14) for $S^c_{k_0,+1}$ and $S^c_{k_0,-1}$ in (4.15), we have

$$\begin{aligned}\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c)\right| &\leq \frac{1}{(c\omega)\, q^{(k_0+1)(\mu-1)}} \sum_{\ell \geq k_0\mu-2} 2\left(\frac{2}{c\omega q^{\mu+1}(1-q)}\right)^n q^{\ell}e^{-\frac{\varphi_1}{\omega}}(1-q) \\ &= \frac{2^{n+1}q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{(c\omega)^{n+1}(q^{\mu+1})^n q^{\mu}q^{k_0\mu-1}}\left(\frac{1}{1-q}\right)^{n-1} \sum_{\ell \geq k_0\mu-2} q^{\ell} \\ &= \frac{2^{n+1}q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{(c\omega)^{n+1}(q^{\mu+1})^n q^{\mu}}\left(\frac{1}{1-q}\right)^{n-1} \frac{q^{\lceil k_0\mu-2\rceil}}{q^{k_0\mu-1}} \frac{1}{1-q}\end{aligned}$$

With $\varepsilon(k_0)$ as above, we obtain

$$\begin{aligned}
\left| S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c) \right| &\leq \frac{2^{n+1} q^{k_0} e^{-\frac{\varphi_1}{\omega}} q^{\varepsilon(k_0)}}{(c\omega)^{n+1} (q^{\mu+1})^n q^{\mu}} \left(\frac{1}{1-q}\right)^n \\
&\leq \frac{2^{n+1} q^{k_0} e^{-\frac{\varphi_1}{\omega}} q^{-1}}{(c\omega)^{n+1} (q^{\mu+1})^n q^{\mu}} \left(\frac{1}{1-q}\right)^n \\
&= q^{k_0} e^{-\frac{\varphi_1}{\omega}} (1-q) \left(\frac{2}{c\omega q^{\mu+1}(1-q)}\right)^{n+1},
\end{aligned}$$

so the assertion is true for $n+1$ and hence, the proof of Lemma 4.5 is completed. $\square$

Remark 4.6. Let $c \in \left(\frac{2}{\pi}, 1\right)$ and $\mu > 1$. Then $\frac{1}{c} + \pi(\mu+1) \leq \frac{c\pi^2(2\mu+3)}{2(c\pi-2)}$.

Proof. Let $c \in \left(\frac{2}{\pi}, 1\right)$. Then

$$\frac{1}{c} + \pi(\mu+1) = \frac{1 + c\pi\mu + c\pi}{c} = \frac{\pi + c\pi^2\mu + c\pi^2}{c\pi} \leq \frac{2c\pi^2\mu + 2c\pi^2 + 2\pi}{2(c\pi-2)}.$$

Since $c\pi > 2$, we have $2\pi < c\pi^2$ and hence

$$\frac{1}{c} + \pi(\mu+1) \leq \frac{2c\pi^2\mu + 3c\pi^2}{2(c\pi-2)} = \frac{c\pi^2(2\mu+3)}{2(c\pi-2)}. \qquad \square$$

***Proof of Theorem* 4.2.** **(i)** From assertion (i) in Lemma 4.4 we see that there exists a point $y_0 \in (S^c_{k_0,s_0} \cap \Omega^c_\infty)$ with sign $f^j(y_0) = s_j$, because $f^j(y_0) \in S^c_{k_j,s_j}$.

(ii) Assume $y_0 \in (S^c_{k_0,s_0} \cap \Omega^c_\infty)$. Assertion (ii) of Lemma 4.4 shows that $\Sigma_j \ni y_0$ and $Z \cap \Sigma_j \neq \emptyset$. Further, assertion (v) of Lemma 4.4 shows that $|\Sigma_j| \to 0$ as $j \to \infty$. This means that there exists a sequence $(z_j) \subset Z$ with $z_j \to y_0$, and this completes the proof.

(iii) Let $c \in \left(\frac{2}{\pi}, 1\right)$ be given. Remark 4.6 shows that the condition $\omega > \frac{c\pi^2(2\mu+3)}{2(c\pi-2)}$ from assertion (iii) of Theorem 4.2 implies the condition $\omega > \frac{1}{c} + \pi(\mu+1)$ of assertion (ii). Hence, $(S^c_{k_0,\pm1} \cap \Omega^c_\infty) \subset \overline{Z}$ for all $k_0 \in \mathbb{N}$. It follows that $\Omega^c_\infty = \bigcup_{k_0 \in \mathbb{N}} (S^c_{k_0,\pm1} \cap \Omega^c_\infty) \subset \overline{Z}$, so $\Omega^c_\infty \subset \overline{Z}$. To prove that Ω^c_∞ has measure zero, we show $\lim_{n\to\infty} |\Omega^c_n \cap S^c_{k_0,\pm1}| = 0$ for every $k_0 \in \mathbb{N}$. For this purpose it is enough to show that under the conditions of assertion (iii) of Theorem 4.2, $c\omega q^{\mu+1}(1-q) > 2$ in (4.9). We use the second order Taylor expansion of e^{-y} around 0 for $y > 0$,

$$e^{-y} = 1 - y + \frac{y^2}{2} + R_3,$$

with $R_3 = \frac{\exp'''(\xi)}{3!}(-y)^3 < 0$ for some $\xi \in (-y, 0)$. Hence, since

$$q = e^{-\frac{\pi}{\omega}} = 1 - \frac{\pi}{\omega} + \frac{(\pi)^2}{2\omega^2} + R_3\left(\frac{\pi}{\omega}\right) < 1 - \frac{\pi}{\omega} + \frac{(\pi)^2}{2\omega^2},$$

we have

$$1 - q = 1 - e^{-\frac{\pi}{\omega}} = \frac{\pi}{\omega} - \frac{\pi^2}{2\omega^2} - R_3\left(\frac{\pi}{\omega}\right) > \frac{\pi}{\omega} - \frac{\pi^2}{2\omega^2}. \tag{4.16}$$

On the other hand, with appropriate ξ,

$$q^{\mu+1} = e^{-\frac{\pi}{\omega}(\mu+1)} = 1 - \frac{\pi(\mu+1)}{\omega} + \frac{\exp''(\xi)}{2!}\left(-\frac{\pi(\mu+1)}{\omega}\right)^2 > 1 - \frac{\pi(\mu+1)}{\omega}. \tag{4.17}$$

Using (4.16) and (4.17), we get

$$
\begin{aligned}
c\omega q^{\mu+1}(1-q) > c\omega\left(1-\frac{\pi(\mu+1)}{\omega}\right)\left(\frac{\pi}{\omega}-\frac{\pi^2}{2\omega^2}\right) &= c\pi\left(1-\frac{\pi\mu+\pi}{\omega}\right)\left(1-\frac{\pi}{2\omega}\right)\\
&= c\pi\left(1-\frac{(3\pi+2\pi\mu)}{2\omega}+\frac{\pi^2(\mu+1)}{2\omega^2}\right)\\
&> c\pi\left(1-\frac{\pi(3+2\mu)}{2\omega}\right). \qquad (4.18)
\end{aligned}
$$

In view of Remark 4.6, and using the assumption which is given in the assertion (iii) of Theorem 4.2 in (4.18), we finally obtain

$$
c\omega q^{\mu+1}(1-q) > c\pi\left(1-\frac{\pi(3+2\mu)}{2\cdot\frac{c\pi^2(2\mu+3)}{2(c\pi-2)}}\right) = c\pi\left(1-\frac{c\pi-2}{c\pi}\right) = 2. \qquad \square
$$

5 The behavior of the points whose orbits follow 'flat-steep-flat' intervals

In chapter three we analyzed the behavior of the points which are mapped from 'flat' intervals to some other 'flat' intervals, and in chapter four we studied the behavior of the points which are mapped from 'steep' intervals to some other 'steep' intervals. Finally in this chapter, as we briefly mentioned in the summary of this thesis, we first construct a specific type of orbit whose points travel from 'flat' intervals to 'steep' intervals, then from 'steep' intervals again to 'flat' intervals under the iteration (see Figure 5.1).

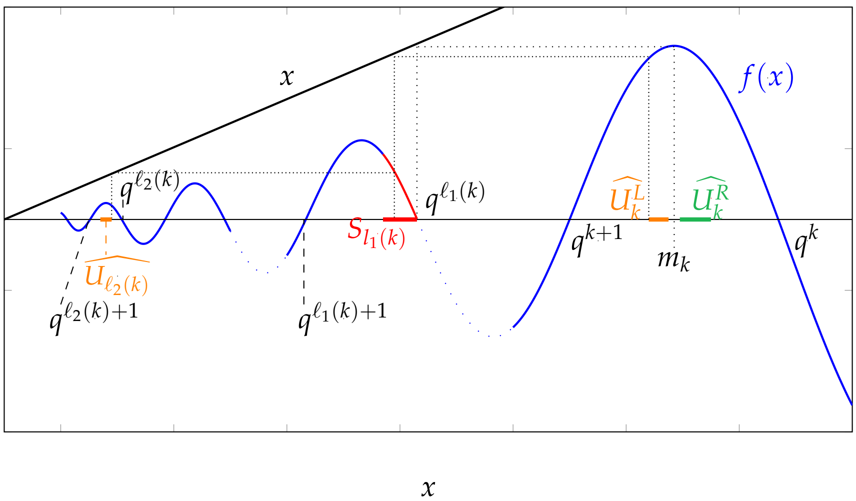

Figure 5.1: The parameters adjusted so that $f(m_k) = q^{\ell_1(k)}$, and the sets $\widehat{U_k^L}$ and $\widehat{U_k^R}$ constructed as counterimages under f^2 of the interval $\widehat{U_{\ell_2(k)}} \subset U_{\ell_2(k)}$ (indicated only for the lower endpoint of $\widehat{U_k^L}$). The dotted parts of the graph indicate possible maxima/minima in between which are not shown.

Besides, to avoid repeating the same expression, we shall use $g_{\omega,\mu+1-j}$ as in Lemma 2.1 and $c \in (0,1)$ for the rest of the paper. For a specific choice of μ, $\omega > 0$, maxima m_k get mapped to zeros $q^{\ell_1(k)}$ of $f_{\mu,\omega} = f$. We shall first introduce 'flat' intervals of the form $U_k = [m_k - \delta_k, m_k + \delta_k]$ for odd k and use the notations $U_k^R = [m_k, m_k + \delta_k]$ and $U_k^L = [m_k - \delta_k, m_k]$ for the right and left part of U_k respectively. We also introduce 'steep' intervals $S_{\ell_1(k)}$, where $\ell_1(k) = k\mu + 1$, of the form $\left[q^{\ell_1(k)} - r_{\ell_1(k)}, q^{\ell_1(k)}\right]$, with a suitable $r_{\ell_1(k)}$. Then we define $U = \bigcup_{\substack{k\in\mathbb{N}\\ k \text{ odd}}} U_k$, $S = \bigcup_{\substack{k\in\mathbb{N}\\ k \text{ odd}}} S_{\ell_1(k)}$, and we construct orbits $\left(f^j(x)\right)_{j\in\mathbb{N}}$, with the properties

$$f^j(x) \in \begin{cases} U, & j \text{ is even,} \\ S, & j \text{ is odd.} \end{cases}$$

Furthermore, for $k, \mu \in \mathbb{N}$, $\omega > 0$ and with φ_1 as in Lemma 2.1, we define

$$\ell_2(k) := \min\left\{ \ell \in \mathbb{N} : q^\ell \le q^{\ell_1(k)\mu} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{c(1-c)\,\omega^3}{4g_{\omega,\mu} \cdot g^2_{\omega,\mu-1}} \right\}.$$

We denote by $\ell_2^j(k)$ the j th iterate of the function ℓ_2 applied to k. Then, given a symbol sequence of the form $\{L, R\}^{n+1}$, where symbols represent the left 'L' or right 'R' hand part of U_k (that is U_k^L, U_k^R), we construct corresponding orbits of f. Given a finite sequence

$$\mathbf{s} = (s_0, s_1, s_2, \ldots, s_n) \in \{L, R\}^{n+1}$$

and $k \in \mathbb{N}$, we first construct the subset of points x in U_k which follow this symbol sequence in the sense that $f^{2j}(x) \in U^L_{\ell_2^j(k)}$ or $f^{2j}(x) \in U^R_{\ell_2^j(k)}$, $j = 0, 1, 2, \ldots, n$ depending on whether $s_j = L$ or $s_j = R$. Hence, we construct the set $I^n_{k,\mathbf{s}} = \bigcap_{j=0}^n f^{-2j}\left(U^{s_j}_{\ell_2^j(k)}\right)$ and the set $\Gamma_k^n = \bigcup_{s\in\{L,R\}^{n+1}} I^n_{k,s}$ which is the set of points following symbol sequences in the set $\{L, R\}^{\{0,1,2,\ldots,n\}}$. Note that strict monotonicity of f^2 on each interval U_k^L and U_k^R implies that the sets $I^n_{k,\mathbf{s}}$ are closed intervals. The corresponding set for infinite symbol sequences is $I^\infty_{k,s} = \bigcap_{j=0}^\infty f^{-2j}\left(U^{s_j}_{\ell_2^j(k)}\right)$. Finally, we analyze the Lebesgue measure of the set Γ_k^n, and consider the limit as $n \to \infty$.

Note that the 'steep' intervals S_k that we use in our calculations in this chapter are some subintervals of $\left(m_k, q^k\right]$, whereas the 'steep' intervals which were used in the fourth chapter are some subintervals of (m_{k+1}, m_k). In the theorem below we restrict ourselves to $\mu \in \mathbb{N}$ for simplicity.

Theorem 5.1. *Let k be a positive odd integer. Let $c \in (0,1)$, and $\mu \in \mathbb{N}$, $\mu \ge \max\left\{\left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right), 15\right\}$ be given. Then there exist an $\omega > 0$ (depending on μ) such that $f = f_{\mu,\omega}$ has the following properties:*

(i) Let a sequence of the form $\mathbf{s} \in \{L, R\}^{\mathbb{N}_0}$ be given. Then, there exists exactly one point $x_{k,s} \in U_k$ with the property:

For all $n \in \mathbb{N}_0$, $f^{2n}(x_{k,s}) \in U_{\ell_2^n(k)}$, and $f^{2n}(x_{k,s})$ is to the left of $m_{\ell_2^n(k)}$ or to the right of $m_{\ell_2^n(k)}$, depending on whether $s_n = L$ or $s_n = R$. That is, $I^\infty_{k,s} = \{x_{k,s}\}$.

(ii) The measure of Γ_k^n as defined above goes to zero, as $n \to \infty$.

The proof requires several lemmas and propositions. The proof of the following lemma is analogous to the proof of Lemma 3.3, but is included for completeness.

Lemma 5.2. *Define φ_1 as in Lemma 2.1. Then the following statements are true.*

(i) Assume $\mu \in \mathbb{N}$, $\mu \geq 15$ and define $\ell_1(k)$ as in the passage before Theorem 5.1. Then there exists an $\omega \in (0,1)$ such that for all $k \in \mathbb{N}$, f has the property

$$|f(m_k)| = q^{\ell_1(k)}, \tag{5.1}$$

which is equivalent to

$$\exp\left(\frac{\pi - \mu \arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}\,. \tag{5.2}$$

(ii) For any choice of ω as in assertion (i), we have $\omega \to 0$ as $\mu \to \infty$.

Proof. **(i)** Let $k \in \mathbb{N}$ be given. With m_k from (2.10), we have from (2.11)

$$|f(m_k)| = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \cdot \sin(\varphi_1).$$

Using (2.3) we obtain

$$|f(m_k)| = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \cdot \frac{1}{\sqrt{1 + \frac{\mu^2}{\omega^2}}}. \tag{5.3}$$

On the other hand, from (2.8) we have

$$q^{\ell_1(k)} = \exp\left(-\frac{\pi\ell_1(k)}{\omega}\right). \tag{5.4}$$

With (5.3) and (5.4) together, we see that (5.1) is equivalent to

$$\exp\left(-\frac{\pi\ell_1(k)}{\omega}\right) = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \frac{1}{\sqrt{1 + \frac{\mu^2}{\omega^2}}}$$

and hence to

$$\exp\left(\frac{\pi(\ell_1(k) - k\mu) - \varphi_1\mu}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}. \tag{5.5}$$

So, if we substitute $\ell_1(k) = k\mu + 1$ and the value of φ_1 given by Lemma 2.1 in (5.5), we finally get that (5.1) is equivalent to

$$\exp\left(\frac{\pi - \mu \arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}},$$

which proves the equivalence of (5.1) and (5.2). Now, we want to find ω and μ such that $|f(m_k)| = q^{\ell_1(k)}$. Define

$$F(\omega, \mu) = \exp\left(\frac{\pi - \mu \arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) - \sqrt{1 + \frac{\mu^2}{\omega^2}}$$

and try to find $F(\omega,\mu)=0$ at least for a special pair of (ω,μ) (see Figure 5.2). On the one hand, for a fixed $\mu>2$, $\arctan\left(\frac{\omega}{\mu}\right)\to 0$ as $\omega\to 0$. Hence, due to the exponential growth, $F(\omega,\mu)\to\infty$ as $\omega\to 0$. On the other hand, for $\omega=1$ we have

$$F(1,\mu)=\exp\left(\pi-\mu\arctan\left(\frac{1}{\mu}\right)\right)-\sqrt{1+\mu^2}. \tag{5.6}$$

From 3.9 we have $\mu\arctan(\frac{1}{\mu})\geq\frac{1}{2}$ for $\mu>2$, and using this estimate in (5.6), we finally have $F(1,\mu)<e^{\pi-\frac{1}{2}}-\sqrt{1+\mu^2}$. From the fact that $e^{\pi-\frac{1}{2}}<15$, we finally have $F(1,\mu)<0$, if we choose $\mu\geq 15$. With the intermediate value theorem, it is clear that there exists at least one $\omega\in(0,1)$ which satisfies $F(\omega,\mu)=0$ for fixed μ. This gives the proof of assertion (i).

(ii) The proof is analogous to the proof of the assertion (ii) of Lemma 3.3. □

In order to find a numerical solution, one can use the bisection method, and we found numerically that there is no solution for $\mu<2.3$. The numerical investigation suggests that ω in Lemma 5.2 is unique. We made no effort to prove that, because part (ii) is true for any possible choice of ω.

The next three propositions (5.3–5.5) give some preparatory calculations.

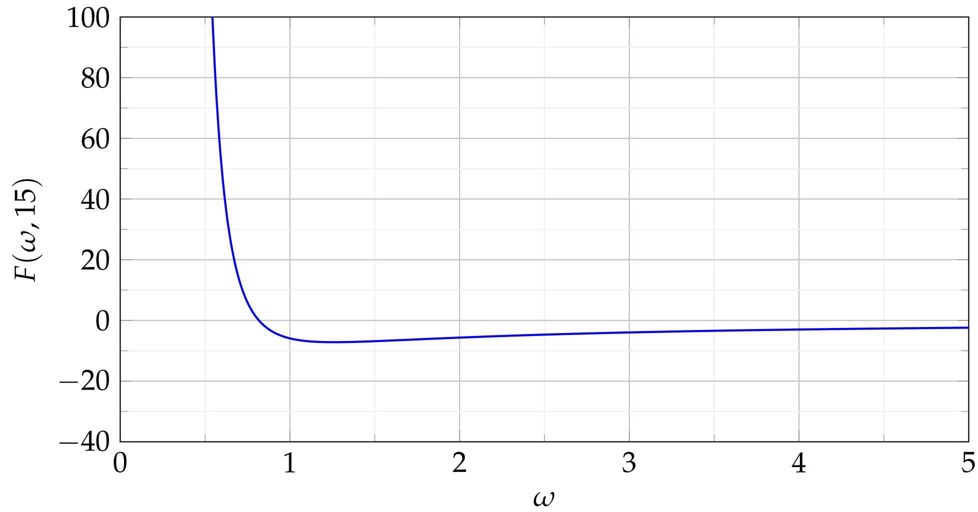

Figure 5.2: Graph of $F(\cdot,\mu)$ for $\mu=15$.

Proposition 5.3. *Let φ_1 be as in Lemma 2.1. Set $\alpha(\omega,\mu,c):=\frac{\exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right)}{g_{\omega,\mu-1}}\sqrt{\frac{1-c}{2c\omega}}$. If $\mu\in\mathbb{N}$,*

$$\mu\geq\max\left\{\left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right),15\right\},$$

and ω is a corresponding value obtained as in Lemma 5.2, then we have $\alpha(\omega,\mu,c)<\frac{1}{2}$.

Proof. Let μ and $\omega\in(0,1)$ be as in the assumption. Then, it is clear that $\frac{3\mu^2}{\omega^2}\geq 1$, and in view of (5.2) we have

$$\frac{2\mu}{\omega}=\sqrt{\frac{3\mu^2}{\omega^2}+\frac{\mu^2}{\omega^2}}\geq\sqrt{1+\frac{\mu^2}{\omega^2}}=\exp\left(\frac{\pi-\mu\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right).$$

Since $\arctan\left(\frac{\omega}{\mu}\right) \leq \frac{\omega}{\mu}$, we get

$$\frac{2\mu}{\omega} \geq \exp\left(\frac{\pi - \mu \cdot \frac{\omega}{\mu}}{\omega}\right) = \exp\left(\frac{\pi}{\omega} - 1\right),$$

and hence we have $2\mu e \geq \omega e^{\frac{\pi}{\omega}}$. Using the second order Taylor expansion of $e^{\frac{\pi}{\omega}}$ in the last inequality, we obtain

$$2\mu e \geq \omega\left(1 + \frac{\pi}{\omega} + \frac{1}{2}\frac{\pi^2}{\omega^2}\right) \geq \frac{1}{2}\frac{\pi^2}{\omega}$$

or

$$4e\mu \geq \frac{\pi^2}{\omega}. \tag{5.7}$$

On the other hand, we know that $\mu \geq \left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right)$, so $\sqrt{\mu} \geq \frac{30e}{7\pi}\sqrt{\frac{1-c}{2c}}$ which implies

$$\frac{1}{2} \geq e^{\frac{1}{2}}\sqrt{\frac{1-c}{2c}}\frac{30e^{\frac{1}{2}}}{14\pi\sqrt{\mu}}.$$

Since $e^{\frac{1}{2}} > e^{\frac{1}{2}-\frac{1}{\mu}}$, it follows that

$$\frac{1}{2} > \exp\left(\frac{1}{2} - \frac{1}{\mu}\right) \cdot \sqrt{\frac{1-c}{2c}} \cdot \frac{15\sqrt{4e}}{14\pi\sqrt{\mu}} = \exp\left(\frac{\omega}{2\mu\omega}(\mu-2)\right) \cdot \sqrt{\frac{1-c}{2c}} \cdot \frac{15\sqrt{4e\mu}}{14\pi\mu}.$$

On the other hand, since $\mu \geq 15$, we have $\frac{1}{\mu-1} \leq \frac{15}{14\mu}$, and with the fact that $\arctan\left(\frac{\omega}{\mu}\right) \leq \frac{\omega}{\mu}$ we get

$$\frac{1}{2} > \exp\left(\frac{(\mu-2)\arctan\left(\frac{\omega}{\mu}\right)}{2\omega}\right) \cdot \sqrt{\frac{1-c}{2c}} \cdot \frac{\sqrt{4e\mu}}{\pi(\mu-1)}.$$

Finally, using (5.7) and the definition of φ_1, we obtain

$$\begin{aligned}\frac{1}{2} &> \exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right) \cdot \sqrt{\frac{1-c}{2c\omega}} \cdot \sqrt{\frac{1}{(\mu-1)^2}} \\ &\geq \exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right) \cdot \sqrt{\frac{1-c}{2c\omega}} \cdot \sqrt{\frac{1}{\omega^2 + (\mu-1)^2}} \\ &= \frac{\exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right)}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}} = \alpha(\omega,\mu,c)\end{aligned}$$

and this completes the proof. □

Proposition 5.4. *Let φ_1 be as in Lemma 2.1 and $c \in (0,1)$ be given. Set $\widetilde{\eta}_1(\omega,\mu) := \frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2g_{\omega,\mu-1}\cdot g_{\omega,\mu-2}}$, $\widetilde{\eta}_2(\omega) := \frac{e^{-\frac{\varphi_1}{\omega}}-q}{2}$, $\widetilde{\eta}_3(\omega) := \frac{1-e^{-\frac{\varphi_1}{\omega}}}{2}$ and $\widetilde{\eta}_4(\omega,\mu) := \frac{\sqrt{(1-c)q\omega}}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}$. There exist $\omega_0 > 0$ and $\mu_0 > 3$ such that for $\omega \leq \omega_0$ and $\mu \in \mathbb{N}$, $\mu \geq \mu_0$, the number*

$$\widetilde{\eta} := \min\left\{\widetilde{\eta}_1(\omega,\mu),\ \widetilde{\eta}_2(\omega),\ \widetilde{\eta}_3(\omega),\ \widetilde{\eta}_4(\omega,\mu)\right\}$$

satisfies

$$\widetilde{\eta} = \widetilde{\eta}_4(\omega,\mu). \tag{5.8}$$

Proof. We prove that $\widetilde{\eta}_4(\omega,\mu) \le \widetilde{\eta}_1(\omega,\mu) \le \min\{\widetilde{\eta}_2(\omega),\widetilde{\eta}_3(\omega)\}$ for μ large enough, ω small enough. For $\omega > 0$, we have $g_{\omega,\mu-1} \ge \sqrt{(\mu-1)^2}$, $g_{\omega,\mu-2} \ge \sqrt{(\mu-2)^2}$ and using these simplifications, we obtain

$$\widetilde{\eta}_1(\omega,\mu) \le \frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2\sqrt{(\mu-1)^2(\mu-2)^2}} = \frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2(\mu-1)(\mu-2)}. \tag{5.9}$$

We have already defined

$$\widetilde{\eta}_2(\omega) = \frac{e^{-\frac{\varphi_1}{\omega}} - q}{2} = \frac{1}{2}e^{-\frac{\varphi_1}{\omega}}\left(1 - e^{\frac{\varphi_1-\pi}{\omega}}\right),$$

and there exists $\widetilde{\omega_1} > 0$ such that for $\omega \in \omega_0 \in (0,\widetilde{\omega_1}]$, the property $\varphi_1 - \pi < 0$ implies $\left(1 - e^{\frac{\varphi_1-\pi}{\omega}}\right) > \frac{1}{2}$. Hence, for such ω we have

$$\widetilde{\eta}_2(\omega) \ge \frac{1}{4}e^{-\frac{\varphi_1}{\omega}}. \tag{5.10}$$

From (5.9) and (5.10) it is obvious that $\widetilde{\eta}_1(\omega,\mu) \le \frac{\omega e^{-\frac{\varphi_1}{\omega}}}{2} \le \widetilde{\eta}_2(\omega)$ for $\mu \ge 3$ and $\omega \le \omega_{12}$, where $\omega_{12} =: \min\left\{\frac{1}{2},\widetilde{\omega_1}\right\}$. Analogously there exists $\omega_{13} > 0$ such that for $\mu \ge 3$ and $\omega \le \omega_{13}$, one has $\widetilde{\eta}_1(\omega,\mu) \le \widetilde{\eta}_3(\omega)$; observe $\widetilde{\eta}_3(\omega) \to \frac{1}{2}$ as $\omega \to 0$. There exist $c_1, c_2 > 0$, and $\widetilde{\mu_0} > 0$ such that for $\mu \ge \widetilde{\mu_0}$ we have $\widetilde{\eta}_1(\omega,\mu) \ge c_1\frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{\mu^2}$ and $\widetilde{\eta}_4(\omega,\mu) \le c_2\frac{\sqrt{\omega e^{-\frac{\pi}{\omega}}}}{\mu^2}$. Hence, we have for $\mu \ge \widetilde{\mu_0}$ and $\omega > 0$

$$\frac{\widetilde{\eta}_4(\omega,\mu)}{\widetilde{\eta}_1(\omega,\mu)} \le \frac{c_2\sqrt{\omega e^{-\frac{\pi}{\omega}}}}{c_1\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}} = \frac{c_2}{c_1}\frac{1}{\sqrt{\omega}}\exp\left(\frac{\varphi_1(\mu-2) - \frac{\pi}{2}}{\omega}\right).$$

Substituting the explicit form of φ_1 as in Lemma 2.1, the last equality turns to

$$\frac{\widetilde{\eta}_4(\omega,\mu)}{\widetilde{\eta}_1(\omega,\mu)} \le \frac{c_2}{c_1}\frac{1}{\sqrt{\omega}}\exp\left(\frac{(\mu-2)\arctan\left(\frac{\omega}{\mu}\right) - \frac{\pi}{2}}{\omega}\right). \tag{5.11}$$

Using the fact that

$$\lim_{\mu\to\infty,\ \omega\to 0}\left(\frac{(\mu-2)\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) = \lim_{\mu\to\infty,\ \omega\to 0}\left(\frac{(\mu-2)\frac{\omega}{\mu}}{\omega}\right) = 1, \text{ and}$$

$\lim_{\omega\to 0}\frac{1}{\sqrt{\omega}}\exp\left(\frac{-\pi}{2\omega}\right) = 0$ in (5.11), we finally have

$$\lim_{\mu\to\infty,\ \omega\to 0}\left(\frac{\widetilde{\eta}_4(\omega,\mu)}{\widetilde{\eta}_1(\omega,\mu)}\right) \le \lim_{\mu\to\infty,\ \omega\to 0}\frac{c_2}{c_1}\frac{1}{\sqrt{\omega}}\exp\left(1 - \frac{\pi}{2\omega}\right) = 0$$

and that shows that there exists $\mu_0 \ge \widetilde{\mu_0}$ and $\omega_0 \in (0,\min\{\omega_{12},\omega_{13}\}]$ such that for $\omega \le \omega_0$ and $\mu \in \mathbb{N}$, $\mu \ge \mu_0$ one has $\widetilde{\eta}_4(\omega,\mu) \le \widetilde{\eta}_1(\omega,\mu) \le \min\{\widetilde{\eta}_2(\omega),\widetilde{\eta}_3(\omega)\}$. □

Now, we aim at finding an interval $U_k := [m_k - \delta_k, m_k + \delta_k]$ as indicated in the passage before Theorem 5.1, which gets mapped to a 'steep' interval $S_{\ell_1(k)}$, but we first provide upper and lower estimates for the second derivative $f''_{\mu,\omega}$ of f.

Proposition 5.5. *Let $k \in \mathbb{N}$. Assume that with μ_0 and ω_0 as in Proposition 5.4, one has $\mu \geq \mu_0$ and $\omega \leq \omega_0$. Define $\widetilde{\eta}$ as in Proposition 5.4 and set $\delta_k := \widetilde{\eta}\delta_k$, $J_k := \left[q^{k+1}, q^k\right]$. Then*

$$U_k := [m_k - \delta_k, m_k + \delta_k] \subset \left[q^{k+1}, q^k\right] = J_k$$

and the following estimates hold:

$$\forall x \in [m_k - \delta_k, m_k + \delta_k] : g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{k(\mu-2)} \geq \left|f''(x)\right| \geq \frac{\left(q^k e^{-\frac{\varphi}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu}}{2}. \tag{5.12}$$

Proof. Let $k \in \mathbb{N}$. With $\overline{\eta}$ from Lemma 3.2, the definition of $\widetilde{\eta}$ given in Proposition 5.4 shows $\widetilde{\eta} \leq \min\{\widetilde{\eta}_2, \widetilde{\eta}_3\} = \overline{\eta}$. Hence, in view of Lemma 3.2, we see that

$$U_k = [m_k - \delta_k, m_k + \delta_k] \subset \left[m_k - \overline{\eta}q^k, m_k + \overline{\eta}q^k\right] \subset \left[q^{k+1}, q^k\right] = J_k.$$

Further, inserting m_k from (2.10) in (2.4) we have

$$\begin{aligned}\left|f''(m_k)\right| &= g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot |m_k|^{\mu-2} \left|\sin\left(\left(\omega \ln(m_k) + \varphi_1\right) + \varphi_2\right)\right| \\ &= g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot |m_k|^{\mu-2} \left|\sin\left(-k\pi + \varphi_2\right)\right|.\end{aligned}$$

Using $\varphi_2 = \arctan\left(\frac{\omega}{\mu-1}\right)$ from Lemma 2.1, we have $\sin(\varphi_2) = \frac{\omega}{g_{\omega,\mu-1}}$ and, inserting this value in the last equality, we have

$$\left|f''(m_k)\right| = g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot |m_k|^{\mu-2} \frac{\omega}{g_{\omega,\mu-1}} = |m_k|^{\mu-2} \omega \cdot g_{\omega,\mu}. \tag{5.13}$$

From (2.5) we have on $\left[q^{k+1}, q^k\right]$

$$\begin{aligned}\left|f'''(x)\right| &= \left|g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot x^{\mu-3} \sin\left(\omega \ln(x) + (\varphi_1 + \varphi_2 + \varphi_3)\right)\right| \\ &\leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot q^{k(\mu-3)}.\end{aligned} \tag{5.14}$$

From (2.13) for $x \in [m_k - \delta_k, m_k + \delta_k]$ and with the definition of δ_k, we also have

$$\begin{aligned}\left|f''(x)\right| &\geq \left|f''(m_k)\right| - \delta_k \cdot \max_{[m_k-\delta_k, m_k+\delta_k]} \left|f'''\right| \\ &= \left|f''(m_k)\right| - \widetilde{\eta}q^k \cdot \max_{[m_k-\delta_k, m_k+\delta_k]} \left|f'''\right| \\ &\geq \left|f''(m_k)\right| - \widetilde{\eta}_1 q^k \cdot \max_{[m_k-\delta_k, m_k+\delta_k]} \left|f'''\right|.\end{aligned} \tag{5.15}$$

With the definition of $\widetilde{\eta}_1$, using (2.10), (5.13) and (5.14) in (5.15), we finally have

$$\begin{aligned}\left|f''(x)\right| &\geq m_k^{\mu-2} \omega \cdot g_{\omega,\mu} - \frac{q^k \omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2 g_{\omega,\mu-1} \cdot g_{\omega,\mu-2}} g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot q^{k(\mu-3)} \\ &= \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu} - \frac{q^{k(\mu-2)} \omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2} g_{\omega,\mu} \\ &= \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu} - \frac{\left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu}}{2} \\ &= \frac{\left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu}}{2}.\end{aligned}$$

This is the lower estimate for $|f''(x)|$; the upper estimate even on the interval $\left[q^{k+1}, q^k\right]$ follows with the formula for f'' in (2.4). $\square$

For $k \in \mathbb{N}$, we specify the boundaries of an associated 'steep' interval $S_{\ell_1(k)}$ with the next proposition.

Proposition 5.6. *Let $k \in \mathbb{N}$. Assume μ and ω are as in Proposition 5.4 and define $\ell_1(k) = k\mu + 1$ as in the passage before the Theorem 5.1. Set $r_{\ell_1(k)} := \frac{(1-c)\omega}{g_{\omega,\mu} \cdot g_{\omega,\mu-1}} q^{\ell_1(k)}$ and $S_{\ell_1(k)} := \left[q^{\ell_1(k)} - r_{\ell_1(k)}, q^{\ell_1(k)}\right]$. Then, $S_{\ell_1(k)} \subset \left(m_{\ell_1(k)}, q^{\ell_1(k)}\right]$ and on $S_{\ell_1(k)}$ we have*

$$\left|f'\right| \geq c\omega q^{\ell_1(k)(\mu-1)}. \tag{5.16}$$

Proof. Let $k \in \mathbb{N}$. From the upper estimate of (5.12), on $S_{\ell_1(k)}$ we have

$$\|f''\|_{\infty, S_{\ell_1(k)}} \leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\ell_1(k)(\mu-2)}. \tag{5.17}$$

From (2.13) we have

$$\forall x \in S_{\ell_1(k)} : \left|f'(x)\right| \geq \left|f'\left(q^{\ell_1(k)}\right)\right| - \|f''\|_{\infty, S_{\ell_1(k)}} \cdot r_{\ell_1(k)}, \tag{5.18}$$

and from (2.9) we also have $\left|f'\left(q^{\ell_1(k)}\right)\right| = \omega q^{\ell_1(k)(\mu-1)}$. Using (5.17) and substituting the explicit values of both $\left|f'\left(q^{\ell_1(k)}\right)\right|$ and $r_{\ell_1(k)}$ in (5.18), we get

$$\begin{aligned}
\forall x \in S_{\ell_1(k)} : \left|f'(x)\right| &\geq \left|f'\left(q^{\ell_1(k)}\right)\right| - g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\ell_1(k)(\mu-2)} \cdot r_{\ell_1(k)} \\
&= \omega q^{\ell_1(k)(\mu-1)} - g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\ell_1(k)(\mu-2)} \cdot \frac{(1-c)\,\omega}{g_{\omega,\mu-1} \cdot g_{\omega,\mu}} q^{\ell_1(k)} \\
&= \omega q^{\ell_1(k)(\mu-1)} - (1-c)\,\omega q^{\ell_1(k)(\mu-1)} = c\omega q^{\ell_1(k)(\mu-1)}.
\end{aligned}$$

It follows now from $f'(m_{\ell_1(k)}) = 0$ that $m_{\ell_1(k)} < q^{\ell_1(k)} - r_{\ell_1(k)}$. □

From the graph of the map one can understand that the image of $S_{\ell_1(k)}$ under $f_{\mu,\omega}$ includes many 'steep' and 'flat' intervals, but we continue our calculations with a subinterval $\widetilde{S_{\ell_1(k)}}$ of $S_{\ell_1(k)}$ which is contained in $f(U_k)$. The next lemma gives an estimate for the size of $f(U_k)$ with a relation between $\widetilde{S_{\ell_1(k)}}$ and $S_{\ell_1(k)}$.

Note that for the sake of simplicity we shall use k as a positive odd integer number for the rest of the paper. Note also that in addition to the notations U_k^L, U_k^R which represent to the left 'L' and right 'R' hand part of U_k respectively, we also use the notation $U_k^{L\backslash R}$ in statements which are valid for both U_k^L and U_k^R.

Lemma 5.7. *Let k be a positive odd integer number. Let ω and even integer μ be as in Proposition 5.4 and satisfying (5.2). Define $\widetilde{\eta}$ as in Proposition 5.4, δ_k and U_k as in Proposition 5.5, and $U_k^{L\backslash R}$ as in the passage before Theorem 5.1. Then the following statements are true.*

(i) Define $r_{\ell_1(k)}$ and $S_{\ell_1(k)}$ as in Proposition 5.6. Then we have $f(U_k) \subset S_{\ell_1(k)}$;

(ii) Set

$$\widetilde{r_{\ell_1(k)}} := q^{\ell_1(k)} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{(1-c)\cdot\omega^2}{4 g_{\omega,\mu} \cdot g_{\omega,\mu-1}^2} \tag{5.19}$$

and $\widetilde{S_{\ell_1(k)}} = \left[q^{\ell_1(k)} - \widetilde{r_{\ell_1(k)}}, q^{\ell_1(k)}\right]$. Then we have $f\left(U_k^{L\backslash R}\right) \supset \widetilde{S_{\ell_1(k)}}$.

Proof. Note that due to (5.1), and since k is odd (see (2.11)), $\max\{f(U_k)\} = f(m_k) = q^{\ell_1(k)}$. For the interval U_k we have

$$\begin{aligned} \min & \left\{\left|f(m_k-\delta_k) - q^{\ell_1(k)}\right|, \left|f(m_k+\delta_k) - q^{\ell_1(k)}\right|\right\} \\ & \le |f(U_k)| \\ & \le \max\left\{\left|f(m_k-\delta_k) - q^{\ell_1(k)}\right|, \left|f(m_k+\delta_k) - q^{\ell_1(k)}\right|\right\}. \end{aligned}$$

It follows from second order Taylor expansion of f around the extremum m_k and from (5.1) that

$$\min_{\xi\in[m_k-\delta_k, m_k+\delta_k]} \left|f''(\xi)\right| \frac{\delta_k^2}{2} \le |f(U_k)| \le \max_{\xi\in[m_k-\delta_k, m_k+\delta_k]} \left|f''(\xi)\right| \frac{\delta_k^2}{2}. \tag{5.20}$$

Consequently, using (5.8) and inserting the upper estimate of $|f''|$ given in (5.12) and the value of δ_k in the upper estimate of (5.20), we finally get

$$\begin{aligned} |f(U_k)| & \le \max_{\xi\in[m_k-\delta_k, m_k+\delta_k]} \left|f''(\xi)\right| \frac{\delta_k^2}{2} \le g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\frac{\delta_k^2}{2} \\ & = g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\widetilde{\eta}^2 q^{2k} \\ & \le g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\left(\widetilde{\eta}_4\right)^2 q^{2k} \\ & \le g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\left(\frac{\sqrt{(1-c)\,q\omega}}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}\right)^2 q^{2k} \\ & = q^{k\mu+1}\frac{(1-c)\,\omega}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}} = \frac{(1-c)\,\omega}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}} q^{\ell_1(k)} = r_{\ell_1(k)} = \left|S_{\ell_1(k)}\right|. \end{aligned}$$

From (5.1) we know that $f(m_k) = q^{\ell_1(k)}$. So, $f(U_k) = \left[\min f(U_k), q^{\ell_1(k)}\right]$ and the estimate

$$|f(U_k)| \le r_{\ell_1(k)} \text{ shows } f(U_k) \subset \left[q^{\ell_1(k)} - r_{\ell_1(k)}, q^{\ell_1(k)}\right] = S_{\ell_1(k)}$$

and this completes the proof of assertion (i).

Note that, although there is no symmetry between the graph of $f_{\mu,\omega}$ to the left and right hand side of U_k, we can estimate the size of $f\left(U_k^L\right)$ and $f\left(U_k^R\right)$ in a similar way. Substituting the lower bound of $|f''|$ given by (5.12), the value of δ_k in the analogue of the lower estimate of (5.20) for $U_k^{L\backslash R}$, and using (5.8) we obtain

$$\begin{aligned} \left|f\left(U_k^{L\backslash R}\right)\right| & \ge \min_{\xi\in U_k}\left|f''(\xi)\right|\frac{\delta_k^2}{2} \ge \frac{\left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2}\cdot\omega\cdot g_{\omega,\mu}}{2}\frac{\delta_k^2}{2} \\ & = \frac{\left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2}\cdot\omega\cdot g_{\omega,\mu}}{4}\widetilde{\eta}^2 q^{2k} = \frac{q^{k\mu-2k}\cdot e^{-\frac{\varphi_1(\mu-2)}{\omega}}\omega\cdot g_{\omega,\mu}}{4}\left(\widetilde{\eta}_4\right)^2 q^{2k} \\ & = \frac{q^{k\mu-2k}\cdot e^{-\frac{\varphi_1(\mu-2)}{\omega}}\omega\cdot g_{\omega,\mu}}{4}\left(\frac{\sqrt{(1-c)\,q\omega}}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}\right)^2 q^{2k} \\ & = \frac{q^{k\mu+1}\cdot e^{-\frac{\varphi_1(\mu-2)}{\omega}}\cdot\omega^2}{4}\frac{(1-c)}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}^2} \\ & = q^{\ell_1(k)}\cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}\cdot\frac{(1-c)\cdot\omega^2}{4g_{\omega,\mu}\cdot g_{\omega,\mu-1}^2} = \widetilde{r_{\ell_1(k)}} = \left|\widetilde{S_{\ell_1(k)}}\right|, \end{aligned}$$

and this completes the proof of assertion (ii) and the proof of the lemma. □

We continue analyzing the next 'flat' interval obtained by the second iteration of f.

Lemma 5.8. *Let k be a positive odd integer number. Let ω, μ be as in Lemma 5.7. Define $\ell_1(k)$ and $\ell_2(k)$ as in the passage before Theorem 5.1. Then for $\widetilde{S_{\ell_1(k)}}$ as in Lemma 5.7, we have*

$$f\left(\widetilde{S_{\ell_1(k)}}\right) \supset \left[0, q^{\ell_2(k)}\right].$$

Proof. Using (2.13) on $\widetilde{S_{\ell_1(k)}}$, we obtain

$$\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| \geq \widetilde{r_{\ell_1(k)}} \cdot \min_{x \in \widetilde{S_{\ell_1(k)}}} \left|f'(x)\right|. \tag{5.21}$$

Using (5.16) and (5.19) in (5.21), and also the definition of $\ell_2(k)$ at the beginning of this section, we have

$$\begin{aligned}
\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| &\geq \widetilde{r_{\ell_1(k)}} \cdot \min_{x \in \widetilde{S_{\ell_1(k)}}} \left|f'(x)\right| \\
&= q^{\ell_1(k)} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{(1-c)\cdot\omega^2}{4g_{\omega,\mu}\cdot g^2_{\omega,\mu-1}} \cdot c\omega q^{\ell_1(k)(\mu-1)} \\
&= q^{\ell_1(k)\mu} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{c(1-c)\,\omega^3}{4g_{\omega,\mu}\cdot g^2_{\omega,\mu-1}} \geq q^{\ell_2(k)}.
\end{aligned}$$

Note also that $\ell_1(k) = k\mu + 1$ is odd, since μ is even. Hence, $f \geq 0$ on $\widetilde{S_{\ell_1(k)}}$ and since $f\left(q^{\ell_1(k)}\right) = 0$, $f\left(\widetilde{S_{\ell_1(k)}}\right) = \left[0, \max f\left(\widetilde{S_{\ell_1(k)}}\right)\right]$. The estimate $\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| \geq q^{\ell_2(k)}$ implies that $f\left(\widetilde{S_{\ell_1(k)}}\right) \supset \left[0, q^{\ell_2(k)}\right]$.

From Lemma 5.7 we know $f\left(U_k^{L\setminus R}\right) \supset \widetilde{S_{\ell_1(k)}}$. In Lemma 5.8 we showed that $f\left(\widetilde{S_{\ell_1(k)}}\right) \supset \left[0, q^{\ell_2(k)}\right]$. In particular, $U_{\ell_2(k)} \subset \left[q^{\ell_2(k)+1}, q^{\ell_2(k)}\right] \subset f\left(\widetilde{S_{\ell_1(k)}}\right)$. Now, in the next lemma we estimate the counterimage of subsets of $U_{\ell_2(k)}$ under $\left(f^2|_{U_k^{L\setminus R}}\right)$. □

Lemma 5.9. *Let k be a positive odd integer. Assume μ is an even integer, $\mu \geq \max\left\{\left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right), 15\right\}$ and $\omega \in (0,1)$ is a corresponding value satisfying (5.2) and such that the assertion of Proposition 5.4 is true (this is possible due to assertion (ii) of Lemma 5.2). Define $\alpha(\omega,\mu,c)$ as in Proposition 5.3 and J_k as in Proposition 5.5. Then, for $p \in (0,1]$ and any subinterval $\widehat{U_{\ell_2(k)}}$ of $U_{\ell_2(k)}$ with $\ell_2(k)$ as in the passage before Theorem 5.1, if*

$$\left|\widehat{U_{\ell_2(k)}}\right| = p\left|J_{\ell_2(k)}\right|$$

then $\left(f|_{U_k}\right)^{-2}\left(\widehat{U_{\ell_2(k)}}\right)$ has two parts of the form

$$\widehat{U_k^L} = \left[m_k - \delta^L_{k,2}, m_k - \delta^L_{k,1}\right] \subset U_k^L \quad \textit{and} \quad \widehat{U_k^R} = \left[m_k + \delta^R_{k,1}, m_k + \delta^R_{k,2}\right] \subset U_k^R,$$

where $\delta^L_{k,1}, \delta^L_{k,2} \in \left(0, m_k - q^{k+1}\right)$ and $\delta^R_{k,1}, \delta^R_{k,2} \in \left(0, q^k - m_k\right)$, and each of them has the size

$$\left|\widehat{U_k^{L\setminus R}}\right| \leq \alpha \cdot p \cdot |J_k|. \tag{5.22}$$

Proof. Set $\widehat{S_{\ell_1(k)}} := \left(f|_{\widetilde{S_{\ell_1(k)}}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)$. Note that injectivity of $f|_{S_{\ell_1(k)}}$ and Lemma 5.8 imply that $\left(f|_{S_{\ell_1(k)}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right) = \left(f|_{\widetilde{S_{\ell_1(k)}}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)$. Using (2.13) on $\widehat{S_{\ell_1(k)}}$, we have

$$\left|\left(f|_{\widetilde{S_{\ell_1(k)}}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)\right| = \left|\widehat{S_{\ell_1(k)}}\right| \leq \frac{\left|\widehat{U_{\ell_2(k)}}\right|}{\min_{\widehat{S_{\ell_1(k)}}} |f'|}.$$

On the other hand, from Proposition 5.6 we already know that on $S_{\ell_1(k)}$, $|f'| \geq c\omega q^{\ell_1(k)(\mu-1)}$. Because of $\widehat{S_{\ell_1(k)}} \subset \widetilde{S_{\ell_1(k)}} \subset S_{\ell_1(k)}$, this property also satisfied on $\widehat{S_{\ell_1(k)}}$. Hence, inserting both $\left|\widehat{U_{\ell_2(k)}}\right| = p\left|J_{\ell_2(k)}\right|$ and the estimate of $\min_{S_{\ell_1(k)}} |f'|$ in the last expression, we have

$$\left|\widehat{S_{\ell_1(k)}}\right| \leq \frac{\left|\widehat{U_{\ell_2(k)}}\right|}{\min_{\widehat{S_{\ell_1(k)}}} |f'|} \leq \frac{p\left|J_{\ell_2(k)}\right|}{\min_{S_{\ell_1(k)}} |f'|} \leq \frac{p \cdot q^{\ell_2(k)}(1-q)}{c\omega q^{\ell_1(k)(\mu-1)}}. \tag{5.23}$$

Now, we calculate subintervals of $(m_k - \delta_k, m_k + \delta_k)$ which get mapped bijectively to $\widehat{S_{\ell_1(k)}}$. Note that the counterimage of $\widehat{S_{\ell_1(k)}}$ has two parts in the form $\widehat{U_k^L} \subset U_k^L$, and $\widehat{U_k^R} \subset U_k^R$. It follows from strict monotonicty of f on $[m_k - \delta_k, m_k]$ and $[m_k, m_k + \delta_k]$ and from the fact that $f\left(U_k^{L\setminus R}\right) \supset \widetilde{S_{\ell_1(k)}}$ that there exist $\delta_{k,1}^{L\setminus R}, \delta_{k,2}^{L\setminus R}$ with

$$\begin{aligned} \left|f\left(\widehat{U_k^R}\right)\right| &= \left|f\left(\left[m_k + \delta_{k,1}^R, m_k + \delta_{k,2}^R\right]\right)\right| \\ &= \left|f\left(\left[m_k - \delta_{k,2}^L, m_k - \delta_{k,1}^L\right]\right)\right| = \left|f\left(\widehat{U_k^L}\right)\right| = \left|\widehat{S_{\ell_1(k)}}\right|. \end{aligned} \tag{5.24}$$

We continue our calculations by using the boundaries of $\widehat{U_k^R}$. Note that for the interval $\left[m_k, m_k + \delta_{k,1}^R\right]$ we know that $f\left(m_k + \delta_{k,1}^R\right) = \max \widehat{S_{\ell_1(k)}}$ and $f(m_k) = q^{\ell_1(k)}$. Again from the monotonicity of the map it follows that $f\left(\left[m_k, m_k + \delta_{k,1}^R\right]\right) = \left[\max \widehat{S_{\ell_1(k)}}, q^{\ell_1(k)}\right]$. Consequently, since $f\left(q^{\ell_1(k)}\right) = 0$ and $f\left(\max \widehat{S_{\ell_1(k)}}\right) \in \left[q^{\ell_2(k)+1}, q^{\ell_2(k)}\right]$, from (2.13) we have

$$\left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right| \geq \frac{q^{\ell_2(k)+1}}{\|f'\|_{\infty, S_{\ell_1(k)}}}. \tag{5.25}$$

From (2.1) we also have that $\|f'\|_{\infty, S_{\ell_1(k)}} \leq g_{\omega,\mu} \cdot q^{\ell_1(k)(\mu-1)}$. Inserting this estimate in (5.25), we obtain

$$\left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right| \geq \frac{q^{\ell_2(k)+1}}{g_{\omega,\mu} \cdot q^{\ell_1(k)(\mu-1)}}. \tag{5.26}$$

In addition, from (2.4) we know that

$$\|f''\|_{\infty, U_k} \leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{k(\mu-2)}. \tag{5.27}$$

Now, using the second order Taylor expansion of $f\left(m_k + \delta_{k,1}^R\right)$, we have

$$\left|f\left(m_k + \delta_{k,1}^R\right) - f(m_k)\right| \leq \left|f''(\xi)\frac{\left(\delta_{k,1}^R\right)^2}{2}\right|, \tag{5.28}$$

where $\xi \in \left(m_k, m_k + \delta_{k,1}^R\right)$. Substituting the values of $f\left(m_k + \delta_{k,1}^R\right)$ and $f\left(m_k\right)$ in (5.28), we have

$$\left|f\left(m_k + \delta_{k,1}^R\right) - f\left(m_k\right)\right| = \left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right| \leq \|f''\|_{\infty,U_k} \frac{\left(\delta_{k,1}^R\right)^2}{2},$$

which implies

$$\delta_{k,1}^R \geq \sqrt{2\frac{\left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right|}{\|f''\|_{\infty,U_k}}}. \tag{5.29}$$

Using both estimates (5.26) and (5.27) in (5.29), we finally get

$$\delta_{k,1}^R \geq \sqrt{2\frac{q^{\ell_2(k)+1}}{g_{\omega,\mu}^2 \cdot g_{\omega,\mu-1} \cdot q^{k(\mu-2)} \cdot q^{\ell_1(k)(\mu-1)}}}. \tag{5.30}$$

On the other hand, from Taylor's formula with the integral remainder term we have

$$\begin{aligned} f\left(m_k + \delta\right) &= f\left(m_k\right) + \int_{m_k}^{m_k+\delta} \left(m_k + \delta - t\right) f''\left(t\right) dt \\ &= f\left(m_k\right) + \int_0^{\delta} \left(\delta - t\right) f''\left(m_k + t\right) dt. \end{aligned} \tag{5.31}$$

Consequently, applying (5.31) for the boundaries of $\widehat{U_k^R}$, we have

$$\begin{aligned} \left|\widehat{S_{\ell_1(k)}}\right| &= \left|f\left(\widehat{U_k^R}\right)\right| = \left|f\left(m_k + \delta_{k,2}^R\right) - f\left(m_k + \delta_{k,1}^R\right)\right| \\ &= \left|\int_0^{\delta_{k,2}^R} \left(\delta_{k,2}^R - t\right) f''\left(m_k + t\right) dt - \int_0^{\delta_{k,1}^R} \left(\delta_{k,1}^R - t\right) f''\left(m_k + t\right) dt\right|. \end{aligned}$$

From (5.12) we already know that $M := \min_{x \in U_k} \left|f''\left(x\right)\right| \geq \frac{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega \cdot g_{\omega,\mu}}{2}$. In particular, f'' has constant sign on U_k. Using the fact that $\delta_{k,1}^R < \delta_{k,2}^R$ in the last equality, we obtain

$$\begin{aligned} \left|\widehat{S_{\ell_1(k)}}\right| &= \left|\int_0^{\delta_{k,1}^R} \left(\delta_{k,2}^R - \delta_{k,1}^R\right) f''\left(m_k + t\right) dt + \int_{\delta_{k,1}^R}^{\delta_{k,2}^R} \left(\delta_{k_2}^R - t\right) f''\left(m_k + t\right) dt\right| \\ &\geq \left|\int_0^{\delta_{k,1}^R} \left(\delta_{k,2}^R - \delta_{k,1}^R\right) f''\left(m_k + t\right) dt\right| \geq \left|\delta_{k,2}^R - \delta_{k,1}^R\right| \cdot M \cdot \delta_{k,1}^R, \end{aligned}$$

so

$$\left|\delta_{k,2}^R - \delta_{k,1}^R\right| \leq \frac{\left|\widehat{S_{\ell_1(k)}}\right|}{M \cdot \delta_{k,1}^R}. \tag{5.32}$$

Substituting the estimate of M and the estimate $\delta_{k,1}^R$ given by (5.30) in (5.32), we obtain

$$\left|\delta_{k,2}^R - \delta_{k,1}^R\right| \leq \frac{\left|\widehat{S_{\ell_1(k)}}\right|}{\frac{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega \cdot g_{\omega,\mu}}{2} \cdot \sqrt{2\frac{q^{\ell_2(k)+1}}{q^{k(\mu-2)} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu}^2 \cdot q^{\ell_1(k)(\mu-1)}}}}. \tag{5.33}$$

Combining the estimate of $\widehat{S_{\ell_1(k)}}$ given by (5.23) with (5.33), we finally have

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &= \left|\delta_{k,2}^R - \delta_{k,1}^R\right| \\
&\leq \frac{\sqrt{2}p \cdot q^{\ell_2(k)}\,(1-q)}{c\omega q^{\ell_1(k)(\mu-1)}} \frac{\sqrt{q^{k(\mu-2)} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu}^2 \cdot q^{\ell_1(k)(\mu-1)}}}{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega \cdot g_{\omega,\mu} \cdot \sqrt{q^{\ell_2(k)+1}}} \\
&= \sqrt{2}p \frac{q^{\ell_2(k)}\,(1-q)}{c\omega q^{\ell_1(k)(\mu-1)}} \cdot \frac{\sqrt{q^{k(\mu-2)} g_{\omega,\mu-1}}}{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega} \cdot \frac{\sqrt{q^{\ell_1(k)(\mu-1)}}}{\sqrt{q^{\ell_2(k)+1}}} \\
&= \sqrt{2}p \frac{(1-q)\sqrt{g_{\omega,\mu-1}}}{c\omega^2 \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \frac{q^{\ell_2(k)}}{q^{\frac{\ell_2(k)}{2}}} \cdot \frac{\sqrt{q^{k(\mu-2)}}}{q^{k(\mu-2)+\frac{1}{2}}} \cdot \frac{\sqrt{q^{\ell_1(k)(\mu-1)}}}{q^{\ell_1(k)(\mu-1)}} \\
&= \frac{\sqrt{2}p \cdot q^k\,(1-q) \cdot q^{\frac{\ell_2(k)}{2}}}{q^{\frac{\ell_1(k)(\mu-1)}{2}} \cdot q^{\frac{k\mu+1}{2}} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \frac{\sqrt{g_{\omega,\mu-1}}}{c\omega^2}.
\end{aligned}$$

Here, using the estimate of $q^{\ell_2(k)}$ given in the passage before Theorem 5.1 and $|J_k| = q^k\,(1-q)$, we obtain

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\leq \frac{\sqrt{2}p \cdot |J_k| \cdot \sqrt{q^{\ell_1(k)\mu} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{c(1-c)\omega^3}{4g_{\omega,\mu} \cdot g_{\omega,\mu-1}^2}}}{q^{\frac{\ell_1(k)(\mu-1)}{2}} \cdot q^{\frac{k\mu+1}{2}} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \frac{\sqrt{g_{\omega,\mu-1}}}{c\omega^2} \\
&= \frac{\sqrt{2}}{2} p \cdot |J_k| \cdot \frac{\sqrt{q^{\ell_1(k)\mu}}}{q^{\frac{\ell_1(k)\mu-\ell_1(k)}{2}} \cdot q^{\frac{k\mu+1}{2}}} \cdot \frac{\sqrt{q^{\frac{\varphi_1(\mu-2)}{\pi}}}}{q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \sqrt{\frac{(1-c)}{c\omega g_{\omega,\mu} \cdot g_{\omega,\mu-1}}}.
\end{aligned}$$

Inserting $\ell_1(k) = k\mu + 1$ one gets

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\leq \frac{\sqrt{2}}{2} p \cdot |J_k| \cdot \frac{\sqrt{q^{(k\mu+1)\mu}}}{q^{\frac{(k\mu+1)\mu-(k\mu+1)}{2}} \cdot q^{\frac{k\mu+1}{2}}} \cdot \frac{\sqrt{q^{\frac{\varphi_1(\mu-2)}{\pi}}}}{q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \sqrt{\frac{(1-c)}{c\omega g_{\omega,\mu} \cdot g_{\omega,\mu-1}}} \\
&= p \cdot |J_k| \cdot q^{-\frac{\varphi_1(\mu-2)}{2\pi}} \cdot \sqrt{\frac{(1-c)}{2c\omega g_{\omega,\mu} \cdot g_{\omega,\mu-1}}}.
\end{aligned}$$

Since $g_{\omega,\mu-1} < g_{\omega,\mu}$, we can simplify the last inequality as follows:

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\leq p \cdot |J_k| \cdot q^{-\frac{\varphi_1(\mu-2)}{2\pi}} \cdot \sqrt{\frac{(1-c)}{2c\omega g_{\omega,\mu-1}^2}} \\
&= p \cdot |J_k| \cdot \frac{q^{-\frac{\varphi_1(\mu-2)}{2\pi}}}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}}.
\end{aligned}$$

Inserting $q = e^{-\frac{\pi}{\omega}}$ we have

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\leq p \cdot |J_k| \cdot \frac{\exp\left(\frac{\pi}{\omega} \cdot \frac{(\mu-2)\varphi_1}{2\pi}\right)}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}} \\
&= p \cdot |J_k| \cdot \frac{\exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right)}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}}.
\end{aligned}$$

Finally, using the definition of $\alpha(\mu,\omega,c)$, we get

$$\left|\widehat{U_k^R}\right| \le \alpha \cdot p \cdot |J_k|,$$

and this completes the proof for $\widehat{U_k^R}$. The proof for $\widehat{U_k^L}$ is analogous. □

Corollary 5.10. *If the set $\widehat{U_{\ell_2(k)}}$ in Lemma 5.9 is not only one interval, but a disjoint union of subintervals of $U_{\ell_2(k)}$, and $\left|\widehat{U_{\ell_2(k)}}\right|$ (the measure of $\widehat{U_{\ell_2(k)}}$) satisfies $\left|\widehat{U_{\ell_2(k)}}\right| = p\left|J_{\ell_2(k)}\right|$, then $\left(f|_{U_k}\right)^{-2}\left(\widehat{U_{\ell_2(k)}}\right)$ has two parts (one in U_k^L and the other in U_k^R) and each of them has measure less or equal $\alpha p\,|J_k|$.*

Proof. (By summation over the subintervals.) □

Now, we consider symbol sequences of the form $\{L,R\}^{n+1}$ and construct corresponding orbits of f. For given a finite sequence

$$\mathbf{s} = (s_0, s_1, s_2, \ldots, s_n) \in \{L,R\}^{n+1}$$

and odd $k \in \mathbb{N}$, we now construct the subset of points x in U_k which follow this symbol sequence. Recall the set $I_{k,\mathbf{s}}^n = \bigcap_{j=0}^n f^{-2j}\big(U_{\ell_2^j(k)}^{s_j}\big)$ defined in the passage before Theorem 5.1. We estimate the size of $\left|I_{k,\mathbf{s}}^n\right|$.

Corollary 5.11. *Let $\mathbf{s} = (s_0, s_1, s_2, \ldots, s_n)$ and an odd $k \in \mathbb{N}$ be given. Then, with ω, μ as in Lemma 5.9 and $\alpha(\omega,\mu,c)$ as in Proposition 5.3 we have $\emptyset \ne I_{k,\mathbf{s}}^n$ and*

$$\left|I_{k,\mathbf{s}}^n\right| \le \alpha^n\,|J_k|.$$

Proof. We prove the corollary by induction over n. For $n=0$, $I_{k,\mathbf{s}}^0 = U_k^{s_0} \ne \emptyset$, and

$$\left|I_{k,\mathbf{s}}^0\right| = \left|U_k^{s_0}\right| \le |J_k|.$$

Now, we assume the result is true for n, and we verify it for $n+1$. Let $\mathbf{s} = (s_0, s_1, s_2, \ldots, s_{n+1})$ be given. Define $\tilde{\mathbf{s}} = (s_1, s_2, \ldots, s_{n+1})$. From the induction hypothesis we have $I_{\ell_2(k),\tilde{\mathbf{s}}}^n \ne \emptyset$, $I_{\ell_2(k),\tilde{\mathbf{s}}}^n \subset U_{\ell_2(k)}$, and

$$\begin{aligned}\left|I_{\ell_2(k),\tilde{\mathbf{s}}}^n\right| &= \left|\bigcap_{j=0}^n f^{-2j}\left(U_{\ell_2^j(\ell_2(k))}^{s_{j+1}}\right)\right| \\ &\le \alpha^n \left|J_{\ell_2(k)}\right|.\end{aligned}$$

Note that $I_{k,\mathbf{s}}^{n+1} = f^{-2}\left(I_{\ell_2(k),\tilde{\mathbf{s}}}^n\right) \cap U_k^{s_0}$. Hence, we have

$$\left|I_{k,\mathbf{s}}^{n+1}\right| = \left|f^{-2}\left(I_{\ell_2(k),\tilde{\mathbf{s}}}^n\right) \cap U_k^{s_0}\right|. \tag{5.34}$$

Applying Corollary 5.10 with $p := \alpha^n$ and $I_{\ell_2(k),\tilde{\mathbf{s}}}^n$ instead of $\widehat{U_{\ell_2(k)}}$ in (5.34), and using this p together with 5.22, we finally obtain

$$\left|I_{k,\mathbf{s}}^{n+1}\right| = \left|f^{-2}\left(I_{\ell_2(k),\tilde{\mathbf{s}}}^n\right) \cap U_k^{s_0}\right| \le \alpha \cdot p \cdot |J_k| = \alpha^{n+1}\,|J_k|.$$

This completes the induction and the proof of Corollary 5.11. □

Proof of Theorem 5.1. Assume k, c and μ are as in the assumptions of the Theorem 5.1 and, $\alpha = \alpha(\omega, \mu, c)$ be as in Proposition 5.3, so that $\alpha < \frac{1}{2}$. Choose $\omega \in (0,1)$ as in Lemma 5.2.

(i) Let a symbol sequence $s = (s_0, s_1, s_2, \dots) \in \{L, R\}^{\mathbb{N}_0}$ be given. From Corollary 5.11 one can see that for $n \in \mathbb{N}_0$ the closed interval $I^n_{k,s}$ consists of the points $x \in U_k$ which follow the finite symbol sequence $s = (s_0, s_1, s_2, \dots, s_n) \in \{L, R\}^{n+1}$. Further we have $I^{n+1}_{k,s} \subset I^n_{k,s}$. It follows that $\bigcap_{n\in\mathbb{N}_0} I^n_{k,s} \neq \emptyset$. Since, in view of Corollary 5.11 and $\alpha < \frac{1}{2}$, we have $\left|I^n_{k,s}\right| \to 0$ for $n \to \infty$, the intersection $\bigcap_{n\in\mathbb{N}_0} I^n_{k,s}$ contains exactly one point $x_{k,s}$. This point $x_{k,s}$ has the asserted properties. Any point in U_k with these properties would also be contained in this intersection and thus equal $x_{k,s}$.

(ii) The set $\{L, R\}^{\{0,1,\dots,n\}}$ has 2^{n+1} elements and from Corollary 5.11 we know that each set corresponding to one $s \in \{L, R\}^{\{0,1,2,\dots,n\}}$ satisfies the estimate $\left|I^n_{k,\mathbf{s}}\right| \leq \alpha^n |J_k|$. It follows that $\left|\Gamma^n_k\right| \leq 2^{n+1}\alpha^n |J_k|$, and it turns out that the measure

$$\lim_{n\to\infty} |\Gamma^n_k| = \lim_{n\to\infty} \left| \bigcup_{s\in\{L,R\}^{\{0,1,2,\dots,n\}}} I^n_{k,\mathbf{s}} \right| \leq 2 \lim_{n\to\infty} 2^n \alpha^n |J_k| = 0$$

and this completes the proof. □

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