Electronic Journal of Qualitative Theory of Differential Equations
2022, No. **3**, 1–23; https://doi.org/10.14232/ejqtde.2022.1.3 www.math.u-szeged.hu/ejqtde/

Global existence and blow-up for semilinear parabolic equation with critical exponent in $\mathbb{R}^N$

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Received 24 September 2021, appeared 14 January 2022
Communicated by Roberto Livrea

Abstract. In this paper, we use the self-similar transformation and the modified potential well method to study the long time behaviors of solutions to the classical semilinear parabolic equation associated with critical Sobolev exponent in $\mathbb{R}^N$. Global existence and finite time blowup of solutions are proved when the initial energy is in three cases. When the initial energy is low or critical, we not only give a threshold result for the global existence and blowup of solutions, but also obtain the decay rate of the L^2 norm for global solutions. When the initial energy is high, sufficient conditions for the global existence and blowup of solutions are also provided. We extend the recent results which were obtained in [R. Ikehata, M. Ishiwata, T. Suzuki, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27**(2010), No. 3, 877– 900].

Keywords: Semilinear parabolic equation, critical Sobolev exponent, potential well method, blow-up.

2020 Mathematics Subject Classification: 35A01, 35K15, 35K58.

1 Introduction

This paper deals with the following classical semilinear parabolic equation associated with critical Sobolev exponent in $\mathbb{R}^N$:

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u & \text{in } \mathbb{R}^N \times (0,T), \\ u|_{t=0} = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geqslant 3$ and $p = (N+2)/(N-2)$, the critical exponent associated with the Sobolev imbedding.

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There is a great literature on the existence of global solutions and blow-up for the problem (1.1) on the bounded domain (see e.g. [2,6,9,13,21,22,25,27,28]):

$$\begin{cases} u_t - \Delta u = |u|^{p-2}u & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $p > 1$. It is well known that there exist choices of u_0 for which the corresponding solutions tend to zero as $t \to \infty$ and other choices for which the solutions blow-up in finite time (see e.g. [9]). Tan [25], R. Ikehata and T. Suzuki [6] considered critical problem (1.2). By means of the potential well method, they established the existence of global solutions and studied the asymptotic behavior of solutions which heavily depend on the initial data. Using the the comparison principle and variational methods, Gazzola and Weth [2] obtained global solutions and finite time blow-up solutions with the initial data at high energy level.

The problem (1.1) in $\mathbb{R}^N$ was considered by Ishiwata and Suzuki [4], Ikehata, Ishiwata, and Suzuki [5], Mizoguchi and Yanagida [14–16]. In [14], a sufficient condition on the decay order of initial data, which may change sign, such that the solution of (1.1) blows up infinite time, was gived. Using self-similar transformation, Mizoguchi and Yanagida [15, 16] established the global existence and blow-up results for problem (1.1) in the $\mathbb{R}^1$. In [4,5], the decay and blow-up of the solution with low energy initial data were studied by means of the potential-well and forward self-similar transformation. For a general scope of this topic, we refer the interested readers to the monographs [23] and references therein.

In this article, we consider the problem (1.1) with low initial energy, critical initial energy and high initial energy. The results in our paper will be obtained by the self-similar transformation and the modified potential well method. Potential well method, which was first put forward to consider semi-linear hyperbolic initial boundary value problem by Payne and Sattinger [20,24] around 1970s, is a powerful tool in studying the long time behaviors of solutions of some evolution equations. The potential well is defined by the level set of energy functional and the derivative functional. It is generally true that solutions starting inside the well are global in time, solutions starting outside the well and at an unstable point blow up in finite time. After the pioneer work of Sattinger and Payne, some authors [7,9–12,17–19,26] used the method to study the global existence and nonexistence of solutions for various nonlinear evolution equations with initial boundary value problem. In [11,12], Liu et al. modified and improved the method by introducing a family of potential wells which include the known potential well as a special case. The modified potential well method has been used to study semilinear pseudo-parabolic equations [9] and fourth-order parabolic equation [3]. In this paper, we use the modified potential well method to obtain global existence and blow up in finite time of solutions when the initial energy is low, critical and high, respectively. When the initial energy is low, similar results are obtained in [5], but our result is more general, moreover, we prove a more precise decay rate of the L^2 norm of global solution.

This paper is organized as follows. In Section 2, we give some notations, definitions and lemmas concerning the basic properties of the related functionals and sets. Sections 3 and 4 will be devoted to the cases $E_K(v_0) < d$ and $E_K(v_0) = d$, respectively, where $E_K(v)$ will be introduced in Section 2. In Section 5, we consider the case when the initial energy is high, i.e. $E_K(v_0) > d$.

2 Preliminaries and main lemmas

In this section, we shall introduce the self-similar transformation and the modified potential well method and give a series of their properties for problem (1.1). The self-similar transformation is defined as follow:

$$v(y,s) = (1+t)^{1/(p-1)}u(x,t), \quad t = e^s - 1, \quad x = (1+t)^{1/2}y.$$

We can easily know that

$$\begin{cases} v_s + Lv = |v|^{p-1}v + \dfrac{1}{p-1}v & \text{in } \mathbb{R}^N \times (0,S), \\ v|_{s=0} = v_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $S = \log(1+T)$ and

$$Lf = -\Delta f - \frac{1}{2}y \cdot \nabla f.$$

Letting

$$K(y) = e^{|y|^2/4},$$

we have

$$Lf = -\frac{1}{K}\nabla \cdot (K\nabla f).$$

Let

$$\|f\|_{2,K} = \left\{ \int_{\mathbb{R}^N} |f(y)|^2 K(y) dy \right\}^{1/2} < +\infty.$$

We also take

$$H^m(K) = \left\{ f \in L^2(K) \mid D^\alpha f \in L^2(K) \text{ for any multi-index } \alpha \text{ in } |\alpha| \leqslant m \right\},$$

where $m = 1, 2, \ldots$ It is a Hilbert space provided with the norm

$$\|f\|_{H^m(K)} = \left\{ \sum_{|\alpha| \leqslant m} \|D^\alpha f\|_{2,K}^2 \right\}^{1/2}.$$

The linear operator L is realized as a self-adjoint operator in $L^2(K)$ through the relation

$$\mathcal{A}_K(u,v) := \int_{\mathbb{R}^N} \nabla u(y) \cdot \nabla v(y) K(y) dy = (Lu, v)_K, \quad u \in D(L) \subset H^1(K), v \in H^1(K),$$

where

$$(u,v)_K = \int_{\mathbb{R}^N} u(y)v(y)K(y)dy.$$

From Lemma 2.1 of [8], the domain $D(L)$ of this operator L is the set of $v \in L^2(K)$ satisfying $Lv \in L^2(K)$, and we have $D(L) = H^2(K)$, It holds also that L is positive selfadjoint and has the compact inverse, and in particular, the set of normalized eigenfunctions of L forms a complete ortho-normal equation in $L^2(K)$. The first eigenvalue λ_1 of L is given by $\lambda_1 = N/2$, and hence from Proposition 2.3 of [1]. the following Poincaré inequality holds,

$$\lambda_1 \|v\|_{2,K}^2 \leqslant \|\nabla v\|_{2,K}^2, \quad v \in H^1(K). \tag{2.2}$$

We have

$$\lambda_1 = \frac{N}{2} > \lambda \equiv \frac{1}{p-1} = \frac{N-2}{4}.$$

Then, the operator

$$A = L - \frac{1}{p-1}$$

in $L^2(K)$ is also positive self-adjoint with the domain $D(A) = H^2(K)$. The semigroup $\{e^{-tA}\}_t \geqslant 0$ are thus defined in $L^2(K)$. These characteristics guarantee the well-posedness of (1.1) locally in time.

Now let us define the level set

$$E^\alpha = \left\{ v \in H^1(K) : E_K(v) < \alpha \right\}. \tag{2.3}$$

Furthermore, by the definition of $E_K(v), \mathcal{N}, E^\alpha$ and d, we easily know that

$$\mathcal{N}_\alpha = \mathcal{N} \cap E^\alpha \equiv \left\{ v \in \mathcal{N} : \|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 < \sqrt{\frac{2\alpha(p+1)}{p-1}} \right\} \neq \varnothing \quad \text{for all } \alpha > d. \tag{2.4}$$

Let

$$\lambda_\alpha = \inf\left\{ \|v\|_{2,K} : v \in \mathcal{N}_\alpha \right\}, \quad \Lambda_\alpha = \sup\left\{ \|v\|_{2,K} : u \in \mathcal{N}_\alpha \right\} \quad \text{for all } \alpha > d. \tag{2.5}$$

It is clear that λ_α is nonincreasing and Λ_α is nondecreasing with respect to α . For $0 < \delta < \frac{p+1}{2}$, let us define the modified functional and Nehari manifold as follows:

$$\begin{aligned} E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1}, \\ D_{K,\delta}(v) &= \delta\|\nabla v\|_{2,K}^2 - \lambda\delta\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}, \\ \mathcal{N}_\delta &= \left\{ u \in H^1(K) : D_{K,\delta}(v) = 0, \|v\| \neq 0 \right\}, \\ d_\delta &= \inf_{v \in N_\delta} E_K(v), \\ r(\delta) &= \delta^{\frac{N-2}{2}} S_\lambda^{\frac{N}{2}}. \end{aligned}$$

Then we can define the modified potential wells and their corresponding sets as follows:

$$\begin{aligned} W_\delta &= \left\{ u \in H^1(K) : D_\delta(u) > 0, E(u) < d(\delta) \right\} \cup \{0\}, \\ V_\delta &= \left\{ u \in H^1(K) : D_\delta(u) < 0, E(u) < d(\delta) \right\}, \\ B_\delta &= \left\{ u \in H^1(K) : \|\nabla u\|_{2,K} < r(\delta) \right\}, \\ B_\delta^c &= \left\{ u \in H^1(K) : \|\nabla u\|_{2,K} > r(\delta) \right\}. \end{aligned} \tag{2.6}$$

We also introduce the following sets

$$\begin{aligned} \mathcal{B} &= \left\{ u_0 \in H^1(K) : \text{ the solution } u = u(t) \text{ of (1.2) blows up in finite time } \right\}, \\ \mathcal{G} &= \left\{ u_0 \in H^1(K) : \text{ the solution } u = u(t) \text{ of (1.2) exists for all } t > 0 \right\}, \\ \mathcal{G}_o &= \left\{ u_0 \in G : u(t) \mapsto 0 \text{ in } H^1(K) \text{ as } t \to \infty \right\}. \end{aligned} \tag{2.7}$$

For future convenience, we give some useful lemmas which will play an important role in the proof of our main results.

Let $L^q(K)$ denote the Banach space composed of measurable functions $v = v(y)$ defined in $\mathbb{R}^N$ such that

$$\|v\|_{q,K} = \left\{ \int_{\mathbb{R}^N} |v(y)|^q K(y) dy \right\}^{1/q} < +\infty$$

for $q \in [1, \infty)$ and

$$\|v\|_{\infty,K} = \operatorname*{ess\,sup}_{y \in \mathbb{R}^N} |f(y)| < +\infty$$

for $q = \infty$. The space $L^\infty(K) = L^\infty\left(\mathbb{R}^N\right)$ is thus compatible to the other spaces, i.e.,

$$\lim_{q \uparrow \infty} \|v\|_{q,K} = \|f\|_{\infty,K}, v \in L^1(K) \cap L^\infty\left(\mathbb{R}^N\right)$$

Although the inclusion

$$L^p(K) \subset L^q(K) \quad (1 \leqslant q < p \leqslant \infty)$$

fails, we have

$$H^1(K) \subset L^{2^*}(K)$$

for $2^* = 2N/(N-2) = p+1$. More precisely, Corollary 4.20 of [1] guarantees the following fact, regarded as a Sobolev–Poincaré inequality.

Lemma 2.1 ([1, Corollary 4.20])**.** *It holds that*

$$S_0 \|v\|_{p+1,K}^2 + \lambda_* \|v\|_{2,K}^2 \leqslant \|\nabla v\|_{2,K}^2, \quad v \in H^1(K),$$

where $\lambda_* = \max(1, N/4)$ *and* S_0 *stands for the Sobolev constant:*

$$S_0 = \inf \left\{ \|\nabla v\|_2^2 \mid v \in C_0^\infty\left(\mathbb{R}^N\right), \|v\|_{p+1} = 1 \right\}.$$

Lemma 2.2 ([5, p. 882])**.** *Set*

$$S_\lambda = \inf \left\{ \frac{\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2}{\|v\|_{p+1,K}^2} \mid v \in H^1(K) \right\},$$

We have $S_\lambda = S_0$.

So, it holds that

$$\|v\|_{p+1,K}^{p+1} \leqslant \left(\frac{1}{S_\lambda} (\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2) \right)^{\frac{p+1}{2}}, \quad v \in H^1(K), \tag{2.8}$$

and

$$r(\delta) = \delta^{\frac{N-2}{2}} S_\lambda^{\frac{N}{2}} \geq \|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2.$$

Lemma 2.3. *Let* $u_0 \in H^1(K)$.

(1) *If* $0 < \|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 < r(\delta)$, *then* $D_{K,\delta}(u) > 0$. *In particular, if* $0 < \|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 < r(1)$, *then* $D_K(u) > 0$;

(2) *If* $D_{K,\delta}(u) < 0$, *then* $\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 > r(\delta)$. *In particular, if* $D_K(u) < 0$, *then* $\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 > r(1)$;

(3) *If* $D_{K,\delta}(v)=0$, *then* $\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}\geqslant r(\delta)$ *or* $\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}=0$. *In particular, if* $D_K(v)=0$, *then* $\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}\geqslant r(1)$ *or* $\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}=0$;

(4) *If* $D_{K,\delta}(v)=0$ *and* $\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}\neq 0$, *then* $E_K(v)>0$ *for* $0<\delta<\frac{p+1}{2}$, $E_K(v)=0$ *for* $\delta=\frac{p+1}{2}$, $E(v)<0$ *for* $\delta>\frac{p+1}{2}$.

Proof. (1) Since $0<\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}<r(\delta)$, by the Lemma 2.2 and (2.8), we have from the assumption $0<\|v\|<r(\delta):=\delta^{\frac{N-2}{2}}S_\lambda^{\frac{N}{2}}$, and we obtain

$$\begin{aligned}
D_{K,\delta}(v)&=\delta\|\nabla v\|^2_{2,K}-\lambda\delta\|v\|^2_{2,K}-\|v\|^{p+1}_{p+1,K}\\
&\geq \delta\|\nabla v\|^2_{2,K}-\lambda\delta\|v\|^2_{2,K}-\left(\frac{1}{S_\lambda}(\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})\right)^{\frac{p+1}{2}}\\
&\geq (\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})\left(\delta-\left(\frac{1}{S_\lambda}\right)^{\frac{p+1}{2}}(\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})^{\frac{2}{N-2}}\right)>0. \qquad (2.9)
\end{aligned}$$

(2) By the assumption $D_{K,\delta}(v)<0$ and (2.8), we have

$$\begin{aligned}
0\geq D_{K,\delta}(v)&=\delta\|\nabla v\|^2_{2,K}-\lambda\delta\|v\|^2_{2,K}-\|v\|^{p+1}_{p+1,K}\\
&\geq \delta\|\nabla v\|^2_{2,K}-\lambda\delta\|v\|^2_{2,K}-\left(\frac{1}{S_\lambda}(\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})\right)^{\frac{p+1}{2}}\\
&\geq (\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})\left(\delta-\left(\frac{1}{S_\lambda}\right)^{\frac{p+1}{2}}(\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})^{\frac{2}{N-2}}\right). \qquad (2.10)
\end{aligned}$$

Hence, $\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}>r(\delta)$.

(3) By the assumption $D_{K,\delta}(v)=0$ and (2.8), we have

$$\begin{aligned}
0= D_{K,\delta}(v)&=\delta\|\nabla v\|^2_{2,K}-\lambda\delta\|v\|^2_{2,K}-\|v\|^{p+1}_{p+1,K}\\
&\geq \delta\|\nabla v\|^2_{2,K}-\lambda\delta\|v\|^2_{2,K}-\left(\frac{1}{S_\lambda}(\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})\right)^{\frac{p+1}{2}}\\
&\geq (\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})\left(\delta-\left(\frac{1}{S_\lambda}\right)^{\frac{p+1}{2}}(\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})^{\frac{2}{N-2}}\right). \qquad (2.11)
\end{aligned}$$

Hence, $\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}=r(\delta)$. or $v=0$.

(4) We easily know that

$$\begin{aligned}
E_K(v)&=\frac{1}{2}\|\nabla v\|^2_{2,K}-\frac{\lambda}{2}\|v\|^2_{2,K}-\frac{1}{p+1}\|v\|^{p+1}_{p+1,K}\\
&=\left(\frac{1}{2}-\frac{\delta}{p+1}\right)(\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K})+\frac{1}{p+1}D_{K,\delta}(v)\\
&=\left(\frac{1}{2}-\frac{\delta}{p+1}\right)(\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}). \qquad (2.12)
\end{aligned}$$

Then we can prove the conclusion. □

Lemma 2.4.

(1) $d(\delta) \geqslant a(\delta)r^2(\delta)$ *for* $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}, 0 < \delta < \frac{p+1}{2}$,

(2) $\lim_{\delta \to 0} d(\delta) = 0, d\left(\frac{p+1}{2}\right) = 0$ *and* $d(\delta) < 0$ *for* $\delta > \frac{p+1}{2}$,

(3) $d(\delta)$ *is increasing on* $0 < \delta \leqslant 1$, *decreasing on* $1 \leqslant \delta \leqslant \frac{p+1}{2}$ *and takes the maximum* $d = d(1)$ *at* $\delta = 1$.

Proof. (1) If $u \in \mathcal{N}_\delta$, by Lemma 2.3 (3), then $\|u\| \geqslant r(\delta)$. Moreover, we can deduce

$$\begin{aligned} E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \left(\frac{1}{2} - \frac{\delta}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D_{K,\delta}(v) \\ &= \left(\frac{1}{2} - \frac{\delta}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) \geq a(\delta)r^2(\delta). \end{aligned} \tag{2.13}$$

Hence, $d(\delta) \geqslant a(\delta)r^2(\delta)$.

(2) We easily know that

$$E_K(sv) = \frac{s^2}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda s^2}{2}\|v\|_{2,K}^2 - \frac{s^{p+1}}{p+1}\|v\|_{p+1,K}^{p+1}.$$

Hence,

$$\lim_{s \to 0} E_K(sv) = 0. \tag{2.14}$$

And if we let $sv \in \mathcal{N}_\delta$, then sv satisfies

$$0 = D_{K,\delta}(sv) = \delta s^2\|\nabla v\|_{2,K}^2 - \lambda s^2\delta\|v\|_{2,K}^2 - s^{p+1}\|v\|_{p+1,K}^{p+1}.$$

Then, we obtain

$$s(\delta) = \left(\frac{\delta(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2)}{\|v\|_{p+1,K}^{p+1}}\right)^{\frac{1}{p-1}}, \tag{2.15}$$

which yields

$$\lim_{\delta \to 0} s(\delta) = 0. \tag{2.16}$$

Now (2.14) implies that

$$\lim_{\delta \to 0} E_K(sv) = \lim_{\lambda \to 0} E_K(sv) = 0, \tag{2.17}$$

and

$$\lim_{\delta \to 0} d(\delta) = 0. \tag{2.18}$$

It is easy to see that from (2.13)

$$d\left(\frac{p+1}{2}\right) = 0 \text{ and } d(\delta) < 0 \text{ for } \delta > \frac{p+1}{2}.$$

The proof is complete.

(3) We need to prove that for any $0<\delta'<\delta''<1$ or $1<\delta''<\delta'<\frac{p+1}{2}$ and for any $w\in\mathcal{N}_{\delta''}$, there is a $v\in\mathcal{N}_{\delta'}$ and a constant $\varepsilon(\delta',\delta'')$ such that $E_K(v)<E_K(w)-\varepsilon(\delta',\delta'')$. Indeed, by the definition of (2.15), we easily know that $D_{K,\delta}(s(\delta)u)=0$ and $\lambda(\delta'')=1$. Let $h(s)=E_K(sw)$, we have

$$\begin{aligned}\frac{d}{ds}h(s)&=\frac{1}{s}\left((1-\delta)(\|\nabla sw\|_{2,K}^2-\lambda\|sw\|_{2,K}^2)+D_{K,\delta}(sw)\right)\\&=(1-\delta)s(\|\nabla w\|_{2,K}^2-\lambda\|w\|_{2,K}^2).\end{aligned}\tag{2.19}$$

Take $v=s(\delta')w$, then $v\in\mathcal{N}_{\delta'}$.

For $0<\delta'<\delta''<1$, we obtain

$$\begin{aligned}E_K(w)-E_K(v)&=h(1)-h\left(s\left(\delta'\right)\right)\\&>\left(1-\delta''\right)r^2\left(\delta''\right)s\left(\delta'\right)\left(1-s\left(\delta'\right)\right)\equiv\varepsilon\left(\delta',\delta''\right).\end{aligned}\tag{2.20}$$

For $1<\delta''<\delta'<\frac{p+1}{2}$, we obtain

$$\begin{aligned}E_K(w)-E_K(v)&=h(1)-h\left(s\left(\delta'\right)\right)\\&>\left(\delta''-1\right)r^2\left(\delta''\right)s\left(\delta''\right)\left(s\left(\delta'\right)-1\right)\equiv\varepsilon\left(\delta',\delta''\right).\end{aligned}\tag{2.21}$$

Hence, the proof is complete. □

Lemma 2.5. *Let $u_0\in H^1(K)$ and $0<\delta<\frac{p+1}{2}$. If $E_K(v)\leqslant d(\delta)$, then we have*

(1) *If $D_{K,\delta}(v)>0$, then $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2<\frac{d(\delta)}{a(\delta)}$, where $a(\delta)=\frac{1}{2}-\frac{\delta}{p+1}$. In particular, if $E_K(v)\leqslant d$ and $D_K(v)>0$, then*

$$\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2<\frac{2(p+1)}{p-1}d.\tag{2.22}$$

(2) *If $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2>\frac{d(\delta)}{a(\delta)}$, then $D_{K,\delta}(u)<0$. In particular, if $E_K(v)\leqslant d$ and*

$$\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2>\frac{2(p+1)}{p-1}d,\tag{2.23}$$

then $D_K(v)<0$.

(3) *If $D_{K,\delta}(v)=0$, then $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2\leqslant\frac{d(\delta)}{a(\delta)}$. In particular, if $E_K(v)\leqslant d$ and $D_K(v)=0$, then*

$$\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2\leqslant\frac{2(p+1)}{p-1}d.\tag{2.24}$$

Proof. (1) For $0<\delta<\frac{p+1}{2}$, we see that

$$\begin{aligned}E_K(v)&=\frac{1}{2}\|\nabla v\|_{2,K}^2-\frac{\lambda}{2}\|v\|_{2,K}^2-\frac{1}{p+1}\|v\|_{p+1,K}^{p+1}\\&=\left(\frac{1}{2}-\frac{\delta}{p+1}\right)(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)+\frac{1}{p+1}D_\delta(u)\\&=a(\delta)\|u\|^2\leq d(\delta).\end{aligned}\tag{2.25}$$

Therefore,

$$\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2<\frac{d(\delta)}{a(\delta)}.$$

Finally, (2) and (3) follow from (2.25). □

Lemma 2.6. *Let $v \in H^1(K)$. We have*

(1) *0 is away from both $\mathcal{N}$ and $\mathcal{N}_-$, i.e.* $\operatorname{dist}(0, \mathcal{N}) > 0$, $\operatorname{dist}(0, N_-) > 0$.

(2) *For any $\alpha > 0$, the set $E^\alpha \cap \mathcal{N}_+$ is bounded in $H^1(K)$.*

Proof. (1) If $v \in \mathcal{N}$, then we have

$$\begin{aligned} d \le E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D(u) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2). \end{aligned}$$

If $v \in \mathcal{N}_-$, then we have

$$\begin{aligned} d \le E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D(u) \\ &\le \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2). \end{aligned}$$

Hence, 0 is away from both N and $\mathcal{N}_-$, i.e. $\operatorname{dist}(0, \mathcal{N}) > 0$, $\operatorname{dist}(0, N_-) > 0$.

(2) Since $E_K(v) < \alpha$ and $D_K(v) > 0$, we obtain

$$\begin{aligned} \alpha > E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D(u) \\ &> \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2). \end{aligned}$$

Hence, for any $\alpha > 0$, the set $E^\alpha \cap \mathcal{N}_+$ is bounded in $H^1(K)$. □

3 Low initial energy $E_K(v_0) < d$

The goal of this section is to prove Theorems 3.2–3.6. A threshold result for the global solutions and finite time blowup will be given.

Theorem 3.1. *Assume that $v_0 \in H^1(K)$, T is the maximal existence time of u, and $0 < e < d$, $\delta_1 < \delta_2$ are two roots of equation $d(\delta) = e$. We have*

(1) *If $D_K(v_0) > 0$, all weak solutions u of equation (2.1) with $E_K(v_0) = e$ belong to W_δ for $\delta_1 < \delta < \delta_2, 0 \leqslant t < T$.*

(2) *If $D_K(v_0) < 0$, all weak solutions u of equation (2.1) with $E_K(v_0) = e$ belong to V_δ for $\delta_1 < \delta < \delta_2, 0 \leqslant t < T$.*

Theorem 3.2 (Global existence)**.** *Assume that $v_0 \in H^1(K)$, $E_K(u_0) < d$, $D_K(u_0) > 0$. Then equation (2.1) has a global solution $v(t) \in L^\infty(0, \infty; H^1(K))$ and $v(t) \in W$ for $0 \leqslant t < \infty$.*

Remark 3.3. Result similar to Theorem 3.2 is obtained in [5]. But our proof is different to [5]. In fact, using the modified potential well method we can obtain the more general conclusion:

If the assumption $D_K(u_0) > 0$ is replaced by $D_{K,\delta_2}(u_0) > 0$, where $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(u_0)$, then equation (2.1) admits a global weak solution.

The following result is obtained in [5]. But our proof is different from the proof in [5]. For the reader's convenience, we will give the detailed proof.

Theorem 3.4. *Assume that $v_0 \in H^1(K)$, $E_K(v_0) < d$ and $D(v_0) < 0$. Then the weak solution $v(t)$ of equation (2.1) blows up in finite time, that is, there exists a $T > 0$ such that*

$$\lim_{t\to T}\int_0^t \|v(\tau)\|_{2,K} d\tau = +\infty.$$

Remark 3.5. Assume that $v_0 \in H^1(K)$,$E_K(v_0) < d$. When $D_K(u_0) > 0$, equation (2.1) has a global solution. When $D_K(v_0) < 0$, equation (2.1) does not admit any global weak solution.

Theorem 3.6. *Assume that $v_0 \in H^1(K)$, $E_K(v_0) < d$ and $D_K(v_0) > 0$, $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v_0)$. Then, for the global weak solution v of equation (2.1), it holds*

$$\|v\|_{2,K}^2 \leqslant \|v_0\|_{2,K}^2 e^{-2S_\lambda(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{3.1}$$

Remark 3.7. In comparison with the decay rate in [5], our result concerning the decay rate of $|u|_2$ in Theorem 3.6 is much more precise.

In order to prove Theorems 4.1–4.4, we need the following lemmas:

Lemma 3.8. *For $0 < T \leq \infty$, assume that $v : \Omega \times [0, T) \to \mathbb{R}^3$ is a weak solution to equation (2.1). Then it holds*

$$\int_{t_1}^{t_2} \|v_t\|_{2,K}^2 dt + E_K(v(t_2)) = E_K(v(t_1)), \quad \forall t_1, t_2 \in (0, T). \tag{3.2}$$

Proof. Multiplying (2.1) by v_t and integrating over $\mathbb{R}^N$ via the integration by parts, we get (3.2). □

Lemma 3.9. *If $0 < E_K(v) < d$ for some $v \in H^1(K)$, and $\delta_1 < 1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v)$, then the sign of $D_{K,\delta}(v)$ does not change for $\delta_1 < \delta < \delta_2$.*

Proof. Since $E_K(v) > 0$, we have $\|v\|_{2,K} \neq 0$. If the sign of $D_{K,\delta}(v)$ is changeable for $\delta_1 < \delta < \delta_2$, then we choose $\bar{\delta} \in (\delta_1, \delta_2)$ such that $D_{K,\bar{\delta}}(v) = 0$. Hence, by the definition of $d(\bar{\delta})$, we can obtain $E_K(v) \geqslant d(\bar{\delta})$, which contradicts $E_K(v) = d(\delta_1) = d(\delta_2) < d(\bar{\delta})$ (by Lemma 2.4 (3)). □

Definition 3.10 (Maximal existence time)**.** Assume that $v(t)$ is a weak solution of equation (2.1). The maximal existence time T of $v(t)$ is defined as follows:

(1) If $v(t)$ exists for $0 \leqslant t < \infty$, then $T = +\infty$.

(2) If there is a $t_0 \in (0, \infty)$ such that $v(t)$ exists for $0 \leqslant t < t_0$, but doesn't exist at $t = t_0$, then $T = t_0$.

Proof of Theorem 3.1. (1) Let $v(t)$ be any weak solution of equation (2.1) with $E_K(v_0) = e$, $D_K(v_0) > 0$, and T be the maximal existence time of $v(t)$. Using $E_K(v_0) = e, D_K(v_0) > 0$ and Lemma 3.9, we have $D_{K,\delta}(v_0) > 0$ and $E_K(v_0) < d(\delta)$. So $v_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. We need to prove that $v(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Indeed, if this is not the conclusion, from time continuity of $D_K(v)$ we assume that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such that $v(t_0) \in \partial W_{\delta_0}$, and $D_{\delta_0}(v(t_0)) = 0, \|v(t_0)\| \neq 0$ or $E_K(v(t_0)) = d(\delta_0)$. From the energy equality

$$\int_0^t \int_\Omega |v_\tau|^2 + E_K(v(t)) = E_K(v_0) < d(\delta),\ \delta_1 < \delta < \delta_2, \quad 0 \leqslant t < T, \tag{3.3}$$

we easily know that $E_K(v(t_0)) \neq d(\delta_0)$. If $D_{K,\delta_0}(v(t_0)) = 0, \|v(t_0)\| \neq 0$, then by the definition of $d(\delta)$ we obtain $E_K(v(t_0)) \geqslant d(\delta_0)$, which contradicts (3.3).

(2) Let $v(t)$ be any weak solution of equation (2.1) with $E_K(v_0) = e$, $D_K(v_0) < 0$, and T be the maximal existence time of $v(t)$. Using $E_K(v_0) = e, D_K(v_0) < 0$ and Lemma 3.9, we have $D_\delta(u_0) < 0$ and $E_K(v_0) < d(\delta)$. So $u_0 \in V_\delta$ for $\delta_1 < \delta < \delta_2$. We need to prove that $v(t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Indeed, if this is not the conclusion, from time continuity of $D_K(v)$ we assume that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such that $v(t_0) \in \partial V_{\delta_0}$, and $D_{K,\delta_0}(v(t_0)) = 0$, or $E_K(v(t_0)) = d(\delta_0)$. From the energy equality (3.3), we easily know that $E(v(t_0)) \neq d(\delta_0)$. If $D_{K,\delta_0}(v(t_0)) = 0$, and t_0 is the first time such that $D_{K,\delta_0}(v(t)) = 0$, then $D_{K,\delta_0}(v(t)) < 0$ for $0 \leqslant t < T$. By Lemma (2.3) (2), we have $\|v(t_0)\| > r(\delta_0)$ for $0 \leqslant t < T$. So, $\|v(t_0)\| > r(\delta_0)$ and $E_K(v(t_0)) \neq d(\delta_0)$, which contradicts (3.3). As required. □

Proof of Theorem 3.2. From the standard argument in [5], we can prove the local existence result of (2.1) in a more general case of initial value $v_0 \in H^1(K)$ and $v \in C^0([0, T_0], H^1(K))$.

Using $E_K(v_0) < d, D_K(v_0) > 0$ and Lemma 3.9, we have $D_\delta(v_0) > 0$ and $E_K(v_0) < d(\delta)$. So $v_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. We need to prove that $v(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Indeed, if this is not the conclusion, from time continuity of $D_K(v)$ we assume that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such that $v(t_0) \in \partial W_{\delta_0}$, and $D_{K,\delta_0}(v(t_0)) = 0, \|v(t_0)\| \neq 0$ or $E_K(v(t_0)) = d(\delta_0)$. From the energy equality

$$\int_0^t \int_\Omega |v_\tau|^2 + E_K(v(t)) = E_K(v_0) < d(\delta),\ \delta_1 < \delta < \delta_2, \quad 0 \leqslant t < T, \tag{3.4}$$

we easily know that $E_K(v(t_0)) \neq d(\delta_0)$. If $D_{K,\delta_0}(v(t_0)) = 0, \|v(t_0)\| \neq 0$, then by the definition of $d(\delta)$ we obtain $E_K(v(t_0)) \geqslant d(\delta_0)$, which contradicts (3.3). □

Remark 3.11. If in Theorem 3.2 the condition $D_{\delta_2}(u_0) > 0$ is replaced by $\|u_0\| < r(\delta_2)$, then equation (2.1) has a global weak solution $u(t) \in L^\infty(0, \infty; H^1(K))$ with $u_t(t) \in L^2(0, \infty; H^1(K))$ and the following result holds

$$\|u\| < \frac{d(\delta)}{a(\delta)}, \quad \delta_1 < \delta < \delta_2, 0 \leqslant t < \infty, \tag{3.5}$$

$$\int_0^t |u_\tau|^2 \, d\tau < d(\delta), \quad \delta_1 < \delta < \delta_2, 0 \leqslant t < \infty. \tag{3.6}$$

In particular

$$\|u\|^2 < \frac{d(\delta_1)}{a(\delta_1)}, \tag{3.7}$$

$$\int_0^t |u_\tau|^2 \, d\tau < d(\delta_1), \quad 0 \leqslant t < \infty. \tag{3.8}$$

Proof of Theorem 3.4. We argue by contradiction. Suppose that there would exist a global weak solution $v(t)$. Set

$$f(t) = \int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau, t > 0. \tag{3.9}$$

Multiplying (2.1) by u and integrating over $R^N \times (0,t)$, we get

$$\|v(t)\|_{2,K}^2 - \|v_0\|_{2,K}^2 = -2\int_0^t (\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}). \tag{3.10}$$

According to the definition of $f(t)$, we have $f'(t) = \|v(t)\|_{2,K}^2$ and hence

$$f'(t) = \|v(t)\|_{2,K}^2 = \|v_0\|_{2,K}^2 - 2\int_0^t (\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}), \tag{3.11}$$

and

$$f''(t) = -2(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}) = -2D_K(v). \tag{3.12}$$

Now using (3.2), (3.12) and

$$\begin{aligned} E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \frac{p-1}{2(p+1)}(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D_K(v), \end{aligned}$$

we can obtain

$$\begin{aligned} f''(t) &\geq 2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)\|v\|_{2,K}^2 - 2(p+1)E_K(v_0) \\ &= 2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)f'(t) - 2(p+1)E_K(v_0). \end{aligned} \tag{3.13}$$

Note that

$$\begin{aligned} f(t)f''(t) &= f(t)\left[2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)f'(t) - 2(p+1)E_K(v_0)\right] \\ &= 2(p+1)\int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau \int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)f(t)f'(t) \\ &\quad -2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau. \end{aligned} \tag{3.14}$$

Hence, we have

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 = 2(p+1)&\int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau \int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau \\ &-2(p+1)\int_0^t (v_\tau, v)_K \mathrm{d}\tau + S_\lambda(p-1)f(t)f'(t) \\ &-2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau - (p+1)f'(t)\|v_0\|_{2,K}^2. \end{aligned} \tag{3.15}$$

Making use of the Schwartz inequality, we have

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 &\geq C^*(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2 \\ &\quad -2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau. \end{aligned} \tag{3.16}$$

Next, we distinguish two cases:

(1) If $E_K(u_0) \leqslant 0$, then

$$f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 \geq C^*(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2. \tag{3.17}$$

Now we prove $D_K(v) < 0$ for $t > 0$. If not, we must be allowed to choose a $t_0 > 0$ such that $D_K(v(t_0)) = 0$ and $D_K(v) < 0$ for $0 \leqslant t < t_0$. From Lemma 2.3 (2), we have $\|u\| > r(1)$ for $0 \leqslant t < t_0, \|v(t_0)\| \geqslant r(1)$ and $E_K(v(t_0)) \geqslant d$, which contradicts (3.3). From (3.12) we have $f'(t) > 0$ for $t \geqslant 0$. From $f'(0) = \|v(0)\|_{2,K}^2 \geqslant 0$, we can know that there exists a $t_0 \geqslant 0$ such that $f'(t_0) > 0$. For $t \geqslant t_0$ we have

$$f(t) \geqslant f'(t_0)(t - t_0) > f'(0)(t - t_0). \tag{3.18}$$

Hence, for sufficiently large t , we obtain

$$f(t) > (p+1)\|v_0\|_{2,K}^2, \tag{3.19}$$

then

$$f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 > 0.$$

(2) If $0 < E_K(v_0) < d$, then by Theorem 3.1 we have $v(t) \in V_\delta$ for $1 < \delta < \delta_2, t \geqslant 0$, and $D_\delta(v) < 0$, $\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 > r(\delta)$ for $1 < \delta < \delta_2, t \geqslant 0$, where δ_2 is the larger root of equation $d(\delta) = E_K(v_0)$. Hence, $D_{\delta_2}(v) \leqslant 0$ and $\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 \geqslant r(\delta_2)$ for $t \geqslant 0$. By (3.12), we have

$$\begin{aligned} f''(t) &= -2D_K(v) = 2(\delta_2 - 1)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) - 2D_{\delta_2}(v), \\ &\geqslant 2(\delta_2 - 1)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) \geqslant 2(\delta_2 - 1)r^2(\delta_2), \quad t \geqslant 0, \\ f'(t) &\geqslant 2(\delta_2 - 1)r^2(\delta_2)t + f'(0) \geqslant 2(\delta_2 - 1)r^2(\delta_2)t, \quad t \geqslant 0, \\ f(t) &\geqslant (\delta_2 - 1)r^2(\delta_2)t^2, \quad t \geqslant 0. \end{aligned} \tag{3.20}$$

Therefore, for sufficiently large t, we infer

$$\frac{S_\lambda(p-1)}{2}f(t) > (p+1)\|v_0\|_{2,K}^2, \quad \frac{S_\lambda(p-1)}{2}f'(t) > 2(p+1)E_K(v_0). \tag{3.21}$$

Then, (3.16) implies that

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 &\geq S_\lambda(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2 \\ &\quad - 2(p+1)f(t)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2. \\ &= \left(\frac{S_\lambda(p-1)}{2}f(t) - (p+1)\|v_0\|_{2,K}^2\right)f'(t) \\ &\quad + \left(\frac{S_\lambda(p-1)}{2}f'(t) - 2(p+1)E_K(v_0)\right)f(t) > 0. \end{aligned}$$

The remainder of the proof is the same as that in [12]. □

Proof of Theorem 3.6. Multiplying (2.1) by w, $w \in L^\infty\left(0,\infty; H^1(K)\right)$, we have

$$(v_t, w)_K + (\nabla v, \nabla w)_K = \left(|v|^{p-1}v + \frac{v}{p-1}, w\right)_K. \quad (3.22)$$

Letting $w = v$, (3.22) implies that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{2,K} + D_K(v) = 0, \quad 0 \leqslant t < \infty. \quad (3.23)$$

From $0 < E_K(v_0) < d, D_K(v_0) > 0$ and Lemma 3.1, we have $v(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 \leqslant t < \infty$, where $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v_0)$. Hence, we obtain $D_{K,\delta}(v) \geqslant 0$ for $\delta_1 < \delta < \delta_2$ and $D_{K,\delta_1}(v) \geqslant 0$ for $0 \leqslant t < \infty$. So, (3.23) gives

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{2,K} + (1-\delta_1)(\|\nabla v\|^2_{2,K} - \lambda\|v\|^2_{2,K}) + D_{K,\delta_1}(v) = 0, \quad 0 \leqslant t < \infty. \quad (3.24)$$

Now (3.23) implies that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{2,K} + S_\lambda(1-\delta_1)\|v\|^2_{2,K} \leq 0, \quad 0 \leqslant t < \infty. \quad (3.25)$$

and

$$\|v\|^2_{2,K} \leqslant \|v_0\|^2_{2,K} - 2S_\lambda(1-\delta_1)\int_0^t |v(\tau)|^2 d\tau, \quad 0 \leqslant t < \infty. \quad (3.26)$$

By Gronwall's inequality, we have

$$|v|^2_{2,K} \leqslant |v_0|^2_{2,K}\, e^{-2S_\lambda(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \quad (3.27)$$

This completes the proof. □

4 Critical initial energy $E_K(v_0) = d$

The goal of this section is to prove Theorem 4.1–4.4.

Theorem 4.1 (Global existence). *Assume that $v_0 \in H^1(K)$,$E(v_0) = d$ and $D_K(v_0) \geqslant 0$. Then equation (2.1) has a global weak solution $u(t) \in L^\infty(0,\infty; H^1(K))$ and $v(t) \in \overline{W} = W \cup \partial W$ for $0 \leqslant t < \infty$.*

Lemma 4.2. *Assume that $v \in H^1(K)$, $\|\nabla v\|^2_2 \neq 0$, and $D_K(v) \geq 0$. Then:*

(1) $\lim_{\mu\to 0} E_K(\lambda v) = 0, \lim_{\mu\to+\infty} E_K(\mu v) = -\infty$,

(2) *On the interval $0 < \mu < \infty$, there exists a unique $\mu^* = \mu^*(u)$, such that*

$$\frac{d}{d\mu}E_K(\mu v)|_{\mu=\mu^*} = 0, \quad (4.1)$$

(3) *$E_K(\mu v)$ is increasing on $0 \leqslant \mu \leqslant \mu^*$, decreasing on $\mu^* \leqslant \mu < \infty$ and takes the maximum at $\mu = \mu^*$,*

(4) *$D_K(\mu v) > 0$ for $0 < \mu < \mu^*, D_K(\mu v) < 0$ for $\mu^* < \mu < \infty$, and $D_K(\mu^* v) = 0$.*

Proof. (1) Firstly, from the definition of $E_K(v)$, i.e.

$$E_K(v) = \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1}$$

and we see that

$$E_K(\mu v) = \frac{1}{2}\|\nabla \mu v\|_{2,K}^2 - \frac{\lambda}{2}\|\mu v\|_{2,K}^2 - \frac{1}{p+1}\|\mu v\|_{p+1,K}^{p+1}.$$

Hence, we have

$$\lim_{\mu\to 0} E_K(\mu v) = 0 \quad \text{and} \quad \lim_{\mu\to +\infty} E_K(\mu u) = -\infty. \tag{4.2}$$

(2) It is easy to show that

$$\frac{d}{d\mu}E_K(\mu v) = \mu\|\nabla v\|_{2,K}^2 - \mu\lambda\|v\|_{2,K}^2 - \mu^p\|v\|_{p+1,K}^{p+1},$$

which leads to the conclusion.

(3) By Lemma 4.2 (2), one has

$$\begin{aligned} &\frac{d}{d\mu}E_K(\mu v) > 0 \quad \text{for } 0 < \mu < \mu^*, \\ &\frac{d}{d\mu}E_K(\mu v) < 0 \quad \text{for } \mu^* < \mu < \infty, \end{aligned} \tag{4.3}$$

which leads to the conclusion.

(4) The conclusion follows from

$$D_K(\mu v) = \frac{d}{d\mu}E_K(\mu v) = \mu\|\nabla v\|_{2,K}^2 - \mu\lambda\|v\|_{2,K}^2 - \mu^p\|v\|_{p+1,K}^{p+1}.$$

As desired. □

Proof of Theorem 4.1. Firstly, $E_K(v_0) = d$ implies that $\|v_0\|_{H^1(K)} \neq 0$. Choose a sequence $\{\mu_m\}$ such that $0 < \mu_m < 1$, $m = 1, 2, \dots$ and $\mu_m \to 1$ as $m \to \infty$. Let $v_{0m} = \mu_m v_0$. We consider the following initial problem

$$\begin{cases} v_s + Lv = |v|^{p-1}v + \dfrac{1}{p-1}v & \text{in } \mathbb{R}^N \times (0, S), \\ v|_{s=0} = v_{0m} & \text{in } \mathbb{R}^N. \end{cases} \tag{4.4}$$

From $D_K(v_0) \geqslant 0$ and Lemma 4.2, we have $\mu^* = \mu^*(u_0) \geqslant 1$. Thus, we get $D_K(v_{0m}) = D_K(\mu_m v_0) > 0$ and $E_K(v_{0m}) = E_K(\mu_m v_0) < E_K(v_0) = d$. From Theorem 3.2, it follows that for each m problem (4.4) admits a global weak solution $v_m(t) \in L^\infty(0,\infty; H^1(K))$ with $v_{mt}(t) \in L^2(0,\infty; H^1(K))$ and $v_m(t) \in W$ for $0 \leqslant t < \infty$ satisfying

$$(v_{m,t}, w)_K + (\nabla v_{m,t}, \nabla w)_K = \left(|v|^{p-1}v + \frac{v}{p-1}, w\right)_K, \quad \text{for all } w \in H^1(K), t > 0. \tag{4.5}$$

$$\int_0^t \|v_{m,\tau}\|_{2,K}^2 + E_K(v_m(t)) = E_K(v_{0m}) < d, \quad 0 \leqslant t < \infty, \tag{4.6}$$

which implies that

$$E_K(v_m) = \frac{1}{2}\|\nabla v_m\|_{2,K}^2 - \frac{\lambda}{2}\|v_m\|_{2,K}^2 - \frac{1}{p+1}\|v_m\|_{p+1,K}^{p+1}$$
$$= \frac{p-1}{2(p+1)}(\|\nabla v_m\|_{2,K}^2 - \lambda\|v_m\|_{2,K}^2) + \frac{1}{p+1}D_K(v_m). \tag{4.7}$$

Therefore, one has

$$\int_0^T \|v_{m,\tau}\|_{2,K}^2 \mathrm{d}\tau + \frac{p-1}{2(p+1)}(\|\nabla v_m\|_{2,K}^2 - \lambda\|v_m\|_{2,K}^2) < d, \quad 0 \leqslant t < \infty. \tag{4.8}$$

The remainder of the proof is similar to the proof of Theorem 3.2. □

Theorem 4.3 (Blow-up). *Assume that $v_0 \in H^1(K)$, $E_K(v_0) = d$ and $D(v_0) > 0$, Then the existence time of weak solution for equation (2.1) is finite.*

Proof of Theorem 4.3. Let $v(t)$ be any weak solution of equation (2.1) with $E_K(v_0) = d$ and $D_K(v_0) < 0$, T be the existence time of $v(t)$. We next prove $T < \infty$. We argue by contradiction. Suppose that there would exist a global weak solution $v(t)$. Set

$$f(t) = \int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau, t > 0. \tag{4.9}$$

Multiplying (2.1) by u and integrating over $R^N \times (0,t)$, we get

$$\|v(t)\|_{2,K}^2 - \|v_0\|_{2,K}^2 = -2\int_0^t (\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}). \tag{4.10}$$

According to the definition of $f(t)$, we have $f'(t) = \|v\|_{2,K}^2$ and hence

$$f'(t) = \|v(t)\|_{2,K}^2 = \|v_0\|_{2,K}^2 - 2\int_0^t (\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}), \tag{4.11}$$

and

$$f''(t) = -2(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}) = -2D_K(v). \tag{4.12}$$

Now using (3.2), (4.12) and

$$E_K(v) = \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1}$$
$$= \frac{p-1}{2(p+1)}(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D_K(v),$$

we can obtain

$$f''(t) \geq 2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)\|v\|_{2,K}^2 - 2(p+1)E_K(v_0)$$
$$= 2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)f'(t) - 2(p+1)E_K(v_0). \tag{4.13}$$

Note that

$$f(t)f''(t) = f(t)\left[2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + C^*(p-1)f'(t) - 2(p+1)E_K(v_0)\right]$$
$$= 2(p+1)\int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau \int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)f(t)f'(t)$$
$$-2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau. \tag{4.14}$$

Hence, we have

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 = 2(p+1)\int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau \int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau \\ - 2(p+1)\int_0^t (v_\tau, v)_K \mathrm{d}\tau + S_\lambda(p-1)f(t)f'(t) \\ - 2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau - (p+1)f'(t)\|v_0\|_{2,K}^2. \end{aligned} \tag{4.15}$$

Hence, according to (4.15) and the Schwartz inequality, we obtain

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 &\geq S_\lambda(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2 \\ &\quad - 2(p+1)f(t)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau. \\ &= \left(\frac{S_\lambda(p-1)}{2}f(t) - (p+1)\|v_0\|_{2,K}^2\right) f'(t) \\ &\quad + \left(\frac{S_\lambda(p-1)}{2}f'(t) - 2(p+1)E_K(v_0)\right) f(t). \end{aligned} \tag{4.16}$$

On the other hand, from $E_K(v_0) = d > 0, D_K(v_0) < 0$ and the continuity of $E_K(v)$ and $D_K(v)$ with respect to t, it follows that there exists a sufficiently small $t_1 > 0$ such that $E_K(v(t_1)) > 0$ and $D_K(v) < 0$ for $0 \leqslant t \leqslant t_1$. Hence $(v_t, v)_K = -D_K(v) > 0, \|v_t\|_2 > 0$ for $0 \leqslant t \leqslant t_1$. So, using the continuity of $\int_0^t \|v_\tau\|_{2,K}^2 d\tau$, we can choose a t_1 such that

$$0 < d_1 = d - \int_0^{t_1} \|v_\tau\|_{2,K}^2 d\tau < d. \tag{4.17}$$

And by (3.4), we get

$$0 < E_K(v(t_1)) = d - \int_0^{t_1} \|v_\tau\|_{2,k}^2 d\tau = d_1 < d. \tag{4.18}$$

So we can choose $t = t_1$ as the initial time, then we obtain $v(t) \in V_\delta$ for $\delta \in (\delta_1, \delta_2), t_1 \leqslant t < \infty$, where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > d_1$ for $\delta \in (\delta_1, \delta_2)$. Thus we get $D_{K,\delta}(v) < 0$ and $\|v\| > r(\delta)$ for $\delta \in (1, \delta_2), t_1 \leqslant t < \infty$, and $D_{K,\delta_2}(v) \leqslant 0, \|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 \geqslant r(\delta_2)$ for $t_1 \leqslant t < \infty$. Thus (4.12) implies that

$$\begin{aligned} f''(t) &= -2D_K(v) = 2(\delta_2 - 1)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) - 2D_{\delta_{K,2}}(v), \\ &\geqslant 2(\delta_2 - 1) r(\delta_2), \quad t \geqslant t_1, \\ f'(t) &\geqslant 2(\delta_2 - 1) r(\delta_2)(t - t_1) + f'(t_1) \geqslant 2(\delta_2 - 1) r(\delta_2)(t - t_1), \quad t \geqslant 0, \\ f(t) &\geqslant (\delta_2 - 1) r(\delta_2)(t - t_1)^2 + M(t_1) > (\delta_2 - 1) r(\delta_2)(t - t_1)^2, \quad t \geqslant t_1. \end{aligned} \tag{4.19}$$

Therefore, for sufficiently large t, we infer

$$\frac{S_\lambda(p-1)}{2}f(t) > (p+1)\|v_0\|_{2,K}^2, \quad \frac{S_\lambda(p-1)}{2}f'(t) > 2(p+1)E_K(v_0). \tag{4.20}$$

Then, (4.16) implies that

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 \geq\; & S_\lambda(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2 \\ & - 2(p+1)f(t)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau. \\ =\; & \left(\frac{S_\lambda(p-1)}{2}f(t) - (p+1)\|v_0\|_{2,K}^2\right) f'(t) \\ & + \left(\frac{S_\lambda(p-1)}{2}f'(t) - 2(p+1)E_K(v_0)\right) f(t) > 0. \end{aligned}$$

The remainder of the proof is the same as that in [12]. □

Theorem 4.4. *Assume that $u_0 \in H^1(K)$, $E_K(v_0) = d$ and $D_K(v_0) > 0$, $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v_0)$. Then, for the global weak solution v of equation (2.1), it holds*

$$|v|_2^2 \leqslant |v_0|_2^2 e^{-2S_\lambda(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{4.21}$$

Proof of Theorem 4.4. We first know that equation (2.1) has a global weak solution from Theorem 4.3. Furthermore, Using Theorem 3.4, Theorem 4.3 and (3.3), if $v(t)$ is a global weak solution of equation (2.1) with $E_K(v_0) = d$, $D_K(v_0) > 0$, then must have $D_K(v) \geq 0$ for $0 \leq t < +\infty$. Next, we distinguish two cases:

(1) Suppose that $D_K(v) > 0$ for $0 \leqslant t < \infty$. Multiplying (2.1) by v, $v \in L^\infty\left(0,\infty; H^1(K)\right)$, we have

$$(v_t, w)_K + (\nabla v_t, \nabla w)_K = \left(|v|^{p-1}v + \frac{v}{p-1}, w\right)_K, \quad \text{for all } w \in H^1(K), t > 0. \tag{4.22}$$

Letting $w = v$, (4.22) implies that

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 = -D_K(v) < 0, \quad 0 \leqslant t < \infty. \tag{4.23}$$

Since $\|v_t\|_{2,K} > 0$, we have that $\int_0^t \|v_\tau\|^2 d\tau$ is increasing for $0 \leqslant t < \infty$. By choosing any $t_1 > 0$ such that

$$0 < d_1 = d - \int_0^{t_1} \|v_\tau\|_{2,K}^2 d\tau < d. \tag{4.24}$$

From (3.3), if follows that $0 < E_K(v) \leq d_1 < d$, and $v(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 \leqslant t < \infty$, where $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v_0)$. Hence, we obtain $D_{K,\delta_1}(v) \geqslant 0$ for $\delta_1 < \delta < \delta_2$ and $D_{K,\delta_1}(v) \geqslant 0$ for $t_1 \leqslant t < \infty$. So, (4.23) gives

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 + (1-\delta_1)|v|_2^2 + D_{K,\delta}(v) = 0, \quad t_1 \leqslant t < \infty. \tag{4.25}$$

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 + (1-\delta_1)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + D_{K,\delta_1}(v) = 0, \quad 0 \leqslant t < \infty. \tag{4.26}$$

Now (4.23) implies that

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 + S_\lambda(1-\delta_1)\|v\|_{2,K}^2 \leq 0, \quad 0 \leqslant t < \infty. \tag{4.27}$$

and

$$\|v\|_{2,K}^2 \leqslant \|v_0\|_{2,K}^2 - 2S_\lambda(1-\delta_1)\int_0^t |v(\tau)|^2 d\tau, \quad 0 \leqslant t < \infty. \tag{4.28}$$

and By Gronwall's inequality, we have

$$|v|_{2,K}^2 \leqslant |v_0|_2^2 e^{-2S_\lambda(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{4.29}$$

(2) Suppose that there exists a $t_1 > 0$ such that $D_K(v(t_1)) = 0$ and $D_K(v) > 0$ for $0 \leqslant t < t_1$. Then, $|u_t|_2 > 0$ and $\int_0^t |v_\tau|_2^2 d\tau$ is increasing for $0 \leqslant t < t_1$. By (4.24) we have

$$E_K(v(t_1)) = d - \int_0^{t_1} |v_\tau|_2^2 d\tau < d, \tag{4.30}$$

and $\|v(t_1)\| = 0$. Then, we have that $v(t) \equiv 0$ for $t_1 \leqslant t < \infty$.

Hence, the proof is complete. □

5 High initial energy $E_K(v_0) > d$

In this section, we investigate the conditions to ensure the existence of global solutions or blow-up solutions to system (2.1) with $E_K(v_0) > d$.

Lemma 5.1. *For any $\alpha > d, \lambda_\alpha$ and Λ_α defined in (2.1) satisfy*

$$0 < \lambda_\alpha \leq \Lambda_\alpha < +\infty. \tag{5.1}$$

Proof. (1) By Hölder's inequality, fundamental inequality and $u \in \mathcal{N}$, we have

$$\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 = \|v\|_{p+1,K}^{p+1}. \tag{5.2}$$

Then from Lemma 2.6 (1), we have $\lambda_\alpha > 0$.

Using Lemma 2.1 and $u \in \mathcal{N}$, we have

$$\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 = \|v\|_{p+1,K}^{p+1} \leqslant \left(\frac{1}{S_\lambda} (\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2) \right)^{\frac{p+1}{2}}. \tag{5.3}$$

So we have $\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 \leq \frac{1}{S_\lambda}$ which leads to the conclusion. □

Theorem 5.2. *Suppose that $E_K(v_0) > d$, then we have*

(1) *If $v_0 \in \mathcal{N}_+$ and $\|v_0\|_{2,k} \leq \lambda_{E_K(v_0)}$, then $v_0 \in \mathcal{G}_0$,*

(2) *If $v_0 \in \mathcal{N}_-$ and $\|v_0\|_{2,k} \geq \Lambda_{E_K(v_0)}$, then $v_0 \in \mathcal{B}$.*

Proof. The maximal existence time of the solutions to system (2.1) with initial value v_0 is denoted by T_0. If the solution is global, i.e. $T(v_0) = +\infty$, the limit set of v_0 is denoted by ω_0.

(1) Suppose that $v_0 \in \mathcal{N}_+$ with $|v_0|_2 \leq \lambda_{E_K(v_0)}$. We firstly prove that $v(t) \in \mathcal{N}_+$ for all $t \in [0, T(v_0))$. Assume, on the contrary, that there exists a $t_0 \in (0, T(v_0))$ such that $v(t) \in \mathcal{N}_+$ for $0 \leq t < t_0$ and $v(t_0) \in \mathcal{N}$. It follows from $D_K(v(t)) = -\int_\Omega v_t(x,t)v(x,t)\mathrm{d}x$ that $v_t(x,t) \neq 0$ for $(x,t) \in \Omega \times (0, t_0)$. Recording to(3.2) we then have $E_K(v(t_0)) < E_K(v_0)$, which implies that $u(t_0) \in E_K^{E_K(v_0)}$. Therefore, $v(t_0) \in \mathcal{N}^{E_K(v_0)}$. Recalling the definition of $\lambda_{E_K(v_0)}$, we get

$$|u(t_0)|_2 \geq \lambda_E(v_0). \tag{5.4}$$

Since $D_K(v(t)) > 0$ for $t \in [0, t_0)$, we obtain from (3.23) that

$$|v(t_0)|_2 < |v_0|_2 \le \lambda_{E_K(v_0)}, \tag{5.5}$$

which contradicts (5.4). Hence, $v(t) \in \mathcal{N}_+$ which shows that $v(t) \in E_K^{E_K(v_0)}$ for all $t \in [0, T(v_0))$. Now Lemma 3.9 (2) implies that the orbit $\{v(t)\}$ remains bounded in $H^1(K)$ for $t \in [0, T(v_0))$ so that $T(v_0) = \infty$. Assume that ω is an arbitrary element in $\omega(v_0)$. Then by (3.2) and (3.23) we obtain

$$|\omega|_2 > \Lambda_{E_K(v_0)}, \quad E_K(\omega) < E_K(v_0), \tag{5.6}$$

which, according to the definition of $\lambda_{E_K(v_0)}$ again, implies that $\omega(v_0) \cap N = \emptyset$. So, $\omega(v_0) = \{0\}$, i.e. $v_0 \in \mathcal{G}_0$.

(2) Suppose that $v_0 \in \mathcal{N}_-$ with $|v_0|_2 \ge \Lambda_{E_K(v_0)}$. We now prove that $v(t) \in \mathcal{N}_-$ for all $t \in [0, T(v_0))$. Assume, on the contrary, that there exists a $t^0 \in (0, T(v_0))$ such that $v(t) \in \mathcal{N}_-$ for $0 \le t < t^0$ and $v(t^0) \in \mathcal{N}$. Similarly to case (1), one has $E_K(v(t^0)) < E_K(v_0)$, which implies that $v(t^0) \in E_K^{E_K(v_0)}$. Therefore, $v(t^0) \in \mathcal{N}^{E_K(v_0)}$. Recalling the definition of $\Lambda_{E_K(v_0)}$, we infer

$$|v(t^0)|_2 \le \Lambda_{E_K(v_0)}. \tag{5.7}$$

On the other hand, from (3.23) and the fact that $D_K(v(t)) < 0$ for $t \in [0, t^0)$, we obtain

$$|v(t^0)|_2 > |v_0|_2 \ge \Lambda_{E_K(v_0)}, \tag{5.8}$$

which contradicts (5.7).

Assume that $T(v_0) = \infty$. Then for each $\omega \in \omega(v_0)$, it follows from by (3.2) and (3.23) that

$$\|\omega\|_2 > \Lambda_{E_K(v_0)}, \quad E_K(\omega) < E_K(v_0). \tag{5.9}$$

Noting the definition of $\Lambda_{E_K(v_0)}$ again, we have $\omega(v_0) \cap N = \emptyset$. Hence, it is holded that $\omega(v_0) = \{0\}$, which contradicts Lemma 3.9 (1). Therefore, $T(v_0) < \infty$. This ends the proof. □

Theorem 5.3. *Assume that $v_0 \in H^1(K)$ satisfies*

$$E_K(v_0) \le \|v_0\|_{2,K} < \frac{p}{p+1}\|v_0\|_{p+1,K}^{p+1}, \tag{5.10}$$

Then, $v_0 \in \mathcal{N}_- \cap \mathcal{B}$.

Proof. Firstly, we observe

$$\begin{aligned} E_K(v_0) &= \frac{1}{2}\|\nabla v_0\|_{2,K}^2 - \frac{\lambda}{2}\|v_0\|_{2,K}^2 - \frac{1}{p+1}\|v_0\|_{p+1,K}^{p+1} \\ &= \frac{1}{2}D_K(v_0) + \frac{p}{p+1}\|v_0\|_{p+1,K}^{p+1}. \end{aligned} \tag{5.11}$$

Thus, we have

$$E_K(v_0) - \frac{p}{p+1}\|v_0\|_{p+1,K}^{p+1} = \frac{1}{2}D_K(v_0) < 0, \tag{5.12}$$

which shows that $v_0 \in \mathcal{N}_-$. Then for any $v \in \mathcal{N}_{E_K(v_0)}$, one has

$$\|v\|_{p+1,K}^{p+1} = \|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 \le E_K(v_0) \le \sqrt{\frac{2(p+1)}{p-1}} E_K(v_0).$$

Taking supremum over $\mathcal{N}_{E_K(v_0)}$ and (5.10), by Theorem 5.2 we can deduce

$$\|v_0\|_2 \ge \Lambda_{E_K(v_0)}.$$

Thus, $v_0 \in \mathcal{N}_- \cap \mathcal{B}$. This finishes the proof. □

Acknowledgements

This work is supported by NSFC (No. 11801017, No. 11871199) and the Cultivation Project of Young and Innovative Talents in Universities of Shandong Province.

References

[1] M. Escobedo, O. Kavian, Variational problems related to self-similar solutions of the heat equation, *Nonlinear Anal.* **11**(1987), No. 1, 1103–1133, https://doi.org/10.1016/0362-546X(87)90001-0; MR0913672

[2] F. Gazzola, T. Weth, Finite time blow-up and global solutions for semilinear parabolic equations with initial data at high energy level, *Differential Integral Equations* **18**(2005), No. 9, 961–990. MR2162421

[3] Y. Z. Han, A class of fourth-order parabolic equation with arbitrary initial energy, *Nonlinear Anal. Real World Appl.* **43**(2018), 451–466. https://doi.org/10.1016/j.nonrwa.2018.03.009; MR3799693

[4] M. Ishiwata, T. Suzuki, Positive solution to semilinear parabolic equation associated with critical Sobolev exponent, *NoDEA Nonlinear Differential Equations Appl.* **20**(2013), No. 4, 1553–1576. https://doi.org/10.1007/s00030-013-0221-6; MR3078242

[5] R. Ikehata, M. Ishiwata, T. Suzuki, Semilinear parabolic equation in $\mathbf{R}^N$ associated with critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **27**(2010), No. 3, 877–900. https://doi.org/10.1016/j.anihpc.2010.01.002; MR2629884

[6] R. Ikehata, T. Suzuki, Semilinear parabolic equations involving critical Sobolev exponent: local and asymptotic behavior of solutions, *Differential Integral Equations* **13**(2000), No. 7–9, 869–901. MR1775238

[7] R. Ikehata, Some remarks on the wave equations with nonlinear damping and source terms, *Nonlinear Anal.* **27**(1996), No. 10, 1165–1175. https://doi.org/10.1016/0362-546X(95)00119-G; MR1407454

[8] O. Kavian, Remarks on the large time behaviour of a nonlinear diffusion equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4**(1987), No. 5, 423–452. MR0921547

[9] H. A. Levine, The role of critical exponents in blowup theorems, *SIAM Rev.* **32**(1990), No. 2, 262–288. https://doi.org/10.1137/1032046; MR1056055

[10] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969. MR0259693

[11] Y. C. Liu, On potential wells and vacuum isolating of solutions for semilinear wave equations, *J. Differential Equations* **192**(2003), No. 1, 155–169. https://doi.org/10.1016/S0022-0396(02)00020-7; MR1987088

[12] Y. C. Liu, J. S. Zhao, On potential wells and applications to semilinear hyperbolic equations and parabolic equations, *Nonlinear Anal.* **64**(2006), No. 12, 2665–2687. https://doi.org/10.1016/j.na.2005.09.011; MR2218541

[13] L. Ma, Blow-up for semilinear parabolic equations with critical Sobolev exponent, *Commun. Pure Appl. Anal.* **12**(2013), No. 2, 1103–1110. https://doi.org/10.3934/cpaa.2013.12.1103; MR2982809

[14] N. Mizoguchi, E. Yanagida, Blowup and life span of solutions for a semilinear parabolic equation, *SIAM J. Math. Anal.* **29**(1998), No. 6, 1434–1446. https://doi.org/10.1137/S0036141097324934; MR1638070

[15] N. Mizoguchi, E. Yanagida, Critical exponents for the blow-up of solutions with sign changes in a semilinear parabolic equation, *Math. Ann.* **307**(1997), No. 4, 663–675. https://doi.org/10.1007/s002080050055; MR1464136

[16] N. Mizoguchi, E. Yanagida, Critical exponents for the blowup of solutions with sign changes in a semilinear parabolic equation. II. *J. Differential Equations* **145**(1998), No. 2, 295–331. https://doi.org/10.1006/jdeq.1997.3387; MR1621030

[17] N. Pan, B. Zhang, J. Cao, Degenerate Kirchhoff-type diffusion problems involving the fractional p-Laplacian, *Nonlinear Anal. Real World Appl.* **37**(2017), 56–70. https://doi.org/10.1016/j.nonrwa.2017.02.004; MR3648372

[18] N. Pan, P. Pucci, B. Zhang, Degenerate Kirchhoff-type hyperbolic problems involving the fractional Laplacian, *J. Evol. Equ.* **18**(2018), No. 2, 385–409. https://doi.org/10.1007/s00028-017-0406-2; MR3820409

[19] N. Pan, P. Pucci, R. Xu, B. Zhang, Degenerate Kirchhoff-type wave problems involving the fractional Laplacian with nonlinear damping and source terms, *J. Evol. Equ.* **19**(2019), No. 3, 615–643. https://doi.org/10.1007/s00028-019-00489-6; MR3997239

[20] L. E. Payne, D. H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.* **22**(1975), No. 3–4, 273–303. https://doi.org/10.1007/BF02761595; MR0402291

[21] P. Quittner, Continuity of the blow-up time and a priori bounds for solutions in superlinear parabolic problems, *Houston J. Math.* **29**(2003), No. 3, 757–799. MR1998164

[22] P. Quittner, Universal bound for global positive solutions of a superlinear parabolic problem, *Math. Ann.* **320**(2001), No. 2, 299–305. https://doi.org/10.1007/PL00004475; MR1839765

[23] P. Quittner, P. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhäuser Advanced Texts, Basel/Boston/Berlin, 2007. https://doi.org/10.1007/978-3-7643-8442-5; MR2346798

[24] D. H. Sattinger, On global solution of nonlinear hyperbolic equations, *Arch. Ration. Mech. Anal.* **30**(1968), 148–172. https://doi.org/10.1007/BF00250942; MR0227616

[25] Z. Tan, Global solution and blowup of semilinear heat equation with critical Sobolev exponent, *Comm. Partial Differential Equations* **26**(2001), No. 3–4, 717–741. https://doi.org/10.1081/PDE-100001769; MR1842047

[26] M. Tsutsumi, Existence and nonexistence of global solutions for nonlinear parabolic equations, *Publ. Res. Inst. Math. Sci.* **8**(1972), 211–229. https://doi.org/10.2977/prims/1195193108; MR0312079

[27] F. B. Weissler, Local existence and nonexistence for semilinear parabolic equations in L^p, *Indiana Univ. Math. J.* **29**(1980), No. 1, 79–102. https://doi.org/10.1512/iumj.1980.29.29007; MR0554819

[28] F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, *Israel J. Math.* **38**(1981), No. 1–2, 29–40. https://doi.org/10.1007/BF02761845; MR0599472