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Qualitative properties and global bifurcation of solutions for a singular boundary value problem

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. This paper deals with a singular, nonlinear Sturm–Liouville problem of the form $\{A(x)u'(x)\}' + \lambda u(x) = f(x, u(x), u'(x))$ on $(0, 1)$ where A is positive on $(0, 1]$ but decays quadratically to zero as x approaches zero. This is the lowest level of degeneracy for which the problem exhibits behaviour radically different from the regular case. In this paper earlier results on the existence of bifurcation points are extended to yield global information about connected components of solutions.

Keywords: singular Sturm–Liouville problem, global bifurcation, Hadamard differentiable mapping.

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1 Introduction

The aim of this paper is to investigate the set of solutions of the boundary value problem,

$$-\{A(x)u'(x)\}' + V(x)u(x) + n(x, u'(x)) + g(x, u(x)) = \lambda u(x) \quad \text{for } 0 < x < 1, \tag{1.1}$$

$$u(1) = 0 \quad \text{and} \quad \int_0^1 A(x)u'(x)^2 dx < \infty, \tag{1.2}$$

for an unknown function u such that $u \in C^1((0, 1])$ and Au' is absolutely continuous on the compact subintervals of $(0, 1]$. The differential equation is singular at $x = 0$ because we suppose that the coefficient A satisfies the following condition.

(A) $A \in C([0, 1])$ with $A(x) > 0$ for $x > 0$ and $\lim_{x\to 0} \frac{A(x)}{x^2} = a > 0$.

Hence there exist constants $C_2 \geq C_1 > 0$ such that $C_1 x^2 \leq A(x) \leq C_2 x^2$ for all $x \in [0, 1]$.

As we have shown in previous work on the problem in [31], this level of degeneracy leads to behaviour that does not occur for regular problems nor problems with weaker degeneracy.

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For example, solutions can become unbounded as x tends to zero and there may be no bifurcation at a simple eigenvalue of the linearisation lying below the essential spectrum. For a more detailed presentation of the critical character of quadratic degeneracy we refer to [33] concerning the analogous elliptic problem in higher dimensions. Other aspects of criticality have been emphasised in some work on the asymptotic behaviour of solutions for a porous medium equation with degeneracy [17,18]. In the stability analysis for the parabolic problem associated with the higher dimensional analogue of (1.1)(1.2) it is shown in [32] that the principle of linearised stability can fail at the stationary solution $u \equiv 0$ when the degeneracy is critical. For subcritical degeneracy, i.e. when $\liminf_{x \to 0} x^{-d} A(x) > 0$ for some $d < 2$, global bifurcation of positive stationary solutions and their stability are proved in [20] for a parabolic problem corresponding to the higher dimensional analogue of (1.1)(1.2).

Before proceeding to describe other aspects of the problem some information about the lower order terms in (1.1) is necessary. The potential V in (1.1) is bounded and has a well-defined limit as $x \to 0$.

(V) $V \in L^\infty(0,1)$ and there exists $V_0 \in \mathbb{R}$ such that $\lim_{z \to 0} \|V - V_0\|_{L^\infty(0,z)} = 0$.

The nonlinear terms n and g are of higher order in the sense that

$$\lim_{s \to 0} \frac{n(x,s)}{s} = \lim_{s \to 0} \frac{g(x,s)}{s} = 0 \quad \text{for all } x \in (0,1) \tag{1.3}$$

and they satisfy some additional conditions introduced in Subection 2.2. Under these hypotheses $u \equiv 0$ is a solution of (1.1)(1.2) and the parameter $\lambda \in \mathbb{R}$ is treated as an eigenvalue. The sense in which the equation (1.1) is satisfied is made precise in Section 2.3. In this form the problem has been studied in some detail in [31,33] and Section 2 contains the conclusions from those papers that are needed here.

In view of (1.3) the linearisation of (1.1) is the singular Sturm–Liouville problem

$$-\{A(x)u'(x)\}' + V(x)u(x) = \lambda u(x), \quad \text{where } u \in L^2(0,1) \text{ and } u(1) = 0, \tag{1.4}$$

and its spectrum is discussed in Section 2.4. It is in the limit point case at $x = 0$ when (A) and (V) are satisfied but

$$\lim_{x \to 0} A(x)u'(x) = 0, \tag{1.5}$$

appears as a natural boundary condition. In fact, it is noted in Section 2 that the expression $-(Au')' + Vu$ defines a self-adjoint operator, $S_A + V$, acting in $L^2(0,1)$ with domain

$$D_A = \{u \in L^2(0,1) : (Au')' \in L^2(0,1) \text{ and } u(1) = 0\}$$

and all elements of D_A satisfy (1.2) and (1.5). The eigenvalues of $S_A + V$ are all simple and its essential spectrum is the interval $[\frac{a}{4} + V_0, \infty)$. In Section 2.4 some special cases treated in [33] are recalled showing that $S_A + V$ may or may not have eigenvalues.

The main results of this paper give information about the global behaviour of components of solutions $(\lambda, u) \in \mathbb{R} \times D_A$ of the singular problem (1.1)(1.2) in the spirit of the regular case treated in [7,24]. This involves confronting two principal difficulties arising from the degeneracy. First of all, the presence of a non-trivial essential spectrum of the linearisation indicates that the problem cannot be reduced to an equation for a compact perturbation of the identity. Secondly, previous work on the existence of bifurcation points for problem (1.1)(1.2) has shown that, under reasonable assumptions about the nonlinear terms, Fréchet differentiability at the trivial solution $u \equiv 0$ cannot be obtained. Indeed, there are cases in [31,33] where there is no bifurcation at an eigenvalue of $S_A + V$ lying below its essential spectrum, a situation which could not occur if the nonlinearity were Fréchet differentiable at $u \equiv 0$.

The conclusions obtained here concerning problem (1.1)(1.2) are established by following what has become a standard path since the classic paper by Rabinowitz [23]. First of all an abstract result is formulated under hypotheses that accommodate the two main difficulties just mentioned. This result is then applied to the boundary value problem and the nodal properties of solutions are used to sharpen the information about components of solutions given by the abstract theory.

Let X and Y be real Banach spaces and consider a mapping $F : \mathbb{R} \times X \to Y$ having the properties that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$ and $F(\lambda, \cdot) : X \to Y$ is at least Hadamard differentiable at 0. For the equation $F(\lambda, u) = 0$, local results concerning bifurcation at isolated singular points of the derivative $D_u F(\lambda, 0)$ were established in [29] using the Brouwer degree after reduction to a finite dimensional space. In a similar setting global conclusions about connected components of solutions have been obtained recently in [34] using a topological degree for continuous perturbations of C^1-Fredholm maps constructed by Benevieri, Calamai and Furi [3,4], combined with techniques from in [29]. In these contributions a considerable amount of rather specialised terminology is required in order the formulate the hypotheses. The class of admissible perturbations for the existence of the degree defined in [3,4] is specified using notions related to the Kuratowski measure of non-compactness and the conditions for bifurcation involve the parity of the path $\lambda \mapsto D_u F(\lambda, 0)$ as defined by Fitzpatrick and Pejsachowicz, [15]. Instead of recalling these results in their fully generality with all the requisite terminology, we formulate two special cases concerning equations of a simpler form in Hilbert space. With the exception of Hadamard and w-Hadamard differentiability which are defined in Section 4.1, these results can be stated using only well-known concepts and problem (1.1)(1.2) can be dealt with in this context.

The Hilbert space theory, as set out in Section 4, is applied to problem (1.1)(1.2) in Section 5. As for regular Sturm–Liouville problems, the nodal properties of solutions and comparison principles for self-adjoint operators can be used to refine the conclusions coming directly from the abstract theory. However the strong degeneracy of equation (1.1) at $x = 0$ means that the behaviour of solutions as $x \to 0$ requires some care and various aspects of this are investigated in Section 3, generalizing results of a similar nature in [31]. As special cases of the main results in Section 5, hypotheses are provided under which the following somewhat unusual phenomena occur. Consider problem (1.1)(1.2) with $n \equiv 0$. Given any $n \in \mathbb{N}$, there are coefficients A and V such that the linearisation (1.4) has exactly n simple eigenvalues $\lambda_1 < \lambda_2, \ldots < \lambda_n$ below its essential spectrum which is $[m_e, \infty)$ where $m_e = \frac{a}{4} + V_0$.

(1) For any $k \in \{1, \ldots, n\}$ there is a class of nonlinearities with $g(x,s)s \leq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$ for which an unbounded component of non-trivial solutions bifurcates from $(\lambda_i, 0)$ for each $i \leq k$, but there is no bifurcation from $(\lambda_i, 0)$ for $i > k$. (See Remark 5.4.)

(2) There is another class of nonlinearities with $g(x,s)s \geq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$ for which a component $\mathcal{C}_i$ of non-trivial solutions bifurcates from $(\lambda_i, 0)$ for every $i \in \{1, ..., n\}$ and $\{\lambda : (\lambda, u) \in \mathcal{C}_i\} = [\lambda_i, m_e)$. If $(\lambda, u) \in \mathcal{C}_i$ with λ near λ_i, $u \in C^1((0,1]) \cap L^\infty(0,1)$, whereas for λ near m_e, $u \in C^1((0,1])$ but $u(x) \to \infty$ as $n \to \infty$. (See Remark 5.6.)

Many references to problems of the type studied here can be found in the papers [17,18,20,31,33] and, as shown in an appendix in [31], several other types of equation can be reduced to the form (1.1) by a change of variable. The radially symmetric version of the analogous problem in higher dimensions can also be transformed to (1.1)(1.2). Following what was done in [13] for a closely related case, this is mentioned in [31] and more details are given in Section 6.6 of [33] where local results on bifurcation are formulated.

The line of research pursued here on bifurcation for problems like (1.1)(1.2) was stimulated by the unusual behaviour revealed in [28] concerning the buckling of a critically tapered rod which is modelled by an equation having the same kind of degeneracy. Using variational methods it is shown in [28] that an unbounded curve of positive solutions bifurcates from the lowest point Λ of the spectrum of the linearisation, even if it is not an eigenvalue. In fact, bifurcation occurs at every point in the interval $[\Lambda, \infty)$. For the same problem, global bifurcation at all eigenvalues lying below the essential spectrum was established in collaboration with G. Vuillaume [35,36] using a topological approach. In this buckling problem the full nonlinear equation involves a compact perturbation of the identity but it is not Fréchet differentiable at the trivial solution and its linearisation is not a compact perturbation of the identity. In work with G. Evéquoz [13,14] on a more general class of degenerate problems a variational method was used show that bifurcation can occur at points which are not necessarily eigenvalues of the linearisation and singular behaviour of the bifurcating solutions was demonstrated in the radially symmetric case. Some of the abstract results on bifurcation for non-Fréchet differentiable problems are summarised in [30] together with references to applications to uniformly elliptic equations on $\mathbb{R}^N$.

2 A class of singular boundary value problems

Throughout this section it is assumed that the function A satisfies condition (A). The first step is to define the domain of a positive self-adjoint operator, S_A, in $L^2(0,1)$ associated with the singular differential operator $-(Au')'$ and the boundary condition $u(1) = 0$. In addition to noting some crucial properties of functions in this domain, D_A, it is also necessary to investigate the domain, H_A, of the positive, self-adjoint square-root, $S_A^{\frac{1}{2}}$. Although the set D_A depends upon A, it turns out that H_A is the same set for all coefficients satisfying condition (A). Most of the results mentioned in this section are proved in [31].

2.1 The spaces D_A and H_A

From the results in Section 2 of [31] the set D_A can be defined as

$$D_A = \{u \in C^1((0,1]) \cap L^2(0,1) : (Au')' \in L^2(0,1) \text{ and } u(1) = 0\},$$

where $(Au')'$ is the generalized derivative on $(0,1)$ of the continuous function Au'. It is also shown in [31] that

$$S_A : D_A \subset L^2(0,1) \to L^2(0,1) \quad \text{with} \quad S_A u = -(Au')' \quad \text{for } u \in D_A$$

is a self-adjoint operator having the following properties. See Lemmas 2.1 and 2.2 and Corollary 2.3 in [31].

(D1) $(S_A u, v)_{L^2} = \int_0^1 Au'v'\,dx$ for all $u, v \in D_A$.

(D2) $(S_A u, u)_{L^2} \geq \frac{C_1}{4}\|u\|_{L^2}^2$ and $\|u\|_{L^2} \leq \frac{2}{\sqrt{C_1}}\|A^{\frac{1}{2}}u'\|_{L^2} \leq \frac{4}{C_1}\|S_A u\|_{L^2}$ for all $u \in D_A$.

(D3) $S_A : D_A \to L^2$ is an isomorphism and $S_A^{-1}w = \int_x^1 \frac{1}{A(y)}[\int_0^y w(z)\,dz]\,dy$ for all $w \in L^2(0,1)$.

Henceforth, $L^2 = L^2(0,1)$ and $a \geq C_1 \equiv \inf\left\{\frac{A(x)}{x^2} : 0 < x \leq 1\right\} > 0$ by (A). By (D2), $\|S_A u\|_{L^2}$ defines a norm on D_A that is equivalent to the graph norm of S_A. Elements of D_A enjoy the following properties which are proved in Lemmas 2.4 and 2.5 in [31].

(P1) $x^{\frac{3}{2}}u'(x) \to 0$ as $x \to 0$ and $\|x^{\frac{3}{2}}u'\|_{L^\infty} \leq \frac{1}{C_1}\|S_A u\|_{L^2}$ for all $u \in D_A$.

(P2) $x^{\frac{1}{2}}u(x) \to 0$ as $x \to 0$ and $\|x^{\frac{1}{2}}u\|_{L^\infty} \leq \frac{1}{\sqrt{C_1}}\|A^{\frac{1}{2}}u'\|_{L^2} \leq \frac{2}{C_1}\|S_A u\|_{L^2}$ for all $u \in D_A$.

By condition (A) and (P1), $A(x)u'(x) \to 0$ as $x \to 0$ for all $u \in D_A$ showing that (1.5) is a natural boundary condition for the operator S_A. If $u \in D_A$ and $u(z) = 0$ for some $z \in (0,1]$, it follows from (P1) and (P2) that

$$\int_0^z [S_A u(x)]u(x)\,dx = \int_0^z A(x)u'(x)^2 dx. \tag{2.1}$$

Let

$$H = \left\{u \in L^2(0,1) : \int_0^1 x^2 u'(x)^2 dx < \infty\right\}$$

where u' is the generalized derivative of u on $(0,1)$. If $u \in H$, its restriction to $(\eta,1)$ belongs to the usual Sobolev space $H^1((\eta,1))$ for all $\eta \in (0,1)$ and so, with the usual abuse to terminology, we can consider that $u \in C((0,1])$. The space H_A is defined by

$$H_A = \{u \in H : u(1) = 0\} = \left\{u \in L^2(0,1) : \int_0^1 A(x)u'(x)^2 dx < \infty \text{ and } u(1) = 0\right\}.$$

It is a Hilbert space for the norm defined by $\|u\|_A = \|A^{\frac{1}{2}}u'\|_{L^2}$ and the corresponding scalar product is denoted by

$$\langle u, v\rangle_A = \int_0^1 A(x)u'(x)v'(x)\,dx \quad \text{for } u, v \in H_A.$$

Denoting the unique positive, self-adjoint square root of S_A by $S_A^{\frac{1}{2}} : D(S_A^{\frac{1}{2}}) \subset L^2(0,1) \to L^2(0,1)$, it is also shown in [31] that $H_A = D(S_A^{\frac{1}{2}})$. In particular, D_A is a dense subspace of $(H_A, \|\cdot\|_A)$ and so (D1), (D2) and (P2) imply the following properties.

(H1) $\|u\|_{L^2} \leq \frac{2}{\sqrt{C_1}}\|u\|_A$ and $\|u\|_A = \|S_A^{\frac{1}{2}}u\|_{L^2}$ for all $u \in H_A$.

(H2) $x^{\frac{1}{2}}u(x) \to 0$ as $x \to 0$ and $\|x^{\frac{1}{2}}u\|_{L^\infty} \leq \frac{1}{\sqrt{C_1}}\|u\|_A$ for all $u \in H_A$.

Using (H1) with $A(x) = x^2$ and a simple rescaling, we have that

$$\int_0^z u(x)^2 dx \leq 4\int_0^z x^2 u'(x)^2 dx \quad \text{if } u \in H_A \text{ and } u(z) = 0 \text{ for some } z \in (0,1]. \tag{2.2}$$

By (P1) and (H2),

$$\int_0^1 [S_A u(x)]v(x)\,dx = \int_0^1 A(x)u'(x)v'(x)\,dx \quad \text{for all } u \in D_A \text{ and } v \in H_A. \tag{2.3}$$

The following compactness property is justified in Remark 2.2 in [31].

(H3) If $\{u_n\} \subset H_A$ is a sequence converging weakly to u in H_A, $\{u_n\} \subset C([\eta,1])$ and it converges uniformly to u on $[\eta,1]$ for all $\eta \in (0,1)$.

2.2 Properties of the nonlinearities

The Nemytskii operator associated with a Caratheodory function $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ will be denoted by $\tilde{f}$. Thus $\tilde{f}(u)(x) = f(x, u(x))$ for a measurable function $u : (0,1) \to \mathbb{R}$ and $x \in (0,1)$.

We now formulate the assumptions which will be used to deal with the nonlinear terms in equation (1.1). They ensure that the corresponding operators are well-defined and map the spaces D_A and H_A into $L^2(0,1)$. For the continuity and differentiability properties of these operators it is understood that D_A and H_A are considered with the norms $\|S_A\|_{L^2}$ and $\|u\|_A$, respectively.

(F) $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

- (i) $\lim_{s\to 0} \frac{f(x,s)}{s} = 0$ for all $x \in (0,1)$,
- (ii) for some $\ell \in [0,\infty)$, $|f(x,s) - f(x,t)| \le \ell|s-t|$ for all $x \in (0,1)$ and $s,t \in \mathbb{R}$.

For a function satisfying condition (F), let

$$\ell_f = \sup\left\{ \frac{|f(x,s) - f(x,t)|}{|s-t|} : 0 < x < 1 \text{ and } s \neq t \right\}. \tag{2.4}$$

The next result refers to Hadamard and w-Hadamard differentiability of a mapping. The definitions of these notions are recalled in Section 4.1.

Proposition 2.1. *Let condition (F) be satisfied by a function f.*

(i) Then the associated Nemytskii operator maps $L^2 = L^2(0,1)$ into itself and $\tilde{f} : L^2 \to L^2$ is uniformly Lipschitz continuous with

$$\|\tilde{f}(u) - \tilde{f}(v)\|_{L^2} \le \ell_f \|u - v\|_{L^2} \quad \textit{for all } u, v \in L^2 \tag{2.5}$$

Furthermore, $\tilde{f} : L^2 \to L^2$ is Gâteaux differentiable at 0 with derivative 0.

(ii) $\tilde{f} : L^2 \to L^2$ is Hadamard differentiable at 0 and $\tilde{f} : H_A \to L^2$ is w-Hadamard differentiable at 0 with derivative 0.

(iii) In addition to condition (F) suppose that there is a constant α with the property that, for all $\delta > 0$, there exist $x(\delta) \in (0,1)$ and $M(\delta)$ such that $|f(x,s) - \alpha s| \le M(\delta) + \delta|s|$ for all $(x,s) \in (0, x(\delta)) \times \mathbb{R}$. Then the mapping $\tilde{f} - \alpha I : H_A \to L^2$ is compact.

Proof. For parts (i) and (ii) see Lemma 3.1 in [31]. Part (iii) appears as Lemma 4.3 (b) in [34]. □

Remark 2.2. Since D_A is continuously embedded in L^2, $\tilde{f} : D_A \to L^2$ is also Hadamard differentiable at 0. However, it is important to emphasise that an assumption like (F) does not imply Fréchet differentiability of $\tilde{f}$ at 0, even when $f \in C^\infty([0,1] \times \mathbb{R})$. For example, it is shown in Example 3.1 in [31] that when $f(x,s) = h(s)$, where $h \in C^\infty(\mathbb{R})$ with $h(0) = h'(0) = 0$ and $\sup_{s\in\mathbb{R}} |h'(s)| < \infty$, condition (F) is satisfied but $\tilde{f} : D_A \to L^2$ is Fréchet differentiable at 0 if and only if $h \equiv 0$.

Fréchet differentiability of $\tilde{f}$ does hold provided that the function $f(x,s)$ decays in an appropriate way as $x \to 0$, as stipulated in the following condition.

(E) $f = \sum_{i=1}^{k} f_i$ where for each i, $f_i : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

  (i) $f_i(x,\cdot) \in C^1(\mathbb{R})$ and $f_i(x,0) = 0$ for all $x \in (0,1)$,

  (ii) there exist K_i and $\alpha_i > \frac{\sigma_i}{2}$ such that $|\partial_s f_i(x,s)| \le K_i x^{\alpha_i} |s|^{\sigma_i}$ for all $x \in (0,1)$ and $s \in \mathbb{R}$ where $0 < \sigma_1 < \cdots < \sigma_k$.

For a function f satisfying condition (E), let $\mathfrak{C}_f(s) = \sum_{i=1}^{k} s^{\sigma_i}$ for $s \ge 0$ and note that for $s,t \ge 0$,

$$\min\{1,t^{\sigma_k}\}\mathfrak{C}_f(s) \le \min\{t^{\sigma_1},t^{\sigma_k}\}\mathfrak{C}_f(s) \le \mathfrak{C}_f(ts) \le \max\{t^{\sigma_1},t^{\sigma_k}\}\mathfrak{C}_f(s) \le \max\{1,t^{\sigma_k}\}\mathfrak{C}_f(s). \quad (2.6)$$

It follows from (E) and property (H2) that for all $u \in H_A$ and $x \in (0,1)$,

$$\left|\frac{f_i(x,u(x))}{u(x)}\right| \le K_i x^{\alpha_i - \frac{\sigma_i}{2}} [x^{\frac{1}{2}} |u(x)|]^{\sigma_i} \le K_i C_1^{-\sigma_i/2} \|u\|_A^{\sigma_i} x^{\alpha_i - \frac{\sigma_i}{2}} \quad \text{if } u(x) \ne 0. \quad (2.7)$$

and

$$|f_i(x,u(x))u(x)| \le K_i C_1^{-\sigma_i/2} \|u\|_A^{\sigma_i} x^{\alpha_i - \frac{\sigma_i}{2}} u(x)^2. \quad (2.8)$$

Hence, setting $\nu = \min\{\alpha_i - \frac{\sigma_i}{2} : 1 \le i \le k\}$, there exists a constant C such that

$$|f(x,u(x))| \le C x^{\nu} \mathfrak{C}_f(\|u\|_A)|u(x)| \quad \text{for all } u \in H_A \text{ and } x \in (0,1). \quad (2.9)$$

Thus $\tilde{f}(u) \in L^2$ for all $u \in H_A$ and the next result shows that condition (E) ensures that $\tilde{f} : H_A \to L^2$ is both continuously Fréchet differentiable on H_A and compact.

Proposition 2.3. *Let f satisfy the condition (E). Then $\tilde{f} \in C^1(H_A, L^2)$ and there is a constant $C > 0$ such that $\|\tilde{f}'(u)\|_{B(H_A,L^2)} \le C\mathfrak{C}_f(\|u\|_A)$. Furthermore, the mapping $\tilde{f} : H_A \to L^2$ is compact.*

Proof. Continuous differentiability is established in Lemma 3.2 in [31]. Compactness is easily proved using the estimate (2.9) on the interval $(0,\eta)$ and property (H3) on $[\eta,1]$ for $\eta \in (0,1)$ in the same way as in Lemma 4.5 of [32] which deals with a similar situation in higher dimensions. □

Remark 2.4. For $u,v \in H_A$, $\|\tilde{f}(u) - \tilde{f}(v)\|_{L^2} \le C\mathfrak{C}_f(\|u\|_A + \|v\|_A)\|u-v\|_A$ and, in particular, $\|\tilde{f}(u)\|_{L^2} \le C\mathfrak{C}_f(\|u\|_A)\|u\|_A$ for all $u \in H_A$. It also follows from this lemma that $\tilde{f} \in C^1(D_A, L^2)$ and there is a constant C such that $\|\tilde{f}'(u)\|_{B(D_A,L^2)} \le C\mathfrak{C}_f(\|S_A u\|_{L^2})$ for all $u \in D_A$.

We now turn to the nonlinear term in equation (1.1) containing u'. Recalling that $D_A \subset C^1((0,1])$ a mapping N is defined on D_A by setting $N(u)(x) = n(x,u'(x)) = \tilde{n}(u')(x)$ where $n : (0,1) \times \mathbb{R} \to \mathbb{R}$. The following condition ensures that N maps D_A into L^2.

(M) $n = \sum_{i=1}^{j} n_i$ where for each i, $n_i : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

  (i) $n_i(x,\cdot) \in C^1(\mathbb{R})$ and $n_i(x,0) = 0$ for all $x \in (0,1)$,

  (ii) there exist $K_i > 0$ and $\beta_i > \frac{3\gamma_i}{2} + 1$ such that $|\partial_s n_i(x,s)| \le K_i x^{\beta_i} |s|^{\gamma_i}$ for all $x \in (0,1)$ and $s \in \mathbb{R}$ where $0 < \gamma_1 < \cdots < \gamma_j$.

For a function n satisfying condition (M), let $\mathfrak{D}_n(s) = \sum_{i=1}^{j} s^{\gamma_i}$ for $s \ge 0$.

It follows from (M) and property (P1) that for all $u \in D_A$ and $x \in (0,1)$,

$$|n_i(x,u'(x))| \le K_i x^{\beta_i - \frac{3\gamma_i}{2} - 1} |xu'(x)| |x^{\frac{3}{2}} u'(x)|^{\gamma_i} \le K_i C_1^{-\gamma_i/2} \|S_A u\|_{L^2}^{\gamma_i} x^{\beta - \frac{\gamma_i}{2} - 1} |xu'(x)| \tag{2.10}$$

and hence there is a constant K such that

$$|n_i(x,u'(x))u(x)| \le K \|S_A u\|_{L^2}^{\gamma_i} x^{\beta_i - \frac{3\gamma_i}{2} - 1} \{u(x)^2 + x^2 u'(x)^2\}. \tag{2.11}$$

Setting $\nu = \min\{\beta_i - \frac{3\gamma_i}{2} - 1 : 1 \le i \le j\}$ it follows from (2.10) that there exists a constant C such that

$$|n(x,u'(x))| \le C x^{\nu} \mathfrak{D}_n(\|S_A u\|_{L^2}) |xu'(x)| \quad \text{for all } u \in D_A \text{ and } x \in (0,1).$$

Hence $N(u) \in L^2$ for all $u \in D_A$ and the main properties of the mapping $N : D_A \to L^2$ are given in the next result.

Proposition 2.5. *When n satisfies the condition (M), $N \in C^1(D_A, L^2)$ with $N'(u)v = \partial_s n(\cdot, u')v'$ for all $u, v \in D_A$ and there is a constant $C > 0$ such that $\|N'(u)\|_{B(D_A,L^2)} \le C\mathfrak{D}_n(\|S_A u\|_{L^2})$. Furthermore, the mapping $N : D_A \to L^2$ is compact.*

Proof. See Lemma 3.4 in [31] and Lemma 4.3 (a) in [34]. □

2.3 Solutions of problem (1.1)(1.2) and bifurcation points

In dealing with problem (1.1)(1.2) from now on it will be assumed that the following condition is satisfied.

(S) The coefficients A and V satisfy conditions (A) and (V). The function n satisfies condition (M) and g can be written as $g_1 + g_2$ where g_1 satisfies condition (F) and g_2 satisfies condition (E).

Under the assumption (S) it follows from Propositions 2.1 to 2.5 that a continuous mapping $F : \mathbb{R} \times D_A \to L^2$ is defined by

$$F(\lambda, u) = S_A u + Vu + N(u) + \tilde{g}(u) - \lambda u, \tag{2.12}$$

provided that D_A is considered with a norm equivalent to the graph norm of S_A. By property (D2), all elements of D_A satisfy (1.2).

Definition 2.6. Henceforth, a solution of problem (1.1)(1.2) is defined to be an element $(\lambda, u) \in \mathbb{R} \times D_A$ such that $F(\lambda, u) = 0$, where F is given by (2.12).

Clearly $(\lambda, 0)$ is a solution for all $\lambda \in \mathbb{R}$ and

$$\mathcal{E} = \{(\lambda, u) \in \mathbb{R} \times D_A : F(\lambda, u) = 0 \text{ and } u \not\equiv 0\} \tag{2.13}$$

denotes the set of all non-trivial solutions of problem (1.1)(1.2). We recall that for $u \in D_A$, $u \in C^1((0,1])$ and, setting $v = Au'$, we also have that v is absolutely continuous on $[0,1]$, as noted at the beginning of Section 2 in [31]. If (λ, u) is a solution of (1.1)(1.2), $v'(x) = f(\lambda, x, u(x), v(x))$ for almost all $x \in (0,1)$ where $f(\lambda, x, p, q) = [V(x) - \lambda]p + n(x, q/A(x)) + g(x, p)$ for $x \in (0,1]$ and $p, q \in \mathbb{R}$. Thus, when A is not differentiable on $(0,1)$, equation (1.1) is satisfied in the sense of a quasi-differential equation, that is

$$(u(x), v(x))' = (v(x)/A(x), f(\lambda, x, u(x), v(x))) \quad \text{for almost all } x \in (0,1). \tag{2.14}$$

(See III.10.1 in [10] and Chapter 2 of [25], for example.) For any given $\eta \in (0,1]$ and $(p_0, q_0) \in \mathbb{R}$, assumption (S) ensures that there exist $L > 0$ and $\delta \in (0, \eta)$ such that

$$\frac{|q_1 - q_2|}{A(x)} \leq L|q_1 - q_2| \text{ and } |f(\lambda, x, p_1, q_1) - f(\lambda, x, p_2, q_2)| \leq L\|(p_1, q_1) - (p_2, q_2)\|$$

for $x \in [\eta - \delta, 1]$ and $\|(p_i, q_i) - (p_0, q_0)\| < \delta$ for $i = 1$ and 2. Hence for any $x_0 > 0$, local existence and uniqueness of the solution of the initial value problem $u(x_0) = p_0$, $v(x_0) = q_0$ for (2.14) hold by standard arguments applied to the equivalent integral equation. (See Chapter 2 of [6], for example.) In particular, if (λ, u) is a solution of (1.1)(1.2) and $u(x_0) = u'(x_0) = 0$ for some $x_0 \in (0,1]$, then $u(x) = 0$ for all $x \in (0,1]$ and it follows that, if $(\lambda, u) \in \mathcal{E}$, then u has a finite number of zeros in any compact subinterval of $(0,1]$ and that they are all simple zeros.

Having clarified what is meant by a solution of problem (1.1)(1.2), we now turn to the notion of bifurcation point.

Definition 2.7. A real number μ is called a bifurcation point for problem (1.1)(1.2) if and only if $(\mu, 0) \in \overline{\mathcal{E}}$ where $\overline{\mathcal{E}}$ denotes the closure of $\mathcal{E}$ in the space $\mathbb{R} \times D_A$ and D_A is considered with the norm $u \mapsto \|S_A u\|_{L^2}$.

To explore the content of this definition, consider a sequence $\{(\lambda_n, u_n)\}$ in $\mathcal{E}$ such that $\lambda_n \to \mu$ and $\|S_A u_n\|_{L^2} \to 0$ as $n \to \infty$. By properties (P1) and (P2) in Section 2.1 this implies that $\|x^{1/2} u_n\|_{L^\infty} \to 0$ and $\|x^{3/2} u_n'\|_{L^\infty} \to 0$ as $n \to \infty$. Hence, $\{u_n\}$ and $\{u_n'\}$ converge uniformly to zero on all compact subintervals of $(0,1]$, but not necessarily on $(0,1]$. However, by (D2) we do have that $\|u_n\|_{L^2} + \|u_n\|_A \to 0$ as $n \to \infty$. The results in Section 5 provide sufficient conditions for a number μ to be a bifurcation point and under their hypotheses the functions u_n have only a finite number of zeros in $(0,1]$. It follows from this and Proposition 3.5(ii) that there exists $n_0 \in \mathbb{N}$ such that $\lim_{x \to 0} u_n(x) = \pm\infty$ for all $n \geq n_0$ if $\mu \in (V_0 + J_s(g_1), \frac{a}{4} + V_0)$. Further details of situations where this phenomenon occurs are given in Section 5.

The assumption (S) and Propositions 2.1 to 2.5 also imply that for all $\lambda \in \mathbb{R}$ the mapping $F(\lambda, \cdot) : D_A \to L^2$ defined by (2.12) is Hadamard differentiable at 0 with $D_u F(\lambda, 0) = S_A + V \in B(D_A, L^2)$. Hence we expect that bifurcation theory for problem (1.1)(1.2) will require some information about the spectrum of the operator $S_A + V$.

2.4 Spectral theory of the linearization

Conditions (A) and (V) are supposed to be satisfied throughout this subsection. Here we summarize the main features of the self-adjoint operator $S = S_A + V : D(S) = D(S_A) \subset L^2(0,1) \to L^2(0,1)$ that are established in [31] and [33]. More precisely, properties (S1) and (S3) are part of Theorem 4.1 in [33] and (S5) is justified by the discussion preceding Theorem 6.11 in [33]. Property (S2) follows from the comments after Definition 2.6 about solutions of (2.14) with n and g equal to zero. In the same way, properties (S4) and (S6) are special cases of Lemma 3.1 and Proposition 3.5, although similar conclusions also appear in [31,33]. Recall that

$$\sigma(S) = \{\lambda \in \mathbb{R} : S - \lambda I : D(S) \to L^2(0,1) \text{ is not an isomorphism}\}$$

and

$$\sigma_e(S) = \{\lambda \in \mathbb{R} : S - \lambda I : D(S) \to L^2(0,1) \text{ is not a Fredholm operator}\}.$$

Let

$$m = \inf \sigma(S) \quad \text{and} \quad m_e = \inf \sigma_e(s).$$

(S1) $\sigma_e(S) = [\frac{a}{4} + V_0, \infty)$ and $\frac{C_1}{4} + \operatorname{ess\,inf} V \leq m \leq m_e = \frac{a}{4} + V_0$.

(S2) All eigenvalues of S are simple and eigenfunctions have only simple zeros in $(0, 1]$.

(S3) If $m < m_e$, it is an eigenvalue having an eigenfunction ϕ with $\phi(x) > 0$ for $0 < x < 1$.

(S4) If u is an eigenfunction for an eigenvalue in the interval $(-\infty, m_e)$, then u has only a finite number of zeros in $(0, 1]$.

(S5) If the eigenvalues are numbered in increasing order with $m = \mu_1 < \mu_2$ etc. and if $\mu_k < m_e$, then an eigenfunction for μ_k has exactly k zeros in $(0, 1]$.

(S6) If μ is an eigenvalue in $(-\infty, V_0)$ its eigenfunction is bounded on $(0, 1)$ whereas, if $V_0 < \mu < m_e$ it has an eigenfunction ϕ with $\phi(x) \to \infty$ as $x \to 0$.

There are cases where S has no eigenvalues, for example, when $A(x) = x^2$ and $V(x) \equiv 0$. More generally, if A and V have the additional properties that A and $V \in C^1((0, 1])$ with

$$\lim_{x\to 0} A'(x)/x = 2a, \quad \lim_{x\to 0} xV'(x) = 0 \quad \text{and} \quad A(x)/x^2 \text{ and } V \text{ non-decreasing on } (0, 1),$$

then S has no eigenvalues. See Corollary 3.9.

The following special cases, which are treated in Section 4.2 of [33], together with the usual comparison principle for self-adjoint operators, provide examples of situations where S does have eigenvalues in $(-\infty, m_e)$.

Example 2.8. Let $A(x) = x^2$ and for some $\tau \in (0, 1)$ and $L > 0$, let

$$V(x) = 0 \quad \text{for } 0 < x < \tau \quad \text{and} \quad V(x) = -L \quad \text{for } \tau < x < 1.$$

Then $\sigma_e(S) = [\frac{1}{4}, \infty)$ and S has no eigenvalues in this interval.

If $\sqrt{L} \ln \frac{1}{\tau} \leq \frac{\pi}{2}$, S has no eigenvalues.

If $(n - \frac{1}{2})\pi < \sqrt{L} \ln \frac{1}{\tau} \leq (n + \frac{1}{2})\pi$ for some positive integer n, then S has exactly n eigenvalues in $(-\infty, \frac{1}{4})$.

The explicit form of the eigenfunctions and estimates for the eigenvalues are also given in [33].

Example 2.9. For $0 < x < 1$, let $A(x) = x^2$ and $V(x) = -(\frac{n\pi s}{2})^2 x^s$ where $s \in (0, \infty)$ and n is a positive integer.

Then $\sigma_e(S) = [\frac{1}{4}, \infty)$ and S has at least n eigenvalues in $(-\infty, \frac{1}{4})$. In fact, $\mu_n = \frac{1}{4}(1 - \frac{s^2}{4})$ is the n-th eigenvalue and $\phi(x) = x^{-\frac{1}{2}(1+\frac{s}{2})} \sin(n\pi x^{\frac{s}{2}})$ is an eigenfunction for μ_n.

Example 2.10. For $\tau \in (0, 1)$, let

$$A(x) = x^2 \quad \text{for } 0 \leq x \leq \tau \quad \text{and} \quad A(x) = \tau^2 \quad \text{for } \tau < x \leq 1.$$

Then $\sigma_e(S_A) = [\frac{1}{4}, \infty)$. If $\tau \geq \frac{2}{2+\pi}$, S_A has no eigenvalues whereas if $\frac{2}{2+(4n+1)\pi} \leq \tau < \frac{2}{2+(4n-3)\pi}$ for a positive integer n, then S_A has exactly n eigenvalues in $(-\infty, \frac{1}{4})$.

The explicit form of the eigenfunctions and estimates for the eigenvalues are also given in [33].

The operator $S = S_A + V$ is always bounded below and for some proofs it is useful to make a shift so that it becomes positive. For any $c > -m$, the operator $S_c \equiv S + cI$ with domain $D(S_c) = D(S) = D(S_A)$ has many properties similar to those of S_A. It is positive definite and self-adjoint. The graph norms of S and S_c are equivalent to the norm defined by $\|S_A u\|_{L^2}$ on $D(S_A)$. Furthermore, the domain of its positive, self-adjoint square root, $S_c^{\frac{1}{2}}$, is H_A and $\|\cdot\|_A$ is equivalent to the graph norm of $S_c^{\frac{1}{2}}$ on H_A. See Section 4.3 of [33] for more details.

The proof of Theorem 5.5 uses some facts about the spectrum of the self-adjoint operator $W \in B(L^2, L^2)$ defined by $W = I - (\lambda + c - \alpha)S_c^{-1}$ where $\alpha \geq 0$, $c > \max\{0, \alpha - \operatorname{ess\,inf} V\}$ and $\alpha - c < \lambda < m_e$. Note that $c > \alpha - m$ by property (S1) so $\alpha - c < m \leq m_e$ and $c > -m$. Hence $S_c : D_A \to L^2$ is an isomorphism and $S_c^{-1} \in B(L^2, L^2)$ is injective but not surjective. Hence $1 \in \sigma_e(W)$ and it is easy to check that

$$\sigma(W) = \{1\} \cup \left\{1 - \frac{\lambda + c - \alpha}{\mu + c} : \mu \in \sigma(S)\right\} \quad \text{and} \quad \sigma_e(W) = \{1\} \cup \left\{1 - \frac{\lambda + c - \alpha}{\mu + c} : \mu \in \sigma_e(S)\right\}.$$

Since $\lambda + c - \alpha > 0$, it follows that $1 - \frac{\lambda+c-\alpha}{\mu+c}$ is an increasing function of μ and hence

$$\inf \sigma(W) = \frac{m + \alpha - \lambda}{m + c} \quad \text{and} \quad 0 < \inf \sigma_e(W) = \frac{m_e + \alpha - \lambda}{m_e + c} < 1. \tag{2.15}$$

3 Qualitative properties of solutions

As noted in Section 2.3, solutions of (1.1)(1.2) have only a finite number of zeros in any compact subinterval in $(0, 1]$ and all zeros are simple. Most of the results in this section concern the behaviour of solutions as x approaches the singular point $x = 0$. Some integral identities also lead to conclusions about the non-existence of non-trivial solutions of (1.1)(1.2) and the absence of eigenvalues of the operator $S_A + V$. Earlier work on the properties of solutions for a related problem can be found in the paper [5] by Caldiroli and Musina which deals with equations of the form $-\{\omega(x)u'(x)\}' = f(u(x))$ under a variety of assumptions about the decay of $\omega(x)$ as $x \to 0$.

3.1 Nodal properties of solutions

The first results in this part provide conditions under which solutions of (1.1)(1.2) have a finite number of zeros in $(0, 1]$. For a function $u \in C((0, 1])$ having only a finite number of zeros in $(0, 1]$ the number of zeros in $(0, 1]$ will be denoted by $\sharp(u)$. Under the hypotheses of Corollary 3.3 this number is locally constant on $\mathcal{E}$.

Lemma 3.1. *Let condition (S) be satisfied.*

(i) Given $\delta > 0$ and $C > 0$ there exists $\eta \in (0, 1)$ such that $u(x) \neq 0$ for $x \in (0, \eta]$ whenever $(\lambda, u) \in \mathcal{E}$ with $\lambda \leq m_e - \ell_{g_1} - \delta$ and $\|S_A u\|_{L^2} \leq C$.

(ii) If there exists $z \in (0, 1)$ such that either $g(x, s)s \geq 0$ for all $(x, s) \in (0, z) \times \mathbb{R}$, or $g_1(x, s)s \geq 0$ for all $(x, s) \in (0, z) \times \mathbb{R}$, then the conclusion holds for $\lambda \leq m_e - \delta$ and $\|S_A u\|_{L^2} \leq C$.

Proof. (i) Fix δ and C as in the statement of the lemma. By (F), (2.9) and (2.11), there exist a constant $D > 0$ and an exponent $\nu > 0$ for which the following inequalities hold for all

$x \in (0,1)$ and all $u \in D_A$ with $\|S_A u\|_{L^2} \le C$.

$$\widetilde{g_1}(u)(x)u(x) \ge -\ell_{g_1} u(x)^2, \tag{3.1}$$
$$\widetilde{g_2}(u)(x)u(x) \ge -Dx^{\nu} u(x)^2, \tag{3.2}$$
$$N(u)(x)u(x) \ge -Dx^{\nu}\{u(x)^2 + x^2 u'(x)^2\}. \tag{3.3}$$

Set $\varepsilon = \min\left\{\frac{a}{2}, \frac{\delta}{4}\right\}$ and then choose $\eta \in (0,1)$ such that, for $0 < x \le \eta$,

$$A(x) \ge (a-\varepsilon)x^2, \qquad V(x) \ge V_0 - \varepsilon \quad \text{and} \quad Dx^{\nu} \le \varepsilon.$$

Consider $(\lambda, u) \in \mathcal{E}$ with $\lambda \le m_e - \ell_{g_1} - \delta$ and $\|S_A u\|_{L^2} \le C$. If $u(z) = 0$ for some $z \in (0, \eta]$, then using (2.1) and (2.2) we have

$$0 = \int_0^z A(x)u'(x)^2 + V(x)u(x)^2 + N(u)(x)u(x) + \widetilde{g}(u)(x)u(x) - \lambda u(x)^2 dx \tag{3.4}$$

$$\begin{aligned} \ge \int_0^z &(a-\varepsilon)x^2 u'(x)^2 + [V_0 - \varepsilon]u(x)^2 \\ &- \varepsilon\{u(x)^2 + x^2 u'(x)^2\} - \ell_{g_1} u(x)^2 - \varepsilon u(x)^2 - \lambda u(x)^2 dx \end{aligned} \tag{3.5}$$

$$\ge \int_0^z \frac{a-2\varepsilon}{4} u(x)^2 + u(x)^2\left\{V_0 - 3\varepsilon - \ell_{g_1} - \lambda\right\} dx = \int_0^z u(x)^2 \left\{m_e - \ell_{g_1} - \lambda - \frac{7}{2}\varepsilon\right\} dx \tag{3.6}$$

$$\ge \int_0^z u(x)^2 \left\{\delta - \frac{7}{2}\varepsilon\right\} dx \ge \frac{\varepsilon}{2}\int_0^z u(x)^2 dx > 0. \tag{3.7}$$

From this contradiction we may conclude that u has no zeros in the interval $(0, z]$.

(ii) In this case the term $\ell_{g_1} u(x)^2$ in (3.5) and (3.6) and be dropped and (3.7) holds for $\lambda \le m_e - \delta$. □

Lemma 3.2. *For $\eta \in (0,1)$, $C^1_\eta \equiv \{u \in C^1([\eta,1]) : u(1) = 0\}$ with norm $\|u\|_\eta = \max\{|u'(x)| : \eta \le x \le 1\}$ is a Banach space.*

(i) Setting $P_\eta u(x) = u(x)$ for $u \in D_A$ and $x \in [\eta, 1]$, $P_\eta \in B(D_A, C^1_\eta)$ is compact.

(ii) If $u \in C^1_\eta$ has exactly n zeros in $(\eta, 1]$ all of which are simple and $u(\eta) \ne 0$, there exists $\delta > 0$ such that for all $v \in C^1_\eta$ with $\|u - v\|_\eta < \delta$, v has exactly n zeros in $(\eta, 1]$ all of which are simple and $v(\eta) \ne 0$.

Proof. (i) By the definition of D_A, $P_\eta(D_A) \subset C^1_\eta$. Let $\{u_n\}$ be a bounded sequence in D_A and let $v_n = (P_\eta u_n)'$. By the Ascoli–Arzelà Theorem, it suffices to show that the sequence $\{v_n\}$ is uniformly bounded and equi-continuous on $[\eta, 1]$. By property (D3) of S_A,

$$v_n(x) = -\frac{1}{A(x)}\int_0^x w_n(y)\, dy \quad \text{for } \eta \le x \le 1$$

where $w_n = S_A u_n$ and $\{w_n\}$ is a bounded sequence in $L^2(0,1)$. Let $M = \sup \|w_n\|_{L^2}$. Then, since $A(x) \ge C_1 x^2$ on $[0,1]$,

$$|v_n(x)| \le \frac{1}{C_1 x^2} x^{\frac{1}{2}} \left\{\int_0^x w_n(y)^2 dy\right\}^{\frac{1}{2}} \le \frac{M}{C_1 \eta^{\frac{3}{2}}} \quad \text{for } \eta \le x \le 1$$

and for $\eta \le x \le z \le 1$,

$$|v_n(x) - v_n(z)| \le \frac{1}{A(x)} \int_x^z |w_n(y)|\, dy + \left| \frac{1}{A(x)} - \frac{1}{A(z)} \right| \int_0^x |w_n(y)|\, dy$$
$$\le \frac{M(z-x)^{\frac{1}{2}}}{C_1 \eta^2} + \frac{M|A(z) - A(x)|}{C_1^2 \eta^{\frac{7}{2}}}.$$

It follows that $\{v_n\}$ has a subsequence converging in $C([\eta, 1])$ and consequently that $P_\eta : D_A \to C_\eta^1$ is a compact operator.

(ii) This is an easy exercise. The details are given in Lemma 3.1 of [36], for example. □

Corollary 3.3. *Suppose that condition (S) is satisfied and that $(\lambda, u) \in \mathcal{E}$ has the property that there exist $\delta > 0$ and $\eta \in (0,1)$ such that, for all $(\xi, v) \in \mathcal{E}$ with $|\xi - \lambda| + \|S_A(u - v)\|_{L^2} < \delta$, v has no zeros in the interval $(0, \eta]$. Then there exists $\varepsilon > 0$ such that $\sharp(v) = \sharp(u)$ for all $(\xi, v) \in \mathcal{E}$ with $|\xi - \lambda| + \|S_A(v - u)\|_{L^2} < \varepsilon$.*

Proof. By Lemma 3.2 (i), $P_\eta \in B(D_A, C_\eta^1)$ and so the conclusion follows from Lemma 3.2 (ii). □

For $z \in (0,1)$ let

$$E(z) = \left\{ (x,s) \in (0,1) \times \mathbb{R} : 0 < x < z \text{ and } |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\}$$

and

$$D(z) = \left\{ (x,s) \in (0,1) \times \mathbb{R} : 0 < x < z \text{ and } z^{-\frac{1}{2}} \ln \frac{1}{z} < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\}.$$

Then, for a Carathéodory function $g : (0,1) \times \mathbb{R} \to \mathbb{R}$, let

$$J_i(g) = \lim_{z \to 0} \operatorname*{ess\,inf}_{0<x<z} \inf \left\{ \frac{g(x,s)}{s} : 0 < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\} \tag{3.8}$$

$$J_s(g) = \lim_{z \to 0} \operatorname*{ess\,sup}_{0<x<z} \sup \left\{ \frac{g(x,s)}{s} : 0 < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\} \tag{3.9}$$

$$I_i(g) = \lim_{z \to 0} \operatorname*{ess\,inf}_{0<x<z} \inf \left\{ \frac{g(x,s)}{s} : z^{-\frac{1}{2}} \ln \frac{1}{z} < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\} \tag{3.10}$$

$$I_s(g) = \lim_{z \to 0} \operatorname*{ess\,sup}_{0<x<z} \sup \left\{ \frac{g(x,s)}{s} : z^{-\frac{1}{2}} \ln \frac{1}{z} < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\}. \tag{3.11}$$

When dealing with solutions of (1.1)(1.2) these quantities lead to the following properties which will be exploited in Proposition 3.5.

Lemma 3.4. *Let condition (S) be satisfied.*

(i) Then $-\ell_{g_1} \le J_i(g_1) = J_i(g) \le 0 \le J_s(g) = J_s(g_1) \le \ell_{g_1}$ and $J_i(g) \le I_i(g_1) = I_i(g) \le I_s(g_1) = I_s(g) \le J_s(g)$. If g_1 also satisfies the compactness condition in Proposition 2.1 then $I_i(g) = I_s(g) = \alpha$.

(ii) If $(\lambda, u) \in \mathcal{E}$, there exists $z \in (0,1)$ such that $(x, u(x)) \in E(z)$ for all $x \in (0, z)$. Setting

$$B_u(x) = \frac{g(x, u(x))}{u(x)} \text{ if } u(x) \neq 0 \text{ and } B_u(x) = 0 \text{ if } u(x) = 0, \tag{3.12}$$

$J_i(g_1) \le \liminf_{x \to 0} B_u(x) \le \limsup_{x \to 0} B_u(x) \le J_s(g_1)$. If either $u(x) \to \infty$ or $u(x) \to -\infty$ as $x \to 0$, then $I_i(g_1) \le \liminf_{x \to 0} B_u(x) \le \limsup_{x \to 0} B_u(x) \le I_s(g_1)$.

Here $\limsup_{x\to 0} B_u(x) = \lim_{x\to 0} \operatorname{ess\,sup}_{0<y<x} B_u(y)$ and similarly for the lim inf.

Proof. (i) Since $g(x,s)/s \to 0$ as $s \to 0$ for all $x \in (0,1)$, $J_s(g) \geq 0 \geq J_i(g)$. Furthermore, $-\ell_{g_1} \leq g_1(x,s)/s \leq \ell_{g_1}$ for $x \in (0,1)$ and $s \neq 0$ so $-\ell_{g_1} \leq I_i(g_1) \leq I_s(g_1) \leq \ell_{g_1}$.

Let f be a function satisfying condition (E)(ii) for $\alpha > \sigma/2 > 0$. Then, for $0 < x < 1$ and $0 < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x}$,

$$\left|\frac{f(x,s)}{s}\right| \leq Kx^{\alpha}|s|^{\sigma} \leq Kx^{\alpha-\frac{\sigma}{2}} \left(\ln \frac{1}{x}\right)^{\sigma}$$

and hence

$$\lim_{z\to 0} \operatorname*{ess\,sup}_{0<x<z} \sup \left\{ \left|\frac{f(x,s)}{s}\right| : 0 < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\} = 0.$$

Since $g - g_1$ is a finite sum of functions of this type and $D(z) \subset E(z)$ the conclusions follow.

(ii) By property (P2) there exists a constant K such that $x^{\frac{1}{2}}|u(x)| \leq K$ for $0 < x < 1$. Hence there exists $z_0 \in (0,1)$ such that $(x, u(x)) \in E(z)$ for $0 < x < z < z_0$. Furthermore, if $|u(x)| \to \infty$ as $x \to 0$, for all $z \in (0, z_0)$, there exists $\delta_z < z$ such that $(x, u(x)) \in D(z)$ for $0 < x < \delta_z$. The conclusions in part (ii) are easily deduced from these observations. □

We can now establish a number of results concerning the behaviour of a solution of (1.1)(1.2) as $x \to 0$. They generalise and improve similar conclusions in Theorem 5.1 of [31].

Proposition 3.5. *Let condition (S) be satisfied and $n \equiv 0$.*

(i) If $\lambda < m_e + J_i(g_1)$ and $(\lambda, u) \in \mathcal{E}$, there exists $\eta \in (0,1)$ such that u has no zeros in the interval $(0,\eta]$.

(ii) If $\lambda > V_0 + J_s(g_1)$ and $(\lambda, u) \in \mathcal{E}$, then either u has a sequence of zeros converging to 0 or $\lim_{x\to 0} u(x) = \pm\infty$.

(iii) If $\lambda > \max\{V_0 + J_s(g_1), m_e + I_s(g_1)\}$ and $(\lambda, u) \in \mathcal{E}$, then u has a sequence of zeros converging to 0.

(iv) If $\lambda < V_0 + I_i(g_1)$ and $(\lambda, u) \in \mathcal{E}$, then $u \in L^{\infty}(0,1)$.

Remark 3.6. Since $m_e + J_s(g_1) \geq \max\{V_0 + J_s(g_1), m_e + I_s(g_1)\}$ it follows from part (iii) that u has a sequence of zeros converging to 0 if $\lambda > m_e + J_s(g_1)$ and $(\lambda, u) \in \mathcal{E}$.

Taking $g \equiv 0$, Proposition 3.5 gives the following information about an eigenfunction, ϕ, of $S_A + V$ associated with an eigenvalue λ. For $\lambda < m_e$ it has a finite number of zeros whereas for $\lambda > m_e$ it has infinitely many zeros. If $\lambda < V_0$, ϕ is bounded on $(0,1)$ and if $V_0 < \lambda < m_e$, $\phi(x) \to \pm\infty$ as $x \to 0$.

Proof. Recall that $m_e = \frac{a}{4} + V_0$ and, for $(\lambda, u) \in \mathcal{E}$, set $B(\lambda, u)(x) = \lambda - V(x) - B_u(x)$.

Part (i) This can be proved in the same way as Lemma 3.1 since, given any $\varepsilon > 0$, there exists $\eta \in (0,1)$ such that, for $0 < x < \eta$, $A(x) \geq (a - \varepsilon)x^2$, $V(x) \geq V_0 - \varepsilon$ and $g(x, u(x))u(x) = B_u(x)u(x)^2 \geq \{J_i(g_1) - \varepsilon\}u(x)^2$. It suffices to repeat the estimates (3.4) to (3.7) with minor adjustments.

Part (ii) Consider $(\lambda, u) \in \mathcal{E}$ and suppose that u has only a finite number of zeros in $(0,1)$. Since $u \in C((0,1])$ there exists $\eta > 0$ such that either $u > 0$ on $(0,\eta]$ or $u < 0$ on $(0,\eta]$.

Suppose that $u > 0$ on $(0,\eta]$. By property (D3) of D_A,

$$A(x)u'(x) = -\int_0^x B(\lambda, u)(y)u(y)\,dy \quad \text{for } 0 < x \leq \eta.$$

Define $\varepsilon > 0$ by $3\varepsilon = \lambda - V_0 - J_s(g_1)$. By condition (V) and Lemma 3.4, η can be chosen so that $V(x) \leq V_0 + \varepsilon$ and $B_u(x) \leq J_s(g_1) + \varepsilon$ for $0 < x < \eta$. Then for $0 < y < \eta$, $B(\lambda, u)(y) \geq \lambda - V_0 - J(g_1) - 2\varepsilon = \varepsilon$ and so

$$A(x)u'(x) \leq -\int_0^x \varepsilon u(y)\,dy < 0 \quad \text{for } 0 < x < \eta,$$

from which it follows that u is decreasing on $(0, \eta)$ and consequently, $A(x)u'(x) \leq -\varepsilon u(\eta)x$ for $0 < x \leq \eta$. By condition (A),

$$u(\eta) - \lim_{x\to 0} u(x) \leq -\varepsilon u(\eta) \int_0^\eta \frac{y}{A(y)}\,dy = -\infty,$$

proving that $u(x) \to \infty$ as $x \to 0$.

The case where $u < 0$ on $(0, \eta]$ can be dealt with in the same way.

Part (iii) Choose $\gamma > \frac{1}{4}$ such that $\lambda > \gamma a + V_0 + I_s(g_1)$ and then define $\varepsilon > 0$ by $(3+\gamma)\varepsilon = \lambda - \gamma a - V_0 - I_s(g_1)$.

There exists $\eta \in (0,1)$ such that, for $0 < x < \eta$, $A(x) \leq (a+\varepsilon)x^2$ and $V(x) \leq V_0 + \varepsilon$.

Suppose that u has only a finite number of zeros. By part (ii), $u(x) \to \pm\infty$ as $x \to 0$ and so, referring to Lemma 3.4 and reducing η, we may suppose that $B_u(x) \leq I_s(g_1) + \varepsilon$ for $0 < x < \eta$. Then, for $0 < x < \eta$,

$$B(\lambda, u)(x) \geq \lambda - V_0 - I_s(g_1) - 2\varepsilon = \gamma a + (3+\gamma)\varepsilon - 2\varepsilon = \gamma(a+\varepsilon) + \varepsilon.$$

The function w defined by $w(x) = x^{-1/2} \sin\left(\sqrt{\gamma - \frac{1}{4}} \ln x\right)$ for $x > 0$ satisfies the equation $-(x^2 w'(x))' = \gamma w(x)$, which can be written as

$$-(C(x)w'(x))' = Dw(x) \quad \text{where} \quad C(x) = (a+\varepsilon)x^2 \quad \text{and} \quad D = \gamma(a+\varepsilon).$$

On the interval $(0, \eta)$, $-(Au')' = B(\lambda, u)u$, $A \leq C$, $B(\lambda, u) > D$ and w has an infinite sequence of zeros converging to 0. Hence, by the Sturm comparison theorem, u also has a sequence of zeros in $(0, \eta)$ converging to 0. (For the type of coefficients appearing here, the comparison can be established using Picone's identity by the arguments in 10.31 of [19].) This proves part (iii).

Part (iv) Let $\varepsilon > 0$ be defined by $3\varepsilon = V_0 + I_i(g_1) - \lambda$. There exist $\eta \in (0,1)$ and $S > 0$ such that, for $0 < x < \eta$, $V(x) \geq V_0 - \varepsilon$ and, if $|u(x)| > S$, $B_u(x) \geq I_i(g_1) - \varepsilon$ by Lemma 3.4. Then

$$B(\lambda, u)(x) \leq \lambda - V_0 - I_i(g_1) + 2\varepsilon = -\varepsilon \quad \text{on } \omega \equiv \{x \in (0, \eta) : |u(x)| \geq S\}.$$

Let $T = \max\{S, \max_{\eta \leq x \leq 1} |u(x)|\}$ and $\omega^+ = \{x \in (0,1) : u(x) > T\}$. Then $\omega^+ \subset \omega$ and $(u-T)^+ \in H_A$. Hence $\operatorname{supp}(u-T)^+ \subset \omega^+$ and by (2.3),

$$\begin{aligned} 0 &\leq \int_{\omega^+} A(u')^2 dx = \int_0^1 Au'[(u-T)^+]' dx = \int_0^1 S_A u(u-T)^+ dx = \int_0^1 B(\lambda, u)u(u-T)^+ dx \\ &= \int_{\omega^+} B(\lambda, u)u(u-T)\,dx. \end{aligned}$$

But $B(\lambda, u) \leq -\varepsilon$ and $u(u-T)^+ > 0$ on the open set ω^+ which must be empty, since otherwise the final integral would be negative. This proves that $u(x) \leq T$ on $(0, 1]$.

A similar argument using $\omega^- = \{x \in (0,1) : u(x) < -T\}$ and $(u+T)^-$ shows that $u(x) \geq -T$ on $(0, 1]$, completing the proof of part (iv). □

3.2 Integral identities and their consequences

The following identities involving solutions of (1.1)(1.2) lead to new information about their behaviour as $x \to 0$ and also to some conditions under which non-trivial solutions do not exist.

Proposition 3.7. *In addition to the assumption (S) with $n \equiv 0$, suppose that the following condition is satisfied.*

(T1) There exists $\delta \in (0,1]$ such that A and $V \in C^1((0,\delta])$ with $\lim_{x\to 0} \frac{A'(x)}{2x} = a$ and $\lim_{x\to 0} xV'(x) = 0$. Also g_1 and $f_i \in C^1((0,\delta) \times \mathbb{R})$ where $g_2 = \sum_{i=1}^k f_i$ with

$$|x\partial_x g_1(x,s)| \le K|s| \quad \text{and} \quad |x\partial_x f_i(x,s)| \le K_i x^{\sigma_i/2}|s|^{1+\sigma_i} \quad \text{for } (x,s) \in (0,\delta) \times \mathbb{R},$$

where σ_i is given by assumption (E).

Set

$$\Phi(x,s) = \int_0^s g(x,t)\,dt \quad \text{for } (x,s) \in (0,1) \times \mathbb{R}.$$

Suppose that $u(z) = 0$ for some $z \in (0,\delta]$. Then

$$\int_0^z [A - xA'](u')^2 + (\lambda - V - xV')u^2 - 2\{\Phi(x,u) + x\partial_x\Phi(x,u)\}\,dx = zA(z)u'(z)^2, \tag{3.13}$$

$$\int_0^z A(u')^2 + (V-\lambda)u^2 + g(x,u)u\,dx = 0, \tag{3.14}$$

$$\int_0^z [2A - xA'](u')^2 - xV'u^2 + g(x,u)u - 2\{\Phi(x,u) + x\partial_x\Phi(x,u)\}\,dx = zA(z)u'(z)^2. \tag{3.15}$$

Proof. This result is a slight generalization of Lemma 5.2 in [31] and Theorem 7.7 in [33]. The proof requires only minor modifications to the arguments used in these references. □

From the identity (3.13) we can derive an variant of part (i) of Proposition 3.5.

Corollary 3.8. *Under the assumptions (S) with $n \equiv 0$ and (T1), if $(\lambda, u) \in \mathcal{E}$ and $\lambda < m_e + 2\liminf_{x\to 0} \inf_{s\neq 0} \frac{\Phi(x,s)+x\partial_x\Phi(x,s)}{s^2}$, there exists $\eta \in (0,1)$ such that $u(x) \neq 0$ for $x \in (0,\eta]$.*

Proof. By the assumptions about the coefficients A and V, $[A(x) - xA'(x)]/x^2 \to -a$ and $V(x) + xV'(x) \to V_0$ as $x \to 0$. For (λ, u) as in the statement, first choose $\varepsilon > 0$ such that

$$\varepsilon < a \quad \text{and} \quad \lambda + 4\varepsilon < m_e + 2\liminf_{x\to 0} \inf_{s\neq 0} \frac{\Phi(x,s) + x\partial_x\Phi(x,s)}{s^2},$$

and then, for δ as in (T1), choose $\eta \in (0,\delta)$ such that for $0 < x \le \eta$,

$$\frac{A(x) - xA'(x)}{x^2} < -a + \varepsilon, V(x) + xV'(x) > V_0 - \varepsilon$$

and

$$\inf_{s\neq 0} \frac{\Phi(x,s) + x\partial_x\Phi(x,s)}{s^2} > \liminf_{x\to 0} \inf_{s\neq 0} \frac{\Phi(x,s) + x\partial_x\Phi(x,s)}{s^2} - \varepsilon.$$

Suppose now that $u(z) = 0$ for some $z \in (0,\eta]$. By (2.2),

$$\int_0^z [A(x) - xA'(x)]u'(x)^2 dx < -(a-\varepsilon)\int_0^z x^2 u'(x)^2 dx \le -\frac{a-\varepsilon}{4}\int_0^z u(x)^2 dx$$

and hence (3.13) yields

$$zA(z)u'(z)^2 < \int_0^z \left\{ \lambda - \frac{a-\varepsilon}{4} - V_0 + \varepsilon - 2 \liminf_{x\to 0} \inf_{s\neq 0} \frac{\Phi(x,s) + x\partial_x \Phi(x,s)}{s^2} + 2\varepsilon \right\} u(x)^2 dx$$
$$\leq -\frac{3}{4}\varepsilon \int_0^z u(x)^2 dx < 0.$$

Since $A(z) > 0$ and $u'(z) \neq 0$, this is false and so $u(z) \neq 0$ for all $z \in (0, \eta]$. □

Unlike the other results concerning the behaviour of solutions as $x \to 0$ the identity (3.15) yields information without placing any restriction on λ.

Corollary 3.9. *In addition to the assumptions (S) with $n \equiv 0$ and (T1), suppose that*

(T2) there exists $\eta \in (0, \delta]$ such that $\frac{A(x)}{x^2}$ and $V(x)$ are non-decreasing functions of x on $(0, \eta)$ and

$$g(x,s)s \leq 2\{\Phi(x,s) + x\partial_x \Phi(x,s)\} \text{ for } (x,s) \in (0,\eta) \times \mathbb{R}. \tag{3.16}$$

Then for any $(\lambda, u) \in \mathcal{E}$, $u(x) \neq 0$ for $0 < x \leq \eta$ and consequently, $\lambda \leq \max\{V_0 + J_s(g_1), m_e + I_s(g_1)\}$ by part (iii) of Proposition 3.5.

Since $u(1) = 0$ for all $u \in D_A$, if (T1) and (T2) are satisfied with $\delta = \eta = 1$, $\mathcal{E} = \emptyset$ and, taking $g \equiv 0$, the operator $S = S_A + V$ has no eigenvalues.

Proof. If $(\lambda, u) \in \mathcal{E}$ and $u(z) = 0$ for some $z \in (0, \eta]$, then $u'(z) \neq 0$ and $zA(z)u'(z)^2 > 0$. But the hypotheses imply that $2A(x) - xA'(x) \leq 0$ and $V'(x) \geq 0$ on $(0, \eta)$ so (3.15) implies that $zA(z)u'(z)^2 \leq 0$, a contradiction. Hence $u(x) \neq 0$ for $x \in (0, \eta]$. □

Remark 3.10. Consider a function g having the properties required in conditions (S) and (T1). For $(x,s) \in (0,\delta) \times \mathbb{R}$,

$$\Phi(x,s) = \int_0^s \frac{g(x,t)}{t} t\, dt = \frac{1}{2} \left\{ g(x,s)s - \int_0^s t^2 \partial_t \left[\frac{g(x,t)}{t} \right] dt \right\}$$

and so

$$g(x,s)s - 2\{\Phi(x,s) + x\partial_x \Phi(x,s)\} = \int_0^s t \left\{ t\partial_t \left[\frac{g(x,t)}{t} \right] - 2x\partial_x \left[\frac{g(x,t)}{t} \right] \right\} dt.$$

Hence, condition (3.16) is satisfied provided that there exists $\eta \in (0, \delta]$ such that

$$s\partial_s \left[\frac{g(x,s)}{s} \right] \leq 2x\partial_x \left[\frac{g(x,s)}{s} \right] \quad \text{for all } x \in (0,\eta) \text{ and } s \neq 0.$$

A stronger, but more transparent, sufficient condition for (3.16) to hold is

$$s\partial_s \left[\frac{g(x,s)}{s} \right] \leq 0 \leq \partial_x \left[\frac{g(x,s)}{s} \right] \quad \text{for all } x \in (0,\eta) \text{ and } s \neq 0. \tag{3.17}$$

Note that since condition (S) implies that $g(x,s)/s \to 0$ as $s \to 0$ for all $x \in (0,1)$, (3.17) can only be satisfied in cases where $g(x,s)/s \leq 0$ for all $x \in (0, \eta]$ and $s \neq 0$.

4 Global bifurcation in Hilbert space

In this section two results about global bifurcation of solutions for equations in Hilbert space are formulated as Theorems 4.7 and 4.10. They are deduced from recent work in [34] on equations of a more general type in Banach space. It seems worthwhile deriving the special cases given here because their statement avoids a series of not so standard notions which are required for the form treated in [34], but which are not needed here. Of course, the notions in question inevitably appear in the proofs of Theorems 4.7 and 4.10 which amount to verifying that the hypotheses of Theorems 3.4 and 3.5 in [34] are satisfied.

4.1 Preliminaries

In preparation for the subsequent discussion some notation is fixed and a few definitions are recalled.

Let X and Y be two real Banach spaces. As usual, the space of all bounded linear operators from X into Y will be denoted by $B(X,Y)$ and, for $T \in B(X,Y)$, $\|T\| = \sup\{\|Tu\| : u \in X \text{ and } \|u\| = 1\}$.

$\text{Iso}(X,Y) = \{T \in B(X,Y) : T : X \to Y \text{ is an isomorphism }\}$

$\Phi_0(X,Y) = \{T \in B(X,Y) : T : X \to Y \text{ is a Fredholm operator of index } 0\}$

For $(\lambda, u) \in \mathbb{R} \times X$, $\|(\lambda, u)\| = |\lambda| + \|u\|$ and, for $\Omega \subset \mathbb{R} \times X$, $\Omega_\lambda = \{u \in X : (\lambda, u) \in \Omega\}$ and $p(\Omega) = \{\lambda \in \mathbb{R} : \Omega_\lambda \neq \emptyset\}$.

When U and V are subsets of the same Banach space $d(U,V) = \inf\{\|u - v\| : u \in U \text{ and } v \in V\}$ and if $U = \{u\}$ is a singleton, $d(u,V) = d(\{u\},V)$. The boundary of U is denoted by ∂U.

Consider now a Hilbert space $(H, (\cdot,\cdot), \|\cdot\|)$ and a self-adjoint operator $L : D(L) \subset H \to H$ acting in H. The space $D(L)$ equipped with its graph norm, $(\|u\|^2 + \|Lu\|^2)^{1/2}$, is a Hilbert space and $L \in B(D(L),H)$. The spectrum and essential spectrum of L are defined by

$\sigma(L) = \{\lambda \in \mathbb{R} : L - \lambda I \notin \text{Iso}(D(L),H)\}$ and $\sigma_e(L) = \{\lambda \in \mathbb{R} : L - \lambda I \notin \Phi_0(D(L),H)\}$.

When $L \in B(H,H)$ is self-adjoint, $\sigma(L)$ is bounded and $r_e(L) = \max\{|\lambda| : \lambda \in \sigma_e(L)\}$ denotes the radius of its essential spectrum.

Proposition 4.1. *For two bounded self-adjoint operators A and B on a real Hilbert space H, $\inf \sigma_e(A+B) \geq \inf \sigma_e(A) + \inf \sigma_e(B)$.*

Proof. Without further mention it is understood that all the operators introduced in this proof are bounded and self-adjoint. Let $a = \inf \sigma_e(A)$ and $b = \inf \sigma_e(B)$. Choose any $\xi < a + b$ and set $\varepsilon = (a + b - \xi)/2$. Let $T = A - (a - \varepsilon)I$ and $S = B - (b - \varepsilon)I$.

Then $\inf \sigma_e(T) = \varepsilon > 0$ and, from the spectral theory of A, there exists $\eta > 0$ such that T can be written as $D + C$ where $(Du, u) \geq \eta \|u\|^2$ for all $u \in H$ and C has finite rank. (In the notation of Proposition 3.1 in [11], it suffices to take $\eta = \min\{\inf \sigma(T_+), -\sup \sigma(T_-)\}$, $C = 2TP_- - \eta P_0$ and $D = T - C$.) Similarly, $S = E + C_1$ where $(Eu, u) \geq \eta \|u\|^2$ for all $u \in H$ and C_1 has finite rank. For $u \in H$ this yields

$$([A + B - \xi I - C - C_1]u, u) = ([D + E + (a + b - 2\varepsilon - \xi)I]u, u) \geq 2\eta \|u\|^2.$$

Hence $\|[A + B - \xi I - C - C_1]u\| \geq 2\eta \|u\|$ from which it follows that the self-adjoint operator $A + B - \xi I - C - C_1 \in \text{Iso}(H,H)$ and consequently that $A + B - \xi I \in \Phi_0(H,H)$ since $C + C_1$ is compact. This proves that $\inf \sigma_e(A+B) \geq a + b$. □

As pointed out in Remark 2.2, for the simplest types of functions satisfying condition (F), the associated Nemytskii operator is not Fréchet differentiable. The results in Sections 4.2 and 4.3 deal with bifurcation in Hilbert space where differentiability at the trivial solution holds in some weaker sense. To avoid confusion with variants appearing elsewhere the relevant definitions are now recalled in the form used in this paper.

Consider a mapping $G : U \subset X \to Y$ where X and Y are real Banach spaces and U is an open subset of X.

Definition 4.2. The mapping G is said to be Gâteaux differentiable at $u \in U$ if there exists an operator $T \in B(X,Y)$ such that, for all $v \in X$,

$$\left\| \frac{G(u+tv)-G(u)}{t} - Tv \right\| \to 0 \quad \text{as } t \to 0 \text{ in } \mathbb{R}.$$

This notion is quite standard as are variants in which T is not required to be linear. (See [16].) The next definition is less well-known.

Definition 4.3. The mapping G is said to be w-Hadamard differentiable at $u \in U$ if there exists an operator $T \in B(X,Y)$ having the following property. For every $v \in X$,

$$\frac{G(u+t_n v_n)-G(u)}{t_n} \rightharpoonup Tv \text{ weakly in } Y \text{ as } n \to \infty \text{ for all sequences } \{t_n\} \subset \mathbb{R}\backslash\{0\} \text{ and}$$
$$\{v_n\} \subset X \text{ such that } t_n \to 0 \text{ and } v_n \rightharpoonup v \text{ weakly in } X \text{ as } n \to \infty.$$

It was named in this way in [11, 12] where it seems to have been used for the first time in discussing bifurcation, but variants can be found in [2, 21]. The terminology was chosen to reflect the analogy with the better known notion of Hadamard differentiability. (See [16].)

Definition 4.4. The mapping G is said to be Hadamard differentiable at $u \in U$ if there exists an operator $T \in B(X,Y)$ such that, for all $v \in X$,

$$\left\| \frac{G(u+t_n v_n)-G(u)}{t_n} - Tv \right\| \to 0 \text{ as } n \to \infty \text{ for all sequences } \{t_n\} \subset \mathbb{R}\backslash\{0\} \text{ and}$$
$$\{v_n\} \subset X \text{ such that } t_n \to 0 \text{ and } \|v_n - v\| \to 0 \text{ as } n \to \infty.$$

In all these definitions, the linear operator T is unique, if it exists. Furthermore, if G is differentiable at u in more than one sense, the operator T is the same in all cases and it will be denoted by $G'(u)$. If $F : \mathbb{R} \times X \to Y$ and $G = F(\lambda, \cdot)$, $G'(u)$ will be denoted by $D_u F(\lambda, u)$. It is easy to see that Hadamard differentiability at u implies Gâteaux differentiability at u. Also Fréchet differentiability at u implies differentiability in the sense of all three definitions but none of these notions implies Fréchet differentiability.

Example 4.5. Consider a function $f \in C^1(\mathbb{R}, \mathbb{R})$ such that $f(0) = 0$ and $K \equiv \sup\{|f'(s)| : s \in \mathbb{R}\} < \infty$. Since $|f(s)| \leq K|s|$ for all $s \in \mathbb{R}$, $f(u(\cdot)) \in L^2(0,1)$ whenever $u \in L^2(0,1)$ and the associated Nemytskii operator $\tilde{f} : L^2 \to L^2$ is uniformly Lipschitz continuous. In Example 2.3 of [11] and the subsequent remark it is shown that, for all $u \in L^2$, $\tilde{f} : L^2 \to L^2$ is Gâteaux differentiable, w-Hadamard differentiable and Hadamard differentiable at u. On the other hand, if there exists even one element $u \in L^2$ at which $\tilde{f} : L^2 \to L^2$ is Fréchet differentiable, then $f : \mathbb{R} \to \mathbb{R}$ must be linear.

Section 4.3 deals with bifurcation for a problem that is w-Hadamard at the trivial solution, whereas Gâteaux differentiability is assumed in Section 4.2. However in Section 4.2 the problem is also required to be Lipschitz continuous in an open neighbourhood of the trivial solution and this together with Gâteaux differentiability implies Hadamard differentiability at the trivial solution. In fact, the situation treated in Section 4.2 is based on previous work [29] relying heavily on Hadamard differentiability.

Both cases treated here concern bifurcation for an equation $F(\lambda, u) = 0$ at a point μ where $F : \mathbb{R} \times X \to Y$ and $L(\mu) \equiv D_u F(\mu, 0) \in \Phi_0(X, Y)$. In fact, X and Y are Hilbert spaces with $X \subset Y$ and $L(\mu) : X \subset Y \to Y$ is a self-adjoint operator acting in Y. In case 2, $\sigma_e(L(\mu)) \subset (0, \infty)$ but $F(\mu, \cdot) : X \to Y$ need not be Lipschitz continuous, whereas case 1 covers situations where μ may be in a gap in $\sigma_e(L(\mu))$, provided that $d(0, \sigma_e(L(\mu)))$ is sufficiently large relative to the Lipschitz modulus of $F(\mu, \cdot) - L(\mu)$.

4.2 Global bifurcation, case 1

Let $(Y, (\cdot, \cdot), \|\cdot\|)$ be a real Hilbert space and X a subspace of Y that is the domain of some self-adjoint operator acting in Y . Recall from Proposition 5.4 of [29] that the graph norms of all such operators on X are equivalent and let $\|\cdot\|_X$ denote one of these norms. Then $(X, \|\cdot\|_X)$ is a Hilbert space, $\|u\|_Y \leq \|u\|_X$ for all $u \in X$ and X is dense in Y. In this part we consider equations of the form

$$M(u) = \lambda u \quad \text{for } (\lambda, u) \in \mathbb{R} \times X, \tag{4.1}$$

where $M = M_1 + M_2 : X \to Y$ has the following properties.

(m1) $M_1 \in C^1(X, Y)$, $M_1(0) = 0$, $M_1'(0) : X \subset Y \to Y$ is a self-adjoint operator acting in Y and the remainder $R_1 \equiv M_1 - M_1'(0) : X \to Y$ is compact.

(m2) $M_2 : X \to Y$ is Gâteaux differentiable at 0 with $M_2'(0) = 0$ and $M_2(0) = 0$. Furthermore,

$$\ell \equiv \sup \left\{ \frac{\|M_2(u) - M_2(v)\|_Y}{\|u - v\|_Y} : u, v \in X \text{ and } u \neq v \right\} < \infty.$$

Remark 4.6. By (m2), M_2 could be extended to a uniformly Lipschitz continuous mapping of Y into itself. Since X is continuously embedded in Y, $M_2 : X \to Y$ is also uniformly Lipschitz continuous. It follows from these assumptions that $M = M_1 + M_2 : X \to Y$ is locally Lipschitz continuous on X and Gâteaux differentiable at 0 with $M'(0) = M_1'(0)$. In connection with the hypotheses for case 2, it should be noted that (m2) implies that $M_2 : X \to Y$ is Hadamard differentiable at 0 and that (m1) and (m2) imply that $M : X \to Y$ also has this property. However, condition (m2) does not imply that $M_2 : X \to Y$ is w-Hadamard differentiable at 0.

Let $d_\ell = \{\lambda \in \mathbb{R} : d(\lambda, \sigma_e(M'(0))) > \ell\}$ and, for $\mu \in d_\ell$, let $J_\mu(\ell)$ denote the maximal interval in d_ℓ containing μ.

Let $\mathcal{E} = \{(\lambda, u) \in \mathbb{R} \times X : M(u) = \lambda u \text{ and } u \neq 0\}$ denote the set of non-trivial solutions of (4.1) and let $\overline{\mathcal{E}}$ denote its closure in $\mathbb{R} \times X$. The assumptions (m1) and (m2) imply that $(\lambda, 0)$ is a solution of (4.1) for all $\lambda \in \mathbb{R}$ and $\overline{\mathcal{E}} \backslash \mathcal{E} \subset \mathbb{R} \times \{0\}$. A real number μ is a bifurcation point for equation (4.1) if and only if $(\mu, 0) \in \overline{\mathcal{E}}$.

Theorem 4.7. *Consider equation* (4.1) *under the assumptions (m1) and (m2). Suppose that* $\mu \in d_\ell$ *and let* $U = J_\mu(\ell) \times X$.

(1) *If* $\ker\{M'(0) - \mu I\} = \{0\}$, *$\mu$ is not a bifurcation point for equation* (4.1).

(2) *If μ is an eigenvalue of odd multiplicity of $M'(0)$ it is a bifurcation point for equation* (4.1). *The connected component $\mathcal{D}_\mu$ of $\overline{\mathcal{E}} \cap U$ containing $(\mu, 0)$ has at least one of the following properties.*

   (a) $\{|\lambda| + \|u\|_X : (\lambda, u) \in \mathcal{D}_\mu\} = [0, \infty)$.

   (b) $d(p(\mathcal{D}_\mu), \sigma_e(M'(0)) = \ell$.

   (c) $\mathcal{D}_\mu \cap [J_\mu(\ell)\backslash\{\mu\}] \times \{0\} \neq \emptyset$.

(3) *If* $\ker\{M'(0) - \mu I\} = \mathrm{span}\{\phi\}$ *where* $\|\phi\|_Y = 1$ *and* $\{(\lambda_n, u_n)\} \subset \mathcal{E}$ *is such that* $\lambda_n \to \mu$ *and* $\|u_n\|_X \to 0$, *then there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$, $u_n = (u_n, \phi)\{\phi + w_n\}$ *where* $(w_n, \phi) = 0$ *and* $\|w_n\|_X \to 0$ *as* $n \to \infty$.

Remark 4.8. The hypotheses of Theorem 4.7 are similar to those of Corollary 6.11 in [29]. Apart from the fact that they hold on all of X instead of a ball centred at the origin, the compactness of R_1 is added. Parts (1) and (3) are already established in Corollary 6.11 of [29] but part (2) provides new global information. If conditions (m1) and (m2) are satisfied and $M_2 : X \to Y$ is Fréchet differentiable at 0, part (2) of Theorem 4.7 could be deduced from Theorem 1.1 in [27] which was itself based on Theorem 1.6 in [26]. Those results used Nussbaum's degree [22] for k-set contractions and they were applied to a a class of Sturm–Liouville problems on the interval $(0, \infty)$ in [26,27]. If, in addition, $M_2 : X \to Y$ is continuously differentiable on an open neighbourhood of 0, the conclusion in part (3) can be strengthened using the standard result about bifurcation at a simple eigenvalue [8].

Proof. (1) This follows from part (i) of Corollary 6.11 in [29].

(2) It follows from part (ii) of Corollary 6.11 in [29] that μ is a bifurcation point. We suppose now that $\mathcal{D}_\mu$ does not have properties (a) and (b) and use Theorem 3.4 in [34] to show that it must satisfy (c). From the assumption that $\mathcal{D}_\mu$ is bounded it follows that $I_\mu \equiv [\inf p(\mathcal{D}_\mu), \sup p(\mathcal{D}_\mu)]$ is a compact interval and then $d(p(\mathcal{D}_\mu), \sigma_e(M'(0)) \neq \ell$ means that there exists $k > \ell$ such that $I_\mu \subset J_\mu(k)$. Hence $d(\mathcal{D}_\mu, \partial(J_\mu(k) \times X)) > 0$.

The hypotheses of Theorem 3.4 in [34] involve the essential conditioning number, $\gamma(M'(0) - \lambda I)$. By Corollary 5.6 in [29], for all $\lambda \notin \sigma_e(M'(0))$ and all $\varepsilon > 0$,

$$\gamma(M'(0) - \lambda I) \leq \frac{1}{d(\lambda, \sigma_e(M'(0)))} + \varepsilon K_\lambda,$$

provided that the graph norm of $\varepsilon M'(0)$ is used on X and

$$K_\lambda = \max\left\{1, \frac{|p|}{\lambda - p}, \frac{|q|}{q - \lambda}\right\},$$

where (p, q) is the maximal interval in $\mathbb{R}\backslash\sigma_e(M'(0))$ containing λ. If either $p = -\infty$ or $q = \infty$, the corresponding ratio is replaced by 1. We require this estimate for $\lambda \in J_\mu(k)$ and for λ in this interval it is easy to check that $K_\lambda \leq K \equiv \max\left\{1, \frac{|p|}{k}, \frac{|q|}{k}\right\}$, with the same convention concerning the cases $p = -\infty$ and $q = \infty$. Thus, for $\lambda \in J_\mu(k)$,

$$\gamma(M'(0) - \lambda I) \leq \frac{1}{k} + \varepsilon K.$$

For the rest of this proof, choose and fix $\varepsilon > 0$ such that $\varepsilon K < \frac{1}{\ell} - \frac{1}{k}$ and let $\|\cdot\|_X$ denote the graph norm of $\varepsilon M'(0)$. (If $\ell = 0$, $M_2 \equiv 0$ and any $\varepsilon > 0$ is acceptable.) We now have

$$d(\lambda, \sigma_e(M'(0))) > k \quad \text{and} \quad \gamma(M_1'(0) - \lambda I) = \gamma(M'(0) - \lambda I) < \frac{1}{\ell} \quad \text{for all } \lambda \in J_\mu(k). \tag{4.2}$$

Setting $F(\lambda,u) = M(u) - \lambda u$, we aim to show that the hypotheses of Theorem 3.4 in [34] are satisfied with

$$U = J_\mu(\ell) \times X, \Omega = J_\mu(k) \times X, G(\lambda,u) = M_1(u) - \lambda u \quad \text{and} \quad K(\lambda,u) = M_2(u).$$

Using the notation of [34], let $\mathcal{S} = \{(\lambda,u) \in U : F(\lambda,u) = 0 \text{ and } u \neq 0\}$. Then $\mathcal{S} = \mathcal{E} \cap U$ and it is easy to check that $\overline{\mathcal{E}} \cap U$ coincides with the closure of $\mathcal{S}$ in U. Hence $\mathcal{D}_\mu = \mathcal{C}_\mu(U,F)$ in the notation of the Introduction in [34].

Clearly condition (D0) in [34] is satisfied with $J(\Omega) = J_\mu(k)$. Furthermore, $G \in C^1(\Omega, Y)$ and $D_u G(\lambda,u) = M_1'(u) - \lambda I = R_1'(u) + M_1'(0) - \lambda I$. Since $R_1 \in C^1(X,Y)$ and $R_1 : X \to Y$ is compact, it follows from Proposition 8.2 in [9] that $R_1'(u) : X \to Y$ is compact for all $u \in X$. Hence $D_u G(\lambda,u) \in \Phi_0(X,Y)$ if and only if $M_1'(0) - \lambda I \in \Phi_0(X,Y)$. But $J_\mu(k) \cap \sigma_e(M_1'(0)) = \emptyset$ so $M_1'(0) - \lambda I \in \Phi_0(X,Y)$ for all $\lambda \in J_\mu(k)$ and hence condition (D1) in [34] is satisfied. It is an immediate consequence of (m2) that K satisfies condition (D2) with $D_u K(\lambda,0) = 0$ and furthermore

$$\|K(\lambda,u) - K(\lambda,v)\| \leq \ell \|u - v\|_X \quad \text{for all } u,v \in X.$$

In the notation of [34] for the measure of non-compactness, $\alpha(K(\lambda,\cdot),V) \leq \ell$ for every bounded subset V of X for which $\alpha(V)$ is positive. On the other hand, by the compactness of $R_1 : X \to Y$, (4.2) and Proposition 2.1(iv) in [34], for all $\lambda \in J_\mu(k)$,

$$\omega(M_1 - \lambda I, V) = \omega(R_1 + M_1'(0) - \lambda I, V) \geq \omega(M_1'(0) - \lambda I, V) - \alpha(R_1, V) \tag{4.3}$$
$$= \omega(M_1'(0) - \lambda I, V) \geq 1/\gamma(M_1'(0) - \lambda I) > \ell \geq \alpha(K(\lambda,\cdot),V), \tag{4.4}$$

which shows that condition (D3) in [34] is also satisfied.

Setting $L(\lambda) = D_u F(\lambda,0)$ and $\rho(\lambda,u) = K(\lambda,u) - D_u K(\lambda,0)$ as in [34], we have $L(\lambda) = M'(0) - \lambda I \in \Phi_0(X,Y)$, $L_X(\rho,\lambda) \leq \ell$ and $\Delta_r(\rho,\lambda) = 0$ for all $\lambda \in J_\mu(k)$ and $r > 0$. It follows from (4.2) and (4.4) that the conditions (3.15) and (3.16) in [34] are satisfied. Finally, using Criterion I in Section 5.2 of [29], the local parity, $\sigma(L,\mu)$ of the path L at the isolated singular point μ is -1 since $M'(0)$ is self-adjoint and μ has odd multiplicity. At this point, it follows from Theorem 3.4 in [34] that $\mathcal{D}_\mu$ has at least one of the following properties.

(i) $\{|\lambda| + \|u\|_X : (\lambda,u) \in \mathcal{D}_\mu\} = [0,\infty)$.

(ii) $d(\mathcal{D}_\mu, \partial\Omega) = 0$.

(iii) $\mathcal{D}_\mu \cap [\mathbb{R}\backslash\{\mu\}] \times \{0\} \neq \emptyset$.

Recall that we are assuming that $\mathcal{D}_\mu$ does not have the properties (a) and (b) and that Ω has been chosen so that $d(\mathcal{D}_\mu, \partial\Omega) > 0$. Hence $\mathcal{D}_\mu$ must have property (iii) and this implies property (c) since $\mathcal{D}_\mu \subset J_\mu(\ell) \times X$ by definition.

(3) This follows from part (iii) of Corollary 6.11 in [29].

□

4.3 Global bifurcation, case 2

In this part we deal with an equation of the form

$$M(u) = \lambda T(u) \quad \text{for } (\lambda,u) \in \mathbb{R} \times H, \tag{4.5}$$

where $(H, (\cdot,\cdot), \|\cdot\|)$ is a real Hilbert space. The mappings $M = M_1 + M_2$ and T have the following properties.

(W0) $T \in B(H,H)$ is a self-adjoint operator and $(Tu,u) > 0$ for $u \in H\backslash\{0\}$.

(W1) $M_1 \in C^1(H,H)$ with $M_1(0) = 0$ and $M_1'(0)$ is self-adjoint. Furthermore, the remainder $R_1 = M_1 - M_1'(0) : H \to H$ is a compact operator.

(W2) $M_2 \in C(H,H)$ with $M_2(0) = 0$. The mapping $M_2 : H \to H$ is compact and w-Hadamard differentiable at 0 with $M_2'(0)$ self-adjoint. Furthermore,

$$\liminf_{\|u\|\to 0} \frac{(R_2(u),u)}{\|u\|^2} \geq 0, \quad \text{where } R_2 = M_2 - M_2'(0).$$

Remark 4.9. The properties in (W2) do not imply that $M_2'(0) : H \to H$ is a compact linear operator. Since $M_2(0) = 0$ it follows from the w-Hadamard differentiability of M_2 at 0 that

$$\lim_{t\to 0} \frac{(R_2(tu),tu)}{\|tu\|^2} = 0 \quad \text{for all } u \in H\backslash\{0\}$$

and so (W2) implies that

$$\liminf_{\|u\|\to 0} \frac{(R_2(u),u)}{\|u\|^2} = 0.$$

By (W1),

$$\lim_{\|u\|\to 0} \frac{(R_1(u),u)}{\|u\|^2} = 0$$

since $\|R_1(u)\|/\|u\| \to 0$ as $\|u\| \to 0$ and so, when (W1) is satisfied, the assumption about the lim inf in (W2) is equivalent to

$$\liminf_{\|u\|\to 0} \frac{(M(u) - M'(0)u,u)}{\|u\|^2} \geq 0. \tag{4.6}$$

Let $\mathcal{E} = \{(\lambda,u) \in \mathbb{R} \times H : M(u) = \lambda T(u) \text{ and } u \neq 0\}$ denote the set of non-trivial solutions of (4.5) and let $\overline{\mathcal{E}}$ denote its closure in $\mathbb{R} \times H$. As in case 1, μ is a bifurcation point for (4.5) if and only if $(\mu,0) \in \overline{\mathcal{E}}$.

Theorem 4.10. *Under the hypotheses (W0) to (W2), let J be an open interval such that $\inf \sigma_e(M_1'(0) - \lambda T) > r_e(M_2'(0))$ for all $\lambda \in J$. Then $\inf \sigma_e(M'(0) - \lambda T) > 0$ for $\lambda \in J$.*

Consider a point $\mu \in J$ and let $U = J \times H$.

(1) If $\ker\{M'(0) - \mu T\} = \{0\}$, μ is not a bifurcation point for equation (4.5).

(2) If $\dim\ker\{M'(0) - \mu T\}$ is odd, μ is a bifurcation point for the equation (4.5). In fact, the connected component $\mathcal{D}_\mu$ of $\overline{\mathcal{E}} \cap U$ containing $(\mu,0)$ has at least one of the following properties.

(a) $\{|\lambda| + \|u\|_X : (\lambda,u) \in \mathcal{D}_\mu\} = [0,\infty)$.

(b) $d(p(\mathcal{D}_\mu),\partial J) = 0$.

(c) $\mathcal{D}_\mu \cap [J\backslash\{\mu\}] \times \{0\} \neq \emptyset$.

(3) Suppose that $\ker\{M'(0) - \mu T\} = \operatorname{span}\{\phi\}$ where $\|\phi\| = 1$ and that $\{(\lambda_n,u_n)\} \subset \mathcal{E}$ is such that $\lambda_n \to \mu$ and $\|u_n\| \to 0$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|$. Then there exist a subsequence and $c \in \mathbb{R}\backslash\{0\}$ for which $v_{n_k} \rightharpoonup c\phi$ weakly in H as $n_k \to \infty$.

Remark 4.11. By Proposition 4.1, for all $\lambda \in J$

$$\begin{aligned}\inf \sigma_e(M'(0) - \lambda T) &\geq \inf \sigma_e(M_1'(0) - \lambda T) + \inf \sigma_e(M_2'(0)) \\ &\geq \inf \sigma_e(M_1'(0 - \lambda T)) - r_e(M_2'(0)) > 0.\end{aligned}$$

In particular, $M'(0) - \lambda T$ and $M_1'(0) - \lambda T \in \Phi_0(H,H)$ for all $\lambda \in J$.

Proof. Parts (1) and (2) will be deduced from Lemma 3.3 and Theorem 3.5 in [34]. With this in mind, let $F(\lambda, u) = M(u) - \lambda Tu$ for $(\lambda, u) \in U$. Then $\mathcal{S} \equiv \{(\lambda, u) \in U : F(\lambda, u) = 0 \text{ and } u \neq 0\} = \mathcal{E} \cap U$ and it is easy to check that $\overline{\mathcal{E}} \cap U$ coincides with the closure of $\mathcal{S}$ in U. Hence in the notation of the Introduction in [34], $\mathcal{D}_\mu = \mathcal{C}_\mu(U, F)$. Setting

$$\Omega = U = J \times H, G(\lambda, u) = M_1(u) - \lambda Tu \quad \text{and} \quad K(\lambda, u) = M_2(u),$$

we consider first the hypotheses of Lemma 3.3 in [34].

By (W0) to (W2), $F \in C(U, H)$ and $F(\lambda, \cdot) : H \to H$ is w-Hadamard differentiable at 0 with $L(\lambda) \equiv D_u F(\lambda, 0) = M'(0) - \lambda T$ for all $\lambda \in J$. The remainder $R(\lambda, u) = F(\lambda, u) - D_u F(\lambda, 0)u = R_1(u) + R_2(u)$ is independent of λ so the quantity $\Delta_r(F, \lambda) \to 0$ as $r \to 0$ for all $\lambda \in J$. By (W1), (W2) and (4.6),

$$\liminf_{\|u\| \to 0} \frac{(R(\lambda, u), u)}{\|u\|^2} = \liminf_{\|u\| \to 0} \frac{(R_2(u), u)}{\|u\|^2} \geq 0.$$

As noted in Remark 4.11 , $\inf \sigma_e(L(\lambda)) > 0$ and since by Remark 3.2 in [34], $w_l(L(\lambda)) = \inf \sigma_e(L(\lambda))$, it follows that condition (3.14)(a) in [34] is satisfied at $\lambda \in J$ whenever $\ker\{M'(0) - \lambda T\} = \{0\}$. The conclusion in part (1) is now justified by Lemma 3.3(ii) in [34].

For part (2) we use Theorem 3.5 in [34], noting first of all that (D0) is satisfied and that by (W1), $G \in C^1(\Omega, H)$ with $D_u G(\lambda, u) = M_1'(u) - \lambda T = R_1'(u) + M_1'(0) - \lambda T$. By Proposition 8.2 in [9], (W1) also implies that, for all $u \in H$, $R_1'(u) \in B(H, H)$ is compact and so $D_u G(\lambda, u) \in \Phi_0(H, H)$ for all $\lambda \in J$ by Remark 4.11. This proves that condition (d1) in [34] is satisfied and condition (d2) is an immediate consequence of hypothesis (W2). For (d3), consider a bounded subset V of H for which the set-measure of non-compactness, $\alpha(V)$, is positive. Then in the notation of [34], for all $\lambda \in J$,

$$\begin{aligned}\omega(G(\lambda, \cdot), V) = \omega(R_1 + M_1'(0) - \lambda T, V) &\geq \omega(M_1'(0) - \lambda T, V) - \alpha(R_1, V) \\ = \omega(M_1'(0) - \lambda T) &\geq \inf \sigma_e(M_1'(0) - \lambda T) > 0,\end{aligned}$$

by the compactness of R_1 and Remark 2.1 in [34]. Since $\alpha(M_2, V) = 0$ by the compactness of M_2, this shows that condition (d3) in [34] is satisfied. Furthermore, referring again to Remark 2.1 in [34], for $\lambda \in J$,

$$\alpha(D_u K(\lambda, 0)) = \alpha(M_2'(0)) \leq r_e(M_2'(0)) < \inf \sigma_e(M_1'(0 - \lambda T)) \leq \omega(D_u G(\lambda, 0)).$$

Also $\alpha_0(K(\lambda, \cdot)) = 0$ by the compactness of M_2. Hence condition (3.16) in [34] is satisfied because $\rho(\lambda, u) \equiv K(\lambda, u) - D_u K(\lambda, 0)u$ does not depend upon λ.

We have already noted in Remark 4.11 that $L(\lambda) = M'(0) - \lambda T \in \Phi_0(H, H)$ for all $\lambda \in J$. If $u \in \ker L(\lambda)$ and $L'(\lambda)u = -Tu \in \text{range } L(\lambda) = [\ker L(\lambda)]^\perp$, it follows that $(Tu, u) = 0$ and hence $u = 0$ by (W0). Using Criterion I in [29] for the calculation of the local parity, $\sigma(L, \lambda)$, of the path L across λ we find that $\sigma(L, \lambda) = (-1)^n$ where $n = \dim \ker L(\lambda)$. By Remark 3.2 in

[34] and Remark 4.11, $w_l(L(\lambda)) = \inf \sigma_e(M'(0) - \lambda T) > 0$ for all $\lambda \in J$ and so, as in the proof of part (1), (W1), (W2) and (4.6) imply that condition (3.18)(a) in [34] is satisfied.

The conclusion in part (2) now follows from Theorem 3.5 in [34].

(3) For the sequence $\{(\lambda_n, u_n)\}$ in the statement let $t_n = \|u_n\|$. Then $t_n \to 0$ and $u_n = t_n v_n$. Passing to a subsequence, we suppose henceforth that $v_n \rightharpoonup v$ weakly in H. Since $M(0) = 0$ and M is w-Hadamard differentiable at zero it follows that $M(u_n)/\|u_n\| = M(t_n v_n)/t_n \rightharpoonup M'(0)v$ weakly in H as $n \to \infty$. Hence

$$L(\mu)v_n = \{M'(0) - \mu T\}v_n = M'(0)v_n - \frac{M(u_n)}{\|u_n\|} + (\lambda_n - \mu)Tv_n \rightharpoonup 0 \quad \text{weakly in } H \tag{4.7}$$

since $M(u_n) = \lambda_n T u_n$, $\lambda_n \to \mu$ and $M'(0)v_n \rightharpoonup M'(0)v$ weakly in H as $n \to \infty$. This implies that $L(\mu)v = 0$ and so $v = c\phi$ for some $c \in \mathbb{R}$.

If $c = 0$, $v_n \rightharpoonup 0$ weakly in H and so, in the notation of Section 3 of [34], $\{v_n\} \subset \Sigma$ from which it follows that

$$\liminf_{n\to\infty} (L(\mu)v_n, v_n) \geq w_l(L(\mu)) = \inf \sigma_e(L(\mu))$$

by Remark 3.2 in [34], where $\inf \sigma_e(L(\mu)) > 0$ by Remark 4.11. On the other hand from (4.7) and (4.6) we obtain

$$\begin{aligned}\liminf_{n\to\infty} (L(\mu)v_n, v_n) &= \liminf_{n\to\infty} \Big\{ \frac{(M'(0)u_n - M(u_n), u_n)}{\|u_n\|^2} + (\lambda_n - \mu)(Tv_n, v_n) \Big\} \\ &= \liminf_{n\to\infty} \frac{(M'(0)u_n - M(u_n), u_n)}{\|u_n\|^2} \leq 0,\end{aligned}$$

contradicting the earlier conclusion. Hence $c \neq 0$. □

5 Global bifurcation for the boundary value problem

Under the assumption (S) formulated in Section 2.3, Theorems 4.7 and 4.10 will be used to obtain conclusions about the bifurcation of solutions for problem (1.1)(1.2) in the sense of Definition 2.7. The first result is based upon Theorem 4.7 and it deals with that happens for λ in the interval $(-\infty, m_e - \ell_{g_1})$ where $m_e = \inf \sigma_e(S_A + V) = \frac{a}{4} + V_0$ and ℓ_{g_1} is the best Lipschitz for the part g_1 of g which satisfies condition (F). It follows from Theorem 4.7 that there is global bifurcation at every eigenvalue of $S_A + V$ in the interval $(-\infty, m_e - \ell_{g_1})$. Corollary 5.3 deals with a special case, where $n \equiv 0$ and $g(x,s)s \leq 0$ for $(x,s) \in (0,1) \times \mathbb{R}$, in which it can be shown that there may be no bifurcation at eigenvalues of $S_A + V$ lying in the interval $(m_e - \ell_{g_1}, \infty)$. The situation where $g(x,s)s \geq 0$ for $(x,s) \in (0,1) \times \mathbb{R}$ is quite different and bifurcation at all eigenvalues of $S_A + V$ in the interval $(-\infty, m_e)$ can be proved using Theorem 4.10. This case is treated in Section 5.1.

Throughout this section $\mathcal{E}$ denotes the set of all non-trivial solutions of (1.1)(1.2) in $\mathbb{R} \times D_A$ as defined in Section 2.3 and D_A is considered with a norm that is equivalent to the graph norm, $\|\cdot\|_S$, of $S = S_A + V$. Of course, the conclusions do not depend upon the choice of norm. It is often convenient to use the norm defined by $\|S_A u\|_{L^2}$ but the proof of Theorem 5.5 is based on a different choice. The nodal properties of solutions established in Section 3 are used to show that possibility (c) in Theorems 4.7 and 4.10 does not occur.

Theorem 5.1. *Let the assumption (S) be satisfied and consider* $\mu \in (-\infty, m_e - \ell_{g_1})$.

(A) If μ is a bifurcation point for problem (1.1)(1.2), then μ is an eigenvalue of the self-adjoint operator $S = S_A + V$.

(B) If μ is an eigenvalue of S then μ is a bifurcation point for (1.1)(1.2) and the component $\mathcal{C}_\mu$ of $\overline{\mathcal{E}} \cap (-\infty, m_e - \ell_{g_1}) \times D_A$ containing $(\mu, 0)$ has at least one of the following properties.

(i) $\{|\lambda| + \|u\|_S : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$.

(ii) $\sup\{\lambda : (\lambda, u) \in \mathcal{C}_\mu\} = m_e - \ell_{g_1}$.

(C) If μ is the $k-th$ eigenvalue of S, then $\sharp(u) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$, where $\sharp(u)$ denotes the number of zeros of u in $(0, 1]$ and $\mathcal{C}_\mu \cap \mathbb{R} \times \{0\} = \{(\mu, 0)\}$.

Remark 5.2. If assumption (S) is satisfied and $n \equiv 0$, then for $(\lambda, u) \in \mathcal{E}$,

$$\|S_A u\|_{L^2} \leq (|\lambda| + \|V\|_{L^\infty} + \ell_{g_1})\|u\|_{L^2} + C\mathfrak{C}_{g_2}(\|u\|_A)\|u\|_A$$

by Remark 2.4. In this case property (i) in the conclusion can be replaced by $\{|\lambda| + \|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$ and if in addition, $g_2 \equiv 0$, it can be replaced by $\{|\lambda| + \|u\|_{L^2} : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$.

Proof. The first step in this proof is to observe that the hypotheses of Theorem 4.7 are satisfied for the equation $F(\lambda, u) = 0$ where F is defined by (2.12). For this we take $Y = L^2$ and $X = D_A$ equipped with the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_S$, respectively, and set

$$M_1(u) = Su + N(u) + \widetilde{g_2}(u) \quad \text{and} \quad M_2(u) = \widetilde{g_1}(u) \quad \text{for } u \in X.$$

From assumption (S) and Propositions 2.1, 2.3 and 2.5 it follows that the conditions (m1)and (m2) are satisfied with $M'(0) = M_1'(0) = S$ and $R_1 = N + \widetilde{g_2}$. In the notation of Theorem 4.7, $J_\mu(\ell) = (-\infty, m_e - \ell_{g_1})$. From Theorem 4.7, we obtain immediately part (A) and that, if $\mathcal{C}_\mu$ has neither property (i) nor (ii), then there exists an eigenvalue ξ of S in $(-\infty, m_e - \ell_{g_1}) \backslash \{\mu\}$ such that $(\xi, 0) \in \mathcal{C}_\mu$ and hence $\mathcal{C}_\xi = \mathcal{C}_\mu$. To show that this third situation does not occur it suffices to prove part (C).

(C) Since $\mathcal{C}_\mu \subset (-\infty, m_e - \ell_{g_1}) \times D_A$, it follows from Lemma 3.1(i) that u has only a finite number of zeros in $(0, 1]$ if $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$. Setting $Z(\lambda, u) = \sharp(u)$ for $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$, Corollary 3.3 shows that $Z : \mathcal{C}_\mu \cap \mathcal{E} \to \mathbb{N}$ is continuous. Consider now a point $(\xi, 0) \in \mathcal{C}_\mu$. It follows from Lemma 3.1(i) that there exist an open ball B in $\mathbb{R} \times D_A$, centred at $(\xi, 0)$, and $\eta \in (0, 1)$ such that $u(x) \neq 0$ for $0 < x \leq \eta$ if $(\lambda, u) \in B \cap \mathcal{E}$. By part (A), ξ is an eigenvalue of S and an associated eigenfunction ϕ_ξ with $\|\phi_\xi\|_{L^2} = 1$ has a finite number of zeros $\sharp(\phi_\xi)$ in $(0, 1]$ by property (S4) in Section 2.4. Hence η can be chosen so that $\phi_\xi(x) \neq 0$ for $0 < x \leq \eta$. Suppose that there is a sequence $\{(\lambda_n, u_n)\} \subset B \cap \mathcal{E}$ such that $\lambda_n \to \xi$ and $\|u_n\|_S \to 0$ as $n \to \infty$ and, for all $n \in \mathbb{N}$, $\sharp(u_n) \neq \sharp(\phi_\xi)$. By part (3) of Theorem 4.7 we can suppose that $u_n = (u_n, \phi_\xi)\{\phi_\xi + w_n\}$ where $\|w_n\|_S \to 0$ as $n \to \infty$ and, for all n, $(u_n, \phi_\xi) \neq 0$ since $(\lambda_n, u_n) \in \mathcal{E}$. In the notation of Lemma 3.2, $\|P_\eta w_n\|_\eta \to 0$ as $n \to \infty$ and it follows from Lemma 3.2 that there exists n_0 such that $\sharp(\phi_\xi + w_n) = \sharp(\phi_\xi)$ for all $n \geq n_0$, since $\phi_\xi + w_n$ like u_n has no zeros in the interval $(0, \eta]$ because $(\lambda_n, u_n) \in B \cap \mathcal{E}$ and $(u_n, \phi_\xi) \neq 0$. But this implies that $\sharp(u_n) = \sharp(\phi_\xi)$ for all $n \geq n_0$, contradicting the choice of the sequence $\{(\lambda_n, u_n)\}$. Hence there exists an open neighbourhood U_ξ of $(\xi, 0)$ in $\mathbb{R} \times D_A$ such that $Z(\lambda, u) = \sharp(\phi_\xi)$ for all $(\lambda, u) \in U_\xi \cap \mathcal{E}$. Setting $Z(\xi, 0) = \sharp(\phi_\xi)$ for all $(\xi, 0) \in \mathcal{C}_\mu$ we have now proved that $Z : \mathcal{C}_\mu \to \mathbb{N}$ is continuous and hence constant by the connectedness of $\mathcal{C}_\mu$. Since μ is a bifurcation point, it follows that $Z(\lambda, u) = \sharp(\phi_\mu)$ for all $(\lambda, u) \in \mathcal{C}_\mu$. This establishes part (C). □

The following special case sheds some light on the restriction to the interval $(-\infty, m_e - \ell_{g_1})$ in Theorem 5.1. It uses the conditions (T1) and (T2) introduced in Section 3.2 and the quantities defined in (3.8) to (3.11).

Corollary 5.3. *Suppose that conditions (S), (T1) and (T2) are satisfied with $n \equiv 0$ and $g(x,s)s \leq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$. Let $\Theta \equiv \max\{V_0, m_e + I_s(g_1)\}$. Then $-\ell_{g_1} \leq I_s(g_1) \leq J_s(g_1) = 0$ and $m_e - \ell_{g_1} \leq \Theta \leq m_e$.*

(A) A point $\mu \in (-\infty, m_e - \ell_{g_1})$ is a bifurcation point for problem (1.1)(1.2) if and only if it is an eigenvalues of S. When it is an eigenvalue, the component $\mathcal{C}_\mu$ of $(-\infty, m_e - \ell_{g_1}) \times D_A$ containing $(\mu, 0)$ is a subset of $(-\infty, \mu] \times D_A$ and $\{|\lambda| + \|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$. If μ is the k-th eigenvalue of S, $\sharp(u) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$.

(B) There are no bifurcation points for (1.1)(1.2) in the interval (Θ, ∞) since $\mathcal{E} \cap (\Theta, \infty) \times D_A = \emptyset$.

Remark 5.4. If $I_s(g_1) = I_i(g_1) = -\ell_{g_1}$ and $\ell_{g_1} \leq \frac{a}{4}$, then $\Theta = m_e - \ell_{g_1}$.

As an example, suppose that $g_1(x,s) = -r(x)k(s)$ for $(x,s) \in (0,1) \times \mathbb{R}$ where the functions r and k satisfy the following conditions.

(R) $r \in C^1([0,1])$ with $r'(x) \leq 0$ for $0 \leq x \leq 1$, $r(0) > 0$ and $r(1) \geq 0$.

(K) $k \in C^1(\mathbb{R})$ is odd, convex on $[0, \infty)$ and $k'(0) = 0 < k'(\infty) \equiv \lim_{s\to\infty} k'(s) < \infty$.

Then $I_s(g_1) = I_i(g_1) = -\ell_{g_1} = -r(0)k'(\infty)$ and $\Theta = m_e - r(0)k'(\infty)$ if $r(0)k'(\infty) \leq \frac{a}{4}$.

The assumptions (R) and (K) also imply that the function $g_1(x,s) = -r(x)k(s)$ satisfies condition (3.17). Hence, taking $g = g_1$ and S to be as in Example 2.8 or 2.10 we obtain situations where all the hypotheses of Corollary 5.3 are satisfied and $\sigma(S) = \{\lambda_i : 1 \leq i \leq n\} \cup [\frac{1}{4}, \infty)$ where $\lambda_1 > 0$ and $\lambda_i < \lambda_{i+1} < \frac{1}{4} = m_e$ for $1 \leq i \leq n-1$. The quantity Θ is now $\frac{1}{4} - r(0)k'(\infty)$ and it can be placed anywhere in the interval $(0, \frac{1}{4})$ by adjusting $r(0)k'(\infty)$. When $\Theta \notin \{\lambda_i : 1 \leq i \leq n\}$, λ_i is a bifurcation point if and only if $\lambda_i < \Theta$.

Proof. (A) By Theorem 5.1 it suffices to show that $\lambda \leq \mu$ for all $(\lambda, u) \in \mathcal{C}_\mu$. This will be done using the standard comparison principle for the eigenvalues of self-adjoint operators. (See Theorems 1.2 and 1.3 in Chapter XI of [10], for example.) Let $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$ and set $W = W_1 + W_2$ where

$$W_i(x) = \frac{g_i(x, u(x))}{u(x)} \quad \text{if} \quad u(x) \neq 0 \quad \text{and} \quad W_i(x) = 0 \quad \text{if} \quad u(x) = 0 \quad \text{for } i = 1, 2.$$

By assumption (F) for g_1 and (2.9) for g_2, $W_i \in L^\infty(0,1)$ for $i = 1$ and 2 and hence $S + W_1$, $S + W_2$ and $S + W : D_A \subset L^2 \to L^2$ are all self-adjoint operators. By (2.9) and Lemma 2.7 in [31], multiplication by W_2 defines a compact mapping from D_A into L^2 and so $\sigma_e(S + W) = \sigma_e(S + W_1)$. But $W_1(x) \geq -\ell_{g_1}$ on $(0,1)$ so $\inf \sigma_e(S + W_1) \geq \inf \sigma_e(S) - \ell_{g_1} = m_e - \ell_{g_1}$ showing that $\inf \sigma_e(S + W) \geq m_e - \ell_{g_1}$. Also $\mu < m_e - \ell_{g_1}$ is the k-th eigenvalue of S and so it follows that $\lambda_k \leq \mu$ where λ_k is the k-th eigenvalue of $S + W$ since $W(x) \leq 0$ on $(0,1)$. But $Su + Wu = \lambda u$ and $u \not\equiv 0$ since $(\lambda, u) \in \mathcal{E}$ and so $\lambda < m_e$ is an eigenvalue of $S + W$ with u as an eigenfunction. We claim that $\lambda = \lambda_k$ since u has exactly k zeros in $(0,1]$. This is a standard property of regular Strum–Liouville problems and it continues to hold in the present singular situation. A proof is given in Appendix A of [36] for the case $V = W = 0$ but it can be extended to the general case $V + W \in L^\infty(0,1)$ with only notational changes. This being so the proof of part (A) is now complete since $\lambda = \lambda_k \leq \mu$.

(B) From Corollary 3.9 and part (iv) of Proposition 3.5, $\mathcal{E} \cap (\Theta, \infty) \times D_A = \emptyset$. □

5.1 The case where $n \equiv 0$ and $g(x,s)s \geq 0$

When $n \equiv 0$ and $g(x,s)s \geq 0$, Theorem 4.10 can be used to deal with problem (1.1)(1.2) instead of Theorem 4.7. This has the advantage that the size of the Lipschitz constant for g_1 no longer plays a role and so the restriction to the interval $(-\infty, m_e - \ell_{g_1})$ in Theorem 5.1 can be avoided.

Theorem 5.5. *Suppose that assumption (S) is satisfied with $n \equiv 0$ and that the function $g = g_1 + g_2$ has the following additional properties.*

(a) $g(x,s)s \geq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$.

(b) For some $\alpha_{g_1} \geq 0$ and all $\delta > 0$ there exist $x(\delta) \in (0,1)$ and $M(\delta)$ such that $|g_1(x,s) - \alpha_{g_1} s| \leq M(\delta) + \delta|s|$ for $(x,s) \in (0, x(\delta)) \times \mathbb{R}$.

Consider $\mu \in (-\infty, m_e)$.

(A) If μ is a bifurcation point for problem (1.1)(1.2) then μ is an eigenvalue of $S = S_A + V$.

(B) If μ is the k-th eigenvalue of S then μ is a bifurcation point for (1.1)(1.2) and the component $\mathcal{C}_\mu$ of $\overline{\mathcal{E}} \cap (-\infty, m_e) \times D_A$ containing $(\mu, 0)$ is a subset of $[\mu, m_e) \times D_A$ and $\sharp(u) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$. It has at least one of the following properties.

(i) $\{\|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$.

(ii) $\sup\{\lambda : (\lambda, u) \in \mathcal{C}_\mu\} = m_e$.

Proof. For $(\lambda, u) \in \mathcal{E}$,

$$\lambda \int_0^1 u^2 dx = \int_0^1 (Su)u + \tilde{g}(u)u \, dx \geq m \int_0^1 u^2 dx, \tag{5.1}$$

showing that $\mathcal{E} \subset [m, \infty) \times D_A$.

Choose $c > \max\{0, \alpha_{g_1} - \operatorname{ess\,inf} V\}$. Then, by property (S1) in Section 2.4, $m + c > \frac{C_1}{4} + \alpha_{g_1} > 0$ and $S_c \equiv S + c$ is a positive self-adjoint operator with $D(S_c) = D_A$ as discussed at the end of Section 2.4. In particular, $\|S_c u\|_{L^2} \geq (m+c)\|u\|_{L^2}$ for all $u \in D_A$ and $\|u\|_c \equiv \|S_c u\|_{L^2}$ defines a norm, $\|\cdot\|_c$ which is equivalent to the graph norm of S on D_A. Furthermore, $D(S_c^{\frac{1}{2}}) = H_A$ and $\|S_c^{\frac{1}{2}} u\|_{L^2} \geq (m+c)^{1/2}\|u\|_{L^2}$. For $u \in D_A$,

$$\|u\|_A^2 = \int_0^1 A|\nabla u|^2 dx \leq \int_0^1 A|\nabla u|^2 + Vu^2 + cu^2 dx = \int_0^1 (S_c u)u \, dx = \|S_c^{\frac{1}{2}} u\|_{L^2}^2 \tag{5.2}$$

$$\leq \|u\|_A^2 + \|V + c\|_{L^\infty}\|u\|_{L^2}^2 \leq K_c^2 \|u\|_A^2, , \quad \text{where } K_c = \left(1 + \frac{4\|V+c\|_{L^\infty}}{C_1}\right)^{1/2} \tag{5.3}$$

by property (H1) in Section 2.1. Hence $\|u\|_A \leq \|S_c^{\frac{1}{2}} u\|_{L^2} \leq K_c \|u\|_A$ for all $u \in D_A$ and, since D_A is a dense subspace of H_A, these inequalities hold for all $u \in H_A$.

Since $S_c^{-\frac{1}{2}} \in B(L^2, H_A)$ and $\tilde{g} \in C(H_A, L^2)$ by Propositions 2.1 and 2.3, a continuous mapping $f : \mathbb{R} \times L^2 \to L^2$ is defined by

$$f(\lambda, v) = v + S_c^{-\frac{1}{2}} \tilde{g}(S_c^{-\frac{1}{2}} v) - (\lambda + c) S_c^{-1} v \quad \text{for } (\lambda, v) \in \mathbb{R} \times L^2. \tag{5.4}$$

If $f(\lambda, v) = 0$, $v \in H_A$ and consequently $u = S_c^{-\frac{1}{2}} v \in D(S_c) = D_A$ with $F(\lambda, u) = 0$, where F is defined in (2.12). Setting

$$\mathcal{S} = \{(\lambda, v) \in \mathbb{R} \times L^2 : f(\lambda, v) = 0 \text{ and } v \neq 0\},$$

it follows easily that

$$\mathcal{E} = \left\{ \left(\lambda, S_c^{-\frac{1}{2}} v\right) : (\lambda, v) \in \mathcal{S} \right\} \quad \text{and so} \quad \mathcal{S} \subset [m, \infty) \times H_A \quad \text{by (5.1).} \tag{5.5}$$

The rest of this proof involves discussing first bifurcation for the equation $f(\lambda, v) = 0$ and then deducing the desired conclusion about $F(\lambda, u) = 0$ from this.

Step 1. With $H = L^2$, equation (5.4) has the form (4.5) if we set

$$M_1(v) = v - (c - \alpha_{g_1}) S_c^{-1} v + S_c^{-\frac{1}{2}} \widetilde{g_2}(S_c^{-\frac{1}{2}} v), \quad M_2(v) = S_c^{-\frac{1}{2}} [\widetilde{g_1} - \alpha_{g_1}](S_c^{-\frac{1}{2}} v) \quad \text{and} \quad Tv = S_c^{-1} v$$

for $v \in L^2$. We aim to show that the hypotheses of Theorem 4.10 are satisfied on the interval $J = (\alpha_{g_1} - c, m_e)$. We have already shown that $\alpha_{g_1} - c < m$ so $J \neq \emptyset$.

From the choice of c we have that $T \in B(L^2, L^2)$ is a positive self-adjoint operator with $0 = \inf \sigma(T) < \sup \sigma_e(T) = (m_e + c)^{-1} \leq \sup \sigma(T) = (m + c)^{-1} = \|T\|$. If $(Tv, v)_{L^2} = 0$ and $u = S_c^{-1} v$, $0 = (u, S_c u)_{L^2} \geq (m + c)\|u\|_{L^2}^2$ so $u = 0$ and hence $v = 0$. Thus condition (W0) is satisfied and $0 \in \sigma_e(T)$.

By Proposition 2.3, $\widetilde{g_2} \in C^1(H_A, L^2)$ and so $M_1 \in C^1(L^2, L^2)$ since $S_c^{-\frac{1}{2}} \in B(L^2, H_A)$. Also $M_1'(v) = I - (c - \alpha_{g_1}) S_c^{-1} + S_c^{-\frac{1}{2}} \widetilde{g_2}'(S_c^{-\frac{1}{2}} v) S_c^{-\frac{1}{2}}$ for all $v \in L^2$ and, in particular $M_1'(0) = I - (c - \alpha_{g_1})T$ is self-adjoint. Furthermore, $M_1 - M_1'(0) = S_c^{-\frac{1}{2}} \widetilde{g_2}(S_c^{-\frac{1}{2}} \cdot) : L^2 \to L^2$ is compact, since $\widetilde{g_2} : H_A \to L^2$ is compact by Proposition 2.3 and $S_c^{-\frac{1}{2}} \in B(L^2, H_A)$. Thus condition (W1) is satisfied. In the same way it follows easily for Proposition 2.1 that $M_2 \in C(L^2, L^2)$ is compact and w-Hadamard differentiable at 0 with $M_2'(0) = -\alpha_{g_1} T$. For $v \in L^2$, we now have that $M(v) - M'(0)v = S_c^{-\frac{1}{2}} \widetilde{g}(S_c^{-\frac{1}{2}} v)$ and so

$$(M(v) - M'(0)v, v) = \int_0^1 [S_c^{-\frac{1}{2}} \widetilde{g}(S_c^{-\frac{1}{2}} v)] v \, dx = \int_0^1 \widetilde{g}(S_c^{-\frac{1}{2}} v) S_c^{-\frac{1}{2}} v \, dx \geq 0,$$

since $S_c^{-\frac{1}{2}} : L^2 \to L^2$ is self-adjoint. In view of (4.6), this shows that condition (W2) is satisfied.

Since $\lambda + c - \alpha_{g_1} > 0$ for all $\lambda \in J$, it follows from (2.15) that

$$\inf \sigma_e(M_1'(0) - \lambda T) = \inf \sigma_e(I - (\lambda + c - \alpha_{g_1})T) = \frac{m_e - \lambda + \alpha_{g_1}}{m_e + c}$$

whereas $r_e(M_2'(0)) = r_e(\alpha_{g_1} T) = \frac{\alpha_{g_1}}{m_e + c}$. Thus we see that $\inf \sigma_e(M_1'(0) - \lambda T) > r_e(M_2'(0))$ for $\lambda \in J = (\alpha_{g_1} - c, m_e)$. Let $U = J \times L^2$.

We have now verified that the hypotheses of Theorem 4.10 are satisfied in the present context and so $M'(0) - \lambda T \in \Phi_0(L^2, L^2)$ for all $\lambda \in J$ and $J \cap \sigma_e(S) = \emptyset$. Since $M'(0) - \lambda T = S_c^{-\frac{1}{2}} [S - \lambda I] S_c^{-\frac{1}{2}}$, it follows that $\dim \ker[M'(0) - \lambda T] = \dim \ker[S - \lambda I]$. Recalling that $\mathcal{S} \subset [m, \infty) \times H_A$ by (5.5) and that $\inf J < m$, it now follows from Theorem 4.10 that $\mu < m_e$ is a bifurcation point for the equation $f(\lambda, v) = 0$ if and only if $\mu \in \sigma(S)$. Furthermore, when $\mu \in \sigma(S) \cap (-\infty, m_e)$ the component $\mathcal{D}_\mu$ of $\overline{\mathcal{S}} \cap (J \times L^2)$ containing $(\mu, 0)$ has at least one of the properties (a), (b) and (c) in part (2) of Theorem 4.10. Since $\inf J < m \leq \inf p(\mathcal{D}_\mu)$ these properties can be replaced by

(i′) $\{\|v\|_{L^2} : (\lambda, v) \in \mathcal{D}_\mu\} = [0, \infty)$.

(ii′) $\sup p(\mathcal{D}_\mu) = m_e$.

(iii′) $\mathcal{D}_\mu = \mathcal{D}_\nu$ for some $\nu \in \sigma(S) \cap J$ where $\nu \neq \mu$.

Step 2. It has already been observed that $\mathcal{E} = H(\mathcal{S})$ where $H(\lambda, v) = (\lambda, S_c^{-\frac{1}{2}} v)$ for $(\lambda, v) \in$

$\mathbb{R} \times L^2$ and that $\mathcal{S} \subset [m, \infty) \times H_A$. We now show that $H : \mathcal{S} \cup [\mathbb{R} \times \{0\}] \to \mathcal{E} \cup [\mathbb{R} \times \{0\}]$ is a homeomorphism for the metrics induced by $\|\cdot\|_{L^2}$ on L^2 and $\|\cdot\|_c$ on D_A. Clearly, H is a bijection with $H^{-1}(\lambda, u) = (\lambda, S_c^{\frac{1}{2}} u)$. For $(\lambda, v), (\mu, w) \in \mathcal{S} \cup [\mathbb{R} \times \{0\}]$,

$$\begin{aligned}\|H(\lambda,v) - H(\mu,w)\|_{\mathcal{E}} &= |\lambda - \mu| + \|S_c^{-\frac{1}{2}}(v-w)\|_c = |\lambda-\mu| + \|S_c^{\frac{1}{2}}(v-w)\|_{L^2} \\ &= |\lambda-\mu| + \|(\lambda+c)S_c^{-\frac{1}{2}}v - \tilde{g}(S_c^{-\frac{1}{2}}v) - (\mu+c)S_c^{-\frac{1}{2}}w + \tilde{g}(S_c^{-\frac{1}{2}}w)\|_{L^2} \\ &\leq |\lambda-\mu|(1 + \|S_c^{-\frac{1}{2}}v\|_{L^2}) + (|\mu| + c + \ell_{g_1})\|S_c^{-\frac{1}{2}}(v-w)\|_{L^2} \\ &\quad + C\mathfrak{C}_{g_2}(\|S_c^{-\frac{1}{2}}v\|_A + \|S_c^{-\frac{1}{2}}w\|_A)\|S_c^{-\frac{1}{2}}(v-w)\|_A \quad \text{(by Remark 2.4)} \\ &\leq |\lambda-\mu|(1 + (m+c)^{-\frac{1}{2}}\|v\|_{L^2}) + \{(m+c)^{-\frac{1}{2}}(|\mu| + c + \ell_{g_1}) \\ &\quad + C\mathfrak{C}_{g_2}(\|v\|_{L^2} + \|w\|_{L^2})\}\|v-w\|_{L^2} \quad \text{(by (5.2))},\end{aligned}$$

showing that H is continuous. For the continuity of H^{-1}, consider $(\lambda, u), (\mu, z) \in \mathcal{E} \cup [\mathbb{R} \times \{0\}]$. Then

$$\begin{aligned}\|H^{-1}(\lambda,u) - H^{-1}(\mu,z)\|_{\mathcal{S}} &= |\lambda-\mu| + \|S_c^{\frac{1}{2}}(u-z)\|_{L^2} \leq |\lambda-\mu| + (m+c)^{-\frac{1}{2}}\|S_c(u-z)\|_{L^2} \\ &= |\lambda-\mu| + (m+c)^{-\frac{1}{2}}\|u-z\|_c,\end{aligned}$$

as required. At this point we can now assert that $\mathcal{C}_\mu = H(\mathcal{D}_\mu)$ and hence that $\mathcal{C}_\mu$ has at least one of the following properties.

(i′) $\{\|S_c^{\frac{1}{2}} u\|_{L^2} : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$.

(ii′) $\sup p(\mathcal{C}_\mu) = m_e$.

(iii′) $\mathcal{C}_\mu = \mathcal{C}_\nu$ for some $\nu \in \sigma(S) \cap J$ where $\nu \neq \mu$.

Recalling that $\|S_c^{\frac{1}{2}} u\|_{L^2}$ and $\|u\|_A$ define equivalent norms on H_A, it now suffices to show that property (iii′) cannot occur.

Step 3. The proof that $\sharp(u) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$ is essentially the same as for part (C) of Theorem 5.1, using part (ii) of Lemma 3.1 instead of part (i). The only difference occurs in showing that if $(\xi, 0) \in \mathcal{C}_\mu$, there is an open neighbourhood U_ξ of $(\xi, 0)$ in $\mathbb{R} \times D_A$ such that $Z(\lambda, u) = \sharp(\phi_\xi)$ for all $(\lambda, u) \in U_\xi \cap \mathcal{E}$, where ϕ_ξ is a normalised eigenfunction of S associated with ξ. To prove this we again argue by contradiction, supposing that there is a sequence $(\lambda_n, u_n) \in \mathcal{E}$ such that $\lambda_n \to \xi$ and $\|u_n\|_c \to 0$ as $n \to \infty$ and, for all $n \in \mathbb{N}$, $\sharp(u_n) \neq \sharp(\phi_\xi)$. Setting $v_n = S_c^{\frac{1}{2}} u_n$ and $\psi_\xi = S_c^{\frac{1}{2}} \phi_\xi$, we have that $(\lambda_n, v_n) \in \mathcal{S}$, $\|v_n\|_{L^2} \to 0$ as $n \to \infty$ and $M'(0)\psi_\xi = \xi T \psi_\xi$.

Setting $w_n = v_n / \|v_n\|_{L^2}$, it follows from part (C) of Theorem 4.10 that by passing to a further subsequence we can suppose that $w_n \rightharpoonup d\psi_\xi$ weakly in L^2 as $n \to \infty$ where the constant d is not equal to zero. Since $S_c^{-\frac{1}{2}} \in B(L^2, H_A)$, this implies that $S_c^{-\frac{1}{2}} w_n \rightharpoonup d S_c^{-\frac{1}{2}} \psi_\xi$ weakly in H_A. By Propositions 2.1 and 2.3, $\tilde{g} : H_A \to L^2$ is w-Hadamard differentiable at 0 with $\tilde{g}'(0) = 0$. Hence

$$\frac{\tilde{g}(S_c^{-\frac{1}{2}} v_n)}{\|v_n\|_{L^2}} = \frac{\tilde{g}(\|v_n\|_{L^2} S_c^{-\frac{1}{2}} w_n)}{\|v_n\|_{L^2}} \rightharpoonup 0 \quad \text{weakly in } L^2 \text{ as } n \to \infty.$$

But $(\lambda_n, v_n) \in \mathcal{S}$ for all n and so

$$S_c^{\frac{1}{2}} w_n = (\lambda_n + c) S_c^{-\frac{1}{2}} w_n - \frac{\tilde{g}(S_c^{-\frac{1}{2}} v_n)}{\|v_n\|_{L^2}} \rightharpoonup (\xi + c) d S_c^{-\frac{1}{2}} \psi_\xi = (\xi + c) d\phi_\xi \quad \text{weakly in } L^2.$$

Let $(\cdot,\cdot)_c$ denote the scalar product associated with the norm $\|\cdot\|_c$ on D_A. We now have that $S_c^{-\frac{1}{2}} w_n \in D_A$ for all n and, for all $u \in D_A$,

$$(S_c^{-\frac{1}{2}} w_n, u)_c = (S_c^{\frac{1}{2}} w_n, S_c u)_{L^2} \to (\xi + c) d(\phi_\xi, S_c u)_{L^2} = d(S_c \phi_\xi, S_c u)_{L^2} = d(\phi_\xi, u)_c \quad \text{as } n \to \infty.$$

Thus $q_n u_n = S_c^{-\frac{1}{2}} w_n \rightharpoonup d\phi_\xi$ weakly in D_A as $n \to \infty$, where $q_n = \|v_n\|_{L^2} > 0$ for all n. Recalling that $\{(\lambda_n, u_n)\} \subset \mathcal{E}$ with $\lambda_n \to \xi$ and $\|u_n\|_c \to 0$ as $n \to \infty$, it follows from Lemma 3.1(ii) that there exists $\eta \in (0,1)$ such that $u_n(x) \neq 0$ for $0 < x \leq \eta$ and all n. By property (S3) in Section 2.4 we can choose η so that $\phi_\xi(x) \neq 0$ for $0 < x \leq \eta$. By part (i) of Lemma 3.2, $\|P_\eta(q_n u_n) - P_\eta(d\phi_\xi)\|_\eta \to 0$ as $n \to \infty$. It now follows from part (ii) of Lemma 3.2 that there exists n_0 such that $\sharp(q_n u_n) = \sharp(d\phi_\xi)$ for all $n \geq n_0$. Since $\sharp(q_n u_n) = \sharp(u_n)$ and $\sharp(d\phi_\xi) = \sharp(\phi_\xi)$, this contradicts the initial choice of the sequence $\{(\lambda_n, u_n)\}$ and establishes the continuity of the mapping Z at $(\xi, 0)$.

As in the proof of Theorem 5.1 we can now conclude that $\sharp(u) = \sharp(\phi_\mu) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$ and consequently that property (iii′) does not occur.

To complete the proof it only remains to show that $\lambda \geq \mu$ for all $(\lambda, u) \in \mathcal{C}_\mu$. This can be done using the comparison principle self-adjoint operators just as in the proof of Corollary 5.3. Note that in this case, $W \geq 0$ on $(0,1)$ so $\inf \sigma_e(S + W) \geq \inf \sigma_e(S) = m_e$. □

Under some additional assumptions an "a priori" bound for solutions in a component $\mathcal{C}_\mu$ can be established and hence $p(\mathcal{C}_\mu) = [\mu, m_e)$.

Remark 5.6. Recall from Lemma 3.4 that assumption (b) of Theorem 5.5 implies that $I_i(g_1) = I_s(g_1) = \alpha_{g_1}$. Hence if $(\lambda, u) \in \mathcal{C}_\mu$ with $\lambda < V_0 + \alpha_{g_1}$, then $u \in L^\infty(0,1)$ by Proposition 3.5(iv). But u has only a finite number of zeros in $(0,1]$ if $(\lambda, u) \in \mathcal{C}_\mu$ and so it follows from Proposition 3.5(ii) that $\lim_{x\to 0} u(x) = \pm\infty$ if $\lambda > V_0 + J_s(g_1)$. Note that $V_0 + J_s(g_1) < m_e$ provided that $J_s(g_1) < \frac{a}{4}$. The next result exhibits a situation where $p(\mathcal{C}_\mu) = [\mu, m_e)$ and hence, if $\mu < V_0 + \alpha_{g_1}$ and $J_s(g_1) < \frac{a}{4}$, the behaviour of solutions in $\mathcal{C}_\mu$ changes as λ increases. If $(\lambda, u) \in \mathcal{C}_\mu$ with λ near μ, $u \in L^\infty(0,1)$ whereas for $\lambda \in (V_0 + J_s(g_1), m_e)$, $\lim_{x\to 0} u(x) = \pm\infty$. If $g_1(x,s) = r(x)k(s)$ where the functions r and k satisfy the conditions (R) and (K) introduced in Remark 5.4, $J_s(g_1) = I_i(g_1) = \ell_{g_1} = \alpha_{g_1} = r(0)k'(\infty)$ and the transition occurs when λ crosses $V_0 + r(0)k'(\infty)$ if $\mu < V_0 + r(0)k'(\infty)$ and $r(0)k'(\infty) < \frac{a}{4}$. Both cases $u(x) \to \infty$ and $u(x) \to -\infty$ as $x \to 0$ occur since k is odd and hence $\mathcal{C}_\mu = \{(\lambda, -u) : (\lambda, u) \in \mathcal{C}_\mu\}$. Noting that $k(s)/s$ is non-decreasing on $(0,\infty)$ with $\lim_{s\to\infty} k(s)/s = k'(\infty)$, condition (3) in Theorem 5.7 and condition (3′) in Proposition 5.8 will be satisfied in this case if $k'(\infty) > \operatorname{ess\,sup}_{0<x<1} \frac{V_0 - V(x)}{r(x)}$.

Let $t_+ = \max\{0, t\}$ for $t \in \mathbb{R}$. Observe that, since $V \in L^\infty(0,1)$, condition (2) in the following result only involves the behaviour of $V(x)$ as $x \to 0$. Assumptions (1) and (2) are satisfied in Examples 2.8 and 2.9.

Theorem 5.7. *In addition to the hypotheses of Theorem 5.5 suppose that the following conditions are satisfied.*

(1) $A \in C^1((0,1))$ *and* $\{x^{\frac{1}{2}} c(x)\}' \geq 0$ *for* $0 < x < 1$ *where* $c(x) = \frac{A(x)}{x^2} - a$.

(2) $\int_0^1 x^{-1}[V_0 - V(x)]_+ dx < \infty$.

(3) There exist $K_2 > K_1 > 0$ *such that* $V_0 \le V(x) + \frac{g(x,s)}{s}$ *for all* $x \in (0,1)$ *and* $K_1 \le x^{\frac{1}{2}}|s| \le K_2$.

For every eigenvalue μ *of* S *in* $(-\infty, m_e)$, $\sup p(\mathcal{C}_\mu) = m_e$ *where* $\mathcal{C}_\mu$ *is defined in Theorem 5.5.*

Proof. Let us suppose that $m_e - \sup p(\mathcal{C}_\mu) = \eta > 0$. In view of Theorem 5.5 it suffices to deduce from this that $\sup\{\|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} < \infty$.

Step 1. We claim that if $(\lambda, u) \in \mathcal{C}_\mu$, then $|u(x)| < K_1 x^{-\frac{1}{x}}$ for all $x \in (0,1)$ where $K_1 > 0$ is given by assumption (3). To justify this assertion, choose $K \in (K_1, K_2)$ and let $U = \{(\lambda, u) \in \mathcal{C}_\mu : x^{\frac{1}{2}}|u(x)| < K$ for all $x \in (0,1]\}$. We now show that U is both open and closed in $\mathcal{C}_\mu$.

If $(\lambda, u) \in U$, it follows that $u(1) = 0$ and, from property (P2) in Section 2.1, $x^{\frac{1}{2}}u(x) \to 0$ as $x \to 0$. Hence there exists $\varepsilon > 0$ such that $x^{\frac{1}{2}}|u(x)| \le K - \varepsilon$ for all $x \in (0,1]$. Referring again to property (P2), there exists $\delta > 0$ such that $\|x^{\frac{1}{2}}(v-u)\|_{L^\infty} < \varepsilon/2$ for $v \in D_A$ with $\|S_A(v-u)\|_{L^2} < \delta$ and hence

$$x^{\frac{1}{2}}|v(x)| \le x^{\frac{1}{2}}|u(x)| + x^{\frac{1}{2}}|v(x) - u(x)| < K - \varepsilon/2.$$

This proves that U is an open subset of $\mathcal{C}_\mu$.

To prove that it is also a closed subset of $\mathcal{C}_\mu$ consider $(\lambda, u) \in \mathcal{C}_\mu$ and a sequence $\{(\lambda_n, u_n)\}$ in U such that $\lambda_n \to \lambda$ and $\|S_A(u_n - u)\|_{L^2} \to 0$ as $n \to \infty$. By (P2), $u_n(x) \to u(x)$ for all $x \in (0,1]$ as $n \to \infty$ and so $x^{\frac{1}{2}}|u(x)| \le K$ for $0 < x \le 1$. Suppose that $\sup_{0<x\le 1} x^{\frac{1}{2}}u(x) = K$ and let (p,q) be a maximal interval such that $x^{\frac{1}{2}}u(x) > K_1$. Since $\lim_{x\to 0} x^{\frac{1}{2}}u(x) = u(1) = 0$ we have $0 < p < q < 1$ and, setting $v(x) = K_1 x^{-\frac{1}{2}}$, $u(p) = v(p)$, $u'(p) \ge v'(p)$, $u(q) = v(q)$ and $u'(q) \le v'(q)$ since $u, v \in C^1([p,q])$. Hence

$$\begin{aligned}\int_p^q (Au')'v - (Av')'u\,dx &= A[u'v - v'u]|_p^q \\ &= A(q)v(q)[u'(q) - v'(q)] - A(p)v(p)[u'(p) - v'(p)] \le 0.\end{aligned}$$

But it is easy to check that assumption (1) implies that $-(Av')' \ge \frac{a}{4}v$ on $(0,1)$. Since $u(x) > 0$ on (p,q) this yields

$$\int_p^q (Au')'v - (Av')'u\,dx \ge \int_p^q u(x)v(x)\left\{V(x) + \frac{g(x,u(x))}{u(x)} - \lambda + \frac{a}{4}\right\}dx,$$

where

$$V(x) + \frac{g(x,u(x))}{u(x)} - \lambda + \frac{a}{4} = m_e - \lambda + V(x) - V_0 + \frac{g(x,u(x))}{u(x)} \ge m_e - \lambda,$$

by assumption (3) because $K_1 < x^{\frac{1}{2}}u(x) \le K < K_2$ on (p,q). This implies that

$$\int_p^q (Au')'v - (Av')'u\,dx \ge (m_e - \lambda)\int_p^q u(x)v(x)\,dx > 0$$

since $\lambda \le \sup p(\mathcal{C}_\mu) \le m_e - \eta$, contradicting the previous conclusion.
Hence $\sup_{0<x\le 1} x^{\frac{1}{2}}u(x) < K$.

A similar argument shows that $x^{\frac{1}{2}}u(x) > -K$ for $0 < x \le 1$ and so $(\lambda, u) \in U$, proving that U is a closed subset of $\mathcal{C}_\mu$.

Clearly $(\mu, 0) \in U$ and we have now shown that U is both open and closed in $\mathcal{C}_\mu$. Since $\mathcal{C}_\mu$ is connected this means that $U = \mathcal{C}_\mu$ and hence $|u(x)| < Kx^{-\frac{1}{2}}$ for all $x \in (0,1]$ and $(\lambda, u) \in \mathcal{C}_\mu$. This completes Step 1.

Step 2. Here we prove that $\|u\|_A^2 \leq \frac{K_1^2}{\varepsilon} \int_0^1 x^{-1}[V_0 - V(x)]_+ \, dx$ for all $(\lambda, u) \in \mathcal{C}_\mu$ where $\varepsilon = \frac{1}{2} \min \left\{1, \frac{4\eta}{a}\right\}$ and $\eta = m_e - \sup p(\mathcal{C}_\mu)$.

For any $(\lambda, u) \in \mathcal{C}_\mu$,

$$\begin{aligned} \varepsilon\|u\|_A^2 &= \int_0^1 A(u')^2 dx - (1-\varepsilon)\|u\|_A^2 \leq \int_0^1 A(u')^2 dx - (1-\varepsilon) \int_0^1 ax^2(u')^2 dx \\ &\leq \int_0^1 A(u')^2 dx - (1-\varepsilon)\frac{a}{4} \int_0^1 u^2 dx \end{aligned}$$

by property (H1) in Section 2.1 since assumption (1) implies that $A(x) \geq ax^2$ for $0 \leq x \leq 1$. But $g(x,s)s \geq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$ so

$$\int_0^1 A(u')^2 dx = \int_0^1 [\lambda - V(x)]u(x)^2 - g(x, u(x))u(x) \, dx \leq \int_0^1 (\lambda - V)u^2 dx.$$

Hence

$$\begin{aligned} \varepsilon\|u\|_A^2 &\leq \int_0^1 \left\{\lambda - V(x) - (1-\varepsilon)\frac{a}{4}\right\} u(x)^2 dx = \int_0^1 \left\{\lambda - m_e + V_0 - V(x) + \frac{a\varepsilon}{4}\right\} u(x)^2 dx \\ &\leq \int_0^1 [V_0 - V(x)]_+ u(x)^2 dx \leq K_1^2 \int_0^1 x^{-1}[V_0 - V(x)]_+ \, dx \end{aligned}$$

by Step 1 since $\lambda - m_e + \frac{a\varepsilon}{4} \leq -\eta + \frac{a\varepsilon}{4} \leq 0$.

From assumption (2) it now follows that $\sup\{\|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} < \infty$ if $\sup p(\mathcal{C}_\mu) < m_e$. The conclusion follows from Theorem 5.5. □

After strengthening assumption (3) the arguments used to prove Theorem 5.7 yield an "a priori" bound for all solutions of (1.1)(1.2) with $\lambda \leq m_e - \eta$ for some $\eta > 0$, not just those in the components $\mathcal{C}_\mu$.

Proposition 5.8. *Suppose that condition (S) is satisfied with $n \equiv 0$ and that $g(x,s)s \geq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$. Assume also that the following conditions are satisfied.*

(1) $A \in C^1((0,1))$ and $\{x^{\frac{1}{2}} c(x)\}' \geq 0$ for $0 < x < 1$ where $c(x) = \frac{A(x)}{x^2} - a$.

(2) $\int_0^1 x^{-1}[V_0 - V(x)]_+ dx < \infty$.

(3') There exists $K > 0$ such that $V_0 \leq V(x) + \frac{g(x,s)}{s}$ for all $x \in (0,1)$ and $x^{\frac{1}{2}}|s| \geq K$.

Then, for every $\eta > 0$,

$$\|u\|_A^2 \leq \frac{K^2}{\delta(\eta)} \int_0^1 x^{-1}[V_0 - V(x)]_+ dx \quad \text{for all } (\lambda, u) \in \mathcal{E}_\eta \equiv \mathcal{E} \cap (-\infty, m_e - \eta) \times D_A,$$

where $\delta(\eta) = \min\{1, \frac{4\eta}{a}\}$. By Remark 5.2, this implies an "a priori" bound for $\|S_A u\|_{L^2}$ also.

Proof. Fix $\eta > 0$ and then take any $\varepsilon \in (0, \delta(\eta))$. Let $v(x) = Kx^{-\frac{1}{2}}$ where K is given by condition (3').

The argument used to prove that U is a closed subset of $\mathcal{C}_\mu$ in the proof of Theorem 5.7 shows that $|u(x)| \leq v(x)$ for all $(\lambda, u) \in \mathcal{E}_\eta$ and all $x \in (0,1)$ when condition (3) is replaced by (3'). The desired conclusion is then obtained by repeating Step 2 of that proof. □

Remark 5.9. The results in this section improve previous conclusions in Theorem 4.5(ii) of [31] about bifurcation at eigenvalues of S in the interval $(-\infty, m_e)$, even at the local level, when $n \equiv 0$ and $g(x,s)s \geq 0$. However, they do not give a complete description of all bifurcation points in this case since, as shown in Theorem 4.5(iii), bifurcation can occur at points in $[m_e, \infty)$ which are not eigenvalues of S. See also Section 6.3 of [33] for generalisations to higher dimensions.

References

[1] A. Ambrosetti, G. Prodi, *A primer of nonlinear analysis*, Cambridge University Press, Cambridge, 1993. MR1225101

[2] V. I. Averbukh, O. G. Smolyanov, The various definitions of the derivative in linear topological spaces, *Russ. Math. Surv.* **23**(1968), 67–113.

[3] P. Benevieri, A. Calamai, M. Furi, A degree theory for a class of perturbed Fredholm maps, *Fixed Point Theory Appl.* **2**(2005), 185–206. https://doi.org/10.1155/fpta.2005.185; MR2199939

[4] P. Benevieri, A. Calamai, M. Furi, A degree theory for a class of perturbed Fredholm maps. II, *Fixed Point Theory Appl.* **3**(2006), Art. ID 27154, 20 pp. https://doi.org/10.1155/FPTA/2006/27154; MR2210913

[5] P. Caldiroli, R. Musina, Existence and nonexistence results for a class of nonlinear, singular Sturm–Liouville equations, *Adv. Differential Equations* **6**(2001), 303–326. MR1799488

[6] E. A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955. MR0069338

[7] M. P. Crandall, P. H. Rabinowitz, Nonlinear Sturm–Liouville eigenvalue problems and topological degree, *J. Math. Mech.* **19**(1970), 1083–1102. MR0259232

[8] M. P. Crandall, P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Functional Analysis* **8**(1971), 321–340. https://doi.org/10.1016/0022-1236(71)90015-2; MR0288640

[9] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985. https://doi.org/10.1007/978-3-662-00547-7; MR787404

[10] D. E. Edmunds, W. D. Evans, *Spectral theory and differential operators*, Oxford Univ. Press, Oxford, 1987. MR929030

[11] G. Evéquoz, C. A. Stuart, On differentiability and bifurcation, *Adv. Math. Econ.* **8**(2006), 155–184. https://doi.org/10.1007/4-431-30899-7_6; MR2766725

[12] G. Evéquoz, C. A. Stuart, Hadamard differentiability and bifurcation, *Proc. Roy. Soc. Edinburgh Sect. A* **137**(2007), 1249–1285. https://doi.org/10.1017/S0308210506000424; MR2376879

[13] G. Evéquoz, C. A. Stuart, Bifurcation and concentration of radial solutions for a nonlinear degenerate elliptic problem, *Adv. Nonlinear Stud.* **6**(2006), 215–232. https://doi.org/10.1515/ans-2006-0206; MR2219836

[14] G. Evéquoz, C. A. Stuart, Bifurcation points of a degenerate elliptic boundary-value problem, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **17**(2006), 309–331. https://doi.org/10.4171/RLM/471; MR2287705

[15] P. M. Fitzpatrick, J. Pejsachowicz, Parity and generalized multiplicity, *Trans. Amer. Math. Soc.* **326**(1991), 281–305. https://doi.org/10.2307/2001865; MR1030507

[16] T. M. Flett, *Differential analysis*, Cambridge Univ. Press, Cambridge, 1980. MR561908

[17] R. G. Iagar, A. Sánchez, Large time behavior for a porous medium equation in a nonhomogeneous medium with critical density, *Nonlinear Anal.* **102**(2014), 226–241. https://doi.org/10.1016/j.na.2014.02.016; MR3182811

[18] R. G. Iagar, A. Sánchez, Asymptotic behaviour for the critical non-homogeneous porous medium equation in low dimensions, *J. Math. Anal. Appl.* **439**(2015), 843–863. https://doi.org/10.1016/j.jmaa.2016.03.015; MR3475955

[19] E. L. Ince, *Ordinary differential equations*, Dover, New York, 1956.

[20] N. I. Karachalios, N. B. Zographopoulos, On the dynamics of a degenerate parabolic equation: global bifurcation of stationary states and convergence, *Calc. Var. Partial Differential Equations* **25**(2006), 361–393. https://doi.org/10.1007/s00526-005-0347-4; MR2201677

[21] M. Z. Nashed, Differentiability and related properties of nonlinear operators: Some aspects of the role of differentials in nonlinear functional analysis, in: *Nonlinear Functional Anal. and Appl. (Proc. Advanced Sem., Math. Res. Center, Univ. of Wisconsin, Madison, Wis., 1970)*, Academic Press, New York, 1971, pp. 103–309. MR0276840

[22] R. D. Nussbaum, The fixed point index for local condensing maps, *Ann. Mat. Pura Appl.* **89**(1971), 217–258. https://doi.org/10.1007/BF02414948; MR312341

[23] P. H. Rabinowitz, Nonlinear Sturm–Liouville problems for second order ordinary differential equations, *Comm. Pure Appl. Math.* **23**(1970), 939–961. https://doi.org/10.1002/cpa.3160230606; MR284642

[24] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Functional Analysis* **7**(1991), 487–513. https://doi.org/10.1016/0022-1236(71)90030-9; MR0301587 .

[25] W. T. Reid, *Sturmian theory for ordinary differential equations*, Springer, Berlin, 1980. MR606199

[26] C. A. Stuart, Some global bifurcation theory for k-set contractions, *Proc. London Math. Soc. (3)* **27**(1973), 531–550. https://doi.org/10.1112/plms/s3-27.3.531; MR333856

[27] C. A. Stuart, Global properties of components of solutions of non-linear second order ordinary differential equations on the half-line, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **2**(1975), 265–286. MR380013

[28] C. A. Stuart, Buckling of a heavy tapered rod, *J. Math. Pures Appl. (9)* **80**(2001), 281–337. https://doi.org/10.1016/S0021-7824(00)01196-X; MR1826347

[29] C. A. Stuart, Bifurcation at isolated singular points of the Hadamard derivative, *Proc. Roy. Soc. Edinburgh Sect. A* **144**(2014), 1027–1065. https://doi.org/10.1017/S0308210513000486; MR3265543

[30] C. A. Stuart, Bifurcation without Fréchet differentiability at the trivial solution, *Math. Math. Appl. Sci.* **28**(2015), 3444–3463. https://doi.org/10.1002/mma.3409; MR3423708

[31] C. A. Stuart, Bifurcation points of a singular boundary-value problem on $(0,1)$, *J. Differential Equations* **260**(2016), 6267–6321. https://doi.org/10.1016/j.jde.2015.12.040; MR3456833

[32] C. A. Stuart, Stability analysis for a family of degenerate semilinear parabolic problems, *Discrete Contin. Dyn. Syst.* **38**(2018), 5297–5337. https://doi.org/10.3934/dcds.2018234; MR3834720

[33] C. A. Stuart, A critically degenerate elliptic Dirichlet problem, spectral theory and bifurcation, *Nonlinear Anal.* **190**(2020), 111620, 84 pp. https://doi.org/10.1016/j.na.2019.111620; MR4008575

[34] C. A. Stuart, Global bifurcation at isolated singular points of the Hadamard derivative, *Phil. Trans. A*, to appear.

[35] C. A. Stuart, G. Vuillaume, Buckling of a critically tapered rod: global bifurcation, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459**(2003), 1863–1889. https://doi.org/10.1098/rspa.2002.1092; MR1993662

[36] C. A. Stuart, G. Vuillaume, Buckling of a critically tapered rod: properties of some global branches of solutions, *Proc. Royal Soc. London A* **460**(2003), 3261–3282. https://doi.org/10.1098/rspa.2004.1355; MR2098717