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Study of a cyclic system of difference equations with maximum

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Abstract. In this paper we study the behaviour of the solutions of the following cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i(n)}{x_{i+1}(n-1)}\right\}, \qquad i = 1, 2, \dots, k-1,$$
$$x_k(n+1) = \max\left\{A_k, \frac{x_k(n)}{x_1(n-1)}\right\}$$

where $n = 0, 1, 2, \dots$, A_i, $i = 1, 2, \dots, k$, are positive constants, $x_i(-1), x_i(0)$, $i = 1, 2, \dots, k$, are real positive numbers. Finally for $k = 2$ under some conditions we find solutions which converge to periodic six solutions.

Keywords: difference equations with maximum, equilibrium, eventually equal to equilibrium, periodic solutions.

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1 Introduction

Max operators play an important role in the study of some problems in automatic control (see [16, 17]). This fact was one, among others, which motivated some authors to consider differences equations with maximum (see [1–7, 10–15, 20, 21, 23–37, 40–42, 45–47]).

In the beginning, majority of the papers in the topic studied special cases of difference equations in the following form:

$$y_{n+1} = \max\left\{\frac{A_0}{y_n}, \frac{A_1}{y_{n-1}}, \dots, \frac{A_k}{y_{n-k}}\right\}, \qquad n = 0, 1, 2, \dots,$$

where k is a natural number, whereas the coefficients A_j, $j = 0, 1, \dots, k$, are real numbers (see, for example, [2, 5, 7, 12–15, 23, 45–47]).

The study of positive solutions of the following difference equation with maximum

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-2}}\right\}, \qquad n = 0, 1, 2, \dots,$$

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conducted in [14] showed that a suitable change of variables transforms it to the difference equation with maximum of the form:

$$y_{n+1} = \max\left\{D, \frac{y_n}{y_{n-1}}\right\} \tag{1.1}$$

where $D = AB^{-1}$, which suggested the investigation of the equation. Among other things, [14] studied the periodicity of positive solutions of equation (1.1).

This also naturally suggested investigations of difference equations in the following form:

$$y_{n+1} = \max\left\{D, \frac{y_{n-k}}{y_{n-m}}\right\}, \qquad n = 0, 1, 2, \ldots,$$

where k and m are nonnegative integers (for some important results on the difference equation see [1]), which was soon after publication of [1] continued in a comprehensive study of the following difference equation

$$y_{n+1} = \max\left\{D, \frac{y_{n-k}^p}{y_{n-m}^q}\right\}, \qquad n = 0, 1, 2, \ldots,$$

and its natural generalizations, by S. Stević and his collaborators (see, for example, [10, 11, 25–31, 35–37, 40, 42]).

On the other hand, equation (1.1) suggested also studying of the corresponding close-to-symmetric systems of difference equations (some related rational ones had been previously studied for example in [18, 19]).

In [6] the authors studied the periodicity of the positive solutions of the system of difference equations with maximum which is a close-to-symmetric cousin of equation (1.1) :

$$x_{n+1} = \max\left\{A, \frac{y_n}{x_{n-1}}\right\},$$
$$x_{n+1} = \max\left\{B, \frac{x_n}{y_{n-1}}\right\},$$

where $n = 0, 1, 2, \ldots$, and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive real numbers.

Some other results on systems of difference equations with maximum can be found in [21, 24, 33–35, 37, 42]. Recall also that many difference equations and systems with maximum are connected with periodicity (see, e.g., [3–5, 12, 29, 32, 34, 41, 45–47]), a typical characteristic of positive solutions of the equations and systems. For some results on the boundedness character of difference equations and systems with maximum see [1, 3, 13, 20, 40]. The paper [1] is interesting since it also considers real solutions to a difference equations with maximum, unlike great majority of other ones.

On the third side, in [8] Iričanin and Stević suggested investigation of cyclic systems of difference equations, which later motivated some further investigations in the direction (see, for example, [9, 22, 38]).

In what follows we use the following convention (see [8]). If i and j are integers such that $i = j \, (\mathrm{mod}\, k)$, then we will regard that $A_i = A_j$ and $x_i(n) = x_j(n)$. For example, we identify the number A_0 with A_k, and identify the sequence $x_{k+1}(n)$ with $x_1(n)$ (the convention is used in the systems which follows).

Motivated by above mentioned facts, in this paper we study the behaviour of the solutions of the following cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i(n)}{x_{i+1}(n-1)}\right\}, \qquad i = 1,2,\dots,k, \tag{1.2}$$

where $n = 0,1,2,\dots$, the coefficients A_i, $i = 1,2,\dots,k$, are positive constants, and the initial values $x_i(-1), x_i(0)$, $i = 1,2,\dots,k$, are positive real numbers. Moreover for $k = 2$ under some conditions we find solutions which converge to periodic six solutions.

2 Study of system (1.2)

First we study the existence of equilibrium point for (1.2).

Proposition 2.1. *Consider system* (1.2) *where A_i, $i = 1,2,\dots,k$, are positive constants and $x_i(-1)$, $x_i(0)$, $i = 1,2,\dots,k$, are positive real numbers. Then the following statements are true:*

I. *Suppose that*

$$A_i > 1, \qquad i = 1,2,\dots,k. \tag{2.1}$$

Then (1.2) *has a unique equilibrium $(x_1, x_2, \dots, x_k) = (A_1, A_2, \dots, A_k)$.*

II. *Suppose that there exists an r, $r \in \{1,2,\dots,k\}$ such that*

$$(A_r - 1)(A_{r+1} - 1) < 0. \tag{2.2}$$

Then (1.2) *has no equilibrium.*

III. *Let*

$$0 < A_i < 1, \qquad i = 1,2,\dots,k \tag{2.3}$$

be satisfied. Then system (1.2) *has a unique equilibrium $(x_1, x_2, \dots, x_k) = (1,1,\dots,1)$.*

Proof. **I.** We consider the system of algebraic equations

$$x_i = \max\left\{A_i, \frac{x_i}{x_{i+1}}\right\}, \qquad i = 1,2,\dots,k. \tag{2.4}$$

We would like to point out that in (2.4) we use the following convention: if i and j are integers, then we regard that $x_i = x_j$ if $i = j \pmod k$ (see the previous section). Since $x_i \geq A_i > 1$, $i = 1,2,\dots,k$ it is obvious that

$$x_i \neq \frac{x_i}{x_{i+1}}, \qquad i = 1,2,\dots,k.$$

From this it easily follows that system (2.4) has a unique solution

$$(x_1, x_2, \dots, x_k) = (A_1, A_2, \dots, A_k).$$

II. Suppose that there exists $r \in \{1,2,\dots,k\}$ such that inequalities (2.2) hold. Then either

$$A_r < 1, \qquad A_{r+1} > 1 \tag{2.5}$$

or

$$A_r > 1, \qquad A_{r+1} < 1 \tag{2.6}$$

are satisfied.

Suppose firstly that (2.5) hold. From (2.4) we get

$$x_r = \max\left\{A_r, \frac{x_r}{x_{r+1}}\right\}. \tag{2.7}$$

Relations (2.4) and (2.5) imply that $x_{r+1} \geq A_{r+1} > 1$. Hence we have

$$\frac{x_r}{x_{r+1}} \leq \frac{x_r}{A_{r+1}} < x_r$$

and so from (2.7) we take $x_r = A_r$. Moreover, from (2.4) we get

$$x_{r-1} = \max\left\{A_{r-1}, \frac{x_{r-1}}{x_r}\right\} = \max\left\{A_{r-1}, \frac{x_{r-1}}{A_r}\right\} \geq \frac{x_{r-1}}{A_r}$$

which is a contradiction since $0 < A_r < 1$, $r = 1, 2, \dots, k$. So (1.2) has no equilibrium.

Assume now that (2.6) is satisfied. Suppose that there exists a $j \in \{1, 2, \dots, r\}$ such that $A_j < 1$. Let $s = \max\{j : A_j < 1,\ j \in \{1, 2, \dots, r\}\}$. Then it is obvious that

$$A_s < 1, \qquad A_{s+1} > 1. \tag{2.8}$$

Then arguing as in the case where (2.5) hold, system (1.2) has no equilibrium. Assume that there exists a $j \in \{r+2, r+3, \dots, k\}$ such that $A_j > 1$. Let $v = \min\{j : A_j > 1,\ j \in \{r+2, r+3, \dots, k\}\}$. Then we get

$$A_{v-1} < 1, \qquad A_v > 1. \tag{2.9}$$

So, arguing again as above we have that (1.2) has no equilibrium.

Finally suppose that

$$A_j > 1, \quad j = 1, 2, \dots, r, \qquad A_v < 1, \quad v = r+1, r+2, \dots, k. \tag{2.10}$$

Then since from (2.10) $A_1 > 1$ we take $\frac{x_k}{x_1} \leq \frac{x_k}{A_1} < x_k$. Thus we get from (2.4)

$$x_k = \max\left\{A_k, \frac{x_k}{x_1}\right\} = A_k < 1. \tag{2.11}$$

Moreover, from (2.4), (2.10) and (2.11) it holds,

$$x_{k-1} = \max\left\{A_{k-1}, \frac{x_{k-1}}{x_k}\right\} = \max\left\{A_{k-1}, \frac{x_{k-1}}{A_k}\right\} \geq \frac{x_{k-1}}{A_k} > x_{k-1}$$

which is a contradiction and so (1.2) has no equilibrium.

III. We claim that there exists $r \in \{1, 2, \dots, k\}$ such that

$$\frac{x_r}{x_{r+1}} \geq 1. \tag{2.12}$$

Suppose on the contrary that

$$\frac{x_i}{x_{i+1}} < 1, \qquad i = 1, 2, \dots, k,$$

(recall that for $i=k$ it means $\frac{x_k}{x_1}<1$). Then we get

$$1=\frac{x_1}{x_2}\frac{x_2}{x_3}\cdots\frac{x_k}{x_1}<1$$

which is not true.

Therefore there exists an r such that (2.12) holds. From (2.3), (2.7) and (2.12) we have

$$x_r=\frac{x_r}{x_{r+1}}.$$

Hence $x_{r+1}=1$. In addition from (2.4) we take

$$1=x_{r+1}=\max\left\{A_{r+1},\frac{x_{r+1}}{x_{r+2}}\right\}=\max\left\{A_{r+1},\frac{1}{x_{r+2}}\right\}.$$

Then from (2.3) it is obvious that $x_{r+2}=1$. Working inductively we take $x_j=1,\ j=r+1,\dots,k$. From (2.4) we get

$$1=x_k=\max\left\{A_k,\frac{x_k}{x_1}\right\}=\max\left\{A_k,\frac{1}{x_1}\right\}$$

and so $x_1=1$. Then we get

$$1=x_1=\max\left\{A_1,\frac{1}{x_2}\right\}.$$

Then since (2.3) is satisfied it is obvious that $x_2=1$. Working inductively we take $x_j=1,\ j=1,2,\dots,r$. This completes the proof of the proposition. □

Proposition 2.2. *Suppose that* (2.1) *is satisfied. Then every solution of* (1.2) *is eventually equal to the unique equilibrium of* (1.2) $(x_1,x_2.,,,,x_k)=(A_1,A_2,\dots,A_k)$.

Proof. Let $(x_1(n),x_2(n),\dots,x_k(n))$ be an arbitrary solution of (1.2). From (1.2) we get

$$x_i(n)\ge A_i,\qquad i=1,2,\dots,k. \tag{2.13}$$

Let $s\in\{1,2,\dots,k\}$. We prove that there exists an $m_s\ge 3$ such that

$$x_s(m_s)=A_s. \tag{2.14}$$

Suppose on the contrary that for all $n\ge 3$

$$x_s(n)>A_s. \tag{2.15}$$

Then from (1.2), (2.13) and (2.15) we take for $n\ge 3$

$$x_s(n)=\max\left\{A_s,\frac{x_s(n-1)}{x_{s+1}(n-2)}\right\}=\frac{x_s(n-1)}{x_{s+1}(n-2)}\le\frac{x_s(n-1)}{A_{s+1}}.$$

Then we take

$$x_s(3)\le\frac{x_s(2)}{A_{s+1}},\ x_s(4)\le\frac{x_s(2)}{A_{s+1}^2},\dots,\ x_s(n)\le\frac{x_s(2)}{A_{s+1}^{n-2}}.$$

Since from (2.1) $A_{s+1}>1$ there exists an $n_0\ge 3$ such that

$$\frac{x_s(2)}{A_{s+1}^{n-2}}<A_s,\qquad n\ge n_0$$

which implies that $x_s(n) < A_s$, $n \geq n_0$. This contradicts to (2.15) and so there exists a $m_s \geq 3$ such that (2.14) holds. From (1.2) we have

$$x_s(m_s+1) = \max\left\{A_s, \frac{x_s(m_s)}{x_{s+1}(m_s-1)}\right\}. \tag{2.16}$$

In addition relations (2.1), (2.14) imply that

$$\frac{x_s(m_s)}{x_{s+1}(m_s-1)} \leq \frac{A_s}{A_{s+1}} < A_s$$

and so from (2.16) it holds

$$x_s(m_s+1) = A_s.$$

Working inductively we can prove that

$$x_s(n) = A_s, \qquad n \geq m_s. \tag{2.17}$$

So, if $m = \max\{m_1, m_2, \ldots, m_k\}$ we have that $x_i(n) = A_i$, $i = 1, 2, \ldots, k$, for $n \geq m$. This completes the proof of the proposition. □

In the following proposition we prove that all solutions of (1.2) are unbounded if (2.2) are satisfied.

Proposition 2.3. *Consider system* (1.2)*. Suppose that there exists an* $r \in \{1, 2, \ldots, k\}$ *such that* (2.2) *hold. Then all the solutions of system* (1.2) *are unbounded.*

Proof. Let $(x_1(n), x_2(n), \ldots, x_k(n))$ be an arbitrary solution of system (1.2).

Suppose firstly that there exists an $r \in \{1, 2, \ldots, k\}$ such that (2.5) is satisfied. Then since $A_{r+1} > 1$, and using the same argument in the proof of relations (2.14) and (2.17) we can prove that there exists an $n_r \geq 3$ such that

$$x_r(n) = A_r, \qquad n \geq n_r. \tag{2.18}$$

Then from (1.2) and (2.18) we obtain

$$x_{r-1}(n_r+2) = \max\left\{A_{r-1}, \frac{x_{r-1}(n_r+1)}{x_r(n_r)}\right\} \geq \frac{x_{r-1}(n_r+1)}{x_r(n_r)} = \frac{x_{r-1}(n_r+1)}{A_r},$$

and working inductively

$$x_{r-1}(n_r+3) \geq \frac{x_{r-1}(n_r+1)}{A_r^2}, \ldots, x_{r-1}(n_r+n) \geq \frac{x_{r-1}(n_r+1)}{A_r^{n-1}}.$$

Since $A_r < 1$ we have that $\lim_{n\to\infty} x_{r-1}(n) = \infty$. So, the solution of (1.2) is unbounded.

Finally suppose that (2.6) hold. If there exists either an s such that (2.8) hold or a v such that (2.9) are satisfied, then arguing as in the case (2.18) we take that the solution is unbounded. Suppose that (2.10) are satisfied. Therefore since $A_1 > 1$, arguing as in (2.17) we take that there exists an n_k such that

$$x_k(n) = A_k, \qquad n \geq n_k$$

and so using the same argument as above we take

$$x_{k-1}(n_k+n) \geq \frac{x_{k-1}(n_k+1)}{A_k^{n-1}}.$$

Thus since $A_k < 1$ it holds and so $\lim_{n\to\infty} x_{k-1}(n) = \infty$. This completes the proof of the proposition. □

In the next proposition we find unbounded solutions for the system (1.2) in the case where (2.3) hold and k is an even number.

Proposition 2.4. *Consider system* (1.2) *where k is an even number and let condition* (2.3) *hold. Let $(x_1(n), x_2(n), \dots, x_k(n))$ be a solution of* (1.2)*. Suppose that there exists an s, $s \in \{0, 1, \dots, \}$ such that either*

$$\begin{aligned}
&\frac{x_{2r}(s)}{x_{2r+1}(s-1)} > 1, \qquad \frac{x_{2r}(s)}{x_{2r+1}(s-1)x_{2r+1}(s)} > 1, \qquad r = 1, 2, \dots, \frac{k-2}{2},\\
&\frac{x_{2r-1}(s)}{x_{2r}(s-1)} < A_{2r-1}, \qquad x_{2r}(s) > 1, \qquad r = 1, 2, \dots, \frac{k}{2},\\
&\frac{x_k(s)}{x_1(s-1)} > 1, \qquad \frac{x_k(s)}{x_1(s-1)x_1(s)} > 1
\end{aligned} \tag{2.19}$$

or

$$\begin{aligned}
&\frac{x_{2r-1}(s)}{x_{2r}(s-1)} > 1, \qquad \frac{x_{2r-1}(s)}{x_{2r}(s-1)x_{2r}(s)} > 1, \qquad x_{2r-1}(s) > 1, \qquad r = 1, 2, \dots, \frac{k}{2},\\
&\frac{x_{2r}(s)}{x_{2r+1}(s-1)} < A_{2r}, \qquad r = 1, 2, \dots, \frac{k-2}{2},\\
&\frac{x_k(s)}{x_1(s-1)} < A_k, \qquad \frac{x_k(s)}{x_1(s-1)x_1(s)} < A_k
\end{aligned} \tag{2.20}$$

are satisfied. Then if (2.19) *holds we get*

$$\lim_{n\to\infty} x_{2r}(n) = \infty, \qquad x_{2r-1}(n) = A_{2r-1}, \qquad n \geq s+1, \;\; r = 1, 2, \dots, \frac{k}{2} \tag{2.21}$$

and if (2.20) *is satisfied we have*

$$\lim_{n\to\infty} x_{2r-1}(n) = \infty, \qquad x_{2r}(n) = A_{2r}, \qquad n \geq s+1, \;\; r = 1, 2, \dots, \frac{k}{2}. \tag{2.22}$$

Proof. Suppose that the conditions in (2.19) are satisfied. Then form (1.2) and (2.19) we get

$$\begin{aligned}
x_{2r-1}(s+1) &= \max\left\{A_{2r-1}, \frac{x_{2r-1}(s)}{x_{2r}(s-1)}\right\} = A_{2r-1}, \qquad r = 1, 2, \dots, \frac{k}{2},\\
x_{2r}(s+1) &= \max\left\{A_{2r}, \frac{x_{2r}(s)}{x_{2r+1}(s-1)}\right\} = \frac{x_{2r}(s)}{x_{2r+1}(s-1)} > 1, \qquad r = 1, 2, \dots, \frac{k-2}{2},\\
x_k(s+1) &= \max\left\{A_k, \frac{x_k(s)}{x_1(s-1)}\right\} = \frac{x_k(s)}{x_1(s-1)} > 1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
x_{2r-1}(s+2) &= \max\left\{A_{2r-1}, \frac{x_{2r-1}(s+1)}{x_{2r}(s)}\right\} = \max\left\{A_{2r-1}, \frac{A_{2r-1}}{x_{2r}(s)}\right\} = A_{2r-1},\\
x_{2r}(s+2) &= \max\left\{A_{2r}, \frac{x_{2r}(s+1)}{x_{2r+1}(s)}\right\} = \max\left\{A_{2r}, \frac{x_{2r}(s)}{x_{2r+1}(s)x_{2r+1}(s-1)}\right\}\\
&= \frac{x_{2r}(s)}{x_{2r+1}(s-1)x_{2r+1}(s)} > 1,\\
x_k(s+2) &= \max\left\{A_k, \frac{x_k(s+1)}{x_1(s)}\right\} = \frac{x_k(s)}{x_1(s)x_1(s-1)} > 1.
\end{aligned}$$

In addition

$$
\begin{aligned}
x_{2r-1}(s+3) &= \max\left\{A_{2r-1}, \frac{x_{2r-1}(s+2)}{x_{2r}(s+1)}\right\} = \max\left\{A_{2r-1}, \frac{A_{2r-1}}{x_{2r}(s+1)}\right\} = A_{2r-1},\\
x_{2r}(s+3) &= \max\left\{A_{2r}, \frac{x_{2r}(s+2)}{x_{2r+1}(s+1)}\right\} = \max\left\{A_{2r}, \frac{x_{2r}(s)}{x_{2r+1}(s)x_{2r+1}(s-1)A_{2r+1}}\right\}\\
&= \frac{x_{2r}(s)}{x_{2r+1}(s-1)x_{2r+1}(s)A_{2r+1}} > 1,\\
x_k(s+3) &= \max\left\{A_k, \frac{x_k(s+2)}{x_1(s+1)}\right\} = \frac{x_k(s)}{x_1(s)x_1(s-1)A_1} > 1.
\end{aligned}
$$

Working inductively we can prove that

$$
\begin{aligned}
x_{2r-1}(s+v) &= A_{2r-1}, \qquad v = 1,2,\ldots,\quad r = 1,2,\ldots,\frac{k}{2},\\
x_{2r}(s+v) &= \frac{x_{2r}(s)}{x_{2r+1}(s-1)x_{2r+1}(s)A_{2r+1}^{v-2}}, \qquad v = 2,3,\ldots,\quad r = 1,2,\ldots,\frac{k-2}{2},\\
x_k(s+v) &= \frac{x_k(s)}{x_1(s)x_1(s-1)A_1^{v-2}}.
\end{aligned}
$$

Then (2.21) is true if inequalities (2.19) hold. Similarly we can prove that if inequalities (2.20) are satisfied, then (2.22) hold. This completes the proof of the proposition. □

Now we find solutions of system (1.2) where $k = 2$ which converge to period six solutions. A related situation appears in [33]. For simplicity we set

$$x_1(n) = x_n, \qquad x_2(n) = y_n.$$

We use a product-type system of difference equations, which is solvable. There has been some considerable recent interest on solvable product-type systems of difference equations (see, for example, [39,43,44], and the related references therein).

Proposition 2.5. *Consider system*

$$x_{n+1} = \max\left\{A, \frac{x_n}{y_{n-1}}\right\}, \qquad y_{n+1} = \max\left\{B, \frac{y_n}{x_{n-1}}\right\} \tag{2.23}$$

where A, B are positive constants which satisfy

$$0 < A < 1, \qquad 0 < B < 1.$$

Let ϵ be a positive number such that

$$0 < \epsilon < \min\{1 - A,\ 1 - B\}. \tag{2.24}$$

Let (x_n, y_n) be a solution of (2.23) such that

$$\frac{x_0}{y_0} = \left(\frac{x_{-1}}{y_{-1}}\right)^{\lambda}, \qquad \lambda = \frac{1-\sqrt{5}}{2} \tag{2.25}$$

and

$$C_n \geq r = \max\left\{\frac{A}{1-\epsilon},\ \frac{B}{1-\epsilon}\right\}, \tag{2.26}$$

where

$$C_n = \begin{cases} (x_0y_0)^{1/2}, & \text{when } n = 6k, \\ (\frac{x_0y_0}{x_{-1}y_{-1}})^{1/2}, & \text{when } n = 6k+1, \\ (x_{-1}y_{-1})^{-1/2}, & \text{when } n = 6k+2, \\ (x_0y_0)^{-1/2}, & \text{when } n = 6k+3, \\ (\frac{x_0y_0}{x_{-1}y_{-1}})^{-1/2}, & \text{when } n = 6k+4, \\ (x_{-1}y_{-1})^{1/2}, & \text{when } n = 6k+5. \end{cases}$$

Then there exists an n_0 such that for $n \geq n_0$ (x_n, y_n) the form

$$x_n = C_n \left(\frac{x_{-1}}{y_{-1}}\right)^{\frac{1}{2}\lambda^{n+1}}, \qquad y_n = C_n \left(\frac{x_{-1}}{y_{-1}}\right)^{-\frac{1}{2}\lambda^{n+1}} \tag{2.27}$$

and so (x_n, y_n) tends to a period six solution of (2.23).

Proof. First of all we prove that there exist x_0, x_{-1}, y_0, y_{-1} such that (2.26) is satisfied. It is obvious that $0 < r < 1$ since (2.24) holds. We choose a number θ such that

$$0 < -r + \sqrt{r} < \theta < 1 - r. \tag{2.28}$$

Let now numbers v, w be such that

$$r < r + \theta < v < (r+\theta)^{-1} < r^{-1}, \qquad r < r + \theta < w < (r+\theta)^{-1} < r^{-1}. \tag{2.29}$$

From (2.28) we get $r < (r+\theta)^2$. So,

$$r < (r+\theta)^2 < \frac{v}{w} < (r+\theta)^{-2} < r^{-1}.$$

Then if we choose x_0, x_{-1}, y_0, y_{-1}, such that the numbers

$$v = (x_0y_0)^{1/2}, \qquad w = (x_{-1}y_{-1})^{1/2}$$

satisfy inequalities (2.29), relation (2.26) is true.

We consider the system of difference equations

$$x_{n+1} = \frac{x_n}{y_{n-1}}, \qquad y_{n+1} = \frac{y_n}{x_{n-1}}, \qquad n = 0, 1, 2, \ldots \tag{2.30}$$

Let (x_n, y_n) be a solution of (2.30) which satisfies (2.25) and (2.26). Then we get

$$x_{n+4} = \frac{x_{n+3}^2 x_n}{x_{n+2}}$$

which implies that

$$\ln x_{n+4} - 2\ln x_{n+3} + \ln x_{n+2} - \ln x_n = 0.$$

By setting

$$z_n = \ln x_n \tag{2.31}$$

we get

$$z_{n+4} - 2z_{n+3} + z_{n+2} - z_n = 0. \tag{2.32}$$

The characteristic equation of (2.32) is the following

$$p^4 - 2p^3 + p^2 - 1 = (p^2 - p - 1)(p^2 - p + 1) = 0. \tag{2.33}$$

Then z_n has the form

$$z_n = d_1\mu^n + d_2\lambda^n + d_3\cos\left(\frac{n\pi}{3}\right) + d_4\sin\left(\frac{n\pi}{3}\right), \tag{2.34}$$

where $\mu = \frac{1+\sqrt{5}}{2}$, $\lambda = \frac{1-\sqrt{5}}{2}$, d_1, d_2, d_3, d_4 are constants.

If we set

$$w_n = \ln y_n \tag{2.35}$$

from (2.30) we get

$$\begin{aligned} w_n = z_{n+1} - z_{n+2} &= d_1(1-\mu)\mu^{n+1} + d_2(1-\lambda)\lambda^{n+1} \\ &+ d_3\left(\cos\left(\frac{(n+1)\pi}{3}\right) - \cos\left(\frac{(n+2)\pi}{3}\right)\right) \\ &+ d_4\left(\sin\left(\frac{(n+1)\pi}{3}\right) - \sin\left(\frac{(n+2)\pi}{3}\right)\right) \\ &= -d_1\mu^n - d_2\lambda^n + d_3\cos\left(\frac{n\pi}{3}\right) + d_4\sin\left(\frac{n\pi}{3}\right). \end{aligned} \tag{2.36}$$

From (2.34) and (2.36) we get

$$\begin{aligned} z_{-1} &= d_1\mu^{-1} + d_2\lambda^{-1} + d_3\frac{1}{2} - d_4\frac{\sqrt{3}}{2}, \\ z_0 &= d_1 + d_2 + d_3, \\ w_{-1} &= -d_1\mu^{-1} - d_2\lambda^{-1} + d_3\frac{1}{2} - d_4\frac{\sqrt{3}}{2}, \\ w_0 &= -d_1 - d_2 + d_3. \end{aligned} \tag{2.37}$$

From (2.37) we have

$$\begin{aligned} d_1 &= \frac{1+\sqrt{5}}{8\sqrt{5}}\left(2(z_0 - w_0) - (1-\sqrt{5})(z_{-1} - w_{-1})\right), \\ d_2 &= \left(\frac{1}{4} - \frac{1}{4\sqrt{5}}\right)(z_0 - w_0) - \frac{1}{2\sqrt{5}}(z_{-1} - w_{-1}), \\ d_3 &= \frac{z_0 + w_0}{2}, \\ d_4 &= \frac{\sqrt{3}}{6}\left(-2(z_{-1} + w_{-1}) + z_0 + w_0\right). \end{aligned} \tag{2.38}$$

Relation (2.25) implies that

$$\begin{aligned} 2(z_0 - w_0) - (1-\sqrt{5})(z_{-1} - w_{-1}) &= 2(\ln x_0 - \ln y_0) - (1-\sqrt{5})(\ln x_{-1} - \ln y_{-1}) \\ &= 2\left(\ln\frac{x_0}{y_0} - \lambda\ln\frac{x_{-1}}{y_{-1}}\right) = 0 \end{aligned}$$

and so $d_1 = 0$.

From (2.25), (2.34), (2.36), (2.38) we get

$$z_n = \frac{1}{2}(z_{-1} - w_{-1})\lambda^{n+1} + \frac{z_0 + w_0}{2}\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{6}\Big(-2(z_{-1} + w_{-1}) + z_0 + w_0\Big)\sin\frac{n\pi}{3}$$
$$w_n = -\frac{1}{2}(z_{-1} - w_{-1})\lambda^{n+1} + \frac{z_0 + w_0}{2}\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{6}\Big(-2(z_{-1} + w_{-1}) + z_0 + w_0\Big)\sin\frac{n\pi}{3}.$$

By using (2.31) and (2.35) we get

$$\ln x_n = \frac{1}{2}\ln\left(\frac{x_{-1}}{y_{-1}}\right)\lambda^{n+1} + \frac{1}{2}\ln(x_0 y_0)\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{6}\left(-2\ln(x_{-1}y_{-1}) + \ln(x_0 y_0)\right)\sin\frac{n\pi}{3},$$
$$\ln y_n = -\frac{1}{2}\ln\left(\frac{x_{-1}}{y_{-1}}\right)\lambda^{n+1} + \frac{1}{2}\ln(x_0 y_0)\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{6}\left(-2\ln(x_{-1}y_{-1}) + \ln(x_0 y_0)\right)\sin\frac{n\pi}{3}.$$

From this and by some standard algebraic calculations we can easily prove that the relations in (2.27) are satisfied, with the constants C_n as defined in above.

Since $|\lambda| < 1$ it is obvious that

$$\lim_{n\to\infty}\left(\frac{x_{-1}}{y_{-1}}\right)^{\frac{1}{2}\lambda^{n+1}} = 1, \qquad \lim_{n\to\infty}\left(\frac{x_{-1}}{y_{-1}}\right)^{-\frac{1}{2}\lambda^{n+1}} = 1.$$

Then if ϵ is a positive number which satisfy (2.24) there exists a n_0 such that for $n \geq n_0$

$$\left(\frac{x_{-1}}{y_{-1}}\right)^{\frac{1}{2}\lambda^{n+1}} > 1 - \epsilon, \qquad \left(\frac{x_{-1}}{y_{-1}}\right)^{-\frac{1}{2}\lambda^{n+1}} > 1 - \epsilon. \tag{2.39}$$

Therefore using (2.27), (2.39) we get for $n \geq n_0$

$$\begin{aligned} x_n &\geq \max\left\{\frac{A}{1-\epsilon}, \frac{B}{1-\epsilon}\right\}(1-\epsilon) = \max\{A, B\}, \\ y_n &\geq \max\left\{\frac{A}{1-\epsilon}, \frac{B}{1-\epsilon}\right\}(1-\epsilon) = \max\{A, B\}. \end{aligned} \tag{2.40}$$

Then from (2.30) and (2.40) we have that (x_n, y_n) is a bounded solution of (2.23) where for $n \geq n_0$ satisfies system (2.30). This completes the proof of the proposition. □

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