



Permanence and exponential stability for generalised nonautonomous Nicholson systems

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Abstract. The paper is concerned with nonautonomous generalised Nicholson systems under conditions which imply their permanence: by refining the assumptions for permanence, explicit lower and upper uniform bounds for all positive solutions are provided, as well as criteria for the global exponential stability of these systems. In particular, for periodic systems, conditions for the existence of a globally exponentially attractive positive periodic solution are derived.

Keywords: delay differential equations, Nicholson systems, exponential stability, permanence.

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1 Introduction

In a recent paper [9], the permanence for a family of multidimensional nonautonomous and noncooperative delay differential equations (DDEs), which includes a large spectrum of structured models used in population dynamics and other fields, was investigated. Once the permanence is established, several questions about the global behaviour of solutions arise. To further analyse the stability and other features of such models, it is, however, clear that the conditions to be imposed depend heavily on the shape and properties of the nonlinear terms.

Nicholson-type systems constitute a specific case included in such family. Here, we consider a nonautonomous generalised Nicholson system with bounded distributed delays given by

$$\begin{aligned} x_i'(t) = & -d_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)x_j(t) \\ & + \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^t \lambda_{ik}(s)x_i(s)e^{-c_{ik}(s)x_i(s)} ds, \quad t \geq t_0, \quad i = 1, \dots, n, \end{aligned} \quad (1.1)$$

where all the coefficients and delays are continuous, nonnegative and satisfy some additional conditions described in the next section.

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Since the introduction of the classic Nicholson's blowflies equation

$$x'(t) = -dx(t) + px(t - \tau)e^{-ax(t-\tau)} \quad (a, d, p, \tau > 0), \quad (1.2)$$

by Gurney et al. [12], as a model based on the experimental data of Nicholson [18] and constructed to study the Australian sheep blowfly pest, the original equation (1.2) as well as a large number of modified and generalised scalar models have been extensively used in population dynamics and other mathematical biology contexts – yet, many open problems concerning the asymptotic behaviour of solutions to scalar Nicholson equations remain unsolved [1]. In recent years, Nicholson-type systems have received much attention in view of their applications as models for populations structured in several patches or classes (see e.g. [2] for some concrete applications). Significant progress has been made, addressing topics such as the extinction, permanence, existence of positive equilibria or periodic solutions, stability of solutions, global attractivity of equilibria or periodic solutions. Systems with autonomous coefficients (and either autonomous or time-dependent delays) were investigated in [2, 3, 6, 7, 11, 14, 25], whereas the works [4, 8–10, 15, 16, 21, 22, 24] were concerned with nonautonomous versions of such systems.

The purpose of this paper is to complement the studies in [8, 9], with more results on the large time behaviour of solutions to (1.1), by providing criteria for their global exponential stability, as well as explicit uniform lower and upper bounds for all positive solutions. The results on stability are obtained by refining the assumptions for permanence established previously in [9]. In [8], the existence of a positive periodic solution for *periodic* Nicholson's blowflies systems was analysed, and, in the case of systems with all discrete delays multiples of the period, criteria for the global attractivity of such a positive periodic solution established. Here, we provide sufficient conditions for the *exponential stability* of any positive solution of (1.1), without any constraint on the type of delays.

We emphasize that, in spite of the recent interest in *nonautonomous* Nicholson systems, only a few authors have exhibited criteria for their stability, usually for periodic or almost periodic Nicholson equations or systems with *discrete* time-delays; see [5, 8, 13, 15–17, 21, 23, 24] and references therein. Typically, conditions have been imposed in such a way that convenient lower and upper bounds for all solutions hold. Here, as we shall see, the permanence is still a key ingredient to prove the stability, however, only an explicit *upper bound* for solutions of such systems will be required. The criteria enhance and extend some recent achievements in the literature in several ways: not only are the imposed assumptions less restrictive than the ones found in recent papers, but (1.1) is much more general: namely, it incorporates *distributed* delays, not all coefficients are required to be bounded and the global exponential stability is studied for a model that is not necessarily periodic or almost periodic.

This paper is organized as follows: Section 2 is devoted to the study of uniform lower and upper bounds for the positive solutions of (1.1). Section 3 addresses the global stability of (1.1). Examples and a comparison with recent results in the literature [13, 16, 21, 23] are also given, in particular for periodic systems. A brief section of conclusions ends the paper.

2 Permanence: uniform bounds for the solutions

For simplicity of exposition, and without loss of generality, take $t_0 = 0$ in (1.1) and let $\tau = \sup\{\tau_{ik}(t) : t \geq 0, i = 1, \dots, n, k = 1, \dots, m_i\} > 0$. Take $C := C([- \tau, 0]; \mathbb{R}^n)$ with the supremum norm $\|\phi\| = \max_{\theta \in [- \tau, 0]} |\phi(\theta)|$ as the phase space. In abstract form, system (1.1) is

written as the DDE

$$x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)x_j(t) + f_i(t, x_{i,t}), \quad t \geq 0, \quad i = 1, \dots, n, \quad (2.1)$$

where the nonlinearities take the form

$$f_i(t, x_{i,t}) = \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^t \lambda_{ik}(s)x_i(s)e^{-c_{ik}(s)x_i(s)} ds, \quad i = 1, \dots, n. \quad (2.2)$$

For (1.1), define the $n \times n$ matrices

$$\begin{aligned} D(t) &= \text{diag}(d_1(t), \dots, d_n(t)), \quad A(t) = [a_{ij}(t)] \\ B(t) &= \text{diag}(\beta_1(t), \dots, \beta_n(t)), \quad t \geq 0, \end{aligned} \quad (2.3)$$

where we may suppose that $a_{ii}(t) \equiv 0$ (since $a_{ii}(t)$ may be incorporated in $d_i(t)$) and $\beta_i(t)$ denotes

$$\beta_i(t) := \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^t \lambda_{ik}(s) ds, \quad t \geq 0, \quad i = 1, \dots, n;$$

The following assumptions will be considered:

- (h1) $d_i(t), a_{ij}(t), b_{ik}(t), \tau_{ik}(t), \lambda_{ik}(t), c_{ik}(t)$ are continuous and nonnegative with $d_i(t) > 0$, $c_{ik}(t) \geq \underline{c}_i > 0$, $\beta_i(t) > 0$, $\tau_{ik}(t) \in [0, \tau]$, $c_{ik}(t)$ are bounded, for $i, j = 1, \dots, n, k = 1, \dots, m_i$ and $t \geq 0$;
- (h2) there is a positive vector u such that $\liminf_{t \rightarrow \infty} [D(t) - A(t)]u > 0$;
- (h3) there are a positive vector v and $T > 0, \alpha > 1$ such that $B(t)v \geq \alpha[D(t) - A(t)]v$ for $t \geq T$.

The particular case of (1.1) with $c_{ik}(t) \equiv 1$ for $1 \leq i \leq n, 1 \leq k \leq m_i$, is expressed by

$$x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^t \lambda_{ik}(s)h(x_i(s)) ds, \quad 1 \leq i \leq n, \quad (2.4)$$

for $h(x) = xe^{-x}$, $x \geq 0$. Note that the nonlinearity h is unimodal, $e^{-1} = h(1) = \max_{x \geq 0} h(x)$, $h(\infty) = 0$ and $x = 2$ is its unique inflexion point.

We now set the usual orders in \mathbb{R}^n and C . \mathbb{R}^n may be seen as the subset of constant functions in C . We suppose that \mathbb{R}^n is equipped with the maximum norm $|\cdot|$. Let $\mathbb{R}^+ = [0, \infty)$. A vector $v \in \mathbb{R}^n$ is nonnegative, with notation $v \geq 0$ (respectively, positive, denoted by $v > 0$), if $v \in (\mathbb{R}^+)^n$ (respectively $v \in (0, \infty)^n$). We denote $\vec{1} = (1, \dots, 1)$. Consider the cone $C^+ = C([- \tau, 0]; (\mathbb{R}^+)^n)$ of nonnegative functions in C and the partial order in C yielded by C^+ : $\phi \leq \psi$ if and only if $\psi - \phi \in C^+$. Thus, $\phi \geq 0$ if and only if $\phi \in C^+$. We write $\phi > 0$ if $\phi(\theta) > 0$ for $-\tau \leq \theta \leq 0$. The relations \leq and $<$ are defined in the obvious way. For $u, v \in \mathbb{R}^n$ with $u \leq v$, $[u, v] \subset C$ denotes the ordered interval $[u, v] = \{\phi \in C : u \leq \phi \leq v\}$.

Due to the real-world interpretation of our models, we take

$$C_0^+ = \{\phi \in C^+ : \phi(0) > 0\}$$

as the set of admissible initial conditions, and only consider solutions $x(t) = x(t, t_0, \phi)$ of (1.1) with initial conditions $x_{t_0} = \phi$, $\phi \in C_0^+$. It is clear that such solutions are defined and positive on \mathbb{R}^+ .

The definitions of permanence and global stability are recalled below.

Definition 2.1. Consider a DDE $x'(t) = f(t, x_t)$ in C for which all solutions $x(t) = x(t, 0, \phi)$ with $\phi \in C_0^+$ are defined on \mathbb{R}^+ . The DDE is said to be **permanent** if there exist positive constants m, M such that all solutions $x(t) = x(t, 0, \phi)$ with $\phi \in C_0^+$ satisfy

$$m \leq \liminf_{t \rightarrow \infty} x_i(t), \quad \limsup_{t \rightarrow \infty} x_i(t) \leq M \quad \text{for } i = 1, \dots, n.$$

For short, we say that $x'(t) = f(t, x_t)$ is **globally attractive** (in C_0^+) if all positive solutions are globally attractive: for any $\phi, \psi \in C_0^+$,

$$x(t, 0, \phi) - x(t, 0, \psi) \rightarrow 0 \quad \text{as } t \rightarrow \infty;$$

and the DDE $x'(t) = f(t, x_t)$ is said to be (eventually) **globally exponentially stable** if there exist $\delta > 0, M > 0$ such that, for any $\phi \in C_0^+$, there is $T \geq 0$ such that

$$|x(t, t_0, \phi) - x(t, t_0, \psi)| \leq M e^{-\delta(t-t_0)} \|\phi - \psi\|, \quad \text{for } t \geq t_0 \geq T, \psi \in C_0^+.$$

Note that δ, M do not depend on t_0, ϕ , though a priori T depends on ϕ .

Although the nonlinear terms in (1.1) are nonmonotone, results for cooperative systems from [19] will be used.

Definition 2.2. A DDE $x'(t) = f(t, x_t)$ is **cooperative** if $f = (f_1, \dots, f_n)$ satisfies the *quasi-monotone condition* (Q) in [19], as follows:

if $\phi, \psi \in C^+$ and $\phi \geq \psi$, then $f_i(t, \phi) \geq f_i(t, \psi)$ for $t \geq 0$, whenever $\phi_i(0) = \psi_i(0)$ for some i .

In [9], the permanence of generalised Nicholson systems was established.

Theorem 2.3 ([9, Corollary 3]). *Assume (h1)–(h3) and that $\beta_i(t)$ are bounded on \mathbb{R}^+ . Then (1.1) is permanent.*

Remark 2.4. When $\liminf_{t \rightarrow \infty} d_i(t) > 0$, for all i , Theorem 2.3 is still valid if one replaces (h2) by the assumptions $D(t)u \geq \alpha A(t)u, t \gg 1$, for some vector $u > 0$ and constant $\alpha > 1$. Similarly, (h3) can be replaced by the condition $\liminf_{t \rightarrow \infty} [B(t) - D(t) + A(t)]v > 0$, for some vector $v > 0$, when $\beta_i(t)$ are all bounded. In fact, if $\beta_i(t)$ are bounded below and above by positive constants, for all i , conditions $\liminf_{t \rightarrow \infty} [B(t) - D(t) + A(t)]v > 0$ and (h3) are equivalent. See [9] for details.

Remark 2.5. In fact, instead of (2.1), more general Nicholson systems with possible delays in the linear terms were considered in [9]:

$$x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^n L_{ij}(t)x_{j,t} + f_i(t, x_{i,t}), \quad t \geq 0, i = 1, \dots, n, \quad (2.5)$$

where f_i are as in (2.2) and $L_{ij}(t)$ are linear bounded functionals, *nonnegative* (i.e. $L_{ij}(t)(\psi) \geq 0$ for $\psi \in C([- \tau, 0]; \mathbb{R}^+)$) and continuous in t . With $\|L_{ij}(t)\| = a_{ij}(t)$, the permanence of such systems was also established in [9], if in addition to (h1)–(h3) $a_{ij}(t)$ are bounded and $\beta_i(t)$ bounded below and above by positive constants.

When (h2) and (h3) are satisfied simultaneously by a same vector $v = (v_1, \dots, v_n) > 0$, there are δ, α such that

$$\liminf_{t \rightarrow \infty} \left(d_i(t)v_i - \sum_j a_{ij}(t)v_j \right) \geq \delta > 0, \quad \liminf_{t \rightarrow \infty} \frac{\beta_i(t)v_i}{d_i(t)v_i - \sum_j a_{ij}(t)v_j} \geq \alpha > 1, \quad i = 1, \dots, n.$$

This motivates the following definition: for $t \geq 0$ and $v = (v_1, \dots, v_n) > 0$ such that $[D(t) - A(t)]v \neq 0$, set

$$\gamma_i(t, v) = \frac{\beta_i(t)v_i}{d_i(t)v_i - \sum_j a_{ij}(t)v_j}, \quad i = 1, \dots, n. \quad (2.6)$$

For the particular case $v = \vec{1} := (1, \dots, 1)$, we obtain

$$\gamma_i(t) := \gamma_i(t, \vec{1}) = \frac{\beta_i(t)}{d_i(t) - \sum_j a_{ij}(t)}, \quad i = 1, \dots, n. \quad (2.7)$$

Next result gives sufficient conditions, expressed in terms of $\gamma_i(t, v)$, for the positive invariance of some specific intervals under (1.1), and also provides explicit uniform lower and upper bounds for all solutions.

Theorem 2.6. For (1.1), assume (h1), and that $c_{ik}(t)$ are bounded below and above on \mathbb{R}^+ by positive constants, and denote $\underline{c}_i, \bar{c}_i$ such that

$$0 < \underline{c}_i \leq c_{ik}(t) \leq \bar{c}_i \quad \text{for } t \in \mathbb{R}^+, \quad 1 \leq i \leq n, \quad 1 \leq k \leq m_i.$$

Suppose that there are constants a, b with $0 < a \leq b$, $t_0 \geq 0$ and a vector $v = (v_1, \dots, v_n) > 0$ such that

$$e^a \leq \gamma_i(t, v) \leq e^b, \quad 1 \leq i \leq n, \quad t \geq t_0, \quad (2.8)$$

and define

$$\underline{C} = \underline{C}(v) := \min_{1 \leq i \leq n} (\underline{c}_i v_i), \quad \bar{C} = \bar{C}(v) := \max_{1 \leq i \leq n} (\bar{c}_i v_i). \quad (2.9)$$

Then:

(a) The ordered interval $[m\bar{C}^{-1}v, \underline{C}^{-1}e^{b-1}v] = \{\phi = (\phi_1, \dots, \phi_n) \in C : m\bar{C}^{-1}v_i \leq \phi_i \leq \underline{C}^{-1}e^{b-1}v_i, i = 1, \dots, n\} \subset C$, where $m\underline{C}^{-1}\bar{C} \in (0, 1)$ is such that

$$m \leq a \quad \text{and} \quad h(m\bar{c}_i v_i \underline{C}^{-1}) \leq h(\bar{c}_i v_i \underline{C}^{-1} e^{b-1}), \quad i = 1, \dots, n, \quad (2.10)$$

is positively invariant for (1.1) and $t \geq t_0$.

(b) If $\beta_i(t)$ are also bounded below and above by positive constants, any positive solution $x(t) = (x_1(t), \dots, x_n(t))$ of (1.1) satisfies

$$m\bar{C}^{-1}v_i \leq \liminf_{t \rightarrow \infty} x_i(t) \leq \limsup_{t \rightarrow \infty} x_i(t) \leq e^{b-1}\underline{C}^{-1}v_i, \quad i = 1, \dots, n. \quad (2.11)$$

Proof. (a) Write (1.1) as $x'(t) = F(t, x_t)$, with the components F_i of F given by

$$F_i(t, \phi) = -d_i(t)\phi_i(0) + \sum_j a_{ij}(t)\phi_j(0) + \sum_{k=1}^{m_i} b_{ik}(t) \int_{-\tau_{ik}(t)}^0 \lambda_{ik}(t+s) h_{ik}(t+s, \phi_i(s)) ds, \quad i = 1, \dots, n,$$

where $h_{ik}(s, x) := xe^{-c_{ik}(s)x}$. Let $\underline{c}_i, \bar{c}_i$ be such $0 < \underline{c}_i \leq c_{ik}(s) \leq \bar{c}_i$ for $s \in \mathbb{R}^+$ and all i, k , and take the functions $h_i^-(x) = xe^{-\bar{c}_i x}, h_i^+(x) = xe^{-\underline{c}_i x}$. For $h(x) = xe^{-x}$ as before, we have $h_i^-(x) = (\bar{c}_i)^{-1}h(\bar{c}_i x), h_i^+(x) = (\underline{c}_i)^{-1}h(\underline{c}_i x)$. Clearly, $h_i^-(x) \leq h_{ik}(s, x) \leq h_i^+(x) \leq (\underline{c}_i e)^{-1}$, for $s, x \geq 0$.

We know already that the set $(0, \infty)^n$ is forward invariant. We now compare the solutions of (1.1) from above with the solutions of the *cooperative* system $x'(t) = F^u(t, x_t)$, where the components of F^u are given by $F_i^u(t, \phi) = -d_i(t)\phi_i(0) + \sum_j a_{ij}(t)\phi_j(0) + \beta_i(t)(\underline{c}_i e)^{-1}$. Clearly $F_i(t, \phi) \leq F_i^u(t, \phi)$ for all $\phi \in C^+$. From [19], this implies that $x(t, t_0, \phi, F) \leq x(t, t_0, \phi, F^u)$, where $x(t, t_0, \phi, F)$ and $x(t, t_0, \phi, F^u)$ are the solutions of $x'(t) = F(t, x_t)$ and $x'(t) = F^u(t, x_t)$ with initial condition $x_{t_0} = \phi \in C_0^+$, respectively. If $\phi \in [0, \underline{C}^{-1}e^{b-1}v]$ and $\phi_i(0) = \underline{C}^{-1}e^{b-1}v_i$ for some i , the use of (2.8) implies

$$\begin{aligned} F_i^u(t, \phi) &\leq \underline{C}^{-1}e^{b-1} \left[-d_i(t)v_i + \sum_j a_{ij}(t)v_j \right] + \beta_i(t)(\underline{c}_i e)^{-1} \\ &\leq \left[d_i(t)v_i - \sum_j a_{ij}(t)v_j \right] \left[-\underline{C}^{-1}e^{b-1} + \gamma_i(t, v)(\underline{c}_i v_i e)^{-1} \right] \\ &\leq e^{b-1} \left[d_i(t)v_i - \sum_j a_{ij}(t)v_j \right] (-\underline{C}^{-1} + (\underline{c}_i v_i)^{-1}) \leq 0. \end{aligned}$$

From [19, p. 82], the set $(0, \underline{C}^{-1}e^{b-1}v] \subset C$ is positively invariant for (1.1).

Next, we start by observing that, for any $a, b > 0$ with $a \leq b$, we have $a < e^{a-1} \leq e^{b-1}$ for all $a \neq 1$. By considering the cases $a < e^{b-1} \leq 1, a < 1 \leq e^{b-1}$ or $1 \leq a < e^{b-1}$, it is possible to choose $m \in (0, \underline{C}^{-1})$ such that conditions (2.10) are fulfilled. We get

$$h_i^-(\underline{C}^{-1}e^{b-1}v_i) = (\bar{c}_i)^{-1}h(\underline{C}^{-1}e^{b-1}\bar{c}_i v_i) \geq (\bar{c}_i)^{-1}h(m\underline{C}^{-1}\bar{c}_i v_i) = h_i^-(m\underline{C}^{-1}v_i).$$

As $1 > m\bar{c}_i v_i \underline{C}^{-1}$ and h is increasing on $(0, 1)$, for ϕ_i such that $m\underline{C}^{-1}v_i \leq \phi_i(s) \leq \underline{C}^{-1}e^{b-1}v_i$ we therefore obtain

$$h_i^-(\phi_i(s)) \geq h_i^-(m\underline{C}^{-1}v_i)$$

and $F_i(t, \phi) \geq F_i^l(t, \phi) := -d_i(t)\phi_i(0) + \sum_j a_{ij}(t)\phi_j(0) + \beta_i(t)h_i^-(m\underline{C}^{-1}v_i)$ for $i = 1, \dots, n$.

Consider the interval $\hat{I} = [m\underline{C}^{-1}v, \underline{C}^{-1}e^{b-1}v] \subset C$. For $\phi \in \hat{I}$ with $\phi_i(0) = m\underline{C}^{-1}v_i$ for some i , the lower bound in (2.8) leads to

$$\begin{aligned} F_i^l(t, \phi) &\geq \left[d_i(t)v_i - \sum_j a_{ij}(t)v_j \right] \left[-m\underline{C}^{-1} + \gamma_i(t, v)v_i^{-1}h_i^-(m\underline{C}^{-1}v_i) \right] \\ &\geq \left[d_i(t)v_i - \sum_j a_{ij}(t)v_j \right] \left[-m\underline{C}^{-1} + \gamma_i(t, v)(v_i \bar{c}_i)^{-1}h(m\underline{C}^{-1}\bar{c}_i v_i) \right] \\ &= m\underline{C}^{-1} \left[d_i(t)v_i - \sum_j a_{ij}(t)v_j \right] \left[-1 + \gamma_i(t, v)e^{-m\underline{C}^{-1}\bar{c}_i v_i} \right] \\ &\geq m\underline{C}^{-1} \left[d_i(t)v_i - \sum_j a_{ij}(t)v_j \right] [-1 + e^a e^{-m}] \geq 0. \end{aligned}$$

Hence, from [19] it follows that \hat{I} is positively invariant for (1.1).

(b) Next, assume also that $0 < \underline{\beta} \leq \beta_i(t) \leq \bar{\beta}$ for $t \geq 0$. From (2.8),

$$d_i(t)v_i - \sum_j a_{ij}(t)v_j \geq e^{-b}\underline{\beta}v_i, \quad \beta_i(t)v_i \geq e^a[d_i(t)v_i - \sum_j a_{ij}(t)v_j],$$

hence (h2)–(h3) are satisfied. From Theorem 2.3, (2.4) is permanent.

Fix any positive solution $x(t)$ of (2.4), define

$$\bar{x}_i = \limsup_{t \rightarrow \infty} x_i(t), \quad i = 1, \dots, n,$$

and let $\max_j (v_j^{-1} \bar{x}_j) = v_i^{-1} \bar{x}_i$ for some i . By the fluctuation lemma, there exists a sequence $t_k \rightarrow \infty$ such that $x_i(t_k) \rightarrow \bar{x}_i$ and $x'_i(t_k) \rightarrow 0$. Without loss of generality, we can also suppose that $v_i^{-1} x_i(t_k) = \max_{1 \leq j \leq n} v_j^{-1} x_j(t_k)$ for k large – otherwise, we choose $t_k \rightarrow \infty$ such that, for some subsequence, $v_i^{-1} x_i(t_k) = \max_{1 \leq j \leq n} \max_{t \in [k\tau, (k+1)\tau]} v_j^{-1} x_j(t)$. Thus, reasoning as in (a),

$$\begin{aligned} x'_i(t_k) &\leq -d_i(t_k)x_i(t_k) + \sum_{j=1}^n a_{ij}(t_k)v_i^{-1}v_j x_i(t_k) + \beta_i(t_k)(\underline{c}_i e)^{-1} \\ &\leq v_i^{-1} \left(d_i(t_k)v_i - \sum_{j=1}^n a_{ij}(t_k)v_j \right) \left[-x_i(t_k) + \gamma_i(t, v)(\underline{c}_i e)^{-1} \right] \\ &\leq v_i^{-1} \left(d_i(t_k)v_i - \sum_{j=1}^n a_{ij}(t_k)v_j \right) \left[-x_i(t_k) + e^{b-1}(\underline{c}_i v_i)^{-1}v_i \right] \\ &\leq v_i^{-1} \left(d_i(t_k)v_i - \sum_{j=1}^n a_{ij}(t_k)v_j \right) \left[-x_i(t_k) + e^{b-1}\underline{C}^{-1}v_i \right]. \end{aligned} \tag{2.12}$$

Consider a subsequence of (t_k) , still denoted by (t_k) , for which $d_i(t_k) - \sum_{j=1}^n a_{ij}(t_k) \rightarrow \ell > 0$. By letting $k \rightarrow \infty$, we obtain $0 \leq -\bar{x}_i + e^{b-1}\underline{C}^{-1}v_i$, thus $\bar{x}_i \leq e^{b-1}\underline{C}^{-1}v_i$. For $j \neq i$, it follows that $\bar{x}_j \leq v_j v_i^{-1} \bar{x}_i \leq e^{b-1}\underline{C}^{-1}v_j$.

Proceeding as in (a), in a similar way one can now show that $\liminf_{t \rightarrow \infty} x(t) \geq m\bar{C}^{-1}v$ for all positive solutions. This proves (b). \square

Remark 2.7. For the simpler case (2.4), where the nonlinearities are all given in terms of $h(x) = xe^{-x}$, under (h1) and

$$e^a \leq \gamma_i(t) \leq e^b, \quad 1 \leq i \leq n, \quad t \geq t_0 \tag{2.13}$$

(i.e., $v = \bar{1}$ in $\gamma_i(t, v)$), we have $\bar{C} = \underline{C} = 1$; thus, the interval $[m, e^{b-1}]^n$ is forward invariant, where $m > 0$ is chosen so that $m < 1, m \leq a$ and $h(m) \leq h(e^{b-1})$.

We also derive the following auxiliary result.

Lemma 2.8. For (1.1), assume (h1) and that $0 < c_i \leq c_{ik}(t) \leq \bar{c}_i$ for $t \in \mathbb{R}^+, 1 \leq i \leq n, 1 \leq k \leq m_i$. Suppose also that there are a vector $v = (v_1, \dots, v_n) > 0, t \geq t_0$ and a constant γ such that

$$0 < \gamma_i(t, v) \leq \gamma, \quad 1 \leq i \leq n, \quad t \geq t_0. \tag{2.14}$$

For \underline{C}, \bar{C} as in (2.9), the interval $(0, \gamma(\underline{C}e)^{-1}v] \subset C$ is positively invariant for (1.1) ($t \geq t_0$). In particular, if (2.14) holds with

$$\gamma < 2e\underline{C}\bar{C}^{-1},$$

there exist solutions of (1.1) such that $0 < x_i(t) < 2(\bar{c}_i)^{-1}, t \geq t_0, i = 1, \dots, n$.

Proof. The invariance of the interval $I := (0, \gamma(\underline{C}e)^{-1}v]$ for (1.1) was shown in the above proof. If in addition $\gamma < 2e\underline{C}\bar{C}^{-1}$, then $I \subset (0, 2\bar{C}^{-1}v)$, and in particular the solutions with initial conditions $\phi \in I$ satisfy $0 < \bar{c}_i x_i(t) < 2$ for $t \geq 0, 1 \leq i \leq n$. \square

Remark 2.9. Consider e.g. the Nicholson system (2.4). If $0 < \gamma_i(t, v) \leq e^b v_i$ for all i , for some $b > 0$ and a vector $v = (v_1, \dots, v_n) > 0$, from the proof of Theorem 2.6 the interval $(0, e^{b-1}v]$ is positively invariant. With $v = \vec{1}$ and $0 < \gamma_i(t) \leq \gamma < e$ and the boundedness conditions in Theorem 2.6 (b), $\limsup_{t \rightarrow \infty} x_i(t) \leq \gamma e^{-1} < 1$ for all positive solution; this means that (1.1) has a cooperative behaviour, because the nonlinearity $h(x)$ is monotone on $[0, 1]$. Note, however, that (2.8) with e.g. $v = \vec{1}$ and $e < \gamma = e^b$ does not imply that the interval $[1, b]^n$ is positively invariant. In fact, for simplicity take $n = 1$ and consider the Nicholson equation $x'(t) = -d(t)x(t) + e^b d(t)x(t - \tau)e^{-x(t-\tau)}$, for some $b > 1$. For an initial condition $1 \leq \phi \leq b$ such that $\phi(0) = b$ and $\phi(-\tau) = 1$, then $x'(0) = d(0)[-b + e^{b-1}] = d(0)e^b[-h(b) + h(1)] > 0$, thus $x(t) > b$ for $t > 0$ sufficiently small. Nevertheless, we conjecture that if (2.13) is satisfied with $\gamma < e^2$ and all coefficients are bounded, then all positive solutions of (2.4) satisfy $\limsup_{t \rightarrow \infty} x_i(t) < 2$ for all i . See also Remark 3.9.

3 Stability

In this section, sufficient conditions for the global exponential stability of Nicholson systems (1.1) are established.

In the sequel, the following auxiliary lemma will play an important role.

Lemma 3.1 ([8]). Fix $m \in (0, 1)$ and define $G_m : (0, 2) \times [0, \infty) \rightarrow \mathbb{R}$ by

$$G_m(x, y) = \begin{cases} \frac{h(y) - h(x)}{y - x}, & y \neq x \\ (1 - x)e^{-x}, & y = x \end{cases}$$

where $h(x) = xe^{-x}$, $x \geq 0$. Then, $G_m(x, y)$ is continuous and, for any $x \in (0, 2)$, there is $\mathfrak{M}_m(x) := \max_{y \geq m} |G_m(x, y)| < e^{-x}$.

As a consequence, for a function $h_c(x) := xe^{-cx} = c^{-1}h(cx)$ for some $c > 0$, it follows that for any fixed $x \in (0, 2c^{-1})$ and $m \in (0, c^{-1})$, we have

$$|h_c(y) - h_c(x)| \leq \mathfrak{M}_m(cx)|y - x| \quad \text{for all } y \geq m, \quad (3.1)$$

where $\mathfrak{M}_m(x)$ is the function defined in the lemma above. Moreover, $\mathfrak{M}_m : (0, 2) \rightarrow (0, e^{-2})$ is continuous.

We first establish a criterion for the global attractivity of (1.1).

Theorem 3.2. Consider (1.1) under (h1)–(h3) and suppose that the coefficients $\beta_i(t), c_{ik}(t)$ are all bounded below and above by positive constants on \mathbb{R}^+ , for all i, k . Assume in addition that there exists a positive solution $x^*(t)$ such that

$$\limsup_{t \rightarrow \infty} c_{ik}(t)x_i^*(t) < 2, \quad i = 1, \dots, n, \quad k = 1, \dots, m_i. \quad (3.2)$$

Then, any two positive solutions $x(t), y(t)$ of (1.1) satisfy

$$\lim_{t \rightarrow \infty} (x(t) - y(t)) = 0.$$

Proof. From Theorem 2.3, system (1.1) is permanent. Let $h_{ik}(t, x) = xe^{-c_{ik}(t)x}$ for $t, x \geq 0$ and all i, k .

Write $0 < \underline{\beta} \leq \beta_i(t) \leq \bar{\beta}, 0 < \underline{c} \leq c_i \leq c_{ik}(t) \leq \bar{c}_i \leq \bar{c}$ for $t \in \mathbb{R}^+$ and all i, k . From the permanence of (1.1), there are m, M with $0 < m < 1 \leq M$, such that any solution $x(t) = x(t, 0, \phi)$ with $\phi \in C_0^+$ satisfies $m \leq x_i(t) \leq M$ for $i = 1, \dots, n$ and $t \geq T$, for some $T = T(\phi) > 0$. Fix a positive solution $x^*(t)$ as in (3.2), let $m_0 := \underline{c}m$ and $\varepsilon > 0$ small, so that $m_0 \leq c_{ik}(t)x_i^*(t) \leq 2 - \varepsilon$ for all $i = 1, \dots, n, k = 1, \dots, m_i$ and $t \gg 1$. In Lemma 3.1, take the function $\mathfrak{M} := \mathfrak{M}_{m_0}$.

Effecting the changes of variables $z_i(t) = \frac{x_i(t)}{x_i^*(t)} - 1$ ($1 \leq i \leq n$), system (1.1) becomes

$$\begin{aligned} z_i'(t) &= \frac{1}{x_i^*(t)} \left\{ x_i'(t) - (1 + z_i(t))(x_i^*)'(t) \right\} \\ &= \frac{1}{x_i^*(t)} \left\{ -d_i^*(t)z_i(t) + \sum_j a_{ij}(t)x_j^*(t)z_j(t) \right. \\ &\quad \left. + \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^t \lambda_{ik}(s) \left[h_{ik}(s, x_i^*(s)(1 + z_i(s))) - h_{ik}(s, x_i^*(s)) \right] ds \right\}, \end{aligned} \quad (3.3)$$

for $i = 1, \dots, n, t \geq 0$, where

$$d_i^*(t) = \sum_j a_{ij}(t)x_j^*(t) + \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^t \lambda_{ik}(s)h_{ik}(s, x_i^*(s)) ds.$$

Let $z(t) = (z_1(t), \dots, z_n(t))$ be any solution of (3.3) with initial condition $z_0 \geq -1, z(0) > -1$. Define $-v_i = \liminf_{t \rightarrow \infty} z_i(t), u_i = \limsup_{t \rightarrow \infty} z_i(t)$. From the permanence of (1.1), in particular $-1 < -v_i \leq u_i < \infty$ and, as observed, $x_i^*(t) \geq m$ and $x_i^*(t)(1 + z_i(t)) \geq m$ for $t > 0$ large. Consider $u = \max_i u_i, v = \max_i v_i$. A priori, $-v, u$ can be both nonnegative, both nonpositive, or have different signs, nevertheless it is sufficient to show that $\max(u, v) = 0$.

Let $\max(u, v) = u$. In this case, $u \geq 0$. Assume for the sake of contradiction that $u > 0$. Choose i such that $u = u_i$ and take a sequence $t_k \rightarrow \infty$ with $z_i(t_k) \rightarrow u, z_i'(t_k) \rightarrow 0$.

From (3.1), we have

$$\left| h_{ip}(s, x_i^*(s)(1 + z_i(s))) - h_{ip}(s, x_i^*(s)) \right| \leq \mathfrak{M}(c_{ip}(s)x_i^*(s))x_i^*(s)|z_i(s)|,$$

for $1 \leq i \leq n, 1 \leq p \leq m_i$ and $s \geq 0$ sufficiently large. As previously, for k large we may suppose that $z_j(t_k) \leq z_i(t_k)$ for all j , and from (3.3) we get

$$\begin{aligned} z_i'(t_k) &\leq \frac{1}{x_i^*(t_k)} \left[-d_i^*(t_k) + \sum_j a_{ij}(t_k)x_j^*(t_k) \right] z_i(t_k) \\ &\quad + \sum_{p=1}^{m_i} b_{ip}(t_k) \int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s) \mathfrak{M}(c_{ip}(s)x_i^*(s))x_i^*(s)|z_i(s)| ds \\ &= \frac{1}{x_i^*(t_k)} \left\{ -z_i(t_k) \sum_{p=1}^{m_i} b_{ip}(t_k) \int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s)h_{ip}(s, x_i^*(s)) ds \right. \\ &\quad \left. + \sum_{p=1}^{m_i} b_{ip}(t_k) \int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s) \mathfrak{M}(c_{ip}(s)x_i^*(s))x_i^*(s)|z_i(s)| ds \right\} \\ &= \frac{1}{x_i^*(t_k)} \sum_{p=1}^{m_i} b_{ip}(t_k) \int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s)x_i^*(s) \left[-z_i(t_k)e^{-c_{ip}(s)x_i^*(s)} + \mathfrak{M}(c_{ip}(s)x_i^*(s))|z_i(s)| \right] ds. \end{aligned} \quad (3.4)$$

By the mean value theorem for integrals, we obtain

$$z'_i(t_k) \leq \frac{1}{x_i^*(t_k)} \sum_{p=1}^{m_i} x_i^*(s_{k,p}) B_{kp} b_{ip}(t_k) \int_{t_k - \tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s) ds, \quad (3.5)$$

where

$$B_{kp} = -z_i(t_k) e^{-c_{ip}(s_{k,p}) x_i^*(s_{k,p})} + \mathfrak{M}(c_{ip}(s_{k,p}) x_i^*(s_{k,p})) |z_i(s_{k,p})|,$$

for some $s_{k,p} \in [t_k - \tau_{ip}(t_k), t_k]$.

For some subsequence of $(s_{k,p})_{k \in \mathbb{N}}$ ($1 \leq p \leq m_i$), still denoted by $(s_{k,p})$, there exist the limits $\lim_k c_{ip}(s_{k,p}) x_i^*(s_{k,p}) = \zeta_p \in [m_0, 2 - \varepsilon]$ and $\lim_k z_i(s_{k,p}) = w_p \in [-v, u]$. Since $\mathfrak{M}(x)$ is continuous, this leads to

$$\lim_k B_{kp} = -u e^{-\zeta_p} + \mathfrak{M}(\zeta_p) |w_p| \leq (-e^{-\zeta_p} + \mathfrak{M}(\zeta_p)) u < 0,$$

since Lemma 3.1 asserts that $\mathfrak{M}(\zeta) < e^{-\zeta}$ for any $\zeta \in (0, 2)$. In particular, $B_{kp} < 0$ for k large, $p = 1, \dots, m_i$, and from (3.5) we derive that

$$z'_i(t_k) \leq \frac{m}{M} \beta_i(t_k) \max_{1 \leq p \leq m_i} B_{kp} \leq \frac{m}{M} \beta_i \max_{1 \leq p \leq m_i} B_{kp}.$$

By letting $k \rightarrow \infty$, this estimate yields

$$0 \leq \max_{1 \leq p \leq m_i} (-e^{-\zeta_p} + \mathfrak{M}(\zeta_p)) u < 0,$$

which is not possible. Thus, $u = 0$.

Similarly, consider the situation when $\max(u, v) = v$ (which implies $v \geq 0$), and suppose that $v > 0$. By choosing i such that $v = v_i$ and a sequence $t_k \rightarrow \infty$ with $z_i(t_k) \rightarrow -v, z'_i(t_k) \rightarrow 0$, for any $\varepsilon > 0$ small and k sufficiently large, reasoning as above we obtain

$$\begin{aligned} z'_i(t_k) &\geq -\frac{1}{x_i^*(t_k)} \sum_{p=1}^{m_i} b_{ip}(t_k) \int_{t_k - \tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s) x_i^*(s) \left[z_i(t_k) e^{-c_{ip}(s) x_i^*(s)} + \mathfrak{M}(c_{ip}(s) x_i^*(s)) |z_i(s)| \right] ds \\ &\geq -\frac{m}{M} \beta_i(t_k) \max_{1 \leq p \leq m_i} C_{kp}, \end{aligned}$$

where now

$$C_{kp} = z_i(t_k) e^{-c_{ip}(s_{k,p}) x_i^*(s_{k,p})} + \mathfrak{M}(c_{ip}(s_{k,p}) x_i^*(s_{k,p})) |z_i(s_{k,p})|$$

for some subsequences $s_{k,p} \in [t_k - \tau_{ip}(t_k), t_k]$. In an analogous way, by taking convergent subsequences of the sequences $c_{ip}(s_{k,p}) x_i^*(s_{k,p})$ and $z_i(s_{k,p})$, we obtain a contradiction from Lemma 3.1. Consequently, $v = 0$. This completes the proof. \square

Note that hypotheses (h2), (h3) in the statement of Theorem 3.2 were imposed only to derive the permanence of (1.1). In fact, the above proof applies if, instead of the permanence, all solutions are bounded and persistent; in other words, if for any $\phi \in C_0^+$ there are constants $m(\phi), M(\phi)$, such that $0 < m(\phi) \leq \liminf_{t \rightarrow \infty} x(t, 0, \phi) \leq \limsup_{t \rightarrow \infty} x(t, 0, \phi) \leq M(\phi)$.

We are ready to state our main result, on the global exponential stability of (1.1).

Theorem 3.3. *Suppose that the hypotheses of Theorem 3.2 are satisfied. Then, (1.1) is (eventually) globally exponentially stable: there exist $\delta > 0, L > 0$ such that, for any $\phi^* \in C_0^+$, there is $T = T(\phi^*)$ such that*

$$|x(t, t_0, \phi) - x(t, t_0, \phi^*)| \leq L e^{-\delta(t-t_0)} \|x_{t_0}(0, \phi) - x_{t_0}(0, \phi^*)\|, \quad t \geq t_0 \geq T, \phi \in C_0^+. \quad (3.6)$$

Proof. As in the proof of Theorem 3.2, take m, M so that any positive solution $x(t)$ of (1.1) satisfies

$$m < \liminf_{t \rightarrow \infty} x_i(t) \leq \limsup_{t \rightarrow \infty} x_i(t) < M, \quad i = 1, \dots, n,$$

and consider the previous notation for $\underline{\beta}, m_0 := \underline{c}m$ and $\mathfrak{M} := \mathfrak{M}_{m_0}$. Since $\mathfrak{M}(\xi) < e^{-\xi}$ on $(0, 2)$, from the continuity of \mathfrak{M} it follows that, for any $\varepsilon > 0$ small, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\delta + \frac{m}{M} \underline{\beta} \left[\mathfrak{M}(\xi) e^{\delta \tau} - e^{-\xi} \right] < 0 \quad \text{for all } \xi \in [m_0, 2 - \varepsilon]. \quad (3.7)$$

From Theorem 3.2, if $x^*(t)$ is a solution as in (3.2), any positive solution of (1.1) also satisfies (3.2).

Fix any positive solution $x^*(t) = x(t, 0, \phi^*)$ of (1.1) with $\phi^* \in C_0^+$, and take $T = T(\phi^*) \geq \tau$ and $\varepsilon > 0$ in such a way that $m \leq x_i^*(t) \leq M, m_0 \leq c_{ik}(t)x_i^*(t) \leq 2 - \varepsilon$ for all $t \geq T - \tau, i = 1, \dots, n, k = 1, \dots, m_i$. Consider any other positive solution $x(t) = x(t, 0, \phi)$ with $\phi \in C_0^+$, and any $t_0 \geq T$; in particular note that $x^*(t) > 0$ for $t \geq t_0$.

Next, effect the changes of variables $z_i(t) = e^{\delta t} \left(\frac{x_i(t)}{x_i^*(t)} - 1 \right)$ ($1 \leq i \leq n$), where $\delta > 0$ satisfies (3.7). Keeping the notations in Theorem 3.2, the transformed system is

$$\begin{aligned} z_i'(t) = & \delta z_i(t) + \frac{1}{x_i^*(t)} \left\{ -d_i^*(t)z_i(t) + \sum_j a_{ij}(t)x_j^*(t)z_j(t) \right. \\ & \left. + e^{\delta t} \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^t \lambda_{ik}(s) \left[h_{ik}(s, x_i^*(s)(1 + e^{-\delta s}z_i(s))) - h_{ik}(s, x_i^*(s)) \right] ds \right\}, \end{aligned} \quad (3.8)$$

We now claim that the solution of (3.8) with initial condition $z_{t_0} = \psi$ satisfies

$$|z(t, t_0, \psi)| \leq \|\psi\|, \quad t \geq t_0. \quad (3.9)$$

Otherwise, suppose that there exist $t_1 > t_0$ and $i \in \{1, \dots, n\}$ such that

$$|z(t_1)| = |z_i(t_1)| > \|\psi\|, \quad |z_j(t)| < |z_i(t_1)|, \quad \text{for } t \in [t_0 - \tau, t], \quad 1 \leq j \leq n.$$

Consider the case $z_i(t_1) > 0$ (the case $z_i(t_1) < 0$ is analogous). From the definition of t_1 , we have $z_i'(t_1) \geq 0$. On the other hand, from (3.7), (3.8) and reasoning as in (3.4), we obtain

$$\begin{aligned} z_i'(t_1) & \leq \frac{1}{x_i^*(t_1)} \left\{ \left[\delta x_i^*(t_1) - \left(d_i^*(t_1) - \sum_j a_{ij}(t_1)x_j^*(t_1) \right) \right] z_i(t_1) \right. \\ & \quad \left. + e^{\delta t_1} \sum_{k=1}^{m_i} b_{ik}(t_1) \int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s) \mathfrak{M}(c_{ik}(s)x_i^*(s)) e^{-\delta s} x_i^*(s) |z_i(s)| ds \right\}, \\ & \leq \frac{1}{x_i^*(t_1)} \left\{ \left[\delta x_i^*(t_1) - \sum_{k=1}^{m_i} b_{ik}(t_1) \int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s) h_{ik}(s, x_i^*(s)) ds \right] z_i(t_1) \right. \\ & \quad \left. + e^{\delta \tau} \sum_{k=1}^{m_i} b_{ik}(t_1) \int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s) \mathfrak{M}(c_{ik}(s)x_i^*(s)) x_i^*(s) |z_i(s)| ds \right\} \\ & \leq \frac{z_i(t_1)}{x_i^*(t_1)} \left\{ \delta x_i^*(t_1) + \sum_{k=1}^{m_i} b_{ik}(t_1) \int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s) x_i^*(s) \left[-e^{-c_{ik}(s)x_i^*(s)} + e^{\delta \tau} \mathfrak{M}(c_{ik}(s)x_i^*(s)) \right] ds \right\} \\ & < \frac{z_i(t_1)}{x_i^*(t_1)} \left\{ \delta x_i^*(t_1) - \frac{\delta M}{m \underline{\beta}} \sum_{k=1}^{m_i} b_{ik}(t_1) \int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s) x_i^*(s) ds \right\} \\ & \leq \delta M \frac{z_i(t_1)}{x_i^*(t_1)} \left(1 - \frac{\beta_i(t_1)}{\underline{\beta}} \right) \leq 0, \end{aligned}$$

which is a contradiction. Thus (3.9) holds.

Going back to the solution $x(t)$, for $t \geq t_0$ and $i = 1, \dots, n$ we have

$$\begin{aligned} e^{\delta t} |x_i(t) - x_i^*(t)| &= x_i^*(t) |z_i(t)| \leq \|z_{t_0}\| = \sup_{\theta \in [-\tau, 0]} \left\{ e^{\delta(t_0+\theta)} \left(\frac{x_{t_0}(0, \phi)}{x_{t_0}(0, \phi^*)} - 1 \right) \right\} \\ &\leq e^{\delta t_0} \frac{M}{m} \|x_{t_0}(0, \phi) - x_{t_0}(0, \phi^*)\|. \end{aligned}$$

The proof of (3.6) is complete. \square

From the above results, a practical criterion to deduce the global exponential stability of (1.1) is given below.

Theorem 3.4. For (1.1), assume (h1), suppose that $\beta_i(t), c_{ik}(t)$ are all bounded below and above by positive constants. Assume also that there are a vector $v = (v_1, \dots, v_n) > 0$ and constants α, γ such that

$$1 < \alpha \leq \gamma_i(t, v) \leq \gamma < 2e \underline{C} \bar{C}^{-1}, \quad 1 \leq i \leq n, t \gg 1, \quad (3.10)$$

where $0 < \underline{c}_i \leq c_{ik}(t) \leq \bar{c}_i$ for $t \in \mathbb{R}^+$ and all i, k and $\underline{C} = \underline{C}(v)$ $\bar{C} = \bar{C}(v)$ are as in (2.9). Then, (1.1) is (eventually) globally exponentially stable.

Proof. Clearly, $\beta_i(t) \geq \underline{\beta}_i > 0$ on \mathbb{R}^+ and $1 < \alpha \leq \gamma_i(t, v) \leq \gamma$ ($1 \leq i \leq n$) imply that (h2), (h3) hold. The result follows immediately from Theorem 3.3 and Lemma 2.8. \square

Remark 3.5. If all the coefficients are bounded, one can easily check that Theorems 2.6, 3.3 and 3.4 are still valid for systems of the form (2.5).

For Nicholson systems (2.4), the above results are written in a simpler form.

Corollary 3.6. For (2.4), assume (h1) and suppose that there are a vector $v = (v_1, \dots, v_n) > 0$ and a constant $\gamma < 2e|v|^{-1} \min_{1 \leq i \leq n} v_i$ such that

$$0 < \gamma_i(t, v) \leq \gamma, \quad 1 \leq i \leq n, t \gg 1, \quad (3.11)$$

where $|v| = \max_{1 \leq i \leq n} v_i$. Then, there are positive solutions of (1.1) satisfying $x_i(t) < 2$ for all $t \geq 0, i = 1, \dots, n$. If in addition, $\beta_i(t)$ are bounded below and above by positive constants and

$$\gamma_i(t, v) \geq \alpha > 1, \quad 1 \leq i \leq n, t \gg 1,$$

for some α , then (2.4) is (eventually) globally exponentially stable. In particular, this is the case if

$$1 < \alpha \leq \gamma_i(t) \leq \gamma < 2e, \quad 1 \leq i \leq n, t \gg 1. \quad (3.12)$$

Example 3.7. Consider the planar system

$$\begin{aligned} x_1'(t) &= -t^\eta x_1(t) + (t^\eta - 1)x_2(t) + \frac{\beta}{\sigma_1(t)} \int_{t-\sigma_1(t)}^t x_1(s) e^{-x_1(s)} ds, \\ x_2'(t) &= -t^\eta x_2(t) + (t^\eta - 1)x_1(t) + \frac{\beta}{\sigma_2(t)} \int_{t-\sigma_2(t)}^t x_2(s) e^{-x_2(s)} ds, \end{aligned} \quad t \geq 1, \quad (3.13)$$

where $\eta > 0, \beta > 1$, the delays $\sigma_i(t)$ are positive, continuous and bounded, $i = 1, 2$. With the previous notations, $d_i(t) = t^\eta, a_{ii}(t) = 0, \beta_i(t) \equiv \beta > 1, i = 1, 2$ and $a_{12}(t) = a_{21}(t) = t^\eta - 1$, thus $\gamma_1(t) = \gamma_2(t) = \beta$. For this concrete example, if $\beta \in (1, e^2)$, there exists a positive equilibrium $x^* = (\log \beta, \log \beta) < (2, 2)$. From Theorem 3.3, we deduce that all positive solutions $x(t)$ converge exponentially to x^* as $t \rightarrow \infty$.

In the case of *periodic* Nicholson systems, we also obtain the following result.

Corollary 3.8. Consider a periodic Nicholson system (1.1), with $d_i(t)$, $a_{ij}(t)$, $b_{ik}(t)$, $\tau_{ik}(t)$, $\lambda_{ik}(t)$, $c_{ik}(t)$ continuous, nonnegative and ω -periodic functions (for some $\omega > 0$), with $d_i(t)$, $\beta_i(t)$, $c_{ik}(t)$ positive, for all i, j, k . If there exist a vector $v > 0$ such that

$$\begin{cases} \min_{t \in [0, \omega]} \gamma_i(t, v) > 1, \\ \max_{t \in [0, \omega]} \gamma_i(t, v) < 2e \underline{C} \bar{C}^{-1}, \quad 1 \leq i \leq n, \end{cases} \quad (3.14)$$

then there exists a positive ω -periodic solution of (1.1), which is globally exponentially stable.

Proof. By [8], it turns out that the sufficient conditions for permanence also imply the existence of a positive periodic solution. The result is an immediate consequence of Theorem 3.4. \square

Remark 3.9. For the periodic Nicholson system with discrete delays multiple of period given by

$$x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)x_j(t) + \beta_i(t)x_i(t - m_i\omega)e^{-c_i(t)x_i(t - m_i\omega)}, \quad 1 \leq i \leq n. \quad (3.15)$$

with $m_i \in \mathbb{N}$, $\omega > 0$ and $d_i(t) > 0$, $a_{ij}(t) \geq 0$, $\beta_i(t)$, $c_i(t) > 0$ continuous ω -periodic functions, the existence and global attractivity of a positive periodic solution was proven in [8] under the condition

$$\begin{cases} \min_{t \in [0, \omega]} \gamma_i(t, v) > 1, \\ \max_{t \in [0, \omega]} \gamma_i(t, v) < \exp(2 \underline{C} \bar{C}^{-1}), \quad 1 \leq i \leq n, \end{cases} \quad (3.16)$$

for some vector $v > 0$ and \underline{C}, \bar{C} defined as in (2.9). Clearly, $ex \leq e^x$ for $x \geq 0$. We conclude that Corollary 3.8 extends the result in [8] to more general systems (1.1) – with global *exponential* stability, rather than global attractivity –, however, under the more restrictive assumption of $\gamma_i := \max_{t \in [0, \omega]} \gamma_i(t, v) < 2e \underline{C} \bar{C}^{-1}$, instead of $\gamma_i < e^{2 \underline{C} \bar{C}^{-1}}$. The key point to establish the result in [8] under the latter assumption was the following: as the delays are multiple of the period, an ω -periodic solution $x^*(t)$ for (3.15) is also an ω -periodic solution for the corresponding ODE

$$x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)x_j(t) + \beta_i(t)x_i(t)e^{-c_i(t)x_i(t)}, \quad 1 \leq i \leq n. \quad (3.17)$$

From this fact, one easily deduces that $\max_{t \geq 0} (c_i(t)x_i^*(t)) < 2$ for all i , provided that (3.16) holds. Whether Theorem 3.4 is still valid for a general system (1.1) with (3.10) replaced by

$$1 < \alpha \leq \gamma_i(t, v) \leq \gamma < e^{2 \underline{C} \bar{C}^{-1}}, \quad 1 \leq i \leq n, \quad t \gg 1,$$

(conf. Remark 2.9) is an interesting open problem. We conjecture that the answer is affirmative, at least if some further constraints on α are prescribed.

We now apply our results to Nicholson equations and systems with *discrete delays*, and compare the above criteria with some more results in the literature. The corollary below addresses the scalar case, a similar one can be written for systems with $n > 1$.

Corollary 3.10. *Consider the scalar Nicholson equation*

$$x'(t) = -d(t)x(t) + \sum_{k=1}^m \beta_k(t)x(t - \tau_k(t))e^{-c_k(t)x(t - \tau_k(t))}, \quad (3.18)$$

where $d, \beta_k, \tau_k, c_k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are continuous functions and $d(t) > 0$ on \mathbb{R}^+ , $\tau_k(t) \in [0, \tau]$ (for some $\tau > 0$) and $c_k(t), \beta(t) := \sum_{k=1}^m \beta_k(t)$ are bounded above and below by positive constants. If

$$1 < \alpha \leq \frac{\sum_{k=1}^m \beta_k(t)}{d(t)} \leq \gamma < 2e \frac{\min_{1 \leq k \leq m} c_k}{\max_{1 \leq k \leq m} \bar{c}_k}, \quad t \geq 0, \quad (3.19)$$

where $c_k = \inf_{t \geq 0} c_k(t)$, $\bar{c}_k = \sup_{t \geq 0} c_k(t)$, then (3.18) is globally exponentially stable.

If all the coefficients and delays d, β_k, τ_k, c_k are ω -periodic, Corollary 3.8 implies the existence of a globally exponentially stable ω -periodic positive solution to (3.18). We stress that the *periodic* equation (3.18) was studied in [16] and its stability established. Denote $\kappa \in (0, 1)$, $\tilde{\kappa} \in (1, \infty)$ the constants which satisfy

$$h'(\kappa) = -h'(2), \quad h(\kappa) = h(\tilde{\kappa}). \quad (3.20)$$

The approximate values of $\kappa, \tilde{\kappa}$ were evaluated in [23]: $\kappa \approx 0.7215$, $\tilde{\kappa} \approx 1.3423$. Assuming that

$$\frac{\sum_{k=1}^m \beta_k(t)}{d(t)} < e^2, \quad t \in [0, \omega], \quad (3.21)$$

and that there is $M > \kappa$ such that

$$1 \leq \min_{1 \leq k \leq m} c_k \leq \max_{1 \leq k \leq m} \bar{c}_k \leq \frac{\tilde{\kappa}}{M} \quad (3.22)$$

and

$$\frac{1}{eM} \sum_{k=1}^m \frac{\beta_k(t)}{c_k(t)} < d(t) < e^{-\kappa} \sum_{k=1}^m \frac{\beta_k(t)}{c_k(t)}, \quad t \in [0, \omega], \quad (3.23)$$

Liu [16] used a Lyapunov functional to show that there exists an ω -periodic positive solution of (3.18) which is globally exponentially stable. A similar approach was used by Liu in [17], for an almost periodic version of (3.18) with a nonlinear density-dependent mortality term $-d_1(t) + d_2(t)e^{-x(t)}$, instead of $-d(t)x(t)$.

In fact, in order to prove the above exponential stability under the conditions (3.21)–(3.23), Liu [16] started by establishing that the ordered interval $[\kappa, M]$ in $C = ([-\tau, 0]; \mathbb{R})$ is positively invariant. For the periodic case, by itself, the constraint (3.21) is weaker than the second inequality in (3.19). However, not only is the requirement (3.22) a strong restriction to the application of Liu's criterion, but, if (3.22) holds, our assumption (3.19) simply reads as

$$\frac{1}{eM} \frac{\tilde{\kappa}}{2} \sum_{k=1}^m \beta_k(t) < d(t) < \sum_{k=1}^m \beta_k(t), \quad t \in [0, \omega].$$

In this situation, we always have $\sum_{k=1}^m \beta_k(t) > e^{-\kappa} \sum_{k=1}^m \frac{\beta_k(t)}{c_k(t)}$; if one can choose M in (3.22) such that $\tilde{\kappa}^2 < 2M$, i.e., if $\max_{1 \leq k \leq m} \bar{c}_k < 2/\tilde{\kappa} \approx 1.490$, then $\frac{1}{eM} \frac{\tilde{\kappa}}{2} \sum_{k=1}^m \beta_k(t) < \frac{1}{eM} \sum_{k=1}^m \frac{\beta_k(t)}{c_k(t)}$, and our result strongly improves the criterion in [16]. For instance, with $c_k(t) \equiv 1$, our hypothesis (3.19) reads as

$$1 < \frac{\sum_{k=1}^m \beta_k(t)}{d(t)} < 2e \quad \text{for } t \in [0, \omega];$$

on the other hand, we may take $M = \tilde{\kappa}$ in (3.22), and conditions (3.21), (3.23) are equivalent to

$$e^x \leq \frac{\sum_{k=1}^m \beta_k(t)}{d(t)} \leq e\tilde{\kappa} \quad \text{for } t \in [0, \omega],$$

which is much more restrictive than (3.19).

More recently, Wang et al. [23] generalised the scalar version (3.18) by considering the following multi-dimensional model with patch structure:

$$x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1, j \neq i}^n a_{ij}(t)x_j(t) + \sum_{p=1}^m \beta_{ip}(t)x_i(t - \tau_{ip}(t))e^{-c_{ip}(t)x_i(t - \tau_{ip}(t))}, \quad i = 1, \dots, n, \quad (3.24)$$

where $d_i, a_{ij}, \beta_{ip}, \tau_{ip}, c_{ip} : \mathbb{R} \rightarrow \mathbb{R}^+$ are continuous, pseudo almost periodic functions, $d_i(t) > 0$ and satisfies some further properties, and $\inf_{t \geq t_0} \beta_i(t) > 0$ where $\beta_i(t) := \sum_{p=1}^m \beta_{ip}(t)$, for all i, j, p . With $\kappa, \tilde{\kappa}$ defined as in (3.20), in [23] the authors assumed the following set of assumptions, for $1 \leq i \leq n, 1 \leq p \leq m$:

$$\begin{aligned} 1 &\leq \inf_{t \in \mathbb{R}} c_{ip}(t) \leq \sup_{t \in \mathbb{R}} c_{ip}(t) \leq M^{-1}\tilde{\kappa}, \quad \text{for some } M > \kappa, \\ \sup_{t \in \mathbb{R}} \left\{ -d_i(t) + \sum_{j=1, j \neq i}^n a_{ij}(t) + \frac{1}{eM} \sum_{p=1}^m \frac{\beta_{ip}(t)}{c_{ip}(t)} \right\} &< 0, \\ \inf_{t \in \mathbb{R}} \left\{ -d_i(t) + \sum_{j=1, j \neq i}^n a_{ij}(t) + e^{-\kappa} \sum_{p=1}^m \frac{\beta_{ip}(t)}{c_{ip}(t)} \right\} &> 0, \end{aligned} \quad (3.25)$$

and showed that:

(i) all solutions $x(t) = x(t, t_0, \phi)$ of (3.24) with initial conditions $\phi \in C_0^+$ satisfy

$$\kappa \leq \liminf_{t \rightarrow \infty} x_i(t) \leq \limsup_{t \rightarrow \infty} x_i(t) \leq M, \quad i = 1, \dots, n; \quad (3.26)$$

(ii) there exists a positive pseudo almost periodic solution $x^*(t)$ of (3.24), which satisfies $\kappa \leq x_i^*(t) \leq M$ for all $t \in \mathbb{R}$ and $i = 1, \dots, n$;

(iii) $x^*(t)$ is globally exponentially stable.

See also [5, 24] for similar criteria. Recently, some of the constraints in [23] were slightly loosened in [13].

With our methodology, under the condition $\inf_{t \in \mathbb{R}} c_{ip}(t) \geq 1$ and taking e.g. $v = \vec{1}$ in (3.10), from Theorem 3.4 we obtain that system (3.24) is globally exponentially stable provided that

$$\begin{aligned} \inf_{t \in \mathbb{R}} \frac{\sum_{p=1}^m \beta_{ip}(t)}{d_i(t) - \sum_{j=1, j \neq i}^n a_{ij}(t)} &> 1, \\ \sup_{t \in \mathbb{R}} \frac{\sum_{p=1}^m \beta_{ip}(t)}{d_i(t) - \sum_{j=1, j \neq i}^n a_{ij}(t)} &< \frac{2e}{\max_{i,p} \sup_{t \in \mathbb{R}} c_{ip}(t)}, \end{aligned} \quad 1 \leq i \leq n, t \gg 1. \quad (3.27)$$

As in the previous scalar case, one easily verifies that for most situations conditions (3.27) are less restrictive than (3.25).

We finish this section with a couple of simple examples.

Example 3.11. Consider the following ω -periodic Nicholson-type system with discrete delays:

$$\begin{aligned} x'_1(t) &= -d_1(t)x_1(t) + b_1(t)x_2(t - \sigma_1(t)) + c_1(t)x_1(t - \tau_1(t))e^{-x_1(t - \tau_1(t))} \\ x'_2(t) &= -d_2(t)x_2(t) + b_2(t)x_2(t - \sigma_2(t)) + c_2(t)x_2(t - \tau_2(t))e^{-x_2(t - \tau_2(t))}, \end{aligned} \quad (3.28)$$

where $d_i(t), b_i(t), c_i(t), \sigma_i(t), \tau_i(t)$ ($i = 1, 2$) are positive, continuous and ω -periodic functions. Applying Corollary 3.8 with $v = (1, v_2)$ (conf. also Remark 3.5)), we derive that (3.28) has a globally exponentially stable positive ω -periodic solution if there exists a positive constant v_2 such that

$$1 < \frac{c_1(t)}{d_1(t) - b_1(t)v_2} < 2e \frac{\min\{1, v_2\}}{\max\{1, v_2\}}, \quad 1 < \frac{c_2(t)v_2}{d_2(t)v_2 - b_2(t)} < 2e \frac{\min\{1, v_2\}}{\max\{1, v_2\}}, \quad t \in [0, \omega].$$

In particular, this assertion is valid if

$$1 < \frac{c_i(t)}{d_i(t) - b_i(t)} < 2e, \quad t \in [0, \omega], \quad i = 1, 2. \quad (3.29)$$

In the case of (3.28) with $\sigma_i(t) \equiv 0$ and a unique constant delay in the nonlinear part, i.e., $\tau_i(t) \equiv \tau > 0$, by using the continuation theorem of coincidence degree and a Lyapunov functional, Troib [21] established sufficient conditions for the existence and global attractivity of a positive ω -periodic solution. As analysed in [8] with more detail, we can assert that the results in [21] not only do not apply to the framework of nonconstant delays $\tau_i(t)$, nor to other simple situations, but also the assumed constraints are more restrictive than (3.29).

Example 3.12. As a particular case of (3.28), consider the π -periodic system

$$\begin{aligned} x_1'(t) &= -(1 + \cos^2 t)x_1(t) + c_1(1 + \sin^2 t)x_2(t) + \beta_1(1 + \cos^2 t)x_1(t - \tau_1(t))e^{-x_1(t - \tau_1(t))} \\ x_2'(t) &= -(1 + \sin^2 t)x_2(t) + c_2(1 + \cos^2 t)x_1(t) + \beta_2(1 + \sin^2 t)x_2(t - \tau_2(t))e^{-x_2(t - \tau_2(t))} \end{aligned} \quad (3.30)$$

where $c_i, \beta_i > 0$ and the delays $\tau_i(t)$ are π -periodic, continuous and nonnegative, $i = 1, 2$. With the previous notation, for $v = (1, v_2) > 0$ we have

$$\begin{aligned} \gamma_1(t, v) &:= \frac{\beta_1(1 + \cos^2 t)}{1 + \cos^2 t - v_2 c_1(1 + \sin^2 t)} \\ \gamma_2(t, v) &:= \frac{\beta_2 v_2(1 + \sin^2 t)}{v_2(1 + \sin^2 t) - c_2(1 + \cos^2 t)}. \end{aligned} \quad (3.31)$$

If $4c_1c_2 < 1$, choosing v_2 such that $2c_2 < v_2 < (2c_1)^{-1}$, we obtain

$$0 < \alpha_i \leq \gamma_i(t, v) \leq \gamma_i, \quad \text{for } t \in [0, \pi], \quad i = 1, 2,$$

where

$$\alpha_1 = \frac{\beta_1}{1 - \frac{1}{2}v_2c_1}, \quad \gamma_1 = \frac{\beta_1}{1 - 2v_2c_1}, \quad \alpha_2 = \frac{\beta_2}{1 - \frac{1}{2}v_2^{-1}c_2}, \quad \gamma_2 = \frac{\beta_2}{1 - 2v_2^{-1}c_2}.$$

In particular, with $c_i < \frac{1}{2}$, $i = 1, 2$, one can take $v_2 = 1$; if in addition $c_i < 2(2e - 1)(8e - 1)^{-1} \approx 0.4277$ and β_i is chosen so that $1 - \frac{1}{2}c_i < \beta_i < 2e(1 - 2c_i)$ for $i = 1, 2$, we obtain $1 < \alpha_i < \gamma_i < 2e$, $i = 1, 2$, therefore there exists a positive π -periodic solution $x^*(t)$ which is globally exponentially attractive.

4 Conclusions

This paper concerns the global asymptotic behaviour of positive solutions for a very broad family of Nicholson systems (1.1). Uniform lower and upper bounds for all solutions, as

well as their global exponential stability are established, which generalise most of the results in recent literature. We observe that systems (1.1) incorporate distributed delays, whereas most authors only consider systems (3.24) with discrete delays. Moreover, as mentioned in Remarks 2.5 and 3.5, if $a_{ij}(t)$ are bounded, all the results apply to systems (2.5) with delays in the linear terms. The assumptions and proofs presented here rely heavily on the special properties of the Ricker nonlinearity $h(x) = xe^{-x}$, $x \geq 0$.

Some authors [7, 15, 24], have considered autonomous or nonautonomous Nicholson systems with discrete delays under restrictions on the coefficients implying that the systems have a monotone behaviour. In recent papers [5, 13, 16, 21, 23], conditions have been imposed for systems (3.24) in such a way that the estimates (3.26) should hold, where $1 \leq \inf_{t \in \mathbb{R}} c_{ip}(t) \leq \sup_{t \in \mathbb{R}} c_{ip}(t) \leq M^{-1}\tilde{\kappa}$ for some $M > \kappa$, for $\kappa, \tilde{\kappa}$ defined in (3.20), – and thus all positive solutions must satisfy

$$\kappa \leq \liminf_{t \rightarrow \infty} c_{ip}(t)x_i(t) \leq \limsup_{t \rightarrow \infty} c_{ip}(t)x_i(t) \leq \tilde{\kappa},$$

for all i, p . These estimates have been used in order to derive that, since $h(x) \geq h(\kappa)$ and $|h'(x)| \leq e^{-2}$ for $x \in [\kappa, \tilde{\kappa}]$, any two solutions $x(t), y(t)$ must satisfy

$$|h_{ip}(t, x_i(s)) - h_{ip}(t, y_i(s))| \leq e^{-2}|x_i(s) - y_i(s)|, \quad i = 1, \dots, n,$$

for all t and $s \in [t - \tau, t]$, where $h_{ip}(t, x(s)) = x(s)e^{-c_{ip}(t)x(s)}$. Our approach is essentially new: assuming the permanence, the exponential stability of (1.1) is proven using solely an explicit *upper bound* for solutions of such systems. Basically, we only need to assert the existence of (at least) one positive solution satisfying $\limsup_{t \rightarrow \infty} c_{ip}(t)x_i(t) < 2$ for all i, p . In Theorem 3.4 we have imposed condition (3.10), which guarantees that such a solution exists. As mentioned in Remark 3.9, an interesting open problem is whether such a condition can be replaced by the less restrictive assumption $1 < \alpha \leq \gamma_i(t, v) \leq \gamma < e^{2\bar{c}^{-1}}$.

Clearly, the method developed here can be further exploited, to study the global attractivity and exponential stability of other systems with patch structure – such as Mackey–Glass type systems –, or modified Nicholson systems with either nonlinear density-dependent mortality terms or harvesting terms, as in [5, 20, 22, 25]. In other words, under the conditions for permanence established in [9] and with suitable changes, the approach herein carries over to more general settings, and can be used to treat n -dimensional systems

$$x'_i(t) = -d_i(t, x_i(t)) + \sum_{j=1}^n L_{ij}(t)x_{j,t} + f_i(t, x_{i,t}), \quad t \geq 0, \quad i = 1, \dots, n,$$

where $d_i(t, x) \geq 0, d_i(t, x) = O(x)$ at zero, the linear functionals $L_{ij}(t)$ are nonnegative and the nonlinearities f_i incorporate one or several monotone, or unimodal terms.

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