Weak center for a class of $\Lambda$–$\Omega$ differential systems

Zhengxin Zhou

School of Mathematical Sciences, Yangzhou University, China

Received 30 July 2021, appeared 12 May 2022
Communicated by Gabriele Villari

Abstract. In this paper, we give the necessary and sufficient conditions for a class of higher degree polynomial systems to have a weak center. As corollaries, we prove the correctness of the two conjectures about the weak center problem for the $\Lambda$–$\Omega$ differential systems.

Keywords: weak center, $\Lambda$–$\Omega$ system, composition center, center conditions.

2020 Mathematics Subject Classification: 34C07, 34C05, 34C25, 37G15.

1 Introduction

Consider differential system of the form

$$\begin{cases} x' = -y + P, \\ y' = x + Q, \end{cases} \quad (1.1)$$

where $P = \sum_{k=2}^{m} P_k(x,y)$ and $Q = \sum_{k=2}^{m} Q_k(x,y)$, $P_k$ and $Q_k$ are homogeneous polynomials in $x$ and $y$ of degree $k$. The equilibrium point $O(0,0)$ is a center if there exists an open neighborhood $U$ of $O$ where all the orbits contained in $U/O$ are periodic. The center-focus problem asks about the conditions on the coefficients of $P$ and $Q$ under which the origin of system (1.1) is a center. The study of the centers of analytical or polynomial differential system (1.1) has a long history. The first works are due to Poincaré [13] and Dulac [8], and continued by Liapunov [9] and many others. Unfortunately, the center-focus problem has been solved only for quadratic system and some special cubic system and others [2, 6, 7, 12]. Up to now, very little is known about the center conditions for polynomial differential system with arbitrary degree $m$ ($m > 2$).

A center of (1.1) is called a weak center if the Poincaré–Liapunov first integral can be written as $H = \frac{1}{2}(x^2 + y^2)(1 + h.o.t.)$. By literature [10, 11] we know that a center of a polynomial differential system (1.1) is a weak center if and only if it can be written as

$$\begin{cases} x' = -y(1 + \Lambda) + x\Omega, \\ y' = x(1 + \Lambda) + y\Omega, \end{cases} \quad (1.2)$$

Corresponding author. Email: zxzhou@yzu.edu.cn
where \( \Lambda = \Lambda(x, y) \) and \( \Omega = \Omega(x, y) \) are polynomials of degree at most \( m - 1 \) such that \( \Lambda(0, 0) = \Omega(0, 0) = 0 \). The weak centers contain the uniform isochronous centers and the holomorphic isochronous centers [10], they also contain the class of centers studied by Alwash and Lloyd [5], but they do not coincide with all classes of isochronous centers [10].

The class of differential system (1.2) is called the \( \Lambda-\Omega \) system. The reason of called such system in this way is due to the fact that a subclass of these systems already appears in physics [11].

In [11] the authors put forward such conjectures:

**Conjecture 1.1.** The polynomial differential system of degree \( m \)

\[
\begin{aligned}
x' &= -y(1 + \mu(a_2x - a_1y)) + x((a_1x + a_2y) + \Phi_{m-1}), \\
y' &= x(1 + \mu(a_2x - a_1y)) + y((a_1x + a_2y) + \Phi_{m-1}) 
\end{aligned}
\tag{1.3}
\]

where \((\mu + m - 2)(a_1^2 + a_2^2) \neq 0 \) and \( \Phi_{m-1} = \Phi_{m-1}(x, y) \) is a homogeneous polynomial of degree \( m - 1 \), has a weak center at the origin if and only if system (1.3) after a linear change of variables \((x, y) \to (X, Y)\) is invariant under the transformations \((X, Y, t) \to (-X, Y, -t)\).

**Conjecture 1.2.** The polynomial differential system of degree \( m \)

\[
\begin{aligned}
x' &= -y(1 + a_1x + a_2y) + x\Phi_{m-1}, \\
y' &= x(1 + a_1x + a_2y) + y\Phi_{m-1} 
\end{aligned}
\tag{1.4}
\]

has a weak center at the origin if and only if system (1.4) after a linear change of variables \((x, y) \to (X, Y)\) is invariant under the transformations \((X, Y, t) \to (-X, Y, -t)\).

The authors of [11] have used Poincaré–Liapunov first integral and Reeb inverse integrating factor to prove that Conjecture 1.1 and Conjecture 1.2 are correct when \( m = 2, 3, 4, 5, 6 \). They remarked that the only difficulty for proving Conjectures 1.1 and 1.2 for the \( \Lambda-\Omega \) system of degree \( m \) with \( m > 6 \) is the huge number of computations for obtaining the conditions that characterize the centers.

In this paper we will research the weak center problem of the \( \Lambda-\Omega \) system

\[
\begin{aligned}
x' &= -y(1 + \mu(a_2x - a_1y)) + x(v(a_1x + a_2y) + \Lambda_{m-1} + \Omega_{2m-1}), \\
y' &= x(1 + \mu(a_2x - a_1y)) + y(v(a_1x + a_2y) + \Lambda_{m-1} + \Omega_{2m-1}), 
\end{aligned}
\tag{1.5}
\]

in which \( m > 2 \) and \((\mu^2 + v^2)(\mu + v(m - 2))(a_1^2 + a_2^2) \neq 0 \), \( \Lambda_{m-1} = \Lambda_{m-1}(x, y), \Omega_{2m-1} = \Omega_{2m-1}(x, y) \) are respectively homogeneous polynomials of degree \( m - 1 \) and \( 2m-1 \). In the section 3 we will see that by suitable transformation this system can be transformed into

\[
\begin{aligned}
x' &= -y(1 - \mu y) + x(vx + \Phi_{m-1} + \Psi_{2m-1}), \\
y' &= x(1 - \mu y) + y(vx + \Phi_{m-1} + \Psi_{2m-1}). 
\end{aligned}
\tag{1.6}
\]

In the following we use a method different from Llibre [11] and more simply, without huge number of computation, to prove that for system (1.6), under several restrictive conditions, it has a weak center at the origin if and only if

\[
\int_{0}^{2\pi} \sin^i \theta \Phi_{m-1}(\cos \theta, \sin \theta)d\theta = 0 \quad (i = 0, 1, 2, \ldots, m - 1)
\tag{1.7}
\]
and
\[ \int_0^{2\pi} \sin^j \theta \Psi_{2m-1}(\cos \theta, \sin \theta) d\theta = 0 \quad (j = 0, 1, 2, \ldots, 2m - 1). \]  
(1.8)

As corollaries, we also show that for arbitrary \( m (> 2) \), Conjecture 1.1 with \( \mu = 1 \) and Conjecture 1.2 are correct; When \( \mu \neq 1 \) under several restrictive conditions Conjecture 1.1 is correct, too.

2 Several lemmas

In polar coordinates, the system (1.1) becomes
\[ \frac{dr}{d\theta} = \frac{\sum_{k=2}^{m} A_k(\theta) r^k}{1 + \sum_{k=2}^{m} B_k(\theta) r^{k-1}}, \]
where
\[ A_k(\theta) = \cos \theta P_k(\cos \theta, \sin \theta) + \sin \theta Q_k(\cos \theta, \sin \theta), \]
\[ B_k(\theta) = \cos \theta Q_k(\cos \theta, \sin \theta) - \sin \theta P_k(\cos \theta, \sin \theta). \]

By [3, 4], the composition condition is satisfied if there exists a trigonometric polynomial \( u(\theta) \) such that
\[ A_k(\theta) = u'(\theta) \sum a_{kj} u^j(\theta), \quad B_k(\theta) = \sum b_{kj} u^j(\theta) \quad (k = 2, 3, \ldots, m), \]  
(2.1)

where \( a_{kj}, b_{kj} \) are real numbers.

**Lemma 2.1** ([4]). If the conditions (2.1) are satisfied, then the origin point of (1.1) is a center and this center is called composition center.

**Lemma 2.2** ([14]). If
\[ P_n = \sum_{i+j=n} p_{ij} \cos^i \theta \sin^j \theta, \quad p_{ij} \in \mathbb{R}, \]
\[ \hat{P}_1 = p_{10} \sin \theta - p_{01} \cos \theta, \quad p_{10}^2 + p_{01}^2 \neq 0 \]

and
\[ \int_0^{2\pi} \hat{P}_1^k P_n d\theta = 0 \quad (k = 0, 1, 2, \ldots, n), \]
then
\[ P_n = \lambda_1 \hat{P}_1^{n-1}, \]
where \( \lambda_i \) are real numbers.

**Lemma 2.3.** Let \( \Phi_{m-1}(x, y) = \sum_{i+j=m-1} \phi_{ij} x^i y^j \) (\( \phi_{ij} \in \mathbb{R} \)). If relation (1.7) holds, then
\[ \Phi_{m-1}(\cos \theta, \sin \theta) = \cos \theta \sum_{i=1}^{m-1} \lambda_i \sin^{i-1} \theta, \]
where \( \lambda_i \) are real numbers

\[ \lambda_{m-1} = \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \phi_{2i+1} m - 2i. \]  
(2.2)
Proof. In Lemma 2.2, taking $P_1 = \cos \theta$, $\dot{P}_1 = \sin \theta$ we get

$$\Phi_{m-1}(\cos \theta, \sin \theta) = \cos \theta \sum_{i=1}^{m-1} \lambda_i \sin^{i-1} \theta,$$

thus

$$\Phi_{m-1}(x, y) = \sum_{i+j=2n} \phi_{ij} x^i y^j = x \sum_{i=1}^{n} \lambda_{2i} y^{2i-1} (x^2 + y^2)^{n-i}, \quad m - 1 = 2n;$$

$$\Phi_{m-1}(x, y) = \sum_{i+j=2n+1} \phi_{ij} x^i y^j = x \sum_{i=0}^{n} \lambda_{2i+1} y^{2i} (x^2 + y^2)^{n-i}, \quad m - 1 = 2n + 1.$$

Equating the corresponding coefficients of the same power of $x, y$, we obtain

$$\lambda_{m-1} = \sum_{i=0}^{n-1} (-1)^i \phi_{2i+1} 2(2i-1), \quad m - 1 = 2n;$$

$$\lambda_{m-1} = \sum_{i=0}^{n} (-1)^i \phi_{2i+1} 2(2i), \quad m - 1 = 2n + 1.$$

Therefore, the conclusion of the present lemma is valid. \qed

By this lemma, it is easy to deduce the following conclusion.

Lemma 2.4. Let $\Phi_{m-1}(x, y)$ be a homogeneous polynomial of degree $m - 1$. Then it can be written as

$$\Phi_{m-1}(x, y) = x \Phi(x^2 + y^2, y)$$

if and only if the relation (1.7) holds. Where $\Phi$ is a polynomial on $x^2 + y^2$ and $y$.

3 Main results

As $a_1^2 + a_2^2 \neq 0$, taking the linear change:

$$X = a_1 x + a_2 y, \quad Y = -a_2 x + a_1 y,$$

(3.1)

the system (1.5) becomes

$$\begin{cases}
X' = -Y (1 - \mu Y) + X (v X + \Phi_{m-1} + \Psi_{2m-1}), \\
Y' = X (1 - \mu Y) + Y (v X + \Phi_{m-1} + \Psi_{2m-1}),
\end{cases}$$

where $\Phi_{m-1} = \Lambda_{m-1} (a_1 X - a_2 Y, a_1 Y + a_2 X)$, $\Psi_{2m-1} = \Omega_{2m-1} (a_1^2 X^2 + a_1^2 Y^2, a_1 Y + a_2 X)$, and they are respectively homogeneous polynomials of degree $m - 1$ and $2m - 1$.

Obviously, if $\Phi_{m-1} = X \Phi_{m-1} (X^2 + Y^2, Y)$, $\Psi_{2m-1} = X \Psi_{2m-1} (X^2 + Y^2, Y)$, then the $\Lambda - \Omega$ system (1.5) after a linear change of variables $(x, y) \rightarrow (X, Y)$ is invariant under the transformations $(X, Y, t) \rightarrow (-X, Y, -t)$. By Lemma 2.4, in order to find the necessary and sufficient conditions for (1.5) to have a weak center, only need to seek the conditions under which the identities (1.7) and (1.8) are valid.

Case A. If $\nu \neq 0$, applying the transformation $X = \frac{1}{\nu} x$, $Y = \frac{1}{\nu} y$, we get
Theorem 3.1. Suppose that
\[
L_k \neq 0; \quad m-1 \leq k \leq 2m-3
\]
where \( \lambda_{m-1} \) is expressed by (2.2),
\[
L_k := e_k + \sum_{i=0}^{k-m+1} \frac{k+1-2i}{m-1+i} d_i e_{k-m+1-i} \lambda_{m-1} \quad (k = m-1, m, \ldots, 2m-1),
\]
\[
d_k = (m-1) \frac{(m+k-1)^{k-1}}{k!}, \quad e_k = (2m-1) \frac{(2m+k-1)^{k-1}}{k!}
\]
(3.6)
Then the origin point of (3.2) is a center if and only if (1.7) and (1.8) hold.
Moreover, this center is a composition center and weak center.

Proof. In polar coordinates, the system (3.2) can be written as
\[
\frac{dr}{d\theta} = \frac{r^2 \cos \theta + \Phi_{m-1} r^m + \Psi_{2m-1} r^{2m}}{1 - r \sin \theta},
\]
where \( \Phi_{m-1} = \Phi_{m-1}(\cos \theta, \sin \theta), \Psi_{2m-1} = \Psi_{2m-1}(\cos \theta, \sin \theta) \).

Taking \( \rho = \frac{r}{e^{\sin \theta}} \), the above equation becomes
\[
\frac{d\rho}{d\theta} = \rho^m e^{(m-1) r \sin \theta} \Phi_{m-1} + \rho^{2m} e^{(2m-1) r \sin \theta} \Psi_{2m-1}.
\]
(3.7)
Now we recall the Langrange–Bürman formula [1]. If real or complex \( w \) and \( z \) satisfy that \( w = \frac{z}{\phi(z)} \), where \( \phi(0) = 1 \) and \( \phi(z) \) is analytic at \( z = 0 \), then in a neighborhood of \( w = 0 \), the analytic function \( F(z) \) can be expressed as a power series:
\[
F(z) = F(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left. \frac{d^{n-1}}{dx^{n-1}} \left( F'(x) \phi^n(x) \right) \right|_{x=0},
\]
which is analytic at \( w = 0 \).
Applying the Langrange–Bürman formula we have
\[
e^{r(m-1)r\sin\theta} = 1 + (m - 1) \sum_{n=1}^{\infty} \frac{(m+n-1)^{n-1}}{n!} \rho^n \sin^n \theta,
\]
\[
e^{r(2m-1)r\sin\theta} = 1 + (2m - 1) \sum_{n=1}^{\infty} \frac{(2m+n-1)^{n-1}}{n!} \rho^n \sin^n \theta.
\]

Thus the equation (3.7) can be written as
\[
\frac{d\rho}{d\theta} = \Phi_{m-1} \sum_{n=0}^{\infty} d_n \rho^{m+n} \sin^n \theta + \Psi_{2m-1} \sum_{n=0}^{\infty} e_n \rho^{2m+n} \sin^n \theta,
\]
where \(d_n, e_n\ (n = 0, 1, 2, \ldots)\) are expressed by (3.6).

Therefore, the system (3.2) has a center at \((0,0)\) if and only if all the solutions \(\rho(\theta)\) of equation (3.8) near \(\rho = 0\) are periodic [2].

Let \(\rho(\theta, c)\) be the solution of (3.8) such that \(\rho(0,c) = c\) \((0 < c \ll 1)\). We write
\[
\rho(\theta, c) = c \sum_{n=0}^{\infty} a_n(\theta) c^n,
\]
where \(a_0(0) = 1\) and \(a_n(0) = 0\) for \(n \geq 1\). The origin point of (3.2) is a center if and only if \(\rho(\theta + 2\pi, c) = \rho(\theta, c)\), i.e., \(a_0(2\pi) = 1\), \(a_n(2\pi) = 0\) \((n = 1, 2, 3, \ldots)\) [5].

Substituting \(\rho(\theta, c)\) into (3.8) we obtain
\[
c \sum_{i=0}^{\infty} a_i(\theta) c^n = \Phi_{m-1} \sum_{n=0}^{\infty} d_n \sin^n \theta \left(c \sum_{i=0}^{\infty} a_i(\theta) c^i\right)^{m+n} + \Psi_{2m-1} \sum_{n=0}^{\infty} e_n \sin^n \theta \left(c \sum_{i=0}^{\infty} a_i(\theta) c^i\right)^{2m+n}.
\]

Equating the corresponding coefficients of \(c^n\) of (3.9) yields
\[
a_0(\theta) = 1, \ a_i(\theta) = 0, \quad (i = 1, 2, \ldots, m - 2).
\]

Rewriting
\[
\rho = c(1 + c^{m-1} h), \quad h = \sum_{i=0}^{\infty} h_i(\theta) c^i, h_i(0) = 0, \quad (i = 0, 1, 2, \ldots).
\]

Substituting it into (3.8) we get
\[
\sum_{k=0}^{\infty} h'_k(\theta)c^k = \Phi_{m-1} \sum_{k=0}^{\infty} d_k c^k \sin^k \theta \sum_{j=0}^{m+k} C^j_{m+k} h^{(m-1)j} \quad (3.10)
\]
\[
+ \Psi_{2m-1} \sum_{k=0}^{\infty} e_k c^{m+k} \sin^k \theta \sum_{j=0}^{2m+k} C^j_{2m+k} h^{(m-1)j}, \quad h_k(0) = 0 \quad (k = 0, 1, 2, \ldots).
\]

In the following we denote
\[
g_k = d_k \sin^k \theta \Phi_{m-1}, \quad \beta_k = e_k \sin^k \theta \Psi_{2m-1}, \quad (k = 0, 1, 2, \ldots), \quad (3.11)
\]
where
\[
\sin^k \theta \Phi_{m-1} = \int_0^\theta \sin^k \theta \Phi_{m-1} d\theta, \quad \sin^k \theta \Psi_{2m-1} = \int_0^\theta \sin^k \theta \Psi_{2m-1} d\theta.
\]
Equating the corresponding coefficients of $e^k$ of the equation (3.10) we obtain
\[ h'_k = d_k \sin^k \theta \Phi_{m-1}, \quad h_k(0) = 0 \quad (k = 0, 1, 2, \ldots, m - 2), \]
\[ h'_{m-1} = \Phi_{m-1} C_{m1} \sin^{m-1} \theta, \quad h_{m-1}(0) = 0, \]
solving these equations we get
\[ h_k(\theta) = g_k, \quad (k = 0, 1, 2, \ldots, m - 2), \]
\[ h_{m-1}(\theta) = \Phi_{m-1} + \alpha_0, \quad \alpha_0 = \frac{m}{2} \Phi_{m-1}. \]
As $d_k \neq 0 (k = 0, 1, 2 \ldots)$, from $h_k(2\pi) = 0 (k = 0, 1, 2, \ldots, m - 1)$ follow that
\[ \int_0^{2\pi} \sin^k \theta \Phi_{m-1} d\theta = 0 \quad (k = 0, 1, 2, \ldots, m - 1), \]
i.e., the condition (1.7) is a necessary condition for $\rho = 0$ to be a center. By Lemma 2.3 which implies that
\[ \Phi_{m-1} = \cos \theta \sum_{k=1}^{m-1} \lambda_k \sin^{k-1} \theta, \quad \Phi_{m-1} = \int_0^\theta \Phi_{m-1} d\theta = \sum_{k=1}^{m-1} \frac{\lambda_k}{k} \sin^k \theta, \] (3.12)
where $\lambda_k (k = 1, 2, \ldots, m - 1)$ are real numbers and $\lambda_{m-1}$ is expressed by (2.2).
Applying (3.12) we get
\[ \int_0^{2\pi} \sin^k \theta \Phi_{m-1} d\theta = 0, \quad g_k = g_k(\sin \theta), \quad g_k(2\pi) = 0 \quad (k = 0, 1, 2 \ldots). \] (3.13)
Equating the corresponding coefficients of $e^{m-1+k}$ of the equation (3.10) we obtain
\[ h'_{m-1+k} = \Phi_{m-1} \sum_{i=0}^{k} d_i \sin^i \theta \Phi_{m-1} h_{k-i} + d_{m-1+k} \sin^{m-1+k} \theta \Phi_{m-1} + e_{k-1} \sin^{k-1} \theta \Psi_{2m-1}, \]
\[ h_{m-1+k}(0) = 0 \quad (k = 1, 2, \ldots, m - 2), \]
solving these equations we get
\[ h_{m-1+k}(\theta) = g_{m-1+k} + \alpha_k + \beta_{k-1} \quad (k = 1, 2, \ldots, m - 2), \]
where $g_{m-1+k}$ and $\beta_{k-1}$ are expressed by (3.11), $\alpha_k$ is the solution of the following equation
\[ \alpha'_k = \Phi_{m-1} \sum_{i=0}^{k} d_i d_{k-i} \sin^i \theta \Phi_{m-1} C_{m+i} + \sin^{k-1} \theta \Phi_{m-1}, \quad \alpha_k(0) = 0. \] (3.14)
By this we get: when $k = 2n$,
\[ \alpha_k = \sum_{i=0}^{n-1} d_i d_{k-i} \left( C_{m+i} \Phi_{m-1} \sin^{k-1} \theta \Phi_{m-1} + (C_{m+k-i} - C_{m+i}) \sin^{k-1} \theta \Phi_{m-1} \right) \]
\[ + \frac{1}{2} n^2 C_{m+k} \sin^{2n} \Phi_{m-1} \Phi_{m-1} \] (3.15)
when \( k = 2n + 1 \),
\[
\alpha_k = \sum_{i=0}^{n} d_i d_{k-i} \left( C_{m+i}^1 \sin i \theta \Phi_{m-i} \sin^{k-i} \theta \Phi_{m-1} + (C_{m+k-i}^1 - C_{m+i}^1) \sin i \theta \Phi_{m-i} \sin^{k-i} \theta \Phi_{m-1} \right).
\]
(3.16)

By (3.13) we see that \( \alpha_k = \alpha_k(\sin \theta) \), \( \alpha_k(2\pi) = 0 \) \((k = 0, 1, 2, 3, \ldots)\). Then from
\[
h_{m-1+k}(2\pi) = g_{m-1+k}(2\pi) + \alpha_k(2\pi) + \beta_{k-1}(2\pi) = 0 \quad (k = 1, 2, \ldots, m - 2)
\]
imply that
\[
\beta_k(2\pi) = 0 \quad (k = 0, 1, 2, \ldots, m - 3),
\]
in view of \( \epsilon_k \neq 0 \) \((k = 0, 1, 2, \ldots)\), so
\[
\int_{0}^{2\pi} \sin^m \theta \Psi_{2m-1} d\theta = 0 \quad (k = 0, 1, 2, \ldots, m - 3).
\]
(3.17)

Equating the corresponding coefficients of \( e^{2m-2} \) of the equation (3.10) we get
\[
h_{2m-2} = \Phi_{m-1} \sum_{i=0}^{m-1} d_i \sin i \theta C_{m+i}^1 h_{m-1-i} + \Phi_{m-1} \sum_{i=0}^{m-1} d_i \sin i \theta C_{m+i}^1 \sum_{j+l=k-i} h_j h_l
\]
\[+ d_{2m-2} \sin^2 \theta \Phi_{m-1} + e_{m-2} \sin^m \theta \Psi_{2m-1}, \quad h_{2m-2}(0) = 0,
\]
by this we get
\[
h_{2m-2}(\theta) = g_{2m-2} + \alpha_{m-1} + \beta_{m-2} + \delta_0
\]
where
\[
\delta_0 = \frac{m(2m-1)}{6} \Phi_{m-1}^2.
\]
\( \alpha_{m-1} \) is a solution of (3.14) with \( k = m - 1 \) and \( \alpha_{m-1} = \alpha_{m-1}(\sin \theta) \). Thus, using (3.12) and (3.13), from \( h_{2m-2}(2\pi) = 0 \) follows that \( \beta_{m-2}(2\pi) = 0 \), i.e.,
\[
\int_{0}^{2\pi} \sin^m \theta \Psi_{2m-1} d\theta = 0.
\]
(3.18)

Equating the corresponding coefficients of \( e^{2m-2+k} \) of the equation (3.10) we obtain
\[
h_{2m-2+k} = \Phi_{m-1} \sum_{i=0}^{m-1+k} d_i \sin i \theta C_{m+i}^1 h_{m-1-k-i} + \Phi_{m-1} \sum_{i=0}^{m-1+k} d_i \sin i \theta C_{m+i}^1 \sum_{j+l=k-i} h_j h_l
\]
\[+ d_{2m-2+k} \sin^2 \theta \Phi_{m-1} + e_{m-2+k} \sin^m \theta \Psi_{2m-1} + \Psi_{2m-1} \sum_{i=0}^{k-1} e_i \sin i \theta C_{2m+i}^1 h_{k-i},
\]
\[
h_{2m-2+k}(0) = 0 \quad (k = 1, 2, \ldots, m - 2),
\]
solving these equations we get
\[
h_{2m-2+k} = g_{2m-2+k} + \alpha_{k+m-1} + \beta_{k+m-2} + \delta_k + \epsilon_{k-1} \quad (k = 1, 2, \ldots, m - 2),
\]
where \( \alpha_{k+m-1} \) is a solution of (3.14), \( \delta_k \) and \( \epsilon_{k-1} \) are the solutions of the following equations, respectively,
\[
\delta_k^\prime = \Phi_{m-1} \left( \sum_{i=0}^{k} d_i \sin i \theta C_{m+i}^1 \alpha_{k-i} + \sum_{i=0}^{k} C_{m+i}^2 d_i \sin i \theta \sum_{j+l=k-i} h_j h_l \right),
\]
Thus, from simplifying this relation by using (3.17) and (3.18), (3.20) and (3.12) we get

\[ \delta_k = \delta_k(\sin \theta) \text{ and } \delta_k(2\pi) = 0. \]

By (3.12) and (3.13) we see that

\[ \alpha_k(2\pi) + \varepsilon_{k-1}(2\pi) = 0, \quad (k = 1, 2, \ldots, m-2), \]

simplifying this relation by using (3.17) and (3.18), (3.20) and (3.12) we get

\[
\begin{align*}
\varepsilon_{k-1} &= \sum_{i=0}^{k-1} d_i \varepsilon_{k-1-i} \left( C^1_{m+i} \sin^i \theta \Phi_{m-1} \sin^{k-1-i} \theta \Psi_{2m-1} \right. \\
&\quad \left. + (C^1_{2m+k-1-i} - C^1_{m+i}) \sin^i \theta \Phi_{m-1} \sin^{k-1-i} \theta \Psi_{2m-1} \right), \\
\varepsilon_k &= \sum_{i=0}^{k-m+1} k-2i \left( m \right) \sin \theta \Psi_{2m-1} \right. \\
&\quad \left. \int_0^{2\pi} \sin^k \theta \Psi_{2m-1} d\theta = L_k \int_0^{2\pi} \sin^k \theta \Psi_{2m-1} d\theta = 0, \quad (k = m-1, \ldots, 2m-4). \end{align*}
\]

By the hypothesis (3.3), \( L_k \neq 0, \) so

\[
\int_0^{2\pi} \sin^k \theta \Psi_{2m-1} d\theta = 0, \quad (k = m-1, \ldots, 2m-4). \tag{3.21}
\]

Equating the corresponding coefficients of \( e^{3m-3} \) of the equation (3.10) we obtain

\[
h_{3m-3} = g_{3m-3} + a_{2m-2} + \beta_{2m-3} + \delta_{m-1} + \varepsilon_{m-2},
\]

where \( a_{2m-2} \) is a solution of (3.14) with \( k = 2m-2 \) and \( a_{2m-2} = a_{2m-2}(\sin \theta), \) \( \varepsilon_{m-2} \) is expressed by (3.20) with \( k = m-1, \delta_{m-1} \) is a solution of the following equation

\[
\begin{align*}
\delta'_{m-1} &= \Phi^{-1}_{m-1} \left( \sum_{i=0}^{m-1} d_i \sin^i \theta C^1_{m+i} \xi_{m-1-i} + \sum_{i=0}^{m-1} C^2_{m+i} d_i \sin^i \theta g_i g_j + C^1_m \delta_0 + 2C^2_m h_0 a_0 + C^3_m h_0 \right), \\
&\text{By (3.12) and (3.13) we see that } g_k = g_k(\sin \theta) \quad (k = 0, 1, 2, \ldots, m-2), \quad \text{and } \delta_{m-1} = \delta_{m-1}(\sin \theta). \end{align*}
\]

Thus, from \( h_{3m-3}(2\pi) = 0 \) follows that

\[
\beta_{2m-3}(2\pi) + \varepsilon_{m-2}(2\pi) = 0,
\]

simplifying this relation by using (3.17) and (3.18) and (3.21), (3.20) and (3.12) we get

\[
L_{2m+3} \int_0^{2\pi} \sin^{2m-3} \theta \Psi_{2m-1} d\theta = 0,
\]

as \( L_{2m-3} \neq 0, \)

\[
\int_0^{2\pi} \sin^{2m-3} \theta \Psi_{2m-1} d\theta = 0. \tag{3.22}
\]

Equating the corresponding coefficients of \( e^{3m-2} \) of the equation (3.10) we obtain
where \( \alpha_{2m-1} \) is a solution of (3.14) with \( k = 2m - 1 \), \( \epsilon_{m-1} \) is a solution of (3.19) with \( k = m \), \( \delta_m \) is a solution of the following equation

\[
\delta''_m = \Phi_{m-1} \left( \sum_{i=0}^{m} d_i \sin^i \theta C_{m,i+1}^1 \alpha_{m-i} + \sum_{i=0}^{m+1} d_i \sin^i \theta C_{m,i+1}^2 \sum_{j+l=m-i} g_j g_l \right) + \Phi_{m-1} (C_m^1 \delta_1 + d_1 \sin \theta C_m^1 \alpha_0 + C_m^2 (2h_0 \alpha_1 + 2h_1 \alpha_0) + 3C_m^3 h_0^2 h_1 + d_1 \sin \theta C_m^3 h_0^3),
\]

by (3.12) and (3.13) we see that \( \delta_m = \bar{\delta}_m (\sin \theta) \). \( \eta_0 \) is a solution of equation

\[
\eta'_0 = \Phi_{m-1} (C_m^1 \epsilon_0 + 2C_m^2 h_0 \beta_0) + \Psi_{2m-1} (C_{2m}^2 h_0^2 + C_{2m}^1 \alpha_0),
\]

solving this we get

\[
\eta_0 = \frac{1}{2} m(m-1) \Phi_{m-1}^2 \Psi_{2m-1} + m^2 \Phi_{m-1} \Phi_{m-1} \Psi_{2m-1} + \frac{2m^2 - m}{2} \Phi_{m-1}^2 \Psi_{2m-1}. \tag{3.23}
\]

Thus, from \( h_{3m-2}(2\pi) = 0 \) follows that

\[
\beta_{2m-2}(2\pi) + \epsilon_{m-1}(2\pi) + \eta_0(2\pi) = 0,
\]

calculating this relation by using (3.17) and (3.18) and (3.20)–(3.23) and (3.12) we get

\[
\left( L_{2m-2} + \frac{2m^2 - m}{2(m-1)^2} \lambda_{m-1}^2 \right) \int_0^{2\pi} \sin^{2m-2} \psi_{2m-1} d\theta = 0,
\]

in view of the condition (3.4) we have

\[
\int_0^{2\pi} \sin^{2m-2} \psi_{2m-1} d\theta = 0. \tag{3.24}
\]

Equating the corresponding coefficients of \( \epsilon^{3m-1} \) of the equation (3.10) we obtain

\[
h_{3m-1}(\theta) = g_{3m-1} + \alpha_{2m} + \beta_{2m-1} + \delta_{m+1} + \epsilon_m + \eta_1, \tag{3.25}
\]

where \( g_{3m-1}, \alpha_{2m}, \beta_{2m-1} \) and \( \epsilon_m \) are the same as above, \( \delta_{m+1} \) is a solution of the equation

\[
\delta''_{m+1} = \Phi_{m-1} \left( \sum_{i=0}^{m+1} d_i \sin^i \theta C_{m+1,i+1}^1 \alpha_{m+1-i} + \sum_{i=0}^{m+1} d_i \sin^i \theta C_{m+1,i+1}^2 \sum_{j+l=m+i} g_j g_l \right) + \Phi_{m-1} (C_{m+1}^1 \delta_1 + d_1 \sin \theta C_{m+1}^1 \alpha_0 + C_{m+1}^2 (2h_0 \beta_1 + 2h_1 \beta_0) + d_1 \sin \theta C_{m+1}^2 h_0 \beta_0) + \Psi_{2m+1} (C_{2m+1}^0 2h_0 h_1 + e_1 \sin \theta C_{2m+1}^1 h_0^2 + C_{2m+1}^1 \alpha_1 \beta_0 + e_1 \sin \theta C_{2m+1}^2 \alpha_0),
\]

By (3.12) and (3.13) and \( \alpha_k = \alpha_k (\sin \theta) \ (k = 0, 1, 2 \ldots) \), \( \delta_i = \delta_i (\sin \theta) \ (i = 0, 1, 2) \), which imply that \( \delta_{m+1} = \delta_{m+1} (\sin \theta) \). \( \eta_1 \) is a solution of the following equation

\[
\eta'_1 = \Phi_{m-1} (C_m^1 \epsilon_1 + d_1 \sin \theta C_m^1 \epsilon_0 + C_m^2 (2h_0 \beta_1 + 2h_1 \beta_0) + d_1 \sin \theta C_m^2 h_0 \beta_0) + \Psi_{2m+1} (C_{2m+1}^0 2h_0 h_1 + e_1 \sin \theta C_{2m+1}^1 h_0^2 + C_{2m+1}^1 \alpha_1 \beta_0 + e_1 \sin \theta C_{2m+1}^2 \alpha_0),
\]
solving this equation we get
\[ \eta_1 = m \Psi_{2m-1}^2 + c_1 \left( \left( m^2 - \frac{m}{2} \right) \Phi_{m-1}^2 \sin \theta \Psi_{2m-1} + m(m+1)\Phi_{m-1} \Phi_{m-1} \sin \theta \Psi_{2m-1} \right. \\
+ \left. m(m+1)\sin \theta \Psi_{2m-1} \Phi_{m-1}^2 \right) \\
+ d_1 \left( 2m^2 \Phi_{m-1} \Psi_{2m-1} \sin \theta \Phi_{m-1} + m(m-1)\Phi_{m-1} \sin \theta \Phi_{m-1} \Psi_{2m-1} \right. \\
+ \left. m(m+1)\sin \theta \Phi_{m-1} \Phi_{m-1} \Psi_{2m-1} \right) \\
+ \left. 2m \Psi_{2m-1} \Phi_{m-1} \Psi_{2m-1} \sin \theta + 2(m^2 - m)\Phi_{m-1} \Psi_{2m-1} \sin \theta \Phi_{m-1} \right). \tag{3.26} \]

By (3.25) we see that if \( h_{3m-1}(2\pi) = 0 \), then
\[ \beta_{2m-1}(2\pi) + \epsilon_m(2\pi) + \eta_1(2\pi) = 0, \]
simplifying this equation by using (3.17) and (3.18) and (3.20)–(3.24), (3.26) and (3.12) we get
\[ \left( L_{2m-1} + \left( 2d_1 + c_1 \frac{m(m+1)}{(m-1)^2} \right) \lambda_{m-1}^2 \right) \int_0^{2\pi} \sin^{2m-1} \theta \Psi_{2m-1} d\theta = 0, \]
by the hypothesis (3.5) we obtain
\[ \int_0^{2\pi} \sin^{2m-1} \theta \Psi_{2m-1} d\theta = 0. \]

In summary, under the conditions (3.3)–(3.5), the (1.7) and (1.8) are the necessary conditions for \( \rho = 0 \) to be a center of (3.2). Therefore, the necessity has been proved. On the other hand, by Lemma 2.1 and Lemma 2.3, if the conditions (1.7) and (1.8) are satisfied, then \( \rho = 0 \) is a center of equation (3.2), this means that the sufficiency is proved. By Lemma 2.3 this center is a composition center, by Lemma 2.4 this center is a weak center. \( \square \)

**Corollary 3.2.** For arbitrary \( m \ (> 2) \), if \( \mu = 1 \), then the origin point of (1.3) is a center if and only if (1.7) is satisfied.

**Proof.** Under the linear change of variables (3.1) the system (1.3) becomes
\[ \begin{align*}
    x' &= -y(1 - y) + x(x + \Phi_{m-1}), \\
    y' &= x(1 - y) + y(x + \Phi_{m-1}),
\end{align*} \]
which in polar coordinates becomes
\[ \frac{dr}{d\theta} = \frac{r^2 \cos \theta + \Phi_{m-1} r^m}{1 - r \sin \theta}. \]
Taking \( r = \frac{x}{\rho \sin \theta} \) we get
\[ \frac{d\rho}{d\theta} = \Phi_{m-1} \rho^m \sum_{n=0}^{\infty} d_n \rho^n \sin^n \theta, \]
where \( d_0 = 1, d_n = \frac{1}{n^2}(m-1)(m+n-1)^{n-1}, \ (n = 1, 2, 3, \ldots) \). Similar to Theorem 3.1, it can be deduced that the solution \( \rho \) of this equation such that \( \rho(0) = c \ (0 < |c| \ll 1) \) is
\[ \rho = c + c^m \sum_{k=0}^{m-2} \frac{c^k d_k \sin^k \theta \Phi_{m-1}}{k!} + c^{2m-1} \left( \frac{d_{m-1} \sin^{m-1} \theta \Phi_{m-1}}{m} + \frac{m \Phi_{m-1}^2}{2} \right) + o(c^{2m-1}). \]
As \( d_n \neq 0 \ (n = 0, 1, 2, \ldots) \), from \( \rho(2\pi) = c \) it follows that the condition (1.7) is satisfied. Using Lemma 2.3 and Lemma 2.4, the conclusion of the present corollary is valid. \( \square \)
Theorem 3.4. Suppose that

\[
\begin{align*}
\nu & = (1 - \hat{\mu} y) + x(\Phi_{m-1} + \Psi_{2m-1}), \\
\tilde{\nu} & = x(1 - \hat{\mu} y) + y(\Phi_{m-1} + \Psi_{2m-1}).
\end{align*}
\] (3.30)

Case 2. \(\nu \neq 0, \hat{\mu} \neq 1\).

Consider \(\Lambda-\Omega\) system

\[
\begin{align*}
x' &= -y(1 - \hat{\mu} y) + x(\Phi_{m-1} + \Psi_{2m-1}), \\
y' &= x(1 - \hat{\mu} y) + y(\Phi_{m-1} + \Psi_{2m-1}).
\end{align*}
\] (3.27)

Remark 3.3. By Corollary 3.2, when \(\mu = 1\), Conjecture 1.1 is correct for arbitrary \(m > 2\).

Proof. In polar coordinates, the system (3.27) becomes

\[
\begin{align*}
\frac{dr}{d\theta} &= \frac{r^2 \cos \theta + \Phi_{m-1} r^m + \Psi_{2m-1} r^{2m}}{1 - \hat{\mu} r \sin \theta}, \\
\frac{d\theta}{d\phi} &= \left(\frac{2d_1 + \tilde{\epsilon}_1 m(m + 1)}{(m - 1)^2} \right) \lambda_{m-1}^2 \neq 0,
\end{align*}
\]

where \(\lambda_{m-1}\) is expressed by (2.2),

\[
\tilde{L}_k := \tilde{\epsilon}_k + \sum_{i=0}^{k-m+1} \frac{k + 1 - 2i}{m - 1 + i} \tilde{d}_i \tilde{\epsilon}_{k-m+i} \lambda_{m-1}, \quad (k = m - 1, m, \ldots, 2m - 1),
\]

\[
d_n = \frac{\tilde{d}_1}{n!} \prod_{0 \leq r \leq n-2} (\sigma - r(1 - \hat{\mu})), \quad (n = 2, 3, \ldots), \quad \tilde{d}_0 = 1, \quad \tilde{d}_1 = m + \hat{\mu} - 2, \quad \sigma = n + m + 2\hat{\mu} - 3; \quad (3.28)
\]

\[
\tilde{\epsilon}_n = \frac{\tilde{\epsilon}_1}{n!} \prod_{0 \leq r \leq n-2} (\epsilon - r(1 - \hat{\mu})), \quad (n = 2, 3, \ldots), \quad \tilde{\epsilon}_0 = 1, \quad \tilde{\epsilon}_1 = 2m + \hat{\mu} - 2, \quad \epsilon = n + m + 2\hat{\mu} - 3. \quad (3.29)
\]

Then the origin point of (3.27) is a center if and only if (1.7) and (1.8) hold.

Moreover, this center is a composition center and weak center.

Proof. In polar coordinates, the system (3.27) becomes

\[
\frac{dr}{d\theta} = \frac{r^2 \cos \theta + \Phi_{m-1} r^m + \Psi_{2m-1} r^{2m}}{1 - \hat{\mu} r \sin \theta},
\] (3.30)

where \(\Phi_{m-1} = \Phi_{m-1}(\cos \theta, \sin \theta), \Psi_{2m-1} = \Psi_{2m-1}(\cos \theta, \sin \theta)\).

Taking

\[
\rho = \frac{r}{(1 + (1 - \hat{\mu}) r \sin \theta)^{1/\sigma}},
\]

the equation (3.30) can be written as

\[
\frac{d\rho}{d\theta} = \rho^m \Phi_{m-1}(1 + (1 - \hat{\mu}) r \sin \theta)^{m + \hat{\mu} - 2} + \rho^2 \Psi_{2m-1}(1 + (1 - \hat{\mu}) r \sin \theta)^{2m + \hat{\mu} - 2}. \quad (3.31)
\]
Applying the Langrange–Bürman formula we have

\[
(1 + (1 - \mu) r \sin \theta)^{\frac{m+\hat{\mu} - 2}{r - \mu}} = \sum_{n=0}^{\infty} \hat{d}_n \rho^n \sin^n \theta;
\]

\[
(1 + (1 - \mu) r \sin \theta)^{\frac{2m+\hat{\mu} - 2}{r - \mu}} = \sum_{n=0}^{\infty} \hat{e}_n \rho^n \sin^n \theta,
\]

where \( \hat{d}_n, \hat{e}_n \) are expressed by (3.28), (3.29), respectively.

Substituting them into (3.31) we get

\[
\frac{d\rho}{d\theta} = \Phi_{m-1} \rho^m \sum_{n=0}^{\infty} \hat{d}_n \rho^n \sin^n \theta.
\]

Comparing the equations (3.8) and (3.32), we see that they have the same form, only with different coefficients. Similar to Theorem 3.1, the present theorem can be derived.

Remark 3.5. When \( \mu = 0 \), from Theorem 3.4 implies the Theorem 3.1 of [15].

Corollary 3.6. If \( \mu \neq 1 \) and \( \hat{d}_n = \hat{d}_n|_{\mu=\hat{\mu}} \neq 0 \) \((n = 1, 2, \ldots, m - 1) \) \((m > 2) \), where \( \hat{d}_n \) \((n = 1, 2, \ldots, m - 1) \) is expressed by (3.28). Then the origin point of (1.3) is a center if and only if (1.7) is satisfied.

Proof. Similar to Theorem 3.4, when \( \Psi_{2m-1} = 0 \), the equation (1.3) can be transformed as following

\[
\frac{d\rho}{d\theta} = \Phi_{m-1} \rho^m \sum_{n=0}^{\infty} \hat{d}_n \rho^n \sin^n \theta.
\]

Similarly to Theorem 3.1, we get that the solution of (3.33) such that \( \rho(0) = c \) \((0 < |c| \ll 1) \) is

\[
\rho = c + c^m \sum_{k=0}^{m-2} (\hat{d}_k \sin k \theta \Phi_{m-1} + c^{\sum_{j=0}^{k} d(k-j)\Phi_{m-1}} + \frac{m}{2} \Phi_{m-1}) + o(c^{2m-1}).
\]

As \( \hat{d}_n = \hat{d}_n|_{\mu=\hat{\mu}} \neq 0 \) \((n = 1, 2 \ldots, m - 1) \), \( \hat{d}_0 = 1 \), from \( \rho(2\pi) = c \) follows that the condition (1.7) is satisfied. Using Lemma 2.4, the conclusion of the present corollary is valid.

Remark 3.7. By Corollary 3.6, if \( \mu \neq 1 \), Conjecture 1.1 is valid when \( \prod_{1 \leq n \leq m-1} \hat{d}_n \neq 0 \) \((m > 2) \).

Case B. \( \nu = 0, \mu \neq 0 \).

Consider \( \Lambda-\Omega \) system

\[
\begin{align*}
x' &= -y(1 - \mu y) + x(\Phi_{m-1} + \Psi_{2m-1}), \\
y' &= x(1 - \mu y) + y(\Phi_{m-1} + \Psi_{2m-1}).
\end{align*}
\]

Theorem 3.8. Suppose that

\[
\prod_{m-1 \leq k \leq 2m-3} \hat{L}_k \neq 0;
\]

\[
\hat{L}_{2m-2} + \frac{m(2m-1)}{2(m-1)^2} \lambda_{m-1}^2 \neq 0;
\]

\[
\hat{L}_{2m-1} + \mu \left(2 + \frac{m(m+1)}{(m-1)^2}\right) \lambda_{m-1}^2 \neq 0,
\]

where \( \lambda_{m-1} \) is expressed by (2.2), \( \hat{L}_k := \mu^k + \mu^{1-m+k} \sum_{j=0}^{k-m+1} \frac{k+1-2j}{m-1} \lambda_{m-1} \), \((k = m-1, m, \ldots, 2m-1) \).

Then the origin point of (3.34) is a center if and only if (1.7) and (1.8) hold.

Moreover, this center is a composition center and weak center.
Proof. In polar coordinates, the system (3.34) becomes
\[
\frac{dr}{d\theta} = \Phi_{m-1} \sum_{n=0}^{\infty} \mu^n r^{m+n} \sin^n \theta + \Psi_{2m-1} \sum_{n=0}^{\infty} \mu^n r^{2m+n} \sin^n \theta, \tag{3.35}
\]
where \( \Phi_{m-1} = \Phi_{m-1}(\cos \theta, \sin \theta) \), \( \Psi_{2m-1} = \Psi_{2m-1}(\cos \theta, \sin \theta) \).

Obviously, the equation (3.35) has the same form as (3.8), in Theorem 3.1 taking \( d_k = e_k = \mu^k \) \( (k = 0, 1, 2, \ldots) \), the present theorem can be derived directly.

Corollary 3.9. For arbitrary \( m > 2 \), the origin point of (1.4) is a center if and only if (1.7) is satisfied.

Proof. Under the linear changes of variables (3.1) the system (1.4) becomes
\[
\begin{align*}
    x' &= -y(1 - y) + x\Phi_{m-1} , \\
    y' &= x(1 - y) + y\Phi_{m-1} .
\end{align*}
\tag{3.36}
\]
In polar coordinates (3.36) can be written as
\[
\frac{dr}{d\theta} = \Phi_{m-1} \sum_{n=0}^{\infty} r^{m+n} \sin^n \theta . \tag{3.37}
\]
Similar to Theorem 3.1, we get that the solution of (3.37) such that \( r(0) = c \) \( (0 < |c| \ll 1) \) is
\[
r = c + c^m \sum_{i=0}^{m-2} \sin^i \theta \Phi_{m-1} + c^{2m-1} \left( \sin^{m-1} \theta \Phi_{m-1} + \frac{m}{2} \Phi_{m-1}^2 \right) + o(c^{2m-1}) .
\]
Obviously, from \( r(2\pi) = c \) follows that the condition (1.7) is satisfied. Using Lemma 2.4 the conclusion of the present corollary is correct.

Remark 3.10. By Corollary 3.9, Conjecture 1.2 is valid for \( m > 2 \).

Remark 3.11. In the case of \( \mu = \nu = 0, m = 2 \) the center problem of system (1.5) has been discussed by [14].

Acknowledgements

Funding: This study was funded by the National Natural Science Foundation of China (62173292, 12171418).
Conflict of interest: The author declares that she has no conflict of interest.

References


Weak center for a class of $\Lambda$–$\Omega$ differential systems


