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Infinitely many radial positive solutions for nonlocal problems with lack of compactness

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Abstract. We are concerned with the qualitative and asymptotic analysis of solutions to the nonlocal equation

$$(-\Delta)^s u + V(|z|)u = Q(|z|)u^p \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $0 < s < 1$, and $1 < p < \frac{2N}{N-2s}$. As $r \to \infty$, we assume that the potentials $V(r)$ and $Q(r)$ behave as

$$V(r) = V_0 + \frac{a_1}{r^\alpha} + O\left(\frac{1}{r^{\alpha+\theta_1}}\right)$$
$$Q(r) = Q_0 + \frac{a_2}{r^\beta} + O\left(\frac{1}{r^{\beta+\theta_2}}\right)$$

where $a_1,\ a_2 \in \mathbb{R}$, $\alpha,\ \beta > \frac{N+2s}{N+2s+1}$, and $\theta_1,\ \theta_2 > 0,\ V_0,\ Q_0 > 0$. Under various hypotheses on a_1, a_2, α, β, we establish the existence of infinitely many radial solutions. A key role in our arguments is played by the Lyapunov–Schmidt reduction method.

Keywords: fractional Laplacian, radial solution, lack of compactness, Lyapunov–Schmidt reduction method.

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1 Introduction and the main result

We consider the following nonlocal equation driven by the fractional Laplace operator

$$(-\Delta)^s u + V(|z|)u = Q(|z|)u^p, \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

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Fractional powers of the Laplacian arise in various equations in mathematical physics and related fields; see, e.g., [1], [9], and [14]. Numerous results related to equations with fractional Laplace operator sprout in literature. A characterization of the fractional Laplacian through Dirichlet–Neumann maps was given in [3]. Regularity for fractional elliptic equations was investigated in [4] and [17]. Existence of solutions was studied in many papers; see, e.g., [2,7,12].

Along with different results, there are various enlightening approaches. In [10], the author obtained some symmetry results for equations involving the fractional Laplacian in $\mathbb{R}^N$ by the method of moving planes. In [2], symmetry results for nonlinear equations with fractional Laplacian were achieved by the sliding method. Geometric inequality was applied to investigate symmetry properties for a boundary reaction problem in [18]. The method of moving planes and ABP (Aleksandrov–Bakelman–Pucci) estimates for fractional Laplacian were employed in [6] to study radial symmetry and monotonicity properties for positive solutions of fractional Laplacian. We refer the readers to [7] and [11] for very recent new approaches dealing with fractional Laplacian equations, and to [15] for a comprehensive overview of variational methods for nonlocal fractional problems.

Inspired by [19], we obtain the existence of radial positive solutions to (1.1) by Lyapunov–Schmidt reduction. To the best of our knowledge, this method has never been employed in investigating radial solutions to equations as (1.1).

We will use the radial solution of

$$(-\Delta)^s u + u = u^p \quad \text{in } \mathbb{R}^N \tag{1.2}$$

to build up the approximate solutions of problem (1.1). The uniqueness and nondegeneracy of the radial positive solution to problem (1.2) are established in [8].

Our result is based on the following growth assumptions for $V(|z|)$ and $Q(|z|)$ near infinity:

(V): there exist constants $a_1 \in R$, $\alpha > 1$, and $\theta_1 > 0$, such that $V(r) = V_0 + \frac{a_1}{r^\alpha} + O(\frac{1}{r^{\alpha+\theta_1}})$ as $r \to \infty$;

(Q): there exist constants $a_2 \in R$, $\beta > 1$, and $\theta_2 > 0$, such that $Q(r) = Q_0 + \frac{a_2}{r^\beta} + O(\frac{1}{r^{\beta+\theta_2}})$ as $r \to \infty$.

We assume throughout this paper that $V_0 = 1$ and $Q_0 = 1$.

Let

$$x_j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0 \right), \qquad j = 1, \ldots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, $r \in \left[r_0 k^{\frac{N+2s}{N+2s-\tau}}, r_1 k^{\frac{N+2s}{N+2s-\tau}} \right]$, $\tau = \min\{\alpha, \beta\}$, $0 < r_0 < r_1$, and k is the number of the bumps of the solution.

Set $z = (z', z'')$, $z' \in \mathbb{R}^2$, $z'' \in \mathbb{R}^{N-2}$ and define

$$H_{rs} = \Big\{ u : u \in H^s(\mathbb{R}^N), u \text{ is even in } z_h,\ h = 2, \ldots, N,$$
$$u(r\cos\theta, r\sin\theta, z'') = u\left(r\cos\left(\theta + \frac{2\pi j}{k}\right), r\sin\left(\theta + \frac{2\pi j}{k}\right), z'' \right) \Big\},$$

where $H^s(\mathbb{R}^N)$ represents the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}, \qquad 0 < s < 1.$$

Let W be the unique nondegenerate radial positive solution of problem (1.2), Then the result in [8] shows that there exist constants $B_1 > B_2 > 0$, such that

$$\frac{B_2}{1+|z-x_j|^{N+2s}} \le W_{x_j}(z) \le \frac{B_1}{1+|z-x_j|^{N+2s}},$$

where $W_{x_j}(z) = W(z-x_j)$.

Set

$$U_r(z) = \sum_{j=1}^{k} W_{x_j}(z).$$

The main result of this paper establishes the following multiplicity property.

Theorem 1.1. *Assume that $V(r)$, $Q(r)$ satisfy (V) and (Q), while a_1, a_2, α, β satisfy one of the following conditions:*

(i) $a_1 > 0$, $a_2 = 0$, $\alpha < N+2s$, *and* $\alpha \le \beta$;

(ii) $a_1 > 0$, $a_2 > 0$, $\alpha < N+2s$, *and* $\beta \ge N+2s$;

(iii) $a_1 > 0$, $a_2 < 0$, $\alpha < N+2s$, *and* $\alpha > \beta$;

(iv) $a_1 = 0$, $a_2 < 0$, $\alpha \ge \beta$, *and* $\beta < N+2s$;

(v) $a_1 < 0$, $a_2 < 0$, $\alpha \ge N+2s$, *and* $\beta < N+2s$.

Then there exists a positive integer k_0 such that for any $k \ge k_0$, problem (1.1) has a solution U_k of the form

$$U_k(z) = U_{r_k}(z) + w_k,$$

where $w_k \in H_{rs}$, $r_k \in \left[r_0 k^{\frac{N+2s}{N+2s-\tau}}, r_1 k^{\frac{N+2s}{N+2s-\tau}}\right]$, *and as* $k \to +\infty$,

$$\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} w_k|^2 + w_k^2\right) \to 0.$$

For some of the abstract methods used in this paper, we refer to the monographs by Molica Bisci and Pucci [13] and Papageorgiou, Rădulescu and Repovš [16].

2 Reduction

Let

$$P_j = \frac{\partial W_{x_j}}{\partial r}, \qquad j = 1, \dots, k,$$

where

$$x_j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\right), \qquad j = 1, \dots, k$$

and

$$r \in S := \left[\left(\frac{N+2s}{\tau} - \epsilon\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}}, \left(\frac{N+2s}{\tau} + \epsilon\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}}\right].$$

We have denoted $\tau := \min\{\alpha, \beta\}$, where α and β are the constants in the expansions of V and Q, and $\epsilon > 0$ is a small constant.

Define

$$H := \left\{ u : u \in H_{rs}, \int_{\mathbb{R}^N} W_{x_j}^{p-1} P_j u = 0,\ j = 1, \dots, k. \right\}.$$

The norm and the inner product in $H^s(\mathbb{R}^N)$ are defined as

$$\|u\| = \langle u, u \rangle^{\frac{1}{2}}, \qquad u \in H^s(\mathbb{R}^N),$$

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(|z|) uv \right), \qquad u, v \in H^s(\mathbb{R}^N).$$

We can easily check that

$$\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(|z|) uv - pQ(|z|) U_r^{p-1} uv \right), \qquad u, v \in H$$

is a bounded bilinear functional in H. Thus, there exists a bounded linear operator M from H to H satisfying

$$\langle Mu, v \rangle = \int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(|z|) uv - pQ(|z|) U_r^{p-1} uv \right), \qquad u, v \in H. \tag{2.1}$$

We now establish that M is invertible in H.

Lemma 2.1. *There exists a constant $\rho > 0$, independent of k, such that for any $r \in S$,*

$$\|Mu\| \geq \rho \|u\|, \qquad u \in H.$$

Proof. We argue by contradiction. If the thesis does not hold, then for any $\rho_k = \frac{1}{k}(k \to +\infty)$, there exists $r_k \in S$, $u_k \in H$, such that

$$\|Mu_k\| < \rho_k \|u_k\|.$$

It follows that

$$\|Mu_k\| = o(1) \|u_k\|.$$

Then,

$$\langle Mu_k, \varphi \rangle = o(1) \|u_k\| \|\varphi\|, \qquad \forall \varphi \in H. \tag{2.2}$$

We can assume $\|u_k\|^2 = k$.

Let

$$\Omega_j = \left\{ z = (z', z'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{z'}{|z'|}, \frac{x_j}{|x_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

By symmetry and the definition of M, we conclude from (2.2) that for all $\varphi \in H$,

$$\int_{\Omega_1} \left((-\Delta)^{\frac{s}{2}} u_k (-\Delta)^{\frac{s}{2}} \varphi + V(|z|) u_k \varphi - pQ(|z|) U_{r_k}^{p-1} u_k \varphi \right) = \frac{1}{k} \langle Mu_k, \varphi \rangle = o\left(\frac{1}{\sqrt{k}} \right) \|\varphi\|. \tag{2.3}$$

Particularly,

$$\int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|) u_k{}^2 - pQ(|z|) U_{r_k}^{p-1} u_k{}^2 \right) = o(1)$$

and

$$\int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|) u_k{}^2 \right) = 1. \tag{2.4}$$

Let $\widetilde{u_k}(z) = u_k(z + x_1)$. Since

$$|x_2 - x_1| = 2r \sin\frac{\pi}{k} \geq 2r\frac{\pi}{2k} \geq \left(\frac{N+2s}{2\tau}\right)^{\frac{1}{N+2s-\tau}} k^{\frac{\tau}{N+2s-\tau}}\pi,$$

it follows that for any $R > 0$, $B_R(x_1) \subset \Omega_1$. Then from (2.4), we have for all $R > 0$,

$$\int_{B_R(0)} \left(|(-\Delta)^{\frac{s}{2}}\widetilde{u_k}|^2 + V(|z|)\widetilde{u_k}^2\right) \leq 1.$$

So, we can assume that there exists $u \in H^s(\mathbb{R}^N)$, such that as $k \to +\infty$,

$$\widetilde{u_k} \rightharpoonup u, \quad \text{in } H^s_{\text{loc}}(\mathbb{R}^N),$$

and

$$\widetilde{u_k} \to u, \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N).$$

Since $\widetilde{u_k}$ is even in z_h, $h = 2, \ldots, N$, then u is even in z_h, $h = 2, \ldots, N$.

Besides, by

$$\int_{\mathbb{R}^N} W_{x_1}^{p-1} P_1 u_k = 0,$$

we know that

$$\int_{\mathbb{R}^N} W^{p-1}\frac{\partial W}{\partial x_1}\widetilde{u_k} = 0.$$

So, u satisfies

$$\int_{\mathbb{R}^N} W^{p-1}\frac{\partial W}{\partial x_1} u = 0. \tag{2.5}$$

We prove in what follows that u satisfies

$$(-\Delta)^s u + u - pW^{p-1}u = 0, \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

Define

$$\widetilde{H} = \left\{\varphi : \varphi \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} W^{p-1}\frac{\partial W}{\partial x_1}\varphi = 0\right\}.$$

For any $R > 0$, let $\varphi \in C_0^\infty(B_R(0)) \cap \widetilde{H}$ be any function which is even in z_h, $h = 2, \ldots, N$. Then $\varphi_k(z) := \varphi(z - x_1) \in C_0^\infty(B_R(x_1))$. Substituting φ in (2.3) with φ_k, then by Lemma A.1, we get

$$\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}}\varphi + u\varphi - pW^{p-1}u\varphi\right) = 0. \tag{2.7}$$

In addition, since u is even in z_h, $h = 2, \ldots, N$, relation (2.7) holds for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, which is odd in z_h, $h = 2, \ldots, N$. Thus, relation (2.7) is true for any $\varphi \in C_0^\infty(B_R(0)) \cap \widetilde{H}$. By the density of $C_0^\infty(\mathbb{R}^N)$ in $H^s(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}}\varphi + u\varphi - pW^{p-1}u\varphi\right) = 0, \qquad \forall \varphi \in \widetilde{H} \tag{2.8}$$

Meanwhile, relation (2.8) holds for $\varphi = \frac{\partial W}{\partial x_1}$. Therefore, (2.8) holds for any $\varphi \in H^s(\mathbb{R}^N)$. Substituting φ in (2.8) with u yields (2.6).

Since W is non-degenerate, we have $u = C\frac{\partial W}{\partial x_1}$ because u is even in z_h, $h = 2, \ldots, N$. By (2.5), we know that

$$u = 0,$$

which implies

$$\int_{B_R(x_1)} {u_k}^2 = o(1), \qquad \forall R > 0.$$

Besides, it follows from Lemma A.1 that there exists $C' > 0$ such that

$$U_{r_k}(x) \le C', \quad \text{for all } x \in \Omega_1.$$

It follows that

$$\begin{aligned} o(1) &= \int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|){u_k}^2 - pQ(|z|)U_{r_k}^{p-1}{u_k}^2 \right) \\ &= \int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|){u_k}^2 \right) + o(1) - C\int_{\Omega_1} {u_k}^2 \\ &\ge \frac{1}{2}\int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|){u_k}^2 \right) + o(1), \end{aligned}$$

which contradicts (2.4). The proof is now complete. □

Define

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + V(|z|)u^2 \right) - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)|u|^{p+1}. \tag{2.9}$$

Let

$$J(\phi) = I(U_r + \phi), \qquad \phi \in H.$$

Then,

$$\begin{aligned} J(0) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} U_r|^2 + V(|z|){U_r}^2 \right) - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)|U_r|^{p+1}. \\ &= \frac{1}{2}\int_{\mathbb{R}^N} U_r \sum_{j=1}^{k} W_{x_j}^p + \frac{1}{2}\int_{\mathbb{R}^N} (V(|z|)-1){U_r}^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} (Q(|z|)-1)U_r^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^N} U_r^{p+1} \end{aligned}$$

because W_{x_j} solves (1.2).

Lemma 2.2. *There exists a positive integer k_0 such that for each $k \ge k_0$, there is a C^1 map from S to H_{rs}: $\phi_k = \phi_k(r)$, $r = |x_1|$, satisfying $\phi_k \in H$, and*

$$J'(\phi_k)\big|_H = 0.$$

Moreover, there exists a constant $C > 0$, independent of k, such that

$$\|\phi_k\| \le \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}}, \tag{2.10}$$

where $\delta > 0$ is a small constant.

Proof. We expand $J(\phi_k)$ as

$$J(\phi_k) = J(0) + l(\phi_k) + \frac{1}{2}\langle M\phi_k, \phi_k\rangle + R(\phi_k), \qquad \phi_k \in H,$$

where

$$\begin{aligned} l(\phi_k) &= \langle I'(U_r), \phi_k\rangle \\ &= \int_{\mathbb{R}^N}(V(|z|)-1)U_r\phi_k + \int_{\mathbb{R}^N}\left(\sum_{j=1}^{k} W_{x_j}^p - U_r^p\right)\phi_k - \int_{\mathbb{R}^N}(Q(|z|)-1)U_r^p\phi_k. \end{aligned}$$

M is the bounded linear map defined in (2.1) and

$$R(\phi_k) = -\frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)\Big(|U_r+\phi_k|^{p+1} - U_r^{p+1} - (p+1)U_r^p\phi_k - \frac{1}{2}p(p+1)U_r^{p-1}{\phi_k}^2\Big).$$

Since $l(\phi_k)$ is a bounded linear functional in H, there exists $l_k \in H$, such that

$$l(\phi_k) = \langle l_k, \phi_k\rangle.$$

Then, ϕ_k being a critical point of J is equivalent to

$$l_k + M\phi_k + R'(\phi_k) = 0. \tag{2.11}$$

Since M is invertible, we can infer from (2.11) that

$$\phi_k = T(\phi_k) := -M^{-1}(l_k + R'(\phi_k)).$$

Define

$$E = \left\{\phi_k : \phi_k \in H,\ \|\phi_k\| \le \frac{1}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}}}\right\}.$$

Next, we check that T is a contraction map from E to E.

Case 1: $p \le 2$. It is easy to verify that

$$\|R'(\phi_k)\| \le C\|\phi_k\||^p.$$

In fact,

$$\begin{aligned} |\langle R'(\phi_k), v\rangle| &= \left|\int_{\mathbb{R}^N} Q(|z|)(pU_r^{p-1}\phi_k v + U_r^p v - |U_r+\phi_k|^{p-1}(U_r+\phi_k)v)\right| \\ &= \left|\int_{\mathbb{R}^N} Q(|z|)p(|U_r+\theta\phi_k|^{p-1} - U_r^{p-1})\phi_k v\right| \\ &\le C\int_{\mathbb{R}^N} |(|U_r+\theta\phi_k|^{p-1} - U_r^{p-1})|\,|\phi_k|\,|v| \\ &\le C\int_{\mathbb{R}^N} |\theta\phi_k|^{p-1}|\phi_k|\,|v| \\ &\le C\|\phi_k\|^p\|v\| \end{aligned}$$

where $0 < \theta < 1$.

Then, by the boundedness of M and Lemma 2.3,

$$\|T(\phi_k)\| \leq C(\|l_k\| + \|\phi_k\|^p) \leq \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}} + \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}p}} \leq \frac{1}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}}} \tag{2.12}$$

which implies that T maps E to E.

In addition,

$$\|R''(\phi_k)\| \leq C\|\phi_k\|^{p-1}.$$

In fact,

$$\begin{aligned}
|\langle R''(\phi_k)v,h\rangle| &= \left|\int_{\mathbb{R}^N} Q(|z|)(pU_r^{p-1}vh - p|U_r+\phi_k|^{p-1}vh)\right| \\
&= p\left|\int_{\mathbb{R}^N} Q(|z|)(U_r^{p-1}vh - |U_r+\phi_k|^{p-1})vh\right| \\
&\leq C\int_{\mathbb{R}^N} |\phi_k|^{p-1}|v||h| \\
&\leq C\|\phi_k\|^{p-1}\|v\|\,\|h\|.
\end{aligned}$$

Thus, for $\phi_{k_1}, \phi_{k_2} \in E$,

$$\begin{aligned}
\|T(\phi_{k_1}) - T(\phi_{k_2})\| &= \|M^{-1}R'(\phi_{k_1}) - M^{-1}R'(\phi_{k_2})\| \\
&\leq \|M^{-1}\|\,\|R'(\phi_{k_1}) - R'(\phi_{k_2})\| \leq \|M^{-1}\|\,\|R''(\phi_{k_1} + \theta(\phi_{k_2} - \phi_{k_1})\|\,\|\phi_{k_2} - \phi_{k_1}\| \\
&\leq C(\|\phi_{k_1}\|^{p-1} + \|\phi_{k_2}\|^{p-1}\|)\|\phi_{k_1} - \phi_{k_2}\| \leq \frac{1}{2}\|\phi_{k_1} - \phi_{k_2}\|.
\end{aligned}$$

Note that the last inequality holds only when k is large enough, which implies the existence of k_0 in Lemma 2.2. Therefore, T is a contraction map from E to E. Then the contraction mapping theorem implies the existence of ϕ_k as a critical point of J restricted to H.

Case 2: $p > 2$.

Setting $h(t) = |U_r + t\phi_k|^{p-1}(U_r + t\phi_k)v$, then by Taylor's formula,

$$\begin{aligned}
|\langle R'(\phi_k),v\rangle| &= \left|\int_{\mathbb{R}^N} Q(|z|)(pU_r^{p-1}\phi_k v + U_r^p v - |U_r+\phi_k|^{p-1}(U_r+\phi_k)v)\right| \\
&= \left|\int_{\mathbb{R}^N} Q(|z|)(-\frac{1}{2}h''(\theta))\right| \\
&\leq C\left|\int_{\mathbb{R}^N} p(p-1)|U_r+\theta\phi_k|^{p-2}\frac{U_r+\theta\phi_k}{|U_r+\theta\phi_k|}\phi_k^2 v\right| \\
&\leq C\int_{\mathbb{R}^N}(\|\phi_k\|^2 + \|\phi_k\|^p)\|v\| \\
&\leq C\|\phi_k\|^2\|v\|
\end{aligned}$$

which implies that $\|R'(\phi_k)\| \leq C\|\phi_k\|^2$.

By the mean value theorem we obtain

$$\begin{aligned}
|\langle R''(\phi_k)v,h\rangle| &= p\left|\int_{\mathbb{R}^N} Q(|z|)(U_r^{p-1}vh - |U_r+\phi_k|^{p-1})vh\right| \\
&= p(p-1)\left|\int_{\mathbb{R}^N} Q(|z|)|U_r+\theta\phi_k|^{p-3}(U_r+\theta\phi_k)\phi_k vh\right| \\
&\leq C\int_{\mathbb{R}^N}(U_r^{p-2} + |\phi_k|^{p-2})|\phi_k||v||h|
\end{aligned}$$

where $0 < \theta < 1$.

By Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} U_r^{p-2}|\phi_k||v||h| &\le C\left(\int_{\mathbb{R}^N} U_r^{p+1}\right)^{\frac{p-2}{p+1}}\left(\int_{\mathbb{R}^N}|\phi_k|^{p+1}\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^N}|v|^{p+1}\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^N}|h|^{p+1}\right)^{\frac{1}{p+1}}\\ &\le C\|\phi_k\|\,\|v\|\,\|h\|\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}|\phi_k|^{p-1}|v||h| &\le \left(\int_{\mathbb{R}^N}|\phi_k|^{p+1}\right)^{\frac{p-1}{p+1}}\left(\int_{\mathbb{R}^N}|v|^{p+1}\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^N}|h|^{p+1}\right)^{\frac{1}{p+1}}\\ &\le C\|\phi_k\|^{p-1}\|v\|\|h\|\,.\end{aligned}$$

Therefore, $\|R''(\phi_k)\| \le C\|\phi_k\|$.

Arguing similarly as in case 1, we have,

$$\|T(\phi_k)\| \le C(\|l_k\| + \|\phi_k\|^2) \le \frac{1}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}}}. \tag{2.13}$$

and T is a contraction map from E to E. The existence of ϕ_k follows from the contraction mapping theorem, and (2.10) follows from (2.12) and (2.13).

Following the argument employed in [5] to prove Lemma 4.4, we conclude that $\phi_k(r)$ is continuously differentiable in r. □

Lemma 2.3. *If $\tau = \min\{\alpha, \beta\} < N + 2s$, there exists a small constant $\delta > 0$ such that*

$$\|l_k\| \le \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}}.$$

Proof. We have

$$\begin{aligned}\langle l_k, \phi_k\rangle &= l(\phi_k)\\ &= \int_{\mathbb{R}^N}(V(|z|)-1)U_r\phi_k + \int_{\mathbb{R}^N}\left(\sum_{j=1}^k W_{x_j}^p - U_r^p\right)\phi_k - \int_{\mathbb{R}^N}(Q(|z|-1)U_r^p\phi_k.\end{aligned} \tag{2.14}$$

By symmetry,

$$\begin{aligned}\int_{\mathbb{R}^N}(V(|z|)-1)U_r\phi_k &= \int_{\mathbb{R}^N}(V(|z|)-1)\left(\sum_{j=1}^k W_{x_j}\right)\phi_k\\ &= \sum_{j=1}^k\int_{\mathbb{R}^N}(V(|z|)-1)W_{x_j}\phi_k = k\int_{\mathbb{R}^N}(V(|z|)-1)W_{x_1}\phi_k\end{aligned} \tag{2.15}$$

and

$$\int_{\mathbb{R}^N}(V(|z|)-1)W_{x_1}\phi_k = \int_{B_{\frac{r}{2}}(0)}(V(|z|)-1)W_{x_1}\phi_k + \int_{\mathbb{R}^N\backslash B_{\frac{r}{2}}(0)}(V(|z|)-1)W_{x_1}\phi_k.$$

For $z \in \mathbb{R}^N\backslash B_{\frac{r}{2}}(0)$,

$$V(|z|) - 1 = \frac{a_1}{|z|^\alpha} + O\left(\frac{1}{|z|^{\alpha+\theta_1}}\right) \le \frac{C}{|z|^\alpha} \le \frac{2^\alpha C}{|r|^\alpha},$$

$$\int_{\mathbb{R}^N\setminus B_{\frac{r}{2}}(0)}(V(|z|)-1)W_{x_1}\phi_k=\frac{2^\alpha C}{|r|^\alpha}\int_{\mathbb{R}^N\setminus B_{\frac{r}{2}}(0)}W_{x_1}\phi_k$$
$$\leq\frac{2^\alpha C}{|r|^\alpha}\Big(\int_{\mathbb{R}^N\setminus B_{\frac{r}{2}}(0)}W_{x_1}^2\Big)^{\frac12}\Big(\int_{\mathbb{R}^N\setminus B_{\frac{r}{2}}(0)}\phi_k^2\Big)^{\frac12}=O\Big(\frac{1}{r^\alpha}\Big)\|\phi_k\|,$$
$$\int_{B_{\frac{r}{2}}(0)}(V(|z|)-1)W_{x_1}\phi_k\leq C\Big(\int_{B_{\frac{r}{2}}(0)}W_{x_1}^2\Big)^{\frac12}\Big(\int_{B_{\frac{r}{2}}(0)}\phi_k^2\Big)^{\frac12}\leq C\frac{1}{r^{\frac{N}{2}+2s}}\|\phi_k\|.$$

Then we conclude that

$$\int_{\mathbb{R}^N}(V(|z|)-1)W_{x_1}\phi_k\leq O\Big(\frac{1}{r^\alpha}\Big)\|\phi_k\|+C\frac{1}{r^{\frac{N}{2}+2s}}\|\phi_k\|. \tag{2.16}$$

By the mean value theorem and Lemma A.1,

$$\Big|\int_{\mathbb{R}^N}\Big(\sum_{j=1}^k W_{x_j}^p-U_r^p\Big)\phi_k\Big|=k\Big|\int_{\Omega_1}\Big(\sum_{j=1}^k W_{x_j}^p-U_r^p\Big)\phi_k\Big|$$
$$\leq Ck\Big|\int_{\Omega_1}W_{x_1}^{p-1}\Big(\sum_{j=2}^k W_{x_j}\Big)\phi_k\Big|\leq Ck\frac{1}{(k^{-1}r)^{N+2s}}\Big(\int_{\Omega_1}W_{x_1}^p\Big)^{\frac{p-1}{p}}\Big(\int_{\Omega_1}|\phi_k|^p\Big)^{\frac1p}$$
$$\leq Ck\frac{1}{(k^{-1}r)^{N+2s}}\|\phi_k\|. \tag{2.17}$$

By the boundedness of U_r we have

$$\int_{\mathbb{R}^N}(Q(|z|-1)U_r^p\phi_k=\int_{\mathbb{R}^N}(Q(|z|-1)U_r^{p-1}U_r\phi_k$$
$$\leq C\int_{\mathbb{R}^N}(Q(|z|-1)U_r\phi_k\leq Ck\int_{\mathbb{R}^N}(Q(|z|-1)W_{x_1}\phi_k$$
$$\leq Ck\Big(O\Big(\frac{1}{r^\beta}\Big)+\frac{1}{r^{\frac{N}{2}+2s}}\Big)\|\phi_k\|. \tag{2.18}$$

Combining relations (2.14)–(2.18), we obtain

$$\langle l_k,\phi_k\rangle=\int_{\mathbb{R}^N}(V(|z|)-1)U_r\phi_k+\int_{\mathbb{R}^N}\Big(\sum_{i=1}^k W_{x_j}^p-U_r^p\Big)\phi_k-\int_{\mathbb{R}^N}(Q(|z|-1)U_r^p\phi_k$$
$$\leq k\Big(O\Big(\frac{1}{r^\alpha}\Big)+O\Big(\frac{1}{r^\beta}\Big)+C\frac{1}{r^{\frac{N}{2}+2s}}+\frac{C}{(k^{-1}r)^{N+2s}}\Big)\|\phi_k\|$$
$$\leq k\Big(O\Big(\frac{1}{r^\tau}\Big)+\frac{C}{r^{\frac{N}{2}+2s}}+\frac{C}{(k^{-1}r)^{N+2s}}\Big)\|\phi_k\|. \tag{2.19}$$

By $r\in S$, it holds that $r\sim k^{\frac{N+2s}{N+2s-\tau}}$, $\frac{C}{(k^{-1}r)^{N+2s}}\sim O(\frac{1}{r^\tau})$,

$$kO\Big(\frac{1}{r^\tau}\Big)<C\frac{1}{k^{\frac{(N+2s)\tau}{N+2s-\tau}-1}}\leq\frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}},$$

and $k\frac{1}{r^{\frac{N}{2}+2s}}<\frac{1}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}}}$, for $\tau<N+2s$.

We conclude that if $\tau < N+2s$, then

$$\|l_k\| \le \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}}.$$

The proof is now complete. □

3 Proof of the main result

Define

$$G(r) = I(U_r + \phi_k), \qquad \forall r \in S,$$

where $\phi_k = \phi_k(r)$ is the map obtained in Lemma 2.2.

According to Lemma 6.1 in [5], if r is a critical point of $G(r)$, then $U_r + \phi_k(r)$ is a solution of (1.1).

From the energy expansion in the Appendix, we have

$$\begin{aligned} J(0) &= I(U_r) \\ &= k\left(D + \frac{a_1 A_1}{r^\alpha} - \frac{a_2 A_2}{r^\beta} - \frac{B}{(k^{-1}r)^{N+2s}} + O\left(\frac{1}{r^{\alpha+\tau_1}}\right) + O\left(\frac{1}{r^{\beta+\tau_2}}\right) + O\left(\frac{1}{(k^{-1}r)^{N+2s+\sigma}}\right)\right). \end{aligned}$$

Set

$$H(r) = \frac{a_1 A_1}{r^\alpha} - \frac{a_2 A_2}{r^\beta} - \frac{B}{(k^{-1}r)^{N+2s}}.$$

We prove in what follows that in any of the cases in Theorem 1.1, $H(r)$ has a maximum point r_k.

For case (i): if $a_1 > 0$, $a_2 = 0$, $\alpha < N+2s$, and $\alpha \le \beta$ then

$$H'(r) = -\frac{\alpha a_1 A_1}{r^{\alpha+1}} + \frac{B(N+2s)k^{N+2s}}{r^{N+2s+1}}$$

and r_k satisfies

$$\frac{\alpha a_1 A_1}{r_k^{\alpha+1}} = \frac{B(N+2s)k^{N+2s}}{r_k^{N+2s+1}}.$$

Actually, calculating the maximum points in these cases can be summed up as

$$\frac{\tau C}{r_k^{\tau+1}} = \frac{C'(N+2s)k^{N+2s}}{r_k^{N+2s+1}},$$

where $\tau = \min\{\alpha, \beta\}$. Then $H(r)$ has a maximum point

$$r_k = \left(\frac{(N+2s)C'}{\tau C}\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}},$$

which is an interior point of S. Then, there exists a small constant δ such that

$$J(0) = k\left(D + \frac{a_1 A_1}{r^\alpha} - \frac{a_2 A_2}{r^\beta} - \frac{B}{(k^{-1}r)^{N+2s}} + O\left(\frac{1}{k^{\frac{(N+2s)\tau}{N+2s-\tau}+\delta}}\right)\right).$$

Consequently,

$$\begin{aligned} G(r) &= I(W_r+\phi_k) = I(W_r) + l(\phi_k) + \frac{1}{2}\langle M\phi_k, \phi_k\rangle + R(\phi_k) \\ &= J(0) + O(\|l_k\|\|\phi_k\| + \|\phi_k\|^2) \\ &= J(0) + O\left(\frac{1}{k^{\frac{(N+2s)\tau}{N+2s-\tau}-1+\delta}}\right) \\ &= k\left(D + \frac{a_1A_1}{r^\alpha} - \frac{a_2A_2}{r^\beta} - \frac{B}{(k^{-1}r)^{N+2s}} + O\left(\frac{1}{k^{\frac{(N+2s)\tau}{N+2s-\tau}+\delta}}\right)\right). \end{aligned}$$

Since $H(r)$ has a maximum point r_k which is an interior point of S in any of the cases listed, then $G(r)$ has a critical point $\widetilde{r_k}$ in the interior of S. This means that the function

$$U_{\widetilde{r_k}} + \phi_k(\widetilde{r_k})$$

is a solution of problem (1.1). The proof is now complete. □

A Appendix. Energy expansions

In this section, we obtain some energy estimates for the approximate solutions. Recall that

$$\Omega_j = \left\{ z = (z', z'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{z'}{|z'|}, \frac{x_j}{|x_j|} \right\rangle \geq \cos\frac{\pi}{k} \right\},$$

$$x_j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0 \right), \qquad j = 1, \ldots, k,$$

$$r \in S := \left[\left(\frac{N+2s}{\tau} - \epsilon\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}}, \left(\frac{N+2s}{\tau} + \epsilon\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}} \right],$$

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}}u|^2 + V(|z|)u^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)|u|^{p+1},$$

and

$$\begin{aligned} I(U_r) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}}U_r|^2 + V(|z|){U_r}^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)|U_r|^{p+1} \\ &= \frac{1}{2}\int_{\mathbb{R}^N} U_r \sum_{j=1}^{k} W_{x_j}^p + \frac{1}{2}\int_{\mathbb{R}^N} (V(|z|)-1){U_r}^2 \\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^N} (Q(|z|)-1)U_r^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^N} U_r^{p+1}. \end{aligned}$$

Lemma A.1. *For any $z \in \Omega_1$, there exists $C > 0$, such that*

$$\sum_{j=2}^{k} W_{x_j}(z) \leq \frac{C}{(k^{-1}r)^{N+2s}}$$

Proof. By the definition of Ω_j, for any $z \in \Omega_1$,

$$|z - x_j| \geq \frac{1}{2}|x_j - x_1| = r \sin \frac{(j-1)\pi}{k} > 0 \qquad (j \geq 2).$$

Then

$$\begin{aligned}
\sum_{j=2}^{k} W_{x_j}(z) &\leq C \sum_{j=2}^{k} \frac{1}{(r \sin \frac{(j-1)\pi}{k})^{N+2s}} = C \sum_{i=1}^{k-1} \frac{1}{(r \sin \frac{i\pi}{k})^{N+2s}} \\
&= \begin{cases} 2C \sum_{i=1}^{\frac{k-1}{2}} \frac{1}{(r \sin \frac{i\pi}{k})^{N+2s}}, & k \text{ is odd}, \\ 2C\Big(\sum_{i=1}^{\frac{k-2}{2}} \frac{1}{(r \sin \frac{i\pi}{k})^{N+2s}} + \frac{1}{(r \sin \frac{k\pi}{2k})^{N+2s}} \Big), & k \text{ is even}. \end{cases}
\end{aligned}$$

When k is even,

$$\begin{aligned}
\sum_{j=2}^{k} W_{x_j}(z) &\leq 2C\Big(\sum_{i=1}^{\frac{k-2}{2}} \frac{1}{(r \sin \frac{i\pi}{k})^{N+2s}} + \frac{1}{(r \sin \frac{k\pi}{2k})^{N+2s}} \Big) \\
&\leq 2C\Big(\sum_{i=1}^{\frac{k-2}{2}} \frac{1}{(r \frac{2i}{k})^{N+2s}} + \frac{1}{r^{N+2s}} \Big) \\
&\leq 2C\Big(\frac{1}{(k^{-1}r)^{N+2s}} \sum_{i=1}^{\frac{k-2}{2}} \frac{1}{i^{N+2s}} + \frac{1}{r^{N+2s}} \Big) \\
&\leq \frac{C}{(k^{-1}r)^{N+2s}}
\end{aligned}$$

since $\sum_{i=1}^{+\infty} \frac{1}{i^{N+2s}}$ converges.

The proof of the case where k is odd follows with similar arguments. □

Lemma A.2. *We have*

$$\int_{\mathbb{R}^N} (V(|z|) - 1) U_r{}^2 = k\Big(\frac{a_1}{r^\alpha} \int_{\mathbb{R}^N} W^2 + O\Big(\frac{1}{r^{\alpha+\tau_1}}\Big) + O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_1}}\Big)\Big)$$

where $\tau_1 > 0$, and $\sigma_1 > 0$ are small constants.

Proof. By symmetry,

$$\begin{aligned}
\int_{\mathbb{R}^N} (V(|z|) - 1) U_r{}^2 &= k \int_{\Omega_1} (V(|z|) - 1) \Big(W_{x_1} + \sum_{j=2}^{k} W_{x_j} \Big)^2 \\
&= k \int_{\Omega_1} (V(|z|) - 1) \Big(W_{x_1}^2 + 2W_{x_1} \sum_{j=2}^{k} W_{x_j} + \Big(\sum_{j=2}^{k} W_{x_j} \Big)^2 \Big) \qquad \text{(A.1)}
\end{aligned}$$

and

$$\int_{\Omega_1} (V(|z|) - 1) W_{x_1}^2 = \int_{\Omega_1 \setminus B_{\frac{r}{2}}(x_1)} (V(|z|) - 1) W_{x_1}^2 + \int_{B_{\frac{r}{2}}(x_1)} (V(|z|) - 1) W_{x_1}^2. \qquad \text{(A.2)}$$

On the one hand,

$$\int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}^2 \le C\int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)} W_{x_1}^2 \le C\int_{|z-x_1|>\frac{r}{2}}\left(\frac{1}{|z-x_1|^{N+2s}}\right)^2$$
$$= C\int_{\frac{r}{2}}^{+\infty}\frac{t^{N-1}}{t^{2N+4s}}dt = C\int_{\frac{r}{2}}^{+\infty}\frac{1}{t^{N+4s+1}}dt = C\frac{1}{r^{N+4s}} = O\left(\frac{1}{r^{N+4s}}\right). \tag{A.3}$$

On the other hand,

$$\int_{B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}^2$$
$$= \int_{B_{\frac{r}{2}}(x_1)}\left(\frac{a_1}{|z|^\alpha}+O\left(\frac{1}{|z|^{\alpha+\theta_1}}\right)\right)W_{x_1}^2$$
$$= \int_{B_{\frac{r}{2}}(x_1)}\left(\frac{a_1}{r^\alpha}+\frac{a_1}{r^{\alpha+1}}O(|z-x_1|)+\frac{C}{r^{\alpha+\theta_1}}+\frac{C}{r^{\alpha+\theta_1+1}}O(|z-x_1|)\right)W_{x_1}^2$$
$$= \frac{a_1}{r^\alpha}\int_{B_{\frac{r}{2}}(x_1)} W_{x_1}^2 + O\left(\frac{1}{r^{\alpha+\tau_1}}\right) = \frac{a_1}{r^\alpha}\int_{\mathbb{R}^N} W^2 + O\left(\frac{1}{r^{\alpha+\tau_1}}\right) + O\left(\frac{1}{r^{N+4s}}\right), \tag{A.4}$$

where $\tau_1 = \min\{1,\theta_1\}$.

By (A.2)–(A.4),

$$\int_{\Omega_1}(V(|z|)-1)W_{x_1}^2 = \frac{a_1}{r^\alpha}\int_{\mathbb{R}^N} W^2 + O\left(\frac{1}{r^{N+4s}}\right) + O\left(\frac{1}{r^{\alpha+\tau_1}}\right), \tag{A.5}$$

$$\int_{\Omega_1}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j} = \int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j}$$
$$+\int_{B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j} \tag{A.6}$$

By the boundedness of $V(|z|)$ and Lemma A.1,

$$\int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j} \le \frac{C}{(k^{-1}r)^{N+2s}}\int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)} W_{x_1}$$
$$\le \frac{C}{(k^{-1}r)^{N+2s}}\int_{|z-x_1|>\frac{r}{2}}\frac{1}{|z-x_1|^{N+2s}} = \frac{C}{(k^{-1}r)^{N+2s}}\int_{\frac{r}{2}}^{+\infty}\frac{t^{N-1}}{t^{N+2s}}dt$$
$$= \frac{C}{(k^{-1}r)^{N+2s}}\int_{\frac{r}{2}}^{+\infty}\frac{1}{t^{2s+1}}dt = \frac{C}{(k^{-1}r)^{N+2s}}\frac{1}{r^{2s}}$$
$$= O\left(\frac{1}{(k^{-1}r)^{N+4s}}\right). \tag{A.7}$$

Similarly to (A.4),

$$\int_{B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j} \le \frac{C}{(k^{-1}r)^{N+2s}}\int_{B_{\frac{r}{2}}(x_1)}\left(\frac{a_1}{|z|^\alpha}+O\left(\frac{1}{|z|^{\alpha+\theta_1}}\right)\right)W_{x_1}$$
$$= \frac{C}{(k^{-1}r)^{N+2s}}\left(\frac{a_1}{r^\alpha}\int_{B_{\frac{r}{2}}(x_1)} W_{x_1} + O\left(\frac{1}{r^{\alpha+1}}\right) + O\left(\frac{1}{r^{\alpha+\theta_1}}\right)\right)$$
$$\le \frac{C}{(k^{-1}r)^{N+2s}}\frac{C'}{r^\alpha} = O\left(\frac{1}{(k^{-1}r)^{N+2s+\alpha}}\right). \tag{A.8}$$

By relations (A.6)–(A.8) we get

$$\int_{\Omega_1}(V(|z|)-1)W_{x_1}\sum_{j=2}^{k}W_{x_j}=O\Big(\frac{1}{(k^{-1}r)^{N+4s}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\alpha}}\Big), \tag{A.9}$$

$$\begin{aligned}\int_{\Omega_1}(V(|z|)-1)\Big(\sum_{j=2}^{k}W_{x_j}\Big)^2 &\le \frac{C}{(k^{-1}r)^{N+2s}}\sum_{j=2}^{k}\int_{\Omega_1}W_{x_j}\\ &\le \frac{C}{(k^{-1}r)^{N+2s}}\frac{C_1}{(k^{-1}r)^{2s}}=O\Big(\frac{1}{(k^{-1}r)^{N+4s}}\Big).\end{aligned} \tag{A.10}$$

By (A.1), (A.5), (A.9), (A.10), We have

$$\begin{aligned}&\int_{\mathbb{R}^N}(V(|z|)-1){U_r}^2\\ &=k\bigg(\frac{a_1}{r^\alpha}\int_{\mathbb{R}^N}W^2+O\Big(\frac{1}{r^{N+4s}}\Big)+O\Big(\frac{1}{r^{\alpha+\tau_1}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+4s}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\alpha}}\Big)\bigg)\\ &=k\bigg(\frac{a_1}{r^\alpha}\int_{\mathbb{R}^N}W^2+O\Big(\frac{1}{r^{\alpha+\tau 1}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_1}}\Big)\bigg),\end{aligned} \tag{A.11}$$

where $\sigma_1=\min\{2s,\alpha\}$. The proof is now complete. □

Remark A.3. Arguing similarly as in Lemma A.2, we have

$$\int_{\mathbb{R}^N}(Q(|z|)-1){U_r}^{p+1}=k\bigg(\frac{a_2}{r^\beta}\int_{\mathbb{R}^N}W^{p+1}+O\Big(\frac{1}{r^{\beta+\tau_2}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_2}}\Big)\bigg), \tag{A.12}$$

where $\tau_2>0$ and $\sigma_2>0$ are small constants.

Lemma A.4. *There is a small constant $\delta>0$, such that*

$$I(U_r)=k\bigg(D+\frac{a_1}{r^\alpha}A_1-\frac{a_2}{r^\beta}A_2-\frac{B}{(k^{-1}r)^{N+2s}}+O\Big(\frac{1}{k^{\frac{(N+2s)\tau}{(N+2s-\tau)}+\delta}}\Big)\bigg),$$

where

$$D=\Big(\frac12-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N}W^{p+1},\qquad A_1=\frac12\int_{\mathbb{R}^N}W^2,\qquad A_2=\frac{1}{p+1}\int_{\mathbb{R}^N}W^{p+1},$$

and B satisfies

$$\frac12\int_{\Omega_1}W_{x_1}^p\sum_{j=2}^{k}W_{x_j}=\frac{B}{(k^{-1}r)^{N+2s}}.$$

Proof. We first prove that there exists small constant $\sigma_3>0$ such that

$$\begin{aligned}&\frac12\int_{\mathbb{R}^N}U_r\sum_{j=1}^{k}W_{x_j}^p-\frac{1}{p+1}\int_{\mathbb{R}^N}U_r^{p+1}\\ &=k\bigg(\Big(\frac12-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N}W^{p+1}-\frac12\int_{\Omega_1}W_{x_1}^p\sum_{j=2}^{k}W_{x_j}+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_3}}\Big)\bigg).\end{aligned} \tag{A.13}$$

By symmetry,

$$\begin{aligned}\int_{\mathbb{R}^N} U_r \sum_{j=1}^{k} W_{x_j}^p &= k \int_{\Omega_1} U_r \sum_{j=1}^{k} W_{x_j}^p \\ &= k \int_{\Omega_1} \Big(W_{x_1} + \sum_{i=2}^{k} W_{x_i} \Big) \Big(W_{x_1}^p + \sum_{j=2}^{k} W_{x_j}^p \Big) \\ &= k \Big(\int_{\Omega_1} W_{x_1}^{p+1} + W_{x_1}^p \sum_{i=2}^{k} W_{x_i} + W_{x_1} \sum_{j=2}^{k} W_{x_j}^p + \sum_{i=2}^{k} W_{x_i} \sum_{j=2}^{k} W_{x_j}^p \Big). \end{aligned} \tag{A.14}$$

By Lemma A.1,

$$\begin{aligned}\int_{\Omega_1} W_{x_1} \sum_{j=2}^{k} W_{x_j}^p &\le \int_{\Omega_1} W_{x_1} \sum_{j=2}^{k} \Big(\frac{C}{|z - x_j|^{N+2s}} \Big)^p \\ &\le \int_{\Omega_1} W_{x_1} \sum_{j=2}^{k} \frac{C^p}{(\frac{1}{2}|x_j - x_1|)^{(N+2s)p}} = \sum_{j=2}^{k} \frac{C^p}{(\frac{1}{2}|x_j - x_1|)^{(N+2s)p}} \int_{\Omega_1} W_{x_1} \\ &\le \frac{C'}{(k^{-1}r)^{(N+2s)p}} \int_{\Omega_1} W_{x_1} = O\Big(\frac{1}{(k^{-1}r)^{(N+2s)p}} \Big) \end{aligned} \tag{A.15}$$

and

$$\int_{\Omega_1} \sum_{i=2}^{k} W_{x_i} \sum_{j=2}^{k} W_{x_j}^p \le \frac{C}{(k^{-1}r)^{N+2s}} \sum_{j=2}^{k} \int_{\Omega_1} W_{x_j}^p = O\Big(\frac{1}{(k^{-1}r)^{(N+2s)p+2s}} \Big). \tag{A.16}$$

By Taylor's formula,

$$\begin{aligned}&\int_{\Omega_1} U_r^{p+1} - \int_{\Omega_1} W_{x_1}^{p+1} - \int_{\Omega_1} (p+1) W_{x_1}^p \sum_{j=2}^{k} W_{x_j} \\ &= \frac{1}{2} p(p+1) \int_{\Omega_1} \Big(W_{x_1} + \theta \sum_{j=2}^{k} W_{x_j} \Big)^{p-1} \Big(\sum_{j=2}^{k} W_{x_j} \Big)^2 \\ &\le C \int_{\Omega_1} \Big(W_{x_1}^{p-1} \Big(\sum_{j=2}^{k} W_{x_j} \Big)^2 + \Big(\sum_{j=2}^{k} W_{x_j} \Big)^{p+1} \Big). \end{aligned}$$

By Lemma A.1,

$$\int_{\Omega_1} W_{x_1}^{p-1} \Big(\sum_{j=2}^{k} W_{x_j} \Big)^2 \le \frac{C}{(k^{-1}r)^{2N+4s}} \int_{\Omega_1} W_{x_1}^{p-1} \le \frac{C}{(k^{-1}r)^{2N+4s}}$$

and

$$\int_{\Omega_1} \Big(\sum_{j=2}^{k} W_{x_j} \Big)^{p+1} \le \frac{C}{(k^{-1}r)^{(N+2s)p}} \sum_{j=2}^{k} \int_{\Omega_1} W_{x_j} \le \frac{C}{(k^{-1}r)^{(N+2s)p}} \frac{C'}{(k^{-1}r)^{2s}}.$$

Therefore,

$$\int_{\Omega_1} U_r^{p+1} = \int_{\Omega_1} W_{x_1}^{p+1} + (p+1) W_{x_1}^p \sum_{j=2}^{k} W_{x_j} + O\Big(\frac{1}{(k^{-1}r)^{N+4s}} \Big). \tag{A.17}$$

Thus, by (A.13)–(A.17), we conclude that

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} U_r \sum_{j=2}^{k} W_{x_j}^p - \frac{1}{p+1}\int_{\mathbb{R}^N} U_r^{p+1} \\
&= k\Big(\frac{1}{2}\int_{\Omega_1} U_r \sum_{j=2}^{k} W_{x_j}^p - \frac{1}{p+1}\int_{\Omega_1} U_r^{p+1}\Big) \\
&= k\bigg(\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\Omega_1} W_{x_1}^{p+1} - \frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} + O\Big(\frac{1}{(k^{-1}r)^{(N+2s)p}}\Big) + O\Big(\frac{1}{(k^{-1}r)^{N+4s}}\Big)\bigg) \\
&= k\bigg(\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N} W^{p+1} - \frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} + O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_3}}\Big)\bigg) \qquad \text{(A.18)}
\end{aligned}
$$

where $\sigma_3 = \min\{2s, (p-1)(N+2s)\}$.

Next, we claim that there exists a constant $B > 0$, such that

$$\frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} = \frac{B}{(k^{-1}r)^{N+2s}}.$$

It is easy to verify that

$$\frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} \le \frac{B}{(k^{-1}r)^{N+2s}}.$$

Set $G_k = \{z \in \mathbb{R}^N : |z - x_1| < \frac{1}{4}|x_j - x_1|\}$. Then for a fixed $R > 0$, it follows that $B_R(x_1) \subset G_k \subset \Omega_1$. Then

$$
\begin{aligned}
\frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} &\ge \frac{1}{2}\int_{G_k} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} \ge \frac{1}{2}\int_{G_k} W_{x_1}^p \sum_{j=2}^{k} \frac{C}{|z-x_j|^{N+2s}} \\
&\ge C'\int_{G_k} W_{x_1}^p \sum_{j=2}^{k} \frac{1}{|\frac{3}{2}r\sin\frac{(j-1)\pi}{k}|^{N+2s}} = C'\sum_{j=2}^{k} \frac{1}{|\frac{3}{2}r\sin\frac{(j-1)\pi}{k}|^{N+2s}}\int_{G_k} W_{x_1}^p \\
&\ge C'\sum_{j=2}^{k} \frac{1}{|\frac{3}{2}r\sin\frac{(j-1)\pi}{k}|^{N+2s}}\int_{B_R(0)} W^p \ge \frac{B}{(k^{-1}r)^{N+2s}}.
\end{aligned}
$$

Combining this claim with Lemma A.2, Remark A.3, and (A.18), we obtain

$$
\begin{aligned}
I(U_r) = k\bigg(&\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N} W^{p+1} + \frac{1}{2}\frac{a_1}{r^\alpha}\int_{\mathbb{R}^N} W^2 - \frac{1}{p+1}\frac{a_2}{r^\beta}\int_{\mathbb{R}^N} W^{p+1} \\
&- \frac{B}{(k^{-1}r)^{N+2s}} + O\Big(\frac{1}{r^{\alpha+\tau 1}}\Big) + O\Big(\frac{1}{r^{\beta+\tau 2}}\Big) + O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma}}\Big)\bigg),
\end{aligned}
$$

where $\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$.

Denoting

$$D = \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N} W^{p+1}, \qquad A_1 = \frac{1}{2}\int_{\mathbb{R}^N} W^2, \qquad A_2 = \frac{1}{p+1}\int_{\mathbb{R}^N} W^{p+1},$$

and using the fact that $r \in S$, we have

$$I(U_r) = k\bigg(D + \frac{a_1}{r^\alpha}A_1 - \frac{a_2}{r^\beta}A_2 - \frac{B}{(k^{-1}r)^{N+2s}} + O\Big(\frac{1}{k^{\frac{(N+2s)\tau}{(N+2s-\tau)}+\delta}}\Big)\bigg).$$

The proof is now complete. □

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