Dynamical behavior of a parametrized family of one-dimensional maps

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Abstract. The connection of these maps to homoclinic loops acts like an amplifier of the map behavior, and makes it interesting also in the case where all map orbits approach zero (but in many possible ways). We introduce so-called ‘flat’ intervals containing exactly one maximum or minimum, and so-called ‘steep’ intervals containing exactly one zero point of \( f_{\mu,\omega} \) and no zero of \( f'_{\mu,\omega} \). For specific parameters \( \mu \) and \( \omega \), we construct an open set of points with orbits staying entirely in the ‘flat’ intervals in section three. In section four, we describe orbits staying in the ‘steep’ intervals (for open parameter sets), and in section five (for specific parameters) orbits regularly changing between ‘steep’ and ‘flat’ intervals. Both orbit types are described by symbol sequences, and it is shown that their Lebesgue measure is zero.

Keywords: homoclinic behavior, one-dimensional maps, symbolic dynamics, measure.

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1 Introduction

Our aim in this paper is to analyze the dynamics of certain parametrized families \( f_{\mu,\omega} \) of one-dimensional maps. These arise in the dynamics of flows in three dimensions of saddle-focus homoclinic connections which were studied by Šil’nikov [6] and Holmes [2]. Holmes considered maps \( f \) similar to

\[
f_{\mu,\omega} : x \to x^\mu \sin(\omega \ln(x))
\]

for \( \mu > 1, \omega > 0 \) (and odd continuation). The property \( \mu > 1 \) implies that all points \( x \in (-1,1) \) approach 0 under \( f^n \) as \( n \to \infty \). The connection of the map \( f \) to a doubly homoclinic loop (as explained below) implies that the small difference between \( f^n(x) \) being positive or negative corresponds to the ‘macroscopic’ difference that the \( n + 1 \)st return will take place along the upper or lower branch of the homoclinic loop, and is therefore of interest. Holmes claimed that the set \( Z \) of points \( x \) for which there exists an \( n_x \in \mathbb{N} \) such that \( f^{n_x}(x) = 0 \) can be a dense subset of \([0,1]\), but it seems that this proof is not conclusive. (We briefly write \( f \) for \( f_{\mu,\omega} \) now.) In section four, we are interested in the orbit \( x, f(x), f(f(x)), \ldots \) We first assign

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to $x$ a symbolic trajectory $s_0, s_1, s_2, \ldots$ where $s_n = \text{sign}(f^n(x))$. Then we construct sets $\Omega^c_n$ (depending on a parameter $c$ and $n \in \mathbb{N}$) of points with the first $n$ iterates contained in certain ‘steep’ intervals and following arbitrary symbol sequences. We show that $\Omega^c_n$ is contained in the closure of the set $Z$, but $\Omega^c_\infty = \bigcap_{n \in \mathbb{N}} \Omega^c_n$ has measure zero. The remark on the bottom of the page 395 of [2] conjectures that open sets of points with orbit only in the ‘flat’ intervals can exist for certain parameters. (These ‘flat’ intervals are disjoint to $Z$.) We prove this in Section 3.

In the last section, we focus on constructing another type of orbit whose points travel regularly from a ‘flat’ interval to a ‘steep’ interval, then again from the ‘steep’ interval to a ‘flat’ interval. These points form a Cantor type set and are described by sequences of the type $(L, R, R, L, \ldots)$, indicating whether iterates of the initial points are to the left or to the right of corresponding maxima of $f$. Taking counter images $f^{-1}(J)$ of intervals $J$ with $f^{-1}(J)$ close to a quadratic maximum of $f$ involves inversion of the second order Taylor expansion and thus taking square roots. We also show that, despite the expanding effect of the square root, the measure of the points with such orbits (and thus the measure of the Cantor set) is also zero.

### 1.1 Motivation of the map

We consider the differential equation

$$
\begin{align*}
\dot{x} &= sx - vy + F_1(x, y, z) \\
\dot{y} &= vx + sy + F_2(x, y, z) & \text{or} & \quad \dot{X} = F(X), \\
\dot{z} &= \lambda z + F_3(x, y, z)
\end{align*}
$$

where $X = (x, y, z)$, with smooth functions $F_1, F_2, F_3$ which vanish at the origin together with their derivatives and assume that there exists a doubly homoclinic connection associated to a saddle-focus singularity at the origin $(0, 0, 0)$ with eigenvalues $s \pm iv$, $s < 0$, $v \neq 0$, $\lambda > 0$. We also assume that the saddle value satisfies $s + \lambda < 0$ and $F$ possesses symmetry under the change of sign, $F(X) = -F(-X)$. Here, note that while the stable manifold $W^s(0)$ is two-dimensional, the unstable manifold $W^u(0)$ is one-dimensional. The global unstable manifold $W^u(0)$ consists of the homoclinic loops and is contained in $W^s(0)$ (see Figure 1.1). Note also that in case $s + \lambda < 0$ stable periodic orbits bifurcate from the homoclinic loop as described by L. P. Šil’nikov in reference [5], even in cases of only one homoclinic loop.

Furthermore, to obtain expressions for a Poincaré first return map defined by the trajectories close to the homoclinic loop $\Lambda$, we assume that the vector field is linear (i.e. $F_1 = F_2 = F_3 = 0$) in a neighborhood of $(0, 0, 0)$. First, in a neighborhood of $(0, 0, 0)$ we introduce a cross section $\Sigma_0$ that is transversal to $\Lambda$ and has a nonzero projection to the unstable direction. The second property is an automatic consequence of the first in three dimensions. The stable manifold $W^s_{\text{loc}}$ splits $\Sigma_0$ into the upper and lower components $\Sigma^+_0$ and $\Sigma^-_0$ respectively, and the homoclinic loop intersects $\Sigma_0$ at some point $p = (\xi_0, 0, 0) \in \Lambda \cap \Sigma_0$ on $W^s_{\text{loc}}$. We next introduce two cross-sections $\Sigma^+_1$ transversal to $W^u_{\text{loc}}$. Using the trajectories which travel from $\Sigma^+_0$ to $\Sigma^+_1$ we aim at computing local maps $G^+_0 : \Sigma^+_0 \to \Sigma^+_1$ and $G^-_0 : \Sigma^-_0 \to \Sigma^-_1$. These local maps associate to each point $p \in \Sigma_0$ the first intersection with $\Sigma_1$ of the trajectory which starts at $p$. Thus, a local map $G_0$ is defined by the flow on subsets $\Sigma^+_0$ of $\Sigma_0$. Note that since the upper and lower homoclinic orbit of the system have analogous behavior, we shall continue with one (the upper homoclinic loop) of them. For simplification we assume that there exist $\xi > 0$, $\zeta > 0$ such that $\Sigma^+_0 \subset \{(\xi, y, z) : (y, z) \in \mathbb{R}^2 \}$ and $\Sigma^+_1 \subset \{(x, y, \xi) : (x, y) \in \mathbb{R}^2 \}$.

The solution $(x(t), y(t), z(t))$ of (1.1), which starts from a point $(x_0, y_0, z_0) \in \Sigma^+_0$ close to the origin at the time $t = 0$ and ends up at the point $(x_1, y_1, z_1 = \xi) \in \Sigma^+_1$ at the time $t = \tau$, \ldots
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is written (taking into account only the linear terms in (1.1)) as follows:

\[
x(t) + iy(t) = e^{(\alpha+i\gamma)t} (x_0 + iy_0) = e^{(\alpha+i\gamma)t} (\xi + iy_0)
\]

\[
z(t) = z_0 e^{\lambda t}.
\]

The flight time \( \tau \) that the trajectory takes from \( \Sigma_0^+ \) to \( \Sigma_1^+ \) is given by \( \tau = \frac{1}{\lambda} \ln \left( \frac{\xi}{z_0} \right) \). Substituting \( \tau \) and \( \xi \) into formula (1.2), we get the following expression for the local map \( G_0^+ \), in complex notation:

\[
x_1 + iy_1 = e^{(\alpha+i\gamma)\tau(z_0)} (x_0 + iy_0) = e^{(\alpha+i\gamma)\tau(z_0)} (\xi + iy_0).
\]

On the other hand, due to the existence of the homoclinic connection and its transversal intersection with \( \Sigma_0 \) and \( \Sigma_1^+ \), we also have a Poincaré type map

\[
G_1^+ : \Sigma_1^+ \to \Sigma_0
\]

Hence, for \((x_1,y_1,z_1 = \xi) \in \Sigma_1^+\) we have \( G_1^+ (x_1,y_1,z_1 = \xi) = (\xi,y_2,z_2) \in \Sigma_0 \). With \( DG_1^+ (0,0,\xi) \) represented by the matrix

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]

we have for the composite map

\[
(G_1^+ \circ G_0^+) : (\xi,y_0,z_0) \to (\xi,y_2,z_2),
\]

Figure 1.1: Cross sections \( \Sigma_0, \Sigma_1 \) and homoclinic orbit \( \Lambda \).
and finally we get
\[ z_2 = \gamma x_1 + \delta y_1. \]  
Substituting the value of \( \tau (z_0) \), \( x_1 \) and \( y_1 \) in (1.4), in particular for \( y_0 = 0 \), one obtains
\[ z_2 = c z_0 e^{\tau(z_0) [\gamma \cos + \delta \sin] (\nu \tau (z_0))}. \]
Hence, with \( c := \xi \sqrt{\gamma^2 + \delta^2} \) and choosing \( \varphi \) with \( e^{i\varphi} \sqrt{\gamma^2 + \delta^2} = \delta + i\gamma \), the z-component after one return is approximately given by
\[ z_0 \to z_2 = c \left( \frac{\xi}{z_0} \right) \left[ \sin \left( \frac{\nu}{\lambda} \left( \ln \frac{\xi}{z_0} + \varphi \right) \right) \right]. \]
Note that \( \frac{\xi}{\lambda} < -1 \) so \( \mu := -\frac{\xi}{\lambda} > 1 \), with \( x := \frac{\xi}{\xi} \), and \( \omega = \frac{\nu}{\lambda} \) we can rewrite the last equation as
\[ z_2 = cx^{\mu} \left[ \sin \left( -\omega \left( \ln x + \varphi \right) \right) \right]. \]
This motivates the study of the one-dimensional map \( f_{\omega,\mu} : [-1,1] \to [-1,1] \) given by the following simpler expression
\[ f_{\mu,\omega}(x) = \begin{cases} 
  x^{\mu} \sin(\omega \ln(x)), & x > 0, \\
  0, & x = 0, \\
  -f_{\mu,\omega}(-x), & x < 0,
\end{cases} \]
where we use \( x \) instead of \( z \) from now on. Here, note that odd continuation in the definition of \( f_{\mu,\omega} \) is motivated by the corresponding symmetry of vector field. The above process shows how to arrive at this map starting from homoclinic orbits; similar considerations are given in Šil’nikov, L. P. [6], P. J. Holmes [2], or J. Guckenheimer/P. Holmes [1, pp. 320–321]. Analogous infinite-dimensional examples with attracting homoclinic behavior (not necessarily with a double loop) were studied by Walther in [7] and by Ignatenko in [3]. The maps of this kind (see Figure 1.2) were also studied by M. J. Pacifico, A. Rovella and M. Viana [4], but for \( \mu < 1 \), which has expansion properties of \( f_{\mu,\omega} \) as a consequence. Briefly, they proved that a family of one dimensional maps with infinitely many critical points exhibit global chaotic behavior in a persistent way: For a positive Lebesgue measure set of values \( \mu \), the map \( f \) has positive Lyapunov exponent at every critical value and at Lebesgue almost all points in its domain; moreover, \( f \) is topologically transitive, i.e. has dense orbits [4].

After giving some preparatory calculations for the following chapters, we are going to study the orbit \( f_{\omega,\mu}^n(x) = f^n(x); n = 1,2,3,\ldots \) of a typical point \( x \in (0,1) \). If \( f^n(x) = 0 \) for some \( n < \infty \), then it is clear that all \( (f^j(x))_{j \geq n} \) will equal to 0. To orbits of \( f \) we can associate symbol sequences \( (s_j) = (\text{sign } f^j(x))_{j \geq 0} = (+1,+1,-1,\ldots) \). \( f^n(x) = 0 \) implies that \( s_n = 0 \), then \( s_k = 0 \) for all \( k \geq n \). Here +1, −1 and 0 correspond to the upper, to the lower homoclinic branch or to the stable manifold \( W^s(0) \) in terms of the original motivation. Consequently, the following questions arise:

(i) Are all symbol sequences possible or not?

(ii) Does the symbol sequence change in every interval? (Is there chaotic motion?)
(iii) Is it possible to construct open intervals where the symbol sequence does not change?

In the fifth chapter, we shall also consider symbol sequences different from \((\text{sign} f^j(x))\), describing whether \(f^n(x)\) is to the left or to the right hand side of maximum points of \(f\).

![Graph of \(f(x) = x^2 \cdot \sin(10 \cdot \ln(x))\)](image)

**Figure 1.2:** Graph of \(f\) for \(\mu = 2, \omega = 10\).

### 2 Formulas for the derivatives of \(f_{\mu,\omega}\)

**Lemma 2.1.** Define for \(\mu, \omega > 0\) the map

\[
f_{\mu,\omega}(x) = \begin{cases} 
    x^\mu \sin(\omega \ln(x)), & x > 0, \\
    0, & x = 0, \\
    -f_{\mu,\omega}(-x), & x < 0.
\end{cases}
\]

Assume now \(\mu \in (2, \infty), \omega > 0\). Set \(\varphi_j := \arctan\left(\frac{\omega}{\mu+1-j}\right) \in (0, \frac{\pi}{2})\) and

\[
g_{\omega,\mu+1-j} := \sqrt{\left(\mu + 1 - j\right)^2 + \omega^2}
\]

for \(j \in \{1, 2, 3\}\). It is convenient to also define the more general class of functions

\[
f_{\mu,\omega,\varphi}(x) := x^\mu \sin(\omega \ln(x) + \varphi).
\]

Then, the following formulas hold for \(x \in \mathbb{R}\), if \(\mu > 3\):

(i)

\[
f_{\mu,\omega}'(x) = g_{\omega,\mu} \cdot f_{\mu-1,\omega,\varphi_1}(x), \quad (2.1)
\]

\[
\cos(\varphi_1) = \frac{\mu}{\sqrt{\mu^2 + \omega^2}} = \frac{\mu}{g_{\omega,\mu}}, \quad (2.2)
\]

\[
\sin(\varphi_1) = \frac{\omega}{\sqrt{\mu^2 + \omega^2}} = \frac{\omega}{g_{\omega,\mu}}. \quad (2.3)
\]
Proof. (i) From the definition of \( q_1 \), we have \( q_1 = \arctan \left( \frac{\omega}{x} \right) \), and also from the definition of \( g_{\omega,\mu+1-j} \), we have \( g_{\omega,\mu} = \sqrt{\mu^2 + \omega^2} \). It follows that

\[
\cos(q_1) = \frac{\mu}{g_{\omega,\mu}} \quad \text{and} \quad \sin(q_1) = \frac{\omega}{g_{\omega,\mu}}.
\]

This proves (2.2) and (2.3). For \( x > 0 \), we have

\[
f_{\mu,\omega,\phi}'(x) = x^{\nu} \cos \left( \omega \ln(x) + \phi \right) \left( \frac{1}{x} \right) + x^{\nu-1} \mu \sin \left( \omega \ln(x) + \phi \right)
= x^{\nu-1} \left( \mu \sin \left( \omega \ln(x) + \phi \right) + \omega \cos \left( \omega \ln(x) + \phi \right) \right).
\]

By multiplying and dividing the last equation with \( g_{\omega,\mu} \), we have

\[
f_{\mu,\omega,\phi}'(x) = g_{\omega,\mu} \cdot x^{\nu-1} \left( \frac{\mu}{g_{\omega,\mu}} \sin(\omega \ln(x) + \phi) + \frac{\omega}{g_{\omega,\mu}} \cos(\omega \ln(x) + \phi) \right).
\]

Putting (2.2) and (2.3) in (2.6), we finally obtain

\[
f_{\mu,\omega,\phi}'(x) = g_{\omega,\mu} \cdot x^{\nu-1} \left( \cos(q_1) \cdot \sin(\omega \ln(x) + \phi) + \sin(q_1) \cdot \cos(\omega \ln(x) + \phi) \right)
= g_{\omega,\mu} \cdot x^{\nu-1} \left( \mu \sin(\omega \ln(x) + \phi) + \omega \cos(\omega \ln(x) + \phi) \right)
= g_{\omega,\mu} \cdot f_{\mu-1,\omega,\phi+q_1}(x).
\]

(ii) Further, using (2.7) with \( \phi + q_1 \) instead of \( \phi \), and \( \mu - 1 \) instead of \( \mu \), we see that

\[
f_{\mu,\omega}'(x) = f_{\mu,\omega,\phi}'(x) = (g_{\omega,\mu} \cdot f_{\mu-1,\omega,\phi})'(x)
= g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot f_{\mu-2,\omega,\phi+q_1}(x),
\]

which proves (2.4).

(iii) Using (2.7) we obtain (2.5) analogously. \( \square \)

Lemma 2.2. Let \( \mu > 3 \) and \( \omega > 0 \) be given. Define \( q := e^{-\frac{\pi}{\omega}} \) and \( q_j \) as in Lemma 2.1. Then, the following properties are satisfied in \( (0,1] \):

(i) \( f_{\mu,\omega} \) has the zero points

\[
q^k = e^{-\frac{k \pi}{\omega}}, \tag{2.8}
\]

\((k \in \mathbb{N})\) and

\[
f_{\mu,\omega}'(q^k) = (-1)^k \omega q^{k(\mu-1)}. \tag{2.9}
\]

(ii) \( f_{\mu,\omega} \) has the extremal points

\[
m_k = q^k e^{-\frac{q_1}{\omega}}, \tag{2.10}
\]

and

\[
f_{\mu,\omega}(m_k) = (-1)^{k+1} \cdot \exp \left( -\frac{k \pi \mu}{\omega} + \frac{q_1}{\omega} \right) \cdot \sin(q_1). \tag{2.11}
\]
(iii) If $\mu$ is an even integer, and $\beta \in \mathbb{N}$ is odd and $l(k) := k\mu + \beta$, then $f_{\mu,\omega}$ has a maximum at

$$m_{l(k)} = q^{l(k)}e^{-\frac{\varphi_1}{\omega}}.$$  \hspace{1cm} (2.12)

Proof. (i) We first find the zeros of $f_{\mu,\omega}$. For $x \in (0,1)$ one has

$$\sin(\omega \ln(x)) = 0 \iff \exists k \in \mathbb{N} : \omega \ln(x) = -k\pi \iff \exists k \in \mathbb{N} \ln(x) = -\frac{k\pi}{\omega},$$

and hence $x = e^{-\frac{k\pi}{\omega}}$. With $q = e^{-\frac{\varphi_1}{\omega}}$, the zeros of $f_{\mu,\omega}$ in $(0,q]$ are given by $x = e^{-\frac{k\pi}{\omega}} = q^k$. Therefore, by inserting $q^k$ in (2.1), we have

$$f'_{\mu,\omega}(q^k) = q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\omega(\ln q^k) + \varphi_1)$$

$$= q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\omega \left(\frac{k}{\omega} + \frac{\varphi_1}{\omega}\right)$$

$$= (-1)^k q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\varphi_1).$$

Using (2.3) we obtain

$$f'_{\mu,\omega}(q^k) = (-1)^k q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \frac{\omega}{g_{\omega,\mu}}$$

$$= (-1)^k \omega q^{k(\mu-1)}.$$

Hence, assertion (i) is proved.

(ii) Let $k \in \mathbb{N}$. We find the extremum points of $f_{\mu,\omega}$ in the interval $I_k = [q^{k+1}, q^k]$ by solving $f'_{\mu,\omega}(x) = 0$ for $x \in I_k$. Since $x > 0$, $x^{\mu-1} \neq 0$. So, we have

$$\sin(\omega(\ln x) + \varphi_1) = 0,$$

and hence $x = e^{-\frac{k\pi - \varphi_1}{\omega}}$. The last expression equals to $q^k e^{-\frac{\varphi_1}{\omega}} = m_k$, which proves (2.10). Furthermore, for the extremum point $m_k$ of $f_{\mu,\omega}$ in the interval $(q^{k+1}, q^k)$ we have

$$f_{\mu,\omega}(m_k) = m_k^\mu \sin(\omega \ln(m_k))$$

$$= \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^\mu \sin(\omega \ln(q^k e^{-\frac{\varphi_1}{\omega}}))$$

$$= \left(e^{-\frac{k\pi}{\omega}} e^{-\frac{\varphi_1}{\omega}}\right)^\mu \sin(\omega \ln(e^{-\frac{k\pi}{\omega}} e^{-\frac{\varphi_1}{\omega}}))$$

$$= \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\omega \left(-\frac{k\pi - \varphi_1}{\omega}\right))$$

$$= (-1)^k \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\varphi_1).$$

(iii) Substituting $l(k)$ instead of $k$ in (2.11), we have

$$f_{\mu,\omega}\left(m_{l(k)}\right) = (-1)^{l(k)+1} \exp\left(-\frac{l(k) \pi\mu + \varphi_1\mu}{\omega}\right) \sin(\varphi_1)$$

$$= \exp\left(-\frac{l(k) \pi\mu + \varphi_1\mu}{\omega}\right) (-1)^{kp+\beta+1} \sin(\varphi_1)$$

Therefore, it is clear that $f_{\mu,\omega}(m_{l(k)}) > 0$ (and hence $f_{\mu,\omega}$ has a maximum at $m_{l(k)}$), if $\mu$ is even and $\beta$ is odd. \hfill \Box
We shall frequently use the simple lemma below.

**Lemma 2.3.** Assume \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If \(|f'| \geq c\), or \(|f'| \leq d\) \((c \text{ and } d \text{ are constant})\), then we have

\[
|b - a| \leq |f(b) - f(a)| \leq d |b - a|.
\] (2.13)

**Proof.** (Follows from the mean value theorem.) \qed

## 3 The behavior of orbits remaining in some ‘flat’ intervals

In this part we find some parameters \( \mu \) and \( \omega \) such that \( f_{\mu, \omega} \) maps some extremal points \( m_k \) to some other extremal points \( m_{\ell(k)} \) (see Figure 3.1). Then, we construct some open intervals \( U_k \) around \( m_k \) and orbits of \( f_{\mu, \omega} = f \) which are entirely contained in \( \bigcup_{k \in \mathbb{N}} U_k \).

![Figure 3.1](image-url)  \( f(m_k) = m_{\ell(k)} \) for special parameters (picture not produced with realistic parameters, for better visibility).

**Theorem 3.1.** For \( k \in \mathbb{N}, \omega > 0, \) and even integer \( \mu > 5 \), define

\[
\eta := \min \left\{ \frac{q}{g_{\omega, \mu} \cdot g_{\omega, \mu - 1}}, \frac{e^{\frac{q_{\ell(k)}}{\omega}} - q}{2}, \frac{1 - e^{-\frac{q_{\ell(k)}}{\omega}}}{2} \right\},
\] (3.1)

and set \( \ell(k) := k\mu + 1 \) (which corresponds to \( \beta = 1 \) in assertion (iii) of Lemma 2.2), \( \delta_k := \eta q_k^{\ell(k)} \), \( \delta_{\ell(k)} := \eta q_{\ell(k)}^{\ell(k)} \). Then, for every large enough even integer \( \mu \) there exists a corresponding \( \omega \) such that the following properties are satisfied:

(i) With the intervals \( U_k = (m_k - \delta_k, m_k + \delta_k) \) one has \( f(U_k) \subset U_{\ell(k)} \) and

\[
\forall k \in \mathbb{N} : f^{-1}(\{0\}) \cap U_k = \emptyset.
\]

(ii) If \( k \) is odd, then for \( x \in U_k \), the orbits \( (f^j(x))_{j \in \mathbb{N}} \) all have the symbol sequence

\[
(s_j) = \left(\text{sign} f^j(x)\right)_{j \in \mathbb{N}} = (+1, +1, +1, \ldots).
\]
(iii) The set
\[ Z = \{ x \mid \exists n \in \mathbb{N} : f^n(x) = 0 \} \]  
(3.2)
is disjoint to \( \bigcup_k U_k \) and, in particular, is not dense in \([-1, 1]\).

The proof is divided into several lemmas.

**Lemma 3.2.** Let \( k \in \mathbb{N} \) and define \( \varphi_1 \) as in Lemma 2.1. Define \( \eta \) and \( \delta_k \) as in Theorem 3.1, and
\[ \eta := \min \left\{ e^{-\frac{\varphi_1}{\omega}} - q, 1 - e^{-\frac{\varphi_1}{\omega}} \right\}. \]

Then we have
\[ (m_k - \delta_k, m_k + \delta_k) \subset [m_k - \eta q^k, m_k + \eta q^k] \subset (q^{k+1}, q^k). \]

**Proof.** From (3.1) we have \( \eta \leq \eta \). Multiplying both sides with \( q^k \), and using (2.10) we have
\[ \delta_k \leq \eta q^k = \min \left\{ \frac{q^k e^{-\varphi_1/\omega} - q^{k+1}}{2}, q^k - q^k e^{-\varphi_1/\omega} \right\} = \min \left\{ \frac{m_k - q^{k+1}}{2}, q^k - m_k \right\}, \]
it follows that \( (m_k - \delta_k, m_k + \delta_k) \subset [m_k - \eta q^k, m_k + \eta q^k] \subset (q^{k+1}, q^k) \).

**Lemma 3.3.** Define \( \varphi_1 \) as in Lemma 2.1, and define \( \ell(k) \) as in Theorem 3.1. Then the following statements are true.

(i) For every even integer \( \mu \geq 32 \), there exists \( \omega \in (0, 1) \) such that for all \( k \in \mathbb{N} \) \( f \) has the property
\[ f(m_k) = m_{\ell(k)}. \]

(ii) For any choice of \( \omega \) as in assertion (i), one has \( \omega \to 0 \) as \( \mu \to \infty \).

**Proof.** (i) From (2.11) we have for all \( k \in \mathbb{N} \)
\[ |f(m_k)| = \exp \left( -\frac{k\pi \mu + \varphi_1 \mu}{\omega} \right) \sin(\varphi_1). \]  
(3.3)
On the other hand, from the third assertion of Lemma 2.2 we know that for even \( \mu \), \( f \) has a maximum at the point
\[ m_{\ell(k)} = \exp \left( -\frac{\pi \ell(k) + \varphi_1}{\omega} \right). \]
(3.4)
Using (2.3), (3.3) and (3.4), we obtain the following equivalences:
\[ m_{\ell(k)} = f(m_k) \iff \exp \left( -\frac{\pi \ell(k) + \varphi_1}{\omega} \right) = \exp \left( -\frac{k\pi \mu + \varphi_1 \mu}{\omega} \right) \cdot \sin(\varphi_1) \]
\[ \iff \exp \left( -\frac{\pi \ell(k) + \varphi_1}{\omega} \right) = \exp \left( -\frac{k\pi \mu + \varphi_1 \mu}{\omega} \right) \cdot \frac{1}{\sqrt{1 + \frac{\mu^2}{\omega^2}}} \]
\[ \iff \exp \left( -\frac{\pi}{\omega} k\mu - \ell(k) \right) + \frac{\varphi_1 (1 - \mu)}{\omega} = \sqrt{1 + \frac{\mu^2}{\omega^2}}. \]  
(3.5)
Substituting \( \ell (k) = k\mu + 1 \) in (3.5), we have
\[
\exp \left( \frac{\pi + \varphi_1 (1 - \mu)}{\omega} \right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}
\]
or, using the definition of \( \varphi_1 \),
\[
\exp \left( \frac{\pi - (\mu - 1) \arctan \left( \frac{\omega}{\mu} \right)}{\omega} \right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}. \tag{3.6}
\]
In view of (3.6), we define
\[
F (\omega, \mu) = \exp \left( \frac{\pi - (\mu - 1) \arctan \left( \frac{\omega}{\mu} \right)}{\omega} \right) - \sqrt{1 + \frac{\mu^2}{\omega^2}} \tag{3.7}
\]
for all \( \omega > 0 \) and \( \mu > 1 \). We try to find \((\omega, \mu)\) with \( F (\omega, \mu) = 0 \) (see Figure 3.2). Noting that for fixed \( \mu \), \( \lim_{\omega \to 0} F (\omega, \mu) = +\infty \), it is enough to find at least one pair \((\omega, \mu)\) with \( F (\omega, \mu) < 0 \). For \( \omega = 1 \), we have
\[
F (1, \mu) = \exp \left( \frac{\pi - (\mu - 1) \arctan \left( \frac{1}{\mu} \right)}{1} \right) - \sqrt{1 + \mu^2} = \exp \left( \pi - \mu \arctan \left( \frac{1}{\mu} \right) + \arctan \left( \frac{1}{\mu} \right) \right) - \sqrt{1 + \mu^2}. \tag{3.8}
\]
Since \( \arctan' (x) = \frac{1}{1+x^2} \), we have \( \arctan' (x) \geq \frac{1}{2} \) for \(|x| \leq 1\). Hence, (2.13) shows \( \arctan(x) \geq \frac{1}{2} x \) for \( x \in [0, 1] \). It follows that for \( \mu > 1 \),
\[
\mu \arctan \left( \frac{1}{\mu} \right) \geq \frac{1}{2}. \tag{3.9}
\]
Using (3.9) and \( \arctan \left( \frac{1}{\mu} \right) < \frac{\pi}{4} \) for \( \mu > 1 \) in (3.8), we have
\[
F (1, \mu) \leq \exp \left( \pi - \frac{1}{2} + \frac{\pi}{4} \right) - \sqrt{1 + \mu^2} = \exp \left( \frac{5\pi}{4} - \frac{1}{2} \right) - \sqrt{1 + \mu^2}.
\]
From the fact that \( \exp \left( \frac{5\pi}{4} - \frac{1}{2} \right) < 32 \), we have \( F (\omega, \mu) < 0 \), if we set \( \omega = 1 \) and \( \mu \geq 32 \). With the intermediate value theorem, it is trivial that \( F (\omega, \mu) \) has at least one zero point \( \omega \in (0, 1) \). It follows that (3.7) is satisfied with this \( \omega \) depending on the even integer \( \mu \geq 32 \). Hence, the proof of assertion (i) is completed.

(ii) Consider a sequence \( \mu_k, \mu_k \to \infty \) with corresponding \( \omega_k \in (0, 1) \) such that \( F (\mu_k, \omega_k) = 0 \). Then \( \sqrt{1 + \frac{\mu_k^2}{\omega_k^2}} \to \infty \). Further, \((\mu_k - 1) \arctan \left( \frac{\omega_k}{\mu_k} \right) \) is bounded. The exponential term in (3.7) must go to \(+\infty\), so \( \omega_k \to 0 \) necessarily. This completes the proof of (ii) and the proof of Lemma 3.3. \( \square \)

**Remark 3.4.** Consider the equation (3.5). Because \( \mu > 1 \), so \( \frac{\varphi_2 (1 - \mu)}{\omega} < 0 \), and \( \sqrt{1 + \frac{\mu^2}{\omega^2}} > 1 \), the term \( \frac{\omega}{\omega^2} \left[ k\mu - \ell (k) \right] \) must be positive, if we have a solution. Accordingly, \( \ell (k) > k\mu \) must be satisfied. It means (3.6) has no solution for \( \ell (k) \leq k\mu \). Thus \( \ell (k) \geq k\mu + 1 \) necessarily; we made the choice \( \ell (k) = k\mu + 1 \).
**Numerical observations.** In order to find a numerical solution we use two starting points where $F(\cdot, \mu)$ has opposite signs and at the 9th step of a bisection method we obtained $\omega = 0.69895$ and $\mu = 24$ as an appropriate $F(\omega, \mu) = 0$. Although one can obtain some other solution points $\omega$, for some other the parameters $\mu$, we numerically found out that there is no solution for $\mu < 3.1$.

**Lemma 3.5.** Choose an even integer $\mu \geq 32$ and $\omega \in (0, 1)$ with the properties as in Lemma 3.3. Define $\ell(k)$, $\eta$, $\delta_k$ and $\delta_{\ell(k)}$ as in Theorem 3.1. Then with the intervals $U_k = (m_k - \delta_k, m_k + \delta_k)$, we have $f(U_k) \subset U_{\ell(k)}$.

**Proof.** Let $\mu$ and $\omega$ be as in the assumption of the lemma, and $x \in U_k$. With $\ell(k) = k\mu + 1$ we claim that

$$\left| f(x) - m_{\ell(k)} \right| < \delta_{\ell(k)} = \eta q^{\ell(k)}.$$  \hspace{1cm} (3.10)

From the second order Taylor expansion, we have

$$f(x) = f(m_k) + f'(m_k) (x - m_k) + \frac{f''(\xi)}{2} (x - m_k)^2$$  \hspace{1cm} (3.11)

with $\xi \in (m_k - \delta_k, m_k + \delta_k)$. Since $\mu > 2$, note that we also have

$$q^{(k+1)(\mu-2)} \leq |\xi|^{\mu-2} \leq q^{k(\mu-2)}.$$  \hspace{1cm} (3.12)

Substituting the equality (3.11) in the left hand side of (3.10), we get

$$\left| f(x) - m_{\ell(k)} \right| = \left| f(m_k) + f'(m_k) (x - m_k) + \frac{f''(\xi)}{2} (x - m_k)^2 - m_{\ell(k)} \right|.$$  \hspace{1cm} (3.13)

From the fact that we now have fixed parameters $\mu$, $\omega$ with the property $f(m_k) = m_{\ell(k)}$ as in Lemma 3.3 and using $f'(m_k) = 0$ and $(x - m_k) < \delta_k$, the last equality gives

$$\left| f(x) - m_{\ell(k)} \right| \leq \left| f''(\xi) \delta_k^2 \frac{1}{2} \right|.$$  \hspace{1cm} (3.14)
Using (2.4) in the last equality, we obtain
\[
|f(x) - m_{\ell(k)}| = \left| g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot \sin(\omega \ln(\xi) + \varphi_1 + \varphi_2) \right| \frac{|\xi|^\mu - 2 \delta_k^2 |}{2}.
\] (3.13)

Using the upper estimate of (3.12) and substituting the value of \( \delta_k \) in (3.13), we get
\[
|f(x) - m_{\ell(k)}| \leq \left| g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{k+1} \eta \right| \frac{\eta q^{2k}}{2}.
\] (3.14)

Finally, using the definition of \( \eta \) from (3.1) in (3.14), we have
\[
|f(x) - m_{\ell(k)}| \leq \left| g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{k+1} \frac{\eta}{2} \right| \frac{q}{2} \delta_k = \eta q^{k+1} \delta_{\ell(k)}.
\]

\[\square\]

**Proof of Theorem 3.1.** Choose \( \mu, \omega \) as in Lemma 3.3, and let \( \ell(k) \) be as in Theorem 3.1.

(i) Lemma 3.5 shows \( f(U_k) \subset U_{\ell(k)} \) and the definition of \( U_{\ell(k)} \) implies \( 0 \notin U_{\ell(k)}, \) so

\[ f^{-1}(\{0\}) \cap U_k = \emptyset. \]

(ii) If \( k \) is odd and \( \mu \) is as above (therefore even), then all \( \ell(k) (j \geq 0) \) are odd and all \( U_{\ell(k)} \) are intervals around maxima of \( f \), where \( f \) is positive. Hence the assertion is proved.

(iii) For \( k_0 \in \mathbb{N} \), \( x \in U_{k_0} \) and \( n \in \mathbb{N}_0 \), \( f^n(x) \in \bigcup_{k \in \mathbb{N}} U_k \), in particular \( f^n(x) \neq 0 \), which proves assertion (iii). \[\square\]

Note that the possible existence of the orbits which remain close to critical points, i.e. implying non-density has been mentioned as a remark by P. J. Holmes in the bottom of the page 395 of [2] with only a vague indication of proof. With this section we gave a rigorous proof of that idea.

## 4 Behavior of the map \( f_{\mu,\omega} \) in some ‘steep’ intervals

In this section we first construct some orbits whose points stay entirely in so-called ‘steep’ intervals, and then analyze the measure of the set of points which have such orbits. In contrast to Sections 3 and 5, where the parameters \( \mu \) and \( \omega \) are connected by the conditions given in assertion (i) of Lemma 3.3 and in (5.1), in this section both of them can be varied independently.

Consider the interval \((-m_k, -m_{k+1})\) or \((m_{k+1}, m_k)\). From Lemma 2.2 we have
\[
|f'_{\mu,\omega}(q^k)| = \omega \left( q^{k+1} \right)^{\mu-1}.
\]

Since \( f'_{\mu,\omega}(\mp m_k) = f'_{\mu,\omega}(\mp m_{k+1}) = 0 \), continuity of \( f'_{\mu,\omega} \) implies that we can choose a ‘steep’ interval \( S_k \), either as a subset of \((m_{k+1}, m_k)\) or as a subset of \((-m_{k+1}, -m_k)\), on which \( |f'_{\mu,\omega}| \) satisfies a lower estimate. We begin by specifying the boundaries of the ‘steep’ interval \( S_k \) and by giving some new notations.

We use the notation \(|I|\) for the length of an interval \( I \).
Definition 4.1. Let $k \in \mathbb{N}$ and $c \in (0, 1)$. Define

$$a_k := \min \left\{ x \in (m_{k+1}, q^{k+1}] : \left| f'_{\mu, \omega}(x) \right| \geq c\omega \left( q^{k+1} \right)^{(\mu-1)} \text{ on } [x, q^{k+1}] \right\}$$

and

$$b_k := \max \left\{ x \in [q^{k+1}, m_k) : \left| f'_{\mu, \omega}(x) \right| \geq c\omega \left( q^{k+1} \right)^{(\mu-1)} \text{ on } [q^{k+1}, x] \right\}.$$ 

Note that $q^{k+2} < a_k < q^{k+1} < b_k < q^k$ (see Figure 4.1). Given a symbol sequence of the form

$$s = (s_0, s_1, s_2, \ldots) \in \{+1, -1\}^\mathbb{N}_0,$$

where symbols represent the signs of $f_{\mu, \omega}^j(x)$ for some starting value $x$, we construct corresponding orbits of $f_{\mu, \omega}$. Note that in terms of the motivation by the three-dimensional vector field, such orbits correspond to solutions converging to the doubly homoclinic loop, and taking turns along the upper and lower homoclinic orbit according to the symbol sequence. For $0 \leq a \leq b$, define

$$[a, b]_{+1} := [a, b],$$
$$[a, b]_{-1} := [-b, -a],$$

and define ‘steep’ intervals by

$$S^c_{k,s} := [a_k, b_k]_s = \begin{cases} [a_k, b_k], & \text{if } s = +1, \\ [-b_k, -a_k], & \text{if } s = -1. \end{cases}$$

So, we have

$$\left| f'_{\mu, \omega}(x) \right| \geq c\omega \left( q^{k+1} \right)^{(\mu-1)} \text{ for } x \in S^c_{k,s}, \ s \in \{+1, -1\}, \ k \in \mathbb{N}. \quad (4.1)$$

We also define $S_{k,\pm1} := S^c_{k,\pm1} \cup S^c_{k,1-1}$ and define the union of all ‘steep’ intervals by

![Figure 4.1: One interval $(q^{k+2}, q^k)$, with corresponding ‘steep’ interval $S^c_{k,+1} = [a_k, b_k]$.](image-url)
Note that for \( s \in \{ \pm 1 \} \), \( S_{c,k,s} \subset (m_{k+1}, m_k)_s \), and hence
\[
|S_{c,k,s}| \leq m_k - m_{k+1} = q^k e^{-\frac{\omega}{2\pi}} (1 - q) \tag{4.2}
\]
Setting \( f := f_{\mu,\omega} \), we define sets of points with forward orbits which are contained in these ‘steep’ intervals (see Figure 4.2). Namely,
\[
\Omega_n^c = \bigcap_{j=0}^{n} f^{-j}(\Psi^c); \quad \Omega_\infty^c = \bigcap_{j=0}^{\infty} f^{-j}(\Psi^c).
\]

![Figure 4.2: Graphical construction of \( \Sigma_1 \) from \( \Sigma_0 = [q^{k_0+1}, y_0] \) (in case \( s_0 = s_1 = +1 \).)](image)

**Theorem 4.2.** Let \( c \in (0,1) \). Assume \( \mu > 1 \) and define \( S_{c,k,\pm 1} \) and \( \Omega_\infty^c \) as above. Then for \( k_0 \in \mathbb{N} \) the following statements are true:

(i) For every symbol sequence \( s = (s_0, s_1, s_2, \ldots) \) there exists a point \( y_0 \in \left( S_{k_0,s_0}^c \cap \Omega_\infty^c \right) \) with the property that \( \text{sign } f^j(y_0) \) is given by \( s_j \in \{ \pm 1 \}, \) where \( j \in \mathbb{N}_0 \).

(ii) Let \( \omega > \frac{1}{2} + \pi (\mu + 1) \). Then with the set \( Z \) from (3.2) we have \( \left( S_{k_0,s_0}^c \cap \Omega_\infty^c \right) \subset Z \).

(iii) Let \( c \in (\frac{2}{\pi}, 1) \) and \( \omega > \frac{c\pi(2\mu+3)}{2(\pi-2\mu)} \). Then \( \Omega_\infty^c \subset Z \), and \( \Omega_\infty^c \) has Lebesgue measure zero.

**Remark 4.3.** A similar argument is sketched in the page 395 of [2], with the purpose to show that \( Z \) can be dense, but it seems that the method gives density only in a set of measure zero (see part (iii) of the above theorem).

The proof starts with the following lemma.
Lemma 4.4. Let \( k_0 \in \mathbb{N}, c \in (0, 1) \) and \( s = (s_0, s_1, s_2, \ldots) \in \{+1, -1\}^{\mathbb{N}_0} \) be given. Define \( S_{k_0,s_j}^c \) as in the passage given before Theorem 4.2. Then the following statements are true:

(i) There exists a point \( y_0 \in S_{k_0,s_0}^c \) and a sequence \( k_0 < k_1 < k_2 < \cdots \) such that \( \forall j \in \mathbb{N}_0, f^j(y_0) \in S_{k_j,s_j}^c \), in particular, \( y_0 \in \Omega_c^\infty \).

(ii) Let \( y_0 \in (S_{k_0,s_0}^c \cap \Omega_c^\infty) \) be given and define the sequence \( k_0 < k_1 < k_2 < \cdots \) by \( f^j(y_0) = y_j \in S_{k_j,s_j}^c \) \( (j \in \mathbb{N}_0) \). Then there exists a sequence \( (\Sigma_j) \) of intervals in \( S_{k_0,s_0}^c \) with \( \Sigma_j \supset \Sigma_{j+1} \ni y_0 \), \((f^j)' \neq 0 \) on \( \Sigma_j \) and

\[
(f^j)_{|\Sigma_j} = \begin{cases} [q_j^{k_j+1}, y_j], & \text{if } s_j = +1, \\ [y_j - q_j^{k_j+1}], & \text{if } s_j = -1 \end{cases} \subset S_{k_j,s_j}^c \quad \text{for } j \in \mathbb{N}_0, \quad (4.3)
\]

in particular, \( Z \cap \Sigma_j \neq \emptyset \) for all \( j \in \mathbb{N}_0 \).

(iii) For \( y_0 \in S_{k_0,s_0}^c \cap \Omega_c^\infty \) and \( k_0, k_1, k_2, \ldots \) as in assertion (ii) and all \( j \in \mathbb{N} \) we have

\[
|\left((f^j)'(y_0)\right)| \geq (c\omega)^j \left(\prod_{n=0}^{j-1} q_n^{k_n+1}\right)^{\mu-1}. \quad (4.4)
\]

(iv) Let \( y_0 \) and the sequence \( k_0 < k_1 < k_2 < \cdots \) be as in (ii). Then

\[
\forall j \in \mathbb{N} : q_j^{k_j} \geq q_j^{k_{j+1}+2}. \quad (4.5)
\]

(v) Let \( \omega > \frac{1}{c} + \pi (\mu + 1) \). Let \( y_0 \) and the associated \( \Sigma_j \) be as in assertion (ii) and \( \varphi_1 \) be as in Lemma 2.1. Then \( |\Sigma_j| \leq \frac{q_j^{\varphi_1} q_j^j - q_j^{\varphi_1} q_j^j}{(\omega q_j^{\mu+1})^j} \) and \( c\omega q_j^{\mu+1} > 1 \); in particular, \( |\Sigma_j| \to 0 \) as \( j \to \infty \).

Proof. (i) Let \( k_0 \in \mathbb{N} \) and \( s = (s_0, s_1, s_2, \ldots) \) be given. For \( S_{k_0,s_0}^c = [a_{k_0}, b_{k_0}]_{s_0} \) it is clear that \( f(S_{k_0,s_0}^c) \) is an interval which contains \( 0 \) in its interior, and since \( a_k \to 0, b_k \to 0 \) as \( k \to \infty \), there exists \( k_1 > k_0 \) with \( S_{k_1,s_1}^c \subset f(S_{k_0,s_0}^c) \). Further \( f|_{S_{k_0,s_0}^c} \) is injective, and we set

\[
J_1 := \left(f|_{S_{k_0,s_0}^c}\right)^{-1}(S_{k_1,s_1}^c).
\]

(f maps \( J_1 \) bijectively onto \( S_{k_1,s_1}^c \)). Similarly, there exists \( k_2 > k_1 \) with \( S_{k_2,s_2}^c \subset f(S_{k_1,s_1}^c) \), and a closed subinterval \( J_2 \subset J_1 \) such that \( f^2|_{J_2} : J_2 \to S_{k_2,s_2}^c \) is bijective. Thus, we obtain a nested sequence

\[
J_1 \supset J_2 \supset J_3 \supset \cdots
\]

of closed intervals and sequence of numbers

\[
k_0 < k_1 < k_2 < \cdots
\]

with the property that \( f^j(J_j) = S_{k_j,s_j}^c, j = 1, 2, 3, \ldots \). Furthermore, the intersection of nested closed intervals \( \bigcap_{j \in \mathbb{N}} J_j \) is not empty. It means that there exists a point \( y_0 \in \bigcap_{j \in \mathbb{N}} J_j \) which follows the symbol sequence \( s \), and this result completes the proof of assertion (i).
(ii) For the proof of this assertion we use a recursive construction. Define
\[
\Sigma_0 := \left[q^{k_0}, y_0\right]_{s_0} = \begin{cases} 
q^{k_{0}} + 1, y_0, & \text{if } s_0 = +1 \\
y_0 - q^{k_{0}}, & \text{if } s_0 = -1 
\end{cases} \subset S_{k_0, s_0}.
\]
Then \( y_0 \in \Sigma_0 \), and the definition of \( S_{k_0, s_0} \) implies \( f' \neq 0 \) on \( \Sigma_0 \), so (4.3) holds for \( j = 0 \). Assume \( \Sigma_j \) with the properties in (4.3) is constructed and we want to construct \( \Sigma_{j+1} \subset \Sigma_j \) such that (4.3) is also satisfied for \( j + 1 \). We have, observing that \( \text{sign} (y_j) = s_j \),
\[
f \left( \left[q^{k_{j+1}}, y_j\right]_{s_j} \right) = \left[0, f(y_j)\right]_{s_{j+1}} = \left[0, y_{j+1}\right]_{s_{j+1}},
\]
and \( f|_{\Sigma_j} \) as well as \( f|_{\left[q^{k_{j+1}}, y_j\right]_{s_j}} \) are invertible. Hence, we can define
\[
\Sigma_{j+1} = \left(f^{-1}\right)(\Sigma_j) - \left(f\right)^{j+1}\left(\left[q^{k_{j+1}}, y_j\right]_{s_j}\right) - 1 \left(\left[q^{k_{j+1}}, y_j\right]_{s_{j+1}}\right).
\]
Then \( y_0 \in \Sigma_{j+1} \subset \Sigma_j \), the chain rule shows \( (f^{j+1})' \neq 0 \) on \( \Sigma_{j+1} \), and
\[
(f^{j+1})(\Sigma_{j+1}) = \left[q^{k_{j+1}}, y_{j+1}\right]_{s_{j+1}} \subset S_{k_{j+1}, s_{j+1}}.
\]
Hence, the recursive construction is completed. Note also that for \( j \in \mathbb{N} \), \( \Sigma_j \) contains a point \( x_j \) with \( f^j(x_j) = q^{k_{j+1}} \), so \( f^{j+1}(x_j) = f(\left[q^{k_{j+1}}\right]) = 0 \), hence \( x_j \in \Sigma_j \cap Z \).

(iii) By the chain rule the derivative \( (f^j)' \) at \( y_0 \in \bigcap_{j \in \mathbb{N}} \Sigma_j \) can be calculated as the product of the derivatives of \( f \) along the orbit
\[
\left| (f^j)' (y_0) \right| = \left| f' (y_0) \cdot f' (y_1) \cdot \ldots \cdot f' (y_{j-2}) \cdot f' (y_{j-1}) \right| = \prod_{n=0}^{j-1} \left| f' (y_n) \right|.
\]
Using (4.1) for each derivative in the last equality, we have
\[
\left| (f^j)' (y_0) \right| = \prod_{n=0}^{j-1} |f'(y_n)| \geq (c\omega)^j \prod_{n=0}^{j-1} q^{k_{n+1}} \mu^{-1}
\]
\[
= (c\omega)^j \left( \prod_{n=0}^{j-1} q^{k_{n+1}} \right) \mu^{-1}.
\]
This gives the proof of (4.4).

(iv) Let now \( y_0 \in S_{k_0, s_0} \) and sequence \( k_0 < k_1 < k_2 < \cdots \) as in (ii) be given. With \( \Sigma_j \) from (4.3) we have \( f^j (\Sigma_j) \subset S_{k_j, s_j} \), and so
\[
f^{j+1} (y_0) \in S_{k_j, s_j} \cap S_{k_{j+1}, s_{j+1}} \subset f\left( f^j (\Sigma_j) \right) \subset f\left( S_{k_{j}, s_{j}} \right), \quad \text{for } j \in \mathbb{N}_0
\]
which implies \( f(S_{k_j, s_j}) \subset S_{k_{j+1}, s_{j+1}} \neq \emptyset \). Moreover, since \( |f| \leq q^{k_j} \mu \) on \( S_{k_j, s_j} \), we obviously have \( q^{k_j} \mu \geq \max \{ |f(x)| : x \in S_{k_j, s_j} \} \). Together with
\[
\frac{\max \{ |f(x)| : x \in S_{k_j, s_j} \}}{|y| : y \in S_{k_{j+1}, s_{j+1}}} \geq \min \{ |f(x)| : x \in S_{k_{j+1}, s_{j+1}} \},
\]
we conclude

\[ q^{k_j} \geq \max \{ |f(x)| : x \in S_{c}^{\xi} \} \geq \min \{ |y| : y \in S_{c}^{\xi} \} = a_{k_{j+1}} \geq q^{k_{j+1} + 2}. \]

Hence, the proof of (iv) is also completed.

(v) Finally, from (2.13) we know that on \( \Sigma_1 \) we have

\[ \left| \Sigma_j \right| \leq \frac{\left| (f_j) (\Sigma_j) \right|}{\min_{x} \left| (f_j) \right|}. \]  

(4.6)

From (4.3) we have \( \left| (f_j) (\Sigma_j) \right| \leq \left| S_{c}^{\xi} \right| \) and from (4.2) we have \( \left| S_{c}^{\xi} \right| \leq q^{k_j} e^{-\frac{q}{\omega}} (1 - q). \)

Combining both inequalities, we get

\[ \left| (f_j) (\Sigma_j) \right| \leq q^{k_j} e^{-\frac{q}{\omega}} (1 - q). \]  

(4.7)

Using (4.7) and (4.4) in (4.6), we obtain

\[ \left| \Sigma_j \right| \leq \frac{q^{k_j} e^{-\frac{q}{\omega}} (1 - q)}{(c \omega)^j \left( \prod_{n=0}^{k_{j+1}} q^{k_n + 1} \right)^{\mu - 1}}. \]  

(4.8)

By using (4.5) we can estimate the denominator of (4.8) as follows:

\[ (c \omega)^j \left( \prod_{n=0}^{k_{j+1}} q^{k_n + 1} \right)^{\mu - 1} = (c \omega)^j \cdot \left( \prod_{n=0}^{j-1} q^{k_n} \right)^{\mu - 1} \cdot \left( \prod_{n=0}^{j-1} q^{k_n + 1} \right)^{\mu - 1} \]

\[ = (c \omega)^j \cdot q^{j(\mu - 1)} \prod_{n=0}^{j-1} q^{k_n} \quad \geq \quad (c \omega)^j \cdot q^{j(\mu - 1)} \cdot \frac{\prod_{n=0}^{j-1} q^{k_n + 1}}{\prod_{n=0}^{j-1} q^{k_n}} \]

\[ = (c \omega)^j \cdot q^{j(\mu - 1)} \cdot \frac{\prod_{n=0}^{j-1} q^{k_n + 1}}{\prod_{n=0}^{j-1} q^{k_n}} = (c \omega q^{\mu + 1})^j \cdot \frac{q^{k_j}}{q^{k_0}}. \]

Substituting this estimate in (4.8), we finally have

\[ \left| \Sigma_j \right| \leq \frac{q^{k_j} e^{-\frac{q}{\omega}} (1 - q) q^{k_0}}{(c \omega q^{\mu + 1})^j q^{k_j}} = \frac{q^{k_0} e^{-\frac{q}{\omega}} (1 - q)}{(c \omega q^{\mu + 1})^j}. \]

To show that \( \left| \Sigma_j \right| \rightarrow 0 \) as \( j \rightarrow \infty \), it is enough to show \( (c \omega q^{\mu + 1}) > 1 \). Note that the first order Taylor expansion of \( q^{\mu + 1} = \exp \left( -\frac{\pi}{\omega} (\mu + 1) \right) \) is

\[ \exp \left( -\frac{\pi}{\omega} (\mu + 1) \right) = 1 - \frac{\pi}{\omega} (\mu + 1) + R_1 (\xi), \]

where \( R_1 (\xi) = \frac{\exp' (\xi)}{2} \left( \frac{(\pi (\mu + 1))}{\omega} \right)^2 > 0 \), and \( \xi \in \left( -\frac{\pi (\mu + 1)}{\omega}, 0 \right) \). The assumption of (e) gives us \( \frac{1}{\xi} + \pi (\mu + 1) < \omega \), and hence

\[ 1 < c \omega - c \pi (\mu + 1) = c (1 - \frac{\pi (\mu + 1)}{\omega}). \]
Since $R_1(\xi) > 0$, we obtain

$$1 < c\omega \left(1 - \frac{\pi (\mu + 1)}{\omega}\right) < c\omega \left(1 - \frac{\pi (\mu + 1)}{\omega} + R_1(\xi)\right) = c\omega \exp\left(-\frac{\pi}{\omega} (\mu + 1)\right) = c\omega q^{\mu + 1},$$

and this completes the proof of (v).

The next lemma estimates the measure of the points in the ‘steep’ interval $S_{k_0+1}^c$ which have the first $n$ iterates in the union of all ‘steep’ intervals.

**Lemma 4.5.** Let $k_0 \in \mathbb{N}$, $c \in \left(\frac{1}{\pi}, 1\right)$. Let $\Psi^c$ and $S_{k_0+1}^c$ be as in the passage before Theorem 4.2. Define $\varphi_1$ as in Lemma 2.1. Then for $k_0 \in \mathbb{N}$ we have

$$\left|S_{k_0+1}^c \cap \bigcap_{i=1}^n f^{-i}(\Psi^c)\right| \leq \frac{2^\mu q^{k_0} e^{-\frac{\pi}{\omega}} (1 - q)}{(c\omega q^{\mu + 1} (1 - q))^\pi}. \tag{4.9}$$

(The same estimate holds for $S_{k_0-1}^c$.)

**Proof.** Let $k_0 \in \mathbb{N}$ and $c \in \left(\frac{1}{\pi}, 1\right)$ be given. It is clear that $f(S_{k_0+1}^c)$ contains infinitely many ‘steep’ intervals, because it is a neighborhood of zero. Assume $\ell, i \in \mathbb{N}$ are such that $S_{k_0+1}^c \cap f^{-i}(S_{\ell, \pm 1}^c) \neq \emptyset$. Since $|f'(x)| \leq |x|^{\mu'}$ on $S_{k_0+1}^c$, one must have $q^{k_0\mu'} \geq \min\{|y| : y \in S_{\ell, \pm 1}^c\} \geq q^{\ell+2}$. It follows that $\ell \geq k_0\mu' - 2 \geq k_0 - 2$. Thus

$$f^{-i}(\Psi^c) = f^{-i}\left(\bigcup_{\ell \in \mathbb{N}} S_{\ell, \pm 1}^c\right) = f^{-i}\left(\bigcup_{\ell \geq k_0\mu' - 2} S_{\ell, \pm 1}^c\right).$$

Hence, the intersection in (4.9) equals $S_{k_0+1}^c \cap \bigcap_{i=1}^n f^{-i}\left(\bigcup_{\ell \geq k_0\mu' - 2} S_{\ell, \pm 1}^c\right)$. We now prove (4.9) by induction over $n$. For $n = 1$,

$$\left|S_{k_0+1}^c \cap f^{-1}(\Psi^c)\right| = \left|S_{k_0+1}^c \cap f^{-1}\left(\bigcup_{\ell \geq k_0\mu' - 2} S_{\ell, \pm 1}^c\right)\right| = \sum_{\ell \geq k_0\mu' - 2} \left|S_{k_0+1}^c \cap f^{-1}(S_{\ell, \pm 1}^c)\right|. \tag{4.10}$$

From (4.2) we have

$$\left|S_{\ell, \pm 1}^c\right| \leq 2q^\ell e^{-\frac{\pi}{\omega}} (1 - q). \tag{4.11}$$

Using (2.13), (4.1) and (4.11) in (4.10), we have

$$\left|S_{k_0+1}^c \cap f^{-1}(\Psi^c)\right| = \sum_{\ell \geq k_0\mu' - 2} \left|S_{k_0+1}^c \cap f^{-1}(S_{\ell, \pm 1}^c)\right| \leq \sum_{\ell \geq k_0\mu' - 2} \frac{1}{c\omega q^{(k_0+1)(\mu - 1)}} \left|S_{\ell, \pm 1}^c\right| \leq 2e^{-\frac{\pi}{\omega}} (1 - q) \sum_{\ell \geq k_0\mu' - 2} q^\ell. \tag{4.12}$$

Here, note that

$$\sum_{\ell \geq k_0\mu' - 2} q^\ell = \sum_{\ell \geq [k_0\mu - 2]} q^\ell = q^{[k_0\mu - 2]} \frac{1}{1 - q}. \tag{4.13}$$
where \([ \cdot ]\) denotes the ceiling function. Setting 
\(\varepsilon (k_0) := [k_0\mu - 2] - (k_0\mu - 1) \in [-1, 0)\) and using (4.13) in (4.12), we obtain
\[
\left| S_{k_0+1}^c \cap f^{-1} (\Psi^c) \right| \leq \frac{2 e^{-\frac{\varepsilon}{\omega}}}{c \omega q^{(k_0+1)(\mu-1)}} q^{[k_0\mu-2]} = \frac{2 q^{k_0} e^{-\frac{\varepsilon}{\omega}}}{c \omega} \cdot \frac{q^{[k_0\mu-2]} \cdot 1}{q^{k_0\mu-1}} \cdot 1 \\
= 2q^k(k_0) \cdot \frac{q^{k_0} e^{-\frac{\varepsilon}{\omega}}}{c \omega q^\mu} \leq \frac{2q^{-1} q^{k_0} e^{-\frac{\varepsilon}{\omega}}}{c \omega q^\mu} = \frac{2 q^{k_0} e^{-\frac{\varepsilon}{\omega}} (1 - q)}{c \omega q^\mu + (1 - q)}
\]
which proves the case \(n = 1\).

Assume the assertion is true for \(n\), i.e., for all \(k_0 \in \mathbb{N}\) we have
\[
\left| S_{k_0+1}^c \cap \bigcap_{i=1}^n f^{-i} (\Psi^c) \right| \leq q^{k_0} e^{-\frac{\varepsilon}{\omega}} (1 - q) \left( \frac{2}{c \omega q^{\mu+1} (1 - q)} \right)^n,
\]
then the same estimate is true for \(S_{k_0-1}^c\). Now we show that it is true for \(n + 1\). Using (4.10) for the third equality we obtain
\[
\left| S_{k_0+1}^c \cap \bigcap_{i=1}^{n+1} f^{-i} (\Psi^c) \right| = \left| S_{k_0+1}^c \cap f^{-1} (\Psi^c) \cap \cdots \cap f^{-(n-1)} (\Psi^c) \right| \\
= \left| S_{k_0+1}^c \cap f^{-1} \left( \bigcap_{i=0}^n f^{-i} (\Psi^c) \right) \right| \\
= \left| S_{k_0+1}^c \cap f^{-1} \left( \bigcup_{\ell \geq k_0 \mu - 2} S_{\ell, \pm 1}^c \cap \bigcap_{i=0}^n f^{-i} (\Psi^c) \right) \right|.
\]
Note that \(S_{\ell, \pm 1}^c \subset \Psi^c\) implies
\[
S_{\ell, \pm 1}^c \cap \bigcap_{i=0}^n f^{-i} (\Psi^c) = S_{\ell, \pm 1}^c \cap \bigcap_{i=1}^n f^{-i} (\Psi^c).
\]
So, we obtain
\[
\left| S_{k_0+1}^c \cap \bigcap_{i=1}^{n+1} f^{-i} (\Psi^c) \right| = \left| S_{k_0+1}^c \cap f^{-1} \left( \bigcup_{\ell \geq k_0 \mu - 2} \left( S_{\ell, \pm 1}^c \cap \bigcap_{i=1}^n f^{-i} (\Psi^c) \right) \right) \right| (4.15)
\]
Using (2.13), (4.1), (4.11), (4.13) and (4.14) for \(S_{k_0+1}^c\) and \(S_{k_0-1}^c\) in (4.15), we have
\[
\left| S_{k_0+1}^c \cap \bigcap_{i=1}^{n+1} f^{-i} (\Psi^c) \right| \leq \frac{1}{(c \omega) q^{(k_0+1)(\mu-1)}} \sum_{\ell \geq k_0 \mu - 2} 2 \left( \frac{2}{c \omega q^\mu + (1 - q)} \right)^n q^\ell e^{-\frac{\varepsilon}{\omega}} (1 - q) \\
= \frac{2^{n+1} q^{k_0} e^{-\frac{\varepsilon}{\omega}}}{(c \omega)^{n+1} (q^\mu + 1)^n q^\mu} \left( \frac{1}{1 - q} \right)^{n-1} \sum_{\ell \geq k_0 \mu - 2} q^\ell \\
= \frac{2^{n+1} q^{k_0} e^{-\frac{\varepsilon}{\omega}}}{(c \omega)^{n+1} (q^\mu + 1)^n q^\mu} \left( \frac{1}{1 - q} \right)^{n-1} q^{[k_0 \mu - 2]} \frac{1}{q^{k_0 \mu + 1} (1 - q)}
\]
With \( \varepsilon (k_0) \) as above, we obtain

\[
\left| S^c_{k_0,1} \cap \bigcap_{i=1}^{n+1} f^{-i} (\Psi) \right| \leq \frac{2^{n+1} q^{k_0} e^{-\frac{\varepsilon}{\omega}} q^c(k_0)}{(c\omega)^{n+1} (q^{\mu+1})^n q^\mu} \left( \frac{1}{1 - q} \right)^n
\]

\[
\leq \frac{2^{n+1} q^{k_0} e^{-\frac{\varepsilon}{\omega}} q^{-1}}{(c\omega)^{n+1} (q^{\mu+1})^n q^\mu} \left( \frac{1}{1 - q} \right)^n
\]

\[
= q^{k_0} e^{-\frac{\varepsilon}{\omega}} (1 - q) \left( \frac{2}{c\omega q^{\mu+1} (1 - q)} \right)^{n+1},
\]

so the assertion is true for \( n + 1 \) and hence, the proof of Lemma 4.5 is completed.

**Remark 4.6.** Let \( c \in (\frac{2}{\pi}, 1) \) and \( \mu > 1 \). Then \( \frac{1}{c} + \pi (\mu + 1) \leq \frac{c\pi^2 (2\mu + 3)}{2(c\pi - 2)} \).

**Proof.** Let \( c \in (\frac{2}{\pi}, 1) \). Then

\[
\frac{1}{c} + \pi (\mu + 1) = \frac{1 + c\pi \mu + c\pi}{c} = \pi + c\pi^2 \mu + c\pi^2 \leq \frac{2c\pi^2 \mu + 2c\pi^2 + 2\pi}{2(c\pi - 2)}.
\]

Since \( c\pi > 2 \), we have \( 2\pi < c\pi^2 \) and hence

\[
\frac{1}{c} + \pi (\mu + 1) \leq \frac{2c\pi^2 \mu + 3c\pi^2}{2(c\pi - 2)} = \frac{c\pi^2 (2\mu + 3)}{2(c\pi - 2)}.
\]

**Proof of Theorem 4.2.** (i) From assertion (i) in Lemma 4.4 we see that there exists a point \( y_0 \in (S^c_{k_0,1} \cap \Omega^c_{\omega}) \) with sign \( f^1 (y_0) = s_j \), because \( f^1 (y_0) \in S^c_{j,1} \).

(ii) Assume \( y_0 \in (S^c_{k_0,1} \cap \Omega^c_{\omega}) \). Assertion (ii) of Lemma 4.4 shows that \( \Sigma_j \ni y_0 \) and \( Z \cap \Sigma_j \neq \emptyset \). Further, assertion (v) of Lemma 4.4 shows that \( | \Sigma_j | \to 0 \) as \( j \to \infty \). This means that there exists a sequence \( (z_j) \subset Z \) with \( z_j \to y_0 \), and this completes the proof.

(iii) Let \( c \in (\frac{2}{\pi}, 1) \) be given. Remark 4.6 shows that the condition \( \omega > \frac{c\pi^2 (2\mu + 3)}{2(c\pi - 2)} \) from assertion (iii) of Theorem 4.2 implies the condition \( \omega > \frac{1}{c} + \pi (\mu + 1) \) of assertion (ii). Hence, \( (S^c_{k_0,1} \cap \Omega^c_{\omega}) \subset Z \) for all \( k_0 \in \mathbb{N} \). It follows that \( \Omega^c_{\omega} = \bigcup_{k_0 \in \mathbb{N}} (S^c_{k_0,1} \cap \Omega^c_{\omega}) \subset Z \), so \( \Omega^c_{\omega} \subset Z \).

To prove that \( \Omega^c_{\omega} \) has measure zero, we show \( \lim_{n \to \infty} | \Omega^c_{\omega} \cap S^c_{k_0,1} | = 0 \) for every \( k_0 \in \mathbb{N} \). For this purpose it is enough to show that under the conditions of assertion (iii) of Theorem 4.2, \( c\omega q^{\mu+1} (1 - q) > 2 \) in (4.9). We use the second order Taylor expansion of \( e^{-y} \) around 0 for \( y > 0 \),

\[
e^{-y} = 1 - y + \frac{y^2}{2} + R_3,
\]

with \( R_3 = \frac{\exp''(\xi)}{3!} (-y)^3 < 0 \) for some \( \xi \in (-y, 0) \). Hence, since

\[
q = e^{-\frac{\pi}{\omega}} = 1 - \frac{\pi}{\omega} + \frac{(\pi)^2}{2\omega^2} + R_3 \left( \frac{\pi}{\omega} \right) < 1 - \frac{\pi}{\omega} + \frac{(\pi)^2}{2\omega^2},
\]

we have

\[
1 - q = 1 - e^{-\frac{\pi}{\omega}} = \frac{\pi}{\omega} - \frac{\pi^2}{2\omega^2} - R_3 \left( \frac{\pi}{\omega} \right) > \frac{\pi}{\omega} - \frac{\pi^2}{2\omega^2}.
\]

(4.16)

On the other hand, with appropriate \( \xi \),

\[
q^{\mu+1} = e^{-\frac{\pi}{\omega} (\mu+1)} = 1 - \frac{\pi (\mu + 1)}{\omega} + \frac{\exp''(\xi)}{2!} \left( \frac{-\pi (\mu + 1)}{\omega} \right)^2 > 1 - \frac{\pi (\mu + 1)}{\omega}.
\]

(4.17)
Using (4.16) and (4.17), we get

\[ c\omega q^{\mu + 1} (1 - q) > c\omega \left( 1 - \frac{\pi (\mu + 1)}{\omega} \right) \left( \frac{\pi}{2\omega^2} \right) = c\pi \left( 1 - \frac{\pi \mu + \pi}{2\omega} \right) \]

\[ = c\pi \left( 1 - \frac{(3\pi + 2\mu\pi)}{2\omega} + \frac{\pi^2 (\mu + 1)}{2\omega^2} \right) \]

\[ > c\pi \left( 1 - \frac{(3 + 2\mu)}{2\omega} \right). \]  

(4.18)

In view of Remark 4.6, and using the assumption which is given in the assertion (iii) of Theorem 4.2 in (4.18), we finally obtain

\[ c\omega q^{\mu + 1} (1 - q) > c\pi \left( 1 - \frac{\pi (3 + 2\mu)}{2\omega} \right) = c\pi \left( 1 - \frac{2}{c\pi} \right) = 2. \]

5 The behavior of the points whose orbits follow ‘flat-steep-flat’ intervals

In chapter three we analyzed the behavior of the points which are mapped from ‘flat’ intervals to some other ‘flat’ intervals, and in chapter four we studied the behavior of the points which are mapped from ‘steep’ intervals to some other ‘steep’ intervals. Finally in this chapter, as we briefly mentioned in the summary of this thesis, we first construct a specific type of orbit whose points travel from ‘flat’ intervals to ‘steep’ intervals, then from ‘steep’ intervals again to ‘flat’ intervals under the iteration (see Figure 5.1).

Figure 5.1: The parameters adjusted so that \( f(m_k) = q^{f_1(k)} \), and the sets \( \hat{U}_k^L \) and \( \hat{U}_k^R \) constructed as counterimages under \( f^2 \) of the interval \( U_{\ell_2(k)} \subset U_{\ell_2(k)} \) (indicated only for the lower endpoint of \( \hat{U}_k^L \)). The dotted parts of the graph indicate possible maxima/minima in between which are not shown.

Besides, to avoid repeating the same expression, we shall use \( g_{\omega,\mu+1-j} \) as in Lemma 2.1 and \( c \in (0,1) \) for the rest of the paper. For a specific choice of \( \mu, \omega > 0 \), maxima \( m_k \) get
mapped to zeros \( q_{f^i(k)} \) of \( f_{\mu, \omega} = f \). We shall first introduce ‘flat’ intervals of the form \( U_k = [m_k - \delta_k, m_k + \delta_k] \) for odd \( k \) and use the notations \( U_k^L = [m_k, m_k + \delta_k] \) and \( U_k^L = [m_k - \delta_k, m_k] \) for the right and left part of \( U_k \) respectively. We also introduce ‘steep’ intervals \( S_{\ell_1(k)} \), where \( \ell_1(k) = k\mu + 1 \), of the form \( [q_{f^i(k)} - r_{\ell_1(k)}, q_{f^i(k)}] \), with a suitable \( r_{\ell_1(k)} \). Then we define \( U = \bigcup_{k \in \mathbb{N}} U_k \), \( S = \bigcup_{k \in \mathbb{N}} S_{\ell_1(k)} \), and we construct orbits \( (f^j(x)) \) for \( j \in \mathbb{N} \), with the properties

\[
f^j(x) \in \begin{cases} U, & j \text{ is even,} \\ S, & j \text{ is odd.} \end{cases}
\]

Furthermore, for \( k, \mu, \omega \in \mathbb{N}, \omega > 0 \) and with \( \varphi_1 \) as in Lemma 2.1, we define

\[
\ell_2(k) := \min \left\{ \ell \in \mathbb{N} : q_\ell \leq q_{f^i(k)} \mu \cdot q_{\varphi_1(n-2)} \cdot \frac{c(1-c) \omega^3}{4g_{\omega, \mu} \cdot g_{\omega, \mu-1}} \right\}.
\]

We denote by \( \ell_2(k) \) the \( j \)th iterate of the function \( \ell_2 \) applied to \( k \). Then, given a symbol sequence of the form \( \{L, R\}^{n+1} \), where symbols represent the left ‘L’ or right ‘R’ hand part of \( U_k \) (that is \( U_k^L, U_k^R \)), we construct corresponding orbits of \( f \). Given a finite sequence

\[
s = (s_0, s_1, s_2, \ldots, s_n) \in \{L, R\}^{n+1}
\]

and \( k \in \mathbb{N} \), we first construct the subset of points \( x \) in \( U_k \) which follow this symbol sequence in the sense that \( f^{2j}(x) \in U_k^{L_\ell_2(k)} \) or \( f^{2j}(x) \in U_k^{R_\ell_2(k)} \), \( j = 0, 1, 2, \ldots, n \) depending on whether \( s_j = L \) or \( s_j = R \). Hence, we construct the set \( I_{k,s}^{n} = \bigcap_{j=0}^{n} f^{-2j}(U_k^{L_\ell_2(k)}) \) and the set \( I_{k,s}^{\infty} = \bigcup_{s \in \{L, R\}^{n+1}} I_{k,s}^{n} \) which is the set of points following symbol sequences in the set \( \{L, R\}^{0,1,2,\ldots,n} \). Note that strict monotonicity of \( f^2 \) on each interval \( U_k^{L} \) and \( U_k^{R} \) implies that the sets \( I_{k,s}^{n} \) are closed intervals. The corresponding set for infinite symbol sequences is \( I_{k,s}^{\infty} = \bigcap_{j=0}^{\infty} f^{-2j}(U_k^{L_\ell_2(k)}) \). Finally, we analyze the Lebesgue measure of the set \( \Gamma_k^n \), and consider the limit as \( n \to \infty \).

Note that the ‘steep’ intervals \( S_k \) that we use in our calculations in this chapter are some subintervals of \( (m_k, q_k] \), whereas the ‘steep’ intervals which were used in the fourth chapter are some subintervals of \( (m_{k+1}, m_k) \). In the theorem below we restrict ourselves to \( \mu \in \mathbb{N} \) for simplicity.

**Theorem 5.1.** Let \( k \) be a positive odd integer. Let \( c \in (0, 1) \), and \( \mu \in \mathbb{N}, \mu \geq \max\left\{ \left( \frac{3\kappa}{2\pi} \right)^2 \left( \frac{1-\varepsilon}{2\varepsilon} \right), 15 \right\} \) be given. Then there exist an \( \omega > 0 \) (depending on \( \mu \)) such that \( f = f_{\mu, \omega} \) has the following properties:

(i) Let a sequence of the form \( s \in \{L, R\}^{N_0} \) be given. Then, there exists exactly one point \( x_{k,s} \in U_k \) with the property:

For all \( n \in \mathbb{N}_0 \), \( f^{2n}(x_{k,s}) \in U_{\ell_2(k)} \), and \( f^{2n}(x_{k,s}) \) is to the left of \( m_{\ell_2(k)} \) or to the right of \( m_{\ell_2(k)} \), depending on whether \( s_n = L \) or \( s_n = R \). That is, \( I_{k,s}^{\infty} = \{x_{k,s}\} \).

(ii) The measure of \( \Gamma_k^n \) as defined above goes to zero, as \( n \to \infty \).

The proof requires several lemmas and propositions. The proof of the following lemma is analogous to the proof of Lemma 3.3, but is included for completeness.

**Lemma 5.2.** Define \( \varphi_1 \) as in Lemma 2.1. Then the following statements are true.
(i) Assume $\mu \in \mathbb{N}$, $\mu \geq 15$ and define $\ell_1 (k)$ as in the passage before Theorem 5.1. Then there exists an $\omega \in (0, 1)$ such that for all $k \in \mathbb{N}$, $f$ has the property

$$|f (m_k)| = q^{\ell_1(k)},$$

which is equivalent to

$$\exp \left( \frac{\pi - \mu \arctan \left( \frac{\omega}{\mu} \right)}{\omega} \right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}. \tag{5.2}$$

(ii) For any choice of $\omega$ as in assertion (i), we have $\omega \to 0$ as $\mu \to \infty$.

Proof. (i) Let $k \in \mathbb{N}$ be given. With $m_k$ from (2.10), we have from (2.11)

$$|f (m_k)| = \exp \left( - \frac{k \pi \mu + \varphi_1 \mu}{\omega} \right) \cdot \sin(\varphi_1).$$

Using (2.3) we obtain

$$|f (m_k)| = \exp \left( - \frac{k \pi \mu + \varphi_1 \mu}{\omega} \right) \cdot \frac{1}{\sqrt{1 + \frac{\mu^2}{\omega^2}}} \tag{5.3}$$

On the other hand, from (2.8) we have

$$q^{\ell_1(k)} = \exp \left( - \frac{\pi \ell_1 (k)}{\omega} \right). \tag{5.4}$$

With (5.3) and (5.4) together, we see that (5.1) is equivalent to

$$\exp \left( - \frac{\pi \ell_1 (k)}{\omega} \right) = \exp \left( - \frac{k \pi \mu + \varphi_1 \mu}{\omega} \right) \frac{1}{\sqrt{1 + \frac{\mu^2}{\omega^2}}}$$

and hence to

$$\exp \left( \frac{\pi (\ell_1 (k) - k \mu) - \varphi_1 \mu}{\omega} \right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}. \tag{5.5}$$

So, if we substitute $\ell_1 (k) = k \mu + 1$ and the value of $\varphi_1$ given by Lemma 2.1 in (5.5), we finally get that (5.1) is equivalent to

$$\exp \left( \frac{\pi - \mu \arctan \left( \frac{\omega}{\mu} \right)}{\omega} \right) = \sqrt{1 + \frac{\mu^2}{\omega^2}},$$

which proves the equivalence of (5.1) and (5.2). Now, we want to find $\omega$ and $\mu$ such that $|f (m_k)| = q^{\ell_1(k)}$. Define

$$F (\omega, \mu) = \exp \left( \frac{\pi - \mu \arctan \left( \frac{\omega}{\mu} \right)}{\omega} \right) - \sqrt{1 + \frac{\mu^2}{\omega^2}}$$
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and try to find $F(\omega, \mu) = 0$ at least for a special pair of $(\omega, \mu)$ (see Figure 5.2). On the one hand, for a fixed $\mu > 2$, $\arctan \left( \frac{\omega}{\mu} \right) \to 0$ as $\omega \to 0$. Hence, due to the exponential growth, $F(\omega, \mu) \to \infty$ as $\omega \to 0$. On the other hand, for $\omega = 1$ we have

$$F(1, \mu) = \exp \left( \pi - \mu \arctan \left( \frac{1}{\mu} \right) \right) - \sqrt{1 + \mu^2}. \tag{5.6}$$

From 3.9 we have $\mu \arctan \left( \frac{1}{\mu} \right) \geq \frac{1}{2}$ for $\mu > 2$, and using this estimate in (5.6), we finally have $F(1, \mu) < e^{\pi - \frac{1}{2}} - \sqrt{1 + \mu^2}$. From the fact that $e^{\pi - \frac{1}{2}} < 15$, we finally have $F(1, \mu) < 0$, if we choose $\mu \geq 15$. With the intermediate value theorem, it is clear that there exists at least one $\omega \in (0, 1)$ which satisfies $F(\omega, \mu) = 0$ for fixed $\mu$. This gives the proof of assertion (i).

(ii) The proof is analogous to the proof of the assertion (ii) of Lemma 3.3.

In order to find a numerical solution, one can use the bisection method, and we found numerically that there is no solution for $\mu < 2.3$. The numerical investigation suggests that $\omega$ in Lemma 5.2 is unique. We made no effort to prove that, because part (ii) is true for any possible choice of $\omega$.

The next three propositions (5.3–5.5) give some preparatory calculations.

![Figure 5.2: Graph of $F(\cdot, \mu)$ for $\mu = 15$.](image)

**Proposition 5.3.** Let $\varphi_1$ be as in Lemma 2.1. Set $\alpha(\omega, \mu, c) := \exp \left( \frac{\mu - \varphi_1}{\mu \omega} \right) \sqrt{\frac{1 - c}{2\omega}}$. If $\mu \in \mathbb{N}$,

$$\mu \geq \max \left\{ \left( \frac{30e}{7\pi} \right)^2 \left( \frac{1 - c}{2c} \right), 15 \right\},$$

and $\omega$ is a corresponding value obtained as in Lemma 5.2, then we have $\alpha(\omega, \mu, c) < \frac{1}{2}$.

**Proof.** Let $\mu$ and $\omega \in (0, 1)$ be as in the assumption. Then, it is clear that $\frac{3\mu^2}{\omega^2} \geq 1$, and in view of (5.2) we have

$$\frac{2\mu}{\omega} = \sqrt{\frac{3\mu^2}{\omega^2} + \frac{\mu^2}{\omega^2}} \geq \sqrt{1 + \frac{\mu^2}{\omega^2}} = \exp \left( \frac{\pi - \mu \arctan \left( \frac{\omega}{\mu} \right)}{\omega} \right).$$
Since \( \arctan \left( \frac{\omega}{\mu} \right) \leq \frac{\omega}{\mu} \), we get

\[
\frac{2\mu}{\omega} \geq \exp \left( \frac{\pi - \mu \cdot \frac{\omega}{\mu}}{\omega} \right) = \exp \left( \frac{\pi}{\omega} - 1 \right),
\]

and hence we have \( 2\mu e \geq \omega e^{\frac{\pi}{\omega}} \). Using the second order Taylor expansion of \( e^{\frac{\pi}{\omega}} \) in the last inequality, we obtain

\[
2\mu e \geq \omega \left( 1 + \frac{\pi}{\omega} + \frac{1}{2} \frac{\pi^2}{\omega^2} \right) \geq \frac{1}{2} \pi^2
\]
or

\[
4\mu e \geq \frac{\pi^2}{\omega}. \tag{5.7}
\]

On the other hand, we know that \( \mu \geq \left( \frac{30\omega}{7\pi} \right)^2 \left( 1 - \frac{e}{2c} \right) \), so \( \sqrt{\mu} \geq \frac{30\omega}{7\pi} \sqrt{\frac{1 - e}{2c}} \) which implies

\[
\frac{1}{2} \geq e^{\frac{1}{2}} \sqrt{\frac{1 - e}{2c} \frac{30\pi^3}{4\pi \sqrt{\mu}}}
\]

Since \( e^{\frac{1}{2}} > e^{\frac{1 - e}{2c}} \), it follows that

\[
\frac{1}{2} > \exp \left( \frac{1}{2} - \frac{1}{\mu} \right) \cdot \sqrt{\frac{1 - e}{2c}} \cdot \frac{15\pi}{4e} \cdot \sqrt{\frac{1 - e}{2c} \frac{30\pi^3}{4\pi \sqrt{\mu}}}
\]

On the other hand, since \( \mu \geq 15 \), we have \( \frac{1}{\mu - 1} \leq \frac{15}{14\mu} \), and with the fact that \( \arctan \left( \frac{\omega}{\mu} \right) \leq \frac{\omega}{\mu} \) we get

\[
\frac{1}{2} > \exp \left( \frac{(\mu - 2) \arctan \left( \frac{\omega}{\mu} \right)}{2\omega} \right) \cdot \sqrt{\frac{1 - e}{2c} \frac{4e}{\pi (\mu - 1)}}
\]

Finally, using (5.7) and the definition of \( \varphi_1 \), we obtain

\[
\frac{1}{2} > \exp \left( \frac{(\mu - 2) \varphi_1}{2\omega} \right) \cdot \sqrt{\frac{1 - e}{2c} \frac{1}{\omega^2 (\mu - 1)^2}}
\]

\[
\geq \exp \left( \frac{(\mu - 2) \varphi_1}{2\omega} \right) \cdot \sqrt{\frac{1 - e}{2c} \frac{1}{\omega^2 + (\mu - 1)^2}}
\]

\[
= \exp \left( \frac{(\mu - 2) \varphi_1}{2\omega} \right) \cdot \sqrt{\frac{1 - e}{2c} \frac{\varphi_1}{\omega \mu - 1}} = \alpha (\omega, \mu, c)
\]

and this completes the proof. \( \square \)

**Proposition 5.4.** Let \( \varphi_1 \) be as in Lemma 2.1 and \( c \in (0, 1) \) be given. Set \( \tilde{\eta}_1 (\omega, \mu) := \frac{\omega \varphi_1 (\mu - 2)}{\omega \mu - 1} \), \( \tilde{\eta}_2 (\omega) := \frac{e}{2} - \frac{\varphi_1}{2} \), \( \tilde{\eta}_3 (\omega) := 1 - \frac{e}{2} + \frac{\varphi_1}{2} \) and \( \tilde{\eta}_4 (\omega, \mu) := \frac{\sqrt{(1 - e) \omega}}{\omega \mu - 1} \). There exist \( \omega_0 > 0 \) and \( \mu_0 > 3 \) such that for \( \omega \leq \omega_0 \) and \( \mu \geq \mu_0 \), the number

\[
\tilde{\eta} := \min \{ \tilde{\eta}_1 (\omega, \mu), \tilde{\eta}_2 (\omega), \tilde{\eta}_3 (\omega), \tilde{\eta}_4 (\omega, \mu) \}
\]

satisfies

\[
\tilde{\eta} = \tilde{\eta}_4 (\omega, \mu). \tag{5.8}
\]
Proof. We prove that \( \tilde{\eta}_4 (\omega, \mu) \leq \tilde{\eta}_1 (\omega, \mu) \leq \min \{ \tilde{\eta}_2 (\omega), \tilde{\eta}_3 (\omega) \} \) for \( \mu \) large enough, \( \omega \) small enough. For \( \omega > 0 \), we have \( g_{\omega, \mu-1} \geq \sqrt{(\mu - 1)^2}, g_{\omega, \mu-2} \geq \sqrt{(\mu - 2)^2} \) and using these simplifications, we obtain

\[
\tilde{\eta}_1 (\omega, \mu) \leq \frac{\omega e^{-\eta_{(y-2)}}}{2 \sqrt{(\mu - 1)^2 (\mu - 2)^2}} \geq \frac{\omega e^{-\eta_{(y-2)}}}{2 (\mu - 1) (\mu - 2)}. \tag{5.9}
\]

We have already defined

\[
\tilde{\eta}_2 (\omega) = \frac{e^{-\varphi_1}}{2} = \frac{1}{2} e^{-\varphi_1} \left( 1 - e^{-\varphi_1} \right),
\]

and there exists \( \tilde{\omega}_1 > 0 \) such that for \( \omega \in \omega_0 \in (0, \tilde{\omega}_1) \), the property \( \varphi_1 - \pi < 0 \) implies \( (1 - e^{-\varphi_1}) > \frac{1}{2} \). Hence, for such \( \omega \) we have

\[
\tilde{\eta}_2 (\omega) \geq \frac{1}{4} e^{-\varphi_1}. \tag{5.10}
\]

From (5.9) and (5.10) it is obvious that \( \tilde{\eta}_1 (\omega, \mu) \leq \frac{\omega e^{-\eta_{(y-2)}}}{2} \leq \tilde{\eta}_2 (\omega) \) for \( \mu \geq 3 \) and \( \omega \leq \omega_{12} \), where \( \omega_{12} =: \min \{ \frac{1}{2}, \tilde{\omega}_1 \} \). Analogously there exists \( \omega_{13} > 0 \) such that for \( \mu \geq 3 \) and \( \omega \leq \omega_{13} \), one has \( \tilde{\eta}_1 (\omega, \mu) \leq \tilde{\eta}_3 (\omega) \); observe \( \tilde{\eta}_3 (\omega) \rightarrow \frac{\mu}{2} \) as \( \omega \rightarrow 0 \). There exist \( c_1, c_2 > 0 \), and \( \tilde{\mu}_0 > 0 \) such that for \( \mu \geq \tilde{\mu}_0 \) we have \( \tilde{\eta}_1 (\omega, \mu) \geq c_1 \omega e^{-\varphi_1 (\mu-2)} / \mu^2 \) and \( \tilde{\eta}_4 (\omega, \mu) \leq c_2 \sqrt{\omega e^{-\varphi_1}} \). Hence, we have for \( \mu \geq \tilde{\mu}_0 \) and \( \omega > 0 \)

\[
\frac{\tilde{\eta}_4 (\omega, \mu)}{\tilde{\eta}_1 (\omega, \mu)} \leq \frac{c_2 \sqrt{\omega e^{-\varphi_1}}}{c_1 \omega e^{-\varphi_1 (\mu-2)} / \mu^2} = c_2 \frac{1}{c_1 \sqrt{\omega}} \exp \left( \frac{\varphi_1 (\mu-2) - \pi}{2} \right).
\]

Substituting the explicit form of \( \varphi_1 \) as in Lemma 2.1, the last equality turns to

\[
\frac{\tilde{\eta}_4 (\omega, \mu)}{\tilde{\eta}_1 (\omega, \mu)} \leq c_2 \frac{1}{c_1 \sqrt{\omega}} \exp \left( \frac{(\mu - 2) \arctan \left( \frac{\omega}{\mu} \right) - \pi}{2} \right). \tag{5.11}
\]

Using the fact that

\[
\lim_{\mu \to \infty, \omega \to 0} \left( \frac{(\mu - 2) \arctan \left( \frac{\omega}{\mu} \right)}{\omega} \right) = \lim_{\mu \to \infty, \omega \to 0} \left( \frac{(\mu - 2) \omega}{\mu^2} \right) = 1, \quad \text{and}
\]

\[
\lim_{\omega \to 0} \frac{1}{\sqrt{\omega}} \exp \left( -\frac{\pi}{2\omega} \right) = 0
\]

in (5.11), we finally have

\[
\lim_{\mu \to \infty, \omega \to 0} \left( \frac{\tilde{\eta}_4 (\omega, \mu)}{\tilde{\eta}_1 (\omega, \mu)} \right) \leq \lim_{\mu \to \infty, \omega \to 0} \frac{c_2}{c_1 \sqrt{\omega}} \exp \left( 1 - \frac{\pi}{2\omega} \right) = 0
\]

and that shows that there exists \( \mu_0 \geq \tilde{\mu}_0 \) and \( \omega_0 \in (0, \min \{ \omega_{12}, \omega_{13} \} \) such that for \( \omega \leq \omega_0 \) and \( \mu \in \mathbb{N}, \mu \geq \mu_0 \) one has \( \tilde{\eta}_4 (\omega, \mu) \leq \tilde{\eta}_1 (\omega, \mu) \leq \min \{ \tilde{\eta}_2 (\omega), \tilde{\eta}_3 (\omega) \} \).

Now, we aim at finding an interval \( U_k := [m_k - \delta_k, m_k + \delta_k] \) as indicated in the passage before Theorem 5.1, which gets mapped to a ‘steep’ interval \( S_{\ell_1(k)} \), but we first provide upper and lower estimates for the second derivative \( f''_{\mu, \omega} \) of \( f \).
Proposition 5.5. Let $k \in \mathbb{N}$. Assume that with $\mu_0$ and $\omega_0$ as in Proposition 5.4, one has $\mu \geq \mu_0$ and $\omega \leq \omega_0$. Define $\eta$ as in Proposition 5.4 and set $\delta := \eta \delta_1 f_k := [q^{k+1}, q^k]$. Then

$$U_k := [m_k - \delta_k, m_k + \delta_k] \subset [q^{k+1}, q^k]$$

and the following estimates hold:

$$\forall x \in [m_k - \delta_k, m_k + \delta_k] : g_{\omega, \mu} \cdot g_{\omega, \mu-1} \cdot q^{(\mu-2)} \geq |f'''(x)| \geq \left( \frac{q^k e^{-\frac{\pi}{2}}}{2} \cdot \omega \cdot g_{\omega, \mu} \right)^{\mu-2}.$$

(5.12)

Proof. Let $k \in \mathbb{N}$. With $\eta$ from Lemma 3.2, the definition of $\eta$ given in Proposition 5.4 shows $\eta \leq \min \{ \eta_2, \eta_3 \} = \eta$. Hence, in view of Lemma 3.2, we see that

$$U_k = [m_k - \delta_k, m_k + \delta_k] \subset [m_k - \eta q^k, m_k + \eta q^k] \subset [q^{k+1}, q^k] = I_k.$$

Further, inserting $m_k$ from (2.10) in (2.4) we have

$$|f''(m_k)| = g_{\omega, \mu} \cdot g_{\omega, \mu-1} \cdot |m_k|^{\mu-2} \left| \sin (\omega \ln (m_k) + q_1 + q_2) \right|$$

$$= g_{\omega, \mu} \cdot g_{\omega, \mu-1} \cdot |m_k|^{\mu-2} \left| \sin (\omega \ln (x) + q_1 + q_2 + q_3) \right|.$$

Using $q_2 = \arctan(\frac{\omega}{p+1})$ from Lemma 2.1, we have $\sin (q_2) = \frac{\omega}{g_{\omega, \mu-1}}$ and, inserting this value in the last equality, we have

$$|f''(m_k)| = g_{\omega, \mu} \cdot g_{\omega, \mu-1} \cdot |m_k|^{\mu-2} \cdot \frac{\omega}{g_{\omega, \mu-1}} = |m_k|^{\mu-2} \omega \cdot g_{\omega, \mu}. \quad (5.13)$$

From (2.5) we have on $[q^{k+1}, q^k]$

$$|f'''(x)| = |g_{\omega, \mu} \cdot g_{\omega, \mu-1} \cdot g_{\omega, \mu-2} \cdot x^{\mu-3} \sin (\omega \ln (x) + (q_1 + q_2 + q_3))|$$

$$\leq g_{\omega, \mu} \cdot g_{\omega, \mu-1} \cdot g_{\omega, \mu-2} \cdot q^{(\mu-3)}. \quad (5.14)$$

From (2.13) for $x \in [m_k - \delta_k, m_k + \delta_k]$ and with the definition of $\delta_k$, we also have

$$|f''(x)| \geq |f''(m_k)| - \delta_k \cdot \max_{[m_k - \delta_k, m_k + \delta_k]} |f'''|$$

$$= |f''(m_k)| - \eta q^k \cdot \max_{[m_k - \delta_k, m_k + \delta_k]} |f'''|$$

$$\geq |f''(m_k)| - \eta q^k \cdot \max_{[m_k - \delta_k, m_k + \delta_k]} |f'''|. \quad (5.15)$$

With the definition of $\eta_1$, using (2.10), (5.13) and (5.14) in (5.15), we finally have

$$|f''(x)| \geq m_k^{\mu-2} \omega \cdot g_{\omega, \mu} - \frac{q^k e^{-\frac{\pi}{2}}}{2g_{\omega, \mu-1} \cdot g_{\omega, \mu-2} \cdot g_{\omega, \mu-1} \cdot g_{\omega, \mu-2} \cdot q^{(\mu-3)}}$$

$$= \left( q^k e^{-\frac{\pi}{2}} \right)^{\mu-2} \cdot \omega \cdot g_{\omega, \mu} - \frac{q^k e^{-\frac{\pi}{2}}}{2} \cdot \omega \cdot g_{\omega, \mu}$$

$$= \left( q^k e^{-\frac{\pi}{2}} \right)^{\mu-2} \cdot \omega \cdot g_{\omega, \mu} = \frac{2}{2} \cdot \omega \cdot g_{\omega, \mu}.$$

This is the lower estimate for $|f''(x)|$; the upper estimate even on the interval $[q^{k+1}, q^k]$ follows with the formula for $f''$ in (2.4). □
For \( k \in \mathbb{N} \), we specify the boundaries of an associated ‘steep’ interval \( S_{\ell_1(k)} \) with the next proposition.

**Proposition 5.6.** Let \( k \in \mathbb{N} \). Assume \( \mu \) and \( \omega \) are as in Proposition 5.4 and define \( \ell_1(k) = k\mu + 1 \) as in the passage before Theorem 5.1. Set \( r_{\ell_1(k)} := \frac{(1-c)\omega}{g_{\omega,\mu} \cdot g_{\omega,\mu-1}} \) \( q_{\ell_1(k)} \) and \( S_{\ell_1(k)} := \left[ q_{\ell_1(k)} - r_{\ell_1(k)}, q_{\ell_1(k)} \right] \). Then, \( S_{\ell_1(k)} \subset \left( m_{\ell_1(k)}, q_{\ell_1(k)} \right) \) and on \( S_{\ell_1(k)} \) we have

\[
|f'| \geq c_1 q_{\ell_1(k)}(\mu-1). \tag{5.16}
\]

**Proof.** Let \( k \in \mathbb{N} \). From the upper estimate of (5.12), on \( S_{\ell_1(k)} \) we have

\[
\|f''\|_{L^\infty(S_{\ell_1(k)})} \leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q_{\ell_1(k)}(\mu-2). \tag{5.17}
\]

From (2.13) we have

\[
\forall x \in S_{\ell_1(k)} : \left| f'(x) \right| \geq \left| f' \left( q_{\ell_1(k)} \right) \right| - \left| f'' \right|_{L^\infty(S_{\ell_1(k)})} \cdot r_{\ell_1(k)}, \tag{5.18}
\]

and from (2.9) we also have \( \left| f'(q_{\ell_1(k)}) \right| = \omega q_{\ell_1(k)}(\mu-1) \). Using (5.17) and substituting the explicit values of both \( \left| f'(q_{\ell_1(k)}) \right| \) and \( r_{\ell_1(k)} \) in (5.18), we get

\[
\forall x \in S_{\ell_1(k)} : \left| f'(x) \right| \geq \left| f' \left( q_{\ell_1(k)} \right) \right| - g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q_{\ell_1(k)}(\mu-2) \cdot r_{\ell_1(k)}
\]

\[
= \omega q_{\ell_1(k)}(\mu-1) - g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q_{\ell_1(k)}(\mu-2) \cdot \left( 1 - \frac{c_1}{\omega} \right) q_{\ell_1(k)}
\]

\[
= \omega q_{\ell_1(k)}(\mu-1) - (1-c) \omega q_{\ell_1(k)}(\mu-1) = c_1 q_{\ell_1(k)}(\mu-1). \]

It follows now from \( f'(m_{\ell_1(k)}) = 0 \) that \( m_{\ell_1(k)} < q_{\ell_1(k)} - r_{\ell_1(k)} \). \( \square \)

From the graph of the map one can understand that the image of \( S_{\ell_1(k)} \) under \( f_{\mu,\omega} \) includes many ‘steep’ and ‘flat’ intervals, but we continue our calculations with a subinterval \( \tilde{S}_{\ell_1(k)} \) of \( S_{\ell_1(k)} \) which is contained in \( f \left( U_\ell \right) \). The next lemma gives an estimate for the size of \( f \left( U_\ell \right) \) with a relation between \( \tilde{S}_{\ell_1(k)} \) and \( S_{\ell_1(k)} \).

Note that for the sake of simplicity we shall use \( k \) as a positive odd integer number for the rest of the paper. Note also that in addition to the notations \( U_{\ell}^L, U_{\ell}^R \) which represent to the left ‘L’ and right ‘R’ hand part of \( U_\ell \) respectively, we also use the notation \( U_{\ell}^{L,R} \) in statements which are valid for both \( U_{\ell}^L \) and \( U_{\ell}^R \).

**Lemma 5.7.** Let \( k \) be a positive odd integer number. Let \( \omega \) and even integer \( \mu \) be as in Proposition 5.4 and satisfying (5.2). Define \( \tilde{\eta} \) as in Proposition 5.4, \( \delta_\ell \) and \( U_\ell \) as in Proposition 5.5, and \( U_{\ell}^{L,R} \) as in the passage before Theorem 5.1. Then the following statements are true.

(i) Define \( r_{\ell_1(k)} \) and \( S_{\ell_1(k)} \) as in Proposition 5.6. Then we have \( f \left( U_\ell \right) \subset S_{\ell_1(k)} \);

(ii) Set

\[
\tilde{r}_{\ell_1(k)} := q_{\ell_1(k)} \cdot \frac{q_{\ell_2(\mu-2)}}{\alpha} \cdot \frac{(1-c) \cdot \omega^2}{4g_{\omega,\mu} \cdot g_{\omega,\mu-1}} \tag{5.19}
\]

and \( \tilde{S}_{\ell_1(k)} = \left[ q_{\ell_1(k)} - \tilde{r}_{\ell_1(k)}, q_{\ell_1(k)} \right] \). Then we have \( f \left( U_{\ell}^{L,R} \right) \supset \tilde{S}_{\ell_1(k)} \).
Proof. Note that due to (5.1), and since \( k \) is odd (see (2.11)), \( \max \{ f(U_k) \} = f(m_k) = q^f(k) \). For the interval \( U_k \) we have

\[
\min \left\{ \left| f(m_k - \delta_k) - q^f(k) \right|, \left| f(m_k + \delta_k) - q^f(k) \right| \right\} \\
\leq |f(U_k)| \\
\leq \max \left\{ \left| f(m_k - \delta_k) - q^f(k) \right|, \left| f(m_k + \delta_k) - q^f(k) \right| \right\}.
\]

It follows from second order Taylor expansion of \( f \) around the extremum \( m_k \) and from (5.1) that

\[
\min_{\xi \in [m_k - \delta_k, m_k + \delta_k]} |f''(\xi)| \cdot \frac{\delta_k^2}{2} \leq |f(U_k)| \leq \max_{\xi \in [m_k - \delta_k, m_k + \delta_k]} |f''(\xi)| \cdot \frac{\delta_k^2}{2}.
\]

(5.20)

Consequently, using (5.8) and inserting the upper estimate of \( |f''| \) given in (5.12) and the value of \( \delta_k \) in the upper estimate of (5.20), we finally get

\[
|f(U_k)| \leq \max_{\xi \in [m_k - \delta_k, m_k + \delta_k]} |f''(\xi)| \cdot \frac{\delta_k^2}{2} \leq g_{\omega, \mu} \cdot g_{\omega, \mu - 1} \cdot q^{k(\mu - 2)} \frac{\delta_k^2}{2} \\
= g_{\omega, \mu} \cdot g_{\omega, \mu - 1} \cdot q^{k(\mu - 2)} (\eta_4)^2 q^{2k} \\
\leq g_{\omega, \mu} \cdot g_{\omega, \mu - 1} \cdot q^{k(\mu - 2)} \left( \frac{\sqrt{(1 - c) q\omega}}{g_{\omega, \mu} \cdot g_{\omega, \mu - 1}} \right)^2 q^{2k} \\
= q^{k+1} \left( \frac{(1 - c) \omega}{g_{\omega, \mu} \cdot g_{\omega, \mu - 1}} - q^{f(k)} \right) = r_{f(k)} = |S_{f(k)}|.
\]

From (5.1) we know that \( f(m_k) = q^f(k) \). So, \( f(U_k) = \left[ \min f(U_k), q^f(k) \right] \) and the estimate

\[
|f(U_k)| \leq r_{f(k)} \text{ shows } f(U_k) \subset \left[ q^f(k) - r_{f(k)}, q^f(k) \right] = S_{f(k)}
\]

and this completes the proof of assertion (i).

Note that, although there is no symmetry between the graph of \( f_{\mu, \omega} \) to the left and right hand side of \( U_k \), we can estimate the size of \( f(U_k^L) \) and \( f(U_k^R) \) in a similar way. Substituting the lower bound of \( |f''| \) given by (5.12), the value of \( \delta_k \) in the analogue of the lower estimate of (5.20) for \( U_k^L \) and \( U_k^R \), and using (5.8) we obtain

\[
\left| f \left( U_k^{L/R} \right) \right| \geq \min_{\xi \in U_k} |f''(\xi)| \cdot \frac{\delta_k^2}{2} \geq \frac{\left( q^{k \cdot e^{-\eta_4}} \right)^{\mu - 2} \cdot \omega \cdot g_{\omega, \mu} \cdot \delta_k^2}{2} \\
= \frac{4 \cdot q^{k \cdot e^{-\eta_4}} \cdot \omega \cdot g_{\omega, \mu} \cdot \delta_k^2}{2} \cdot \frac{\left( q^{k \cdot e^{-\eta_4}} \cdot \omega \cdot g_{\omega, \mu} \cdot \delta_k^2 \right)}{2} = q^{k+1} \cdot e^{-q^{f(k)}} \cdot \omega \cdot g_{\omega, \mu} \cdot \delta_k^2 \\
= q^{k+1} \cdot e^{-q^{f(k)}} \cdot \omega^2 \cdot \frac{(1 - c) \cdot \omega^2}{g_{\omega, \mu} \cdot g_{\omega, \mu - 1}} = r_{f(k)} = |S_{f(k)}|,
\]

and this completes the proof of assertion (ii) and the proof of the lemma. \( \square \)
We continue analyzing the next ‘flat’ interval obtained by the second iteration of $f$.

**Lemma 5.8.** Let $k$ be a positive odd integer number. Let $\omega, \mu$ be as in Lemma 5.7. Define $\ell_1(k)$ and $\ell_2(k)$ as in the passage before Theorem 5.1. Then for $\widetilde{S_{\ell_1(k)}}$ as in Lemma 5.7, we have

$$f\left(\widetilde{S_{\ell_1(k)}}\right) \supseteq \left[0, q^{\ell_2(k)}\right].$$

**Proof.** Using (2.13) on $\widetilde{S_{\ell_1(k)}}$, we obtain

$$\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| \geq r_{\ell_1(k)} \cdot \min_{x \in \widetilde{S_{\ell_1(k)}}} |f'(x)|. \quad (5.21)$$

Using (5.16) and (5.19) in (5.21), and also the definition of $\ell_2(k)$ at the beginning of this section, we have

$$\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| \geq r_{\ell_1(k)} \cdot \min_{x \in \widetilde{S_{\ell_1(k)}}} |f'(x)|.$$

Using (2.13) on $\widetilde{S_{\ell_1(k)}}$, we obtain

$$\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| \geq r_{\ell_1(k)} \cdot \min_{x \in \widetilde{S_{\ell_1(k)}}} |f'(x)|.$$

Note also that $\ell_1(k) = k\mu + 1$ is odd, since $\mu$ is even. Hence, $f \geq 0$ on $\widetilde{S_{\ell_1(k)}}$ and since $f\left(q^{\ell_1(k)}\right) = 0$, $f\left(\widetilde{S_{\ell_1(k)}}\right) = \left[0, \max f\left(\widetilde{S_{\ell_1(k)}}\right)\right]$. The estimate $\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| \geq q^{\ell_2(k)}$ implies that $f\left(\widetilde{S_{\ell_1(k)}}\right) \supseteq \left[0, q^{\ell_2(k)}\right]$. From Lemma 5.7 we know $f\left(U_k^{L,R}\right) \supseteq \widetilde{S_{\ell_1(k)}}$. In Lemma 5.8 we showed that $f\left(\widetilde{S_{\ell_1(k)}}\right) \supseteq \left[0, q^{\ell_2(k)}\right]$. In particular, $U_{\ell_2(k)} \subset \left[q^{\ell_2(k)+1}, q^{\ell_2(k)}\right] \subset f\left(\widetilde{S_{\ell_1(k)}}\right)$. Now, in the next lemma we estimate the counterimage of subsets of $U_{\ell_2(k)}$ under $f^2|U_k^{L,R}$.

**Lemma 5.9.** Let $k$ be a positive odd integer. Assume $\mu$ is an even integer, $\mu \geq \max\left\{\left(\frac{30}{7\pi}\right)^2, (\frac{1-c}{2c})\right\}, 15\}$ and $\omega \in (0, 1)$ is a corresponding value satisfying (5.2) and such that the assertion of Proposition 5.4 is true (this is possible due to assertion (ii) of Lemma 5.2). Define a $(\omega, \mu, c)$ as in Proposition 5.3 and $J_k$ as in Proposition 5.5. Then, for $p \in (0, 1]$ and any subinterval $\widetilde{U_{\ell_2(k)}}$ of $U_{\ell_2(k)}$ with $\ell_2(k)$ as in the passage before Theorem 5.1, if

$$\left|\widetilde{U_{\ell_2(k)}}\right| = p \left|J_{\ell_2(k)}\right|$$

then $(f|_{U_k})^{-2}\left(\widetilde{U_{\ell_2(k)}}\right)$ has two parts of the form

$$\widetilde{U_k^L} = \left[m_k - \delta_{k,2}, m_k - \delta_{k,1}\right] \subset U_k^L \quad \text{and} \quad \widetilde{U_k^R} = \left[m_k + \delta_{k,1}, m_k + \delta_{k,2}\right] \subset U_k^R,$$

where $\delta_{k,1}, \delta_{k,2} \in (0, m_k - q^{k+1})$ and $\delta_{k,1}, \delta_{k,2} \in (0, q_k - m_k)$, and each of them has the size

$$\left|\widetilde{U_k^{L,R}}\right| \leq c \cdot p \cdot |J_k|. \quad (5.22)$$
Proof. Set $\hat{S}_{\ell_1(k)} := (f|\hat{S}_{\ell_1(k)})^{-1}(U_{\ell_2(k)})$. Note that injectivity of $f|\hat{S}_{\ell_1(k)}$ and Lemma 5.8 imply that $(f|\hat{S}_{\ell_1(k)})^{-1}(U_{\ell_2(k)}) = (f|\hat{S}_{\ell_1(k)})^{-1}(U_{\ell_2(k)})$. Using (2.13) on $\hat{S}_{\ell_1(k)}$, we have

$$
\left|(f|\hat{S}_{\ell_1(k)})^{-1}(U_{\ell_2(k)})\right| = \left|\hat{S}_{\ell_1(k)}\right| \leq \frac{|U_{\ell_2(k)}|}{\min_{\hat{S}_{\ell_1(k)}}|f'|}.
$$

On the other hand, from Proposition 5.6 we already know that on $\hat{S}_{\ell_1(k)}$, $|f'| \geq c\omega q^{\ell_1(k)(\mu-1)}$. Because of $\hat{S}_{\ell_1(k)} \subset \hat{S}_{\ell_1(k)} \subset \hat{S}_{\ell_1(k)}$, this property also satisfied on $\hat{S}_{\ell_1(k)}$. Hence, inserting both $|U_{\ell_2(k)}| = p|J_{\ell_2(k)}|$ and the estimate of $\min_{\hat{S}_{\ell_1(k)}}|f'|$ in the last expression, we have

$$
\left|\hat{S}_{\ell_1(k)}\right| \leq \frac{|U_{\ell_2(k)}|}{\min_{\hat{S}_{\ell_1(k)}}|f'|} \leq \frac{p|J_{\ell_2(k)}|}{\min_{\hat{S}_{\ell_1(k)}}|f'|} \leq \frac{p \cdot q^{\ell_2(k)} (1-q)}{c\omega q^{\ell_1(k)(\mu-1)}}.
$$

(5.23)

Now, we calculate subintervals of $(m_k - \delta_k, m_k + \delta_k)$ which get mapped bijectively to $\hat{S}_{\ell_1(k)}$. Note that the counterimage of $\hat{S}_{\ell_1(k)}$ has two parts in the form $\hat{U}_k^{1,R} \subset \hat{U}_k^L$, and $\hat{U}_k^R \subset \hat{U}_k^R$. It follows from strict monotonicity of $f$ on $[m_k - \delta_k, m_k]$ and $[m_k, m_k + \delta_k]$ and from the fact that $f(U_k^L) \supset \hat{S}_{\ell_1(k)}$ that there exist $\delta_{k,1}^L, \delta_{k,2}^R$ with

$$
|f(\hat{U}_k^R)| = |f\left([m_k + \delta_{k,1}^R, m_k + \delta_{k,2}^R]\right)| = |f\left([m_k - \delta_{k,2}^L, m_k - \delta_{k,1}^L]\right)| = |f(\hat{U}_k^L)| = |\hat{S}_{\ell_1(k)}|.
$$

(5.24)

We continue our calculations by using the boundaries of $\hat{U}_k^R$. Note that for the interval $[m_k, m_k + \delta_{k,1}^R]$ we know that $f(m_k + \delta_{k,1}^R) = \max\hat{S}_{\ell_1(k)}$ and $f(m_k) = q^{\ell_1(k)}$. Again from the monotonicity of the map it follows that $f([m_k, m_k + \delta_{k,1}^R]) = [\max\hat{S}_{\ell_1(k)}, q^{\ell_1(k)}]$. Consequently, since $f(q^{\ell_1(k)}) = 0$ and $f(\max\hat{S}_{\ell_1(k)}) \in [q^{\ell_2(k)+1}, q^{\ell_2(k)}]$, from (2.13) we have

$$
\left|\max\hat{S}_{\ell_1(k)} - q^{\ell_1(k)}\right| \geq \frac{q^{\ell_2(k)+1}}{\|f'\|_{\infty,\hat{S}_{\ell_1(k)}}}.
$$

(5.25)

From (2.1) we also have that $\|f'\|_{\infty,\hat{S}_{\ell_1(k)}} \leq g_{\omega,\mu} \cdot q^{\ell_1(k)(\mu-1)}$. Inserting this estimate in (5.25), we obtain

$$
\left|\max\hat{S}_{\ell_1(k)} - q^{\ell_1(k)}\right| \geq \frac{q^{\ell_2(k)+1}}{g_{\omega,\mu} \cdot q^{\ell_1(k)(\mu-1)}}.
$$

(5.26)

In addition, from (2.4) we know that

$$
\|f''\|_{\infty,U_k} \leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\mu-2}.
$$

(5.27)

Now, using the second order Taylor expansion of $f(m_k + \delta_{k,1}^R)$, we have

$$
|f\left(m_k + \delta_{k,1}^R\right) - f(m_k)| \leq \left|f''(\xi) \left(\frac{\delta_{k,1}^R}{2}\right)^2 \right|,
$$

(5.28)
where \(\xi \in (m_k, m_k + \delta^R_{k,1})\). Substituting the values of \(f(m_k + \delta^R_{k,1})\) and \(f(m_k)\) in (5.28), we have
\[
|f(m_k + \delta^R_{k,1}) - f(m_k)| = \max_{\xi \in (m_k, m_k + \delta^R_{k,1})} \hat{S}_{\xi(k)} - q^{f_1(k)}| \leq \|f''\|_{\infty, U_k} \frac{(\delta^R_{k,1})^2}{2},
\]
which implies
\[
\delta^R_{k,1} \geq \sqrt{\frac{2 \max_{\xi \in (m_k, m_k + \delta^R_{k,1})} \hat{S}_{\xi(k)} - q^{f_1(k)}}{\|f''\|_{\infty, U_k}}}. \tag{5.29}
\]
Using both estimates (5.26) and (5.27) in (5.29), we finally get
\[
\delta^R_{k,1} \geq \sqrt{\frac{q^f_{2(k)}}{\alpha^2 \delta_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{(\mu-2)} \cdot q^{f_1(k)}(\mu-1)}}. \tag{5.30}
\]
On the other hand, from Taylor’s formula with the integral remainder term we have
\[
f(m_k + \delta) = f(m_k) + \int_{m_k}^{m_k+\delta} (m_k + \delta - t) f''(t) \, dt
= f(m_k) + \int_{0}^{\delta} (\delta - t) f''(m_k + t) \, dt. \tag{5.31}
\]
Consequently, applying (5.31) for the boundaries of \(\hat{U}_k^R\), we have
\[
|\hat{S}_{\xi(k)}| = |f(\hat{U}_k^R)| = |f(m_k + \delta^R_{k,2}) - f(m_k + \delta^R_{k,1})| = \left| \int_{0}^{\delta^R_{k,2}} (\delta^R_{k,2} - t) f''(m_k + t) \, dt - \int_{0}^{\delta^R_{k,1}} (\delta^R_{k,1} - t) f''(m_k + t) \, dt \right|.
\]
From (5.12) we already know that \(M := \min_{x \in U_k} |f''(x)| \geq \frac{q^{f_1(x-2)}}{2} \cdot \omega \cdot g_{\omega,\mu}.\) In particular, \(f''\) has constant sign on \(U_k\). Using the fact that \(\delta^R_{k,1} < \delta^R_{k,2}\) in the last equality, we obtain
\[
|\hat{S}_{\xi(k)}| = \left| \int_{0}^{\delta^R_{k,2}} (\delta^R_{k,2} - \delta^R_{k,1}) f''(m_k + t) \, dt + \int_{\delta^R_{k,1}}^{\delta^R_{k,2}} (\delta^R_{k,2} - t) f''(m_k + t) \, dt \right|
\geq \left| \int_{0}^{\delta^R_{k,1}} (\delta^R_{k,2} - \delta^R_{k,1}) f''(m_k + t) \, dt \right| \geq \delta^R_{k,2} - \delta^R_{k,1} |M \cdot \delta^R_{k,1},
\]
so
\[
|\delta^R_{k,2} - \delta^R_{k,1}| \leq \frac{|\hat{S}_{\xi(k)}|}{M \cdot \delta^R_{k,1}}. \tag{5.32}
\]
Substituting the estimate of \(M\) and the estimate \(\delta^R_{k,1}\) given by (5.30) in (5.32), we obtain
\[
|\delta^R_{k,2} - \delta^R_{k,1}| \leq \frac{|\hat{S}_{\xi(k)}|}{q^{f_1(x-2)} \cdot \omega \cdot g_{\omega,\mu}} \cdot \sqrt{\frac{2 \cdot q^f_{2(k)+1}}{q^{(\mu-2)} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu} \cdot q^{f_1(k)}(\mu-1)}}. \tag{5.33}
\]
Combining the estimate of \( \hat{S}_{\ell_1(k)} \) given by (5.23) with (5.33), we finally have

\[
\left| \hat{U}_k^R \right| = \left| \delta_{k,2}^R - \delta_{k,1}^R \right| \\
\leq \sqrt{2p \cdot q^{f_2(k)} (1 - q)} \sqrt{q^{k(\mu - 2)} \cdot g_{\omega,\mu - 1} \cdot g_{\omega,\mu}^2 \cdot q^{f_1(k)(\mu - 1)}} \\
= \sqrt{2p \cdot q^{f_2(k)} (1 - q)} \sqrt{q^{k(\mu - 2)} \cdot g_{\omega,\mu - 1} \cdot \sqrt{q^{f_1(k)(\mu - 1)}}} \\
= \sqrt{2p \cdot \frac{q^{f_2(k)} (1 - q)}{q^{k(\mu - 2)} \cdot g_{\omega,\mu - 1}}} \cdot \frac{\sqrt{q^{k(\mu - 2)}}}{q^{f_1(k)(\mu - 1)}} \\
= \sqrt{2p \cdot \frac{q^{k(1 - q)} \cdot q^{f_2(k)}}{q^{k(\mu - 2)} \cdot g_{\omega,\mu - 1}}} \cdot \frac{\sqrt{\frac{1}{q^{f_1(k)(\mu - 1)}}}}{\sqrt{q^{k(\mu - 2)}}}.
\]

Here, using the estimate of \( q^{f_2(k)} \) given in the passage before Theorem 5.1 and \( |J_k| = q^k (1 - q) \), we obtain

\[
\left| \hat{U}_k^R \right| \leq \sqrt{2p \cdot |J_k|} \cdot \sqrt{q^{f_1(k)\mu} \cdot q^{\frac{\phi_1(\mu - 2)}{\pi}} \cdot \frac{c(1 - c)\omega^2}{4g_{\omega,\mu} \cdot g_{\omega,\mu - 1}}} \cdot \sqrt{q^{k(\mu - 2)} \cdot q^{f_1(k)(\mu - 1)}} \\
= \sqrt{2p \cdot |J_k|} \cdot \sqrt{q^{f_1(k)\mu} \cdot q^{\frac{\phi_1(\mu - 2)}{\pi}} \cdot \sqrt{q^{k(\mu - 2)}} \cdot q^{f_1(k)(\mu - 1)}} \cdot \sqrt{q^{k(\mu - 2)} \cdot q^{f_1(k)(\mu - 1)}} \\
= \frac{\sqrt{2p \cdot |J_k|} \cdot \sqrt{q^{f_1(k)\mu} \cdot q^{\frac{\phi_1(\mu - 2)}{\pi}}}}{\sqrt{q^{k(\mu - 2)}} \cdot \sqrt{q^{f_1(k)(\mu - 1)}}} \cdot \sqrt{q^{k(\mu - 2)}} \cdot \sqrt{q^{f_1(k)(\mu - 1)}} \\
= \frac{\sqrt{2p \cdot |J_k|} \cdot \sqrt{q^{f_1(k)\mu} \cdot q^{\frac{\phi_1(\mu - 2)}{\pi}}}}{\sqrt{q^{k(\mu - 2)}} \cdot \sqrt{q^{f_1(k)(\mu - 1)}}} \cdot \sqrt{(1 - c)} \cdot \frac{1}{\sqrt{2c \cdot g_{\omega,\mu} \cdot g_{\omega,\mu - 1}}}.
\]

Inserting \( \ell_1(k) = k\mu + 1 \) one gets

\[
\left| \hat{U}_k^R \right| \leq \frac{\sqrt{2p \cdot |J_k|} \cdot \sqrt{q^{f_1(k)\mu} \cdot q^{\frac{\phi_1(\mu - 2)}{\pi}}}}{\sqrt{q^{k(\mu - 2)}} \cdot \sqrt{q^{f_1(k)(\mu - 1)}}} \cdot \sqrt{(1 - c)} \cdot \frac{1}{\sqrt{2c \cdot g_{\omega,\mu} \cdot g_{\omega,\mu - 1}}} \\
= \frac{p \cdot |J_k| \cdot q^{\frac{\phi_1(\mu - 2)}{\pi}}}{g_{\omega,\mu - 1}} \cdot \sqrt{\frac{1 - c}{2c \cdot \omega}}.
\]

Since \( g_{\omega,\mu - 1} < g_{\omega,\mu} \), we can simplify the last inequality as follows:

\[
\left| \hat{U}_k^R \right| \leq \frac{p \cdot |J_k| \cdot q^{\frac{\phi_1(\mu - 2)}{\pi}}}{g_{\omega,\mu - 1}} \cdot \sqrt{\frac{1 - c}{2c \cdot \omega}} \\
= p \cdot |J_k| \cdot \frac{q^{\frac{\phi_1(\mu - 2)}{\pi}}}{g_{\omega,\mu - 1}} \cdot \sqrt{\frac{1 - c}{2c \cdot \omega}}.
\]

Inserting \( q = e^{-\frac{\pi}{\omega}} \) we have

\[
\left| \hat{U}_k^R \right| \leq p \cdot |J_k| \cdot \frac{\exp\left(\frac{\pi}{\omega} \cdot \frac{(\mu - 2)\phi_1}{2\pi}\right)}{g_{\omega,\mu - 1}} \cdot \sqrt{\frac{1 - c}{2c \cdot \omega}} \\
= p \cdot |J_k| \cdot \frac{\exp\left(\frac{(\mu - 2)\phi_1}{2\omega}\right)}{g_{\omega,\mu - 1}} \cdot \sqrt{\frac{1 - c}{2c \cdot \omega}}.
\]
Finally, using the definition of \( \alpha (\mu, \omega, c) \), we get

\[
|U_k^R| \leq \alpha \cdot p \cdot |J_k|,
\]
and this completes the proof for \( \hat{U}_k^R \). The proof for \( \hat{U}_k^L \) is analogous.

**Corollary 5.10.** If the set \( \hat{U}_{\ell_2(k)} \) in Lemma 5.9 is not only one interval, but a disjoint union of subintervals of \( U_{\ell_2(k)} \), and \( |U_{\ell_2(k)}| \) (the measure of \( U_{\ell_2(k)} \)) satisfies \( |U_{\ell_2(k)}| = p |J_{\ell_2(k)}| \), then \( (f|_{U_k})^{-2}(U_{\ell_2(k)}) \) has two parts (one in \( U_k^L \) and the other in \( U_k^R \)) and each of them has measure less or equal \( \alpha p \cdot |J_k| \).

**Proof.** (By summation over the subintervals.)

Now, we consider symbol sequences of the form \( \{L, R\}^{n+1} \) and construct corresponding orbits of \( f \). For given a finite sequence

\[
s = (s_0, s_1, s_2, \ldots, s_n) \in \{L, R\}^{n+1}
\]

and odd \( k \in \mathbb{N} \), we now construct the subset of points \( x \) in \( U_k \) which follow this symbol sequence. Recall the set \( I_{k,s}^n = \bigcap_{j=0}^n f^{-2j}(U_{\ell_2(k)}^R) \) defined in the passage before Theorem 5.1.

We estimate the size of \( |I_{k,s}^n| \).

**Corollary 5.11.** Let \( s = (s_0, s_1, s_2, \ldots, s_n) \) and an odd \( k \in \mathbb{N} \) be given. Then, with \( \omega, \mu \) as in Lemma 5.9 and \( \alpha (\omega, \mu, c) \) as in Proposition 5.3 we have \( \emptyset \neq I_{k,s}^0 \) and

\[
|I_{k,s}^n| \leq \alpha^n |J_k|.
\]

**Proof.** We prove the corollary by induction over \( n \). For \( n = 0 \), \( I_{k,s}^0 = U_k^S \neq \emptyset \), and

\[
|I_{k,s}^0| = |U_k^S| \leq |J_k|.
\]

Now, we assume the result is true for \( n \), and we verify it for \( n + 1 \). Let \( s = (s_0, s_1, s_2, \ldots, s_{n+1}) \) be given. Define \( \tilde{s} = (s_1, s_2, \ldots, s_{n+1}) \). From the induction hypothesis we have \( I_{\ell_2(k),\tilde{s}}^n \neq \emptyset \), \( I_{\ell_2(k),\tilde{s}}^n \subset U_{\ell_2(k)}^R \), and

\[
|I_{\ell_2(k),\tilde{s}}^n| = \left| \bigcap_{j=0}^n f^{-2j} \left( U_{\ell_2(k)}^R \right) \right| \leq \alpha^n |J_{\ell_2(k)}|.
\]

Note that \( I_{k,s}^{n+1} = f^{-2} \left( I_{\ell_2(k),\tilde{s}}^n \right) \cap U_k^R \). Hence, we have

\[
|I_{k,s}^{n+1}| = \left| f^{-2} \left( I_{\ell_2(k),\tilde{s}}^n \right) \cap U_k^R \right|.
\]

(5.34)

Applying Corollary 5.10 with \( p := \alpha^n \) and \( I_{\ell_2(k),\tilde{s}}^n \) instead of \( \hat{U}_{\ell_2(k)} \) in (5.34), and using this \( p \) together with 5.22, we finally obtain

\[
|I_{k,s}^{n+1}| = \left| f^{-2} \left( I_{\ell_2(k),\tilde{s}}^n \right) \cap U_k^R \right| \leq \alpha \cdot p \cdot |J_k| = \alpha^{n+1} |J_k|.
\]

This completes the induction and the proof of Corollary 5.11.
Proof of Theorem 5.1. Assume $k$, $c$ and $\mu$ are as in the assumptions of the Theorem 5.1 and, $\alpha = \alpha(\omega, \mu, c)$ be as in Proposition 5.3, so that $\alpha < \frac{1}{2}$. Choose $\omega \in (0,1)$ as in Lemma 5.2.

(i) Let a symbol sequence $s = (s_0, s_1, s_2, \ldots) \in \{L, R\}^\mathbb{N}_0$ be given. From Corollary 5.11 one can see that for $n \in \mathbb{N}_0$ the closed interval $I_{k,s}^n$ consists of the points $x \in U_k$ which follow the finite symbol sequence $s = (s_0, s_1, s_2, \ldots, s_n) \in \{L, R\}^{n+1}$. Further we have $I_{k,s}^{n+1} \subset I_{k,s}^n$. It follows that $\bigcap_{n \in \mathbb{N}_0} I_{k,s}^n \neq \emptyset$. Since, in view of Corollary 5.11 and $\alpha < \frac{1}{2}$, we have $|I_{k,s}^n| \to 0$ for $n \to \infty$, the intersection $\bigcap_{n \in \mathbb{N}_0} I_{k,s}^n$ contains exactly one point $x_{k,s}$. This point $x_{k,s}$ has the asserted properties. Any point in $U_k$ with these properties would also be contained in this intersection and thus equal $x_{k,s}$.

(ii) The set $\{L, R\}^{\{0,1,\ldots,n\}}$ has $2^{n+1}$ elements and from Corollary 5.11 we know that each set corresponding to one $s \in \{L, R\}^{\{0,1,\ldots,n\}}$ satisfies the estimate $|I_{k,s}^n| \leq \alpha^n |J_k|$. It follows that $|\Gamma_{k}^n| \leq 2^{n+1} \alpha^n |J_k|$, and it turns out that the measure

$$
\lim_{n \to \infty} |\Gamma_{k}^n| = \lim_{n \to \infty} \left| \bigcup_{s \in \{L, R\}^{\{0,1,\ldots,n\}}} I_{k,s}^n \right| \leq 2 \lim_{n \to \infty} 2^n \alpha^n |J_k| = 0
$$

and this completes the proof. \(\square\)

References


