Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **46**, 1–14; https://doi.org/10.14232/ejqtde.2020.1.46 www.math.u-szeged.hu/ejqtde/

A sharp oscillation result for second-order half-linear noncanonical delay differential equations

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Received 23 January 2020, appeared 13 July 2020
Communicated by Josef Diblík

Abstract. In the paper, new single-condition criteria for the oscillation of all solutions to a second-order half-linear delay differential equation in noncanonical form are obtained, relaxing a traditionally posed assumption that the delay function is nondecreasing. The oscillation constant is best possible in the sense that the strict inequality cannot be replaced by the nonstrict one without affecting the validity of the theorem. This sharp result is new even in the linear case and, to the best of our knowledge, improves all the existing results reporting in the literature so far. The advantage of our approach is the simplicity of the proof, only based on sequentially improved monotonicities of a positive solution.

Keywords: second-order differential equation, delay, half-linear, oscillation.

2020 Mathematics Subject Classification: 34C10, 34K11.

1 Introduction

Consider the second-order half-linear delay differential equation of the form

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + q(t)y^{\alpha}(\tau(t)) = 0, \qquad t \geq t_0 > 0, \tag{1.1}$$

where we assume that $\alpha > 0$ is a quotient of odd positive integers; functions r, q, and τ are continuous positive functions, $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$. Without further mentioning, we will assume that (1.1) is in so-called noncanonical form, i.e.,

$$\pi(t_0) := \int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\alpha}(t)} < \infty.$$

By a solution of Eq. (1.1) we mean any differentiable function y which does not vanishes eventually such that $r(y')^{\alpha}$ is differentiable, satisfying (1.1) for sufficiently large t. As is customary, a solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor

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eventually negative. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

The oscillation theory of second-order functional differential equations has attracted a great portion of attention, which is evidenced by extensive research in the area. For a compact summary of the most recent results and appearing open problems, the reader is referred to the recent monographs the monographs by Agarwal et al. [2–5], Došlý and Řehák [12] Győri and Ladas [16], and Saker [22].

In the paper, we obtain new single-condition criteria for the oscillation of all solutions to (1.1) with unimprovable constants. This sharp result is new even in the linear case and, to the best of our knowledge, improves all the existing results reported in the literature so far. In the linear case, we also formulate analogous results for canonical equations.

The structure of the paper is the following. In Section 2, we revise the oscillatory properties of various useful equations serving as models for comparison of the obtained results. In Section 3, main results of the paper are stated, and their proofs are given in Section 4.

2 Comparison equations in the oscillation theory

Euler-type differential equations have been of utmost importance in the oscillation theory since Sturm's pioneering work in 1863. Till now, they are commonly used to examine the sharpness of general criteria derived by different methods. The optimal scenario is when the obtained criterion gives a sharp result for the Euler-type equation; or at least it is closer to it for a given set of parameters, compared to another one. Perhaps the most familiar one is the second-order linear Euler equation

$$y''(t) + \frac{q_0}{t^2} y(t) = 0 \tag{2.1}$$

which is oscillatory if and only if

$$q_0 > \frac{1}{4}. \tag{2.2}$$

In 1893, A. Kneser [17] firstly used Sturmian methods and (2.1) to show that the linear equation

$$y''(t) + q(t)y(t) = 0$$

is oscillatory if

$$\liminf_{t\to\infty} t^2 q(t) > \frac{1}{4}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^2 q(t) < \frac{1}{4}.$$

For our purposes, we consider, as a particular case of (1.1), the generalized Euler-type half-linear ordinary differential equation

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + \frac{q_0}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} y^{\alpha}(t) = 0, \qquad q_0 > 0. \tag{2.3}$$

It is well-known that (2.3) is oscillatory if and only if its characteristic equation

$$c_1(m) := \alpha m^{\alpha}(1-m) = q_0$$

has no real roots what happens if

$$q_0 > \max\{c_1(m) : 0 < m < 1\} = c_1\left(\frac{\alpha}{\alpha+1}\right) = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \tag{2.4}$$

cf. [2, Remark 3.7.1] or [12]. If condition (2.4) fails, then (2.3) has a nonoscillatory solution $y(t) = \pi^m(t)$. As an immediate consequence of the Sturmian comparison theorem and the above result concerning (2.3), we get the following version of the classical Kneser oscillation and nonoscillation criterion for the noncanonical equation

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + q(t)y^{\alpha}(t) = 0. \tag{2.5}$$

Proposition 2.1. *Suppose that*

$$\liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{2.6}$$

Then (2.5) is oscillatory. If

$$\limsup_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) < \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

then (2.5) is nonoscillatory.

As another important particular case of (1.1), we consider the linear Euler-type equation with proportional delay, namely,

$$\left(r(t)y'(t)\right)' + \frac{q_0}{r(t)\pi^2(t)}y(kt) = 0, \qquad 0 < k \le 1, \tag{2.7}$$

where $r(t) = t^{p+1}$, $p > 0$. By a simple change of variables

$$s = \frac{1}{\pi(t)} \quad \text{and} \quad y(t) = \frac{u(s)}{s}, \tag{2.8}$$

(2.7) can be rewritten as

$$u''(s) + \frac{q_0}{k^p s^2}u(k^p s) = 0. \tag{2.9}$$

By transforming (2.9) into a constant-coefficient-constant-delay equation, Kulenović [18] showed that (2.9) is oscillatory if and only if the associated characteristic equation

$$c_2(m) := m(1-m)k^{mp} = q_0$$

has no real root what happens if

$$q_0 > \max\{c_2(m) : 0 < m < 1\} = c_2(m_{\max}), \tag{2.10}$$

where

$$m_{\max} = \frac{-\sqrt{r^2+4} + r + 2}{2r}, \qquad r = -p\ln k.$$

It is well-known that the Sturmian comparison theorem fails to extend to the more general equation

$$\left(r(t)y'(t)\right)' + q(t)y(\tau(t)) = 0 \tag{2.11}$$

due to the delayed argument. For delay differential equations, Kusano and Naito established an alternative comparison principle [19] in the sense that oscillation of the studied differential equation can be deduced from the oscillation of a simpler one. Using their result [19, Theorem 3] for (2.7), one can conclude that the equation

$$\left(t^{p+1}y'(t)\right)' + q(t)y(kt) = 0, \qquad p>0,\ 0<k\leq 1, \tag{2.12}$$

is oscillatory if

$$\liminf_{t\to\infty} \frac{t^{1-p}q(t)}{p^2} > \max\{c_2(m) : 0<m<1\}.$$

As a generalized version of (2.7), we consider

$$(r(t)y'(t))' + \frac{q_0}{r(t)\pi^2(t)} y(\tau(t)) = 0, \tag{2.13}$$

with the constant ratio $\pi(\tau(t))/\pi(t) = \lambda$. It can be verified by a direct substitution that (2.13) has a nonoscillatory solution $y(t) = \pi^m(t)$ if

$$q_0 \leq \max\{c_3(m) : 0<m<1\},$$

where

$$c_3(m) := m(1-m)\lambda^{-m}.$$

The "only if" part here is difficult to prove because the transformation to a constant-coefficient-constant-delay form is obviously impossible. To the best of the authors' knowledge, there is no oscillation criterion for (2.11) which would be sharp for (2.13).

Finally, we consider the most general Euler-type half-linear delay differential equation

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + \frac{q_0}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} y^{\alpha}(\tau(t)) = 0, \quad q_0>0, \quad t\geq t_0, \tag{2.14}$$

where the functions r and τ are general and such that $\pi(\tau(t))/\pi(t) = \lambda$. Note that (2.14) includes both (2.3) and (2.13) as particular cases. As previously, we find that (2.14) has a nonoscillatory solution $y(t) = \pi^m(t)$ if there is a positive root of the equation

$$c_4(m) := \alpha m^{\alpha}(1-m)\lambda^{-\alpha m} = q_0, \tag{2.15}$$

what happens if

$$q_0 \leq \max\{c_4(m) : 0<m<1\}. \tag{2.16}$$

In the paper, we will show that (2.16) is not only sufficient but necessary for the existence of nonoscillatory solution of (2.14). Before that, we conclude the introductory section by revising briefly different approaches and oscillation results available for the equation (1.1). Here, it is important to stress that all below-mentioned results require that τ is a nondecreasing function.

Because of its simpler structure of nonoscillatory solutions, (1.1) has been mostly studied in canonical form and much less efforts have been undertaken for noncanonical equations. Since Trench canonical theory [24] fails to extend to half-linear equations, a common approach in the literature for investigation of such equations consists in extending known results for canonical ones, see [1, 11, 13–15, 20, 21, 23, 25]. In 2017, Džurina and Jadlovská [9] revised a variety of existing results by removing a traditionally imposed condition and obtained several one-condition oscillation criteria for (1.1).

In general, there are two main factors contributing to the oscillatory behavior of (1.1): the second-order nature of the equation and the presence of the delay; mostly treated independently by an application of one of the following methods:

1. using comparison with a second-order half-linear ordinary differential equation, directly or indirectly via generalized Riccati generalized inequality
$$u'(t) + q(t) + \alpha r^{-1/\alpha}(t) u^{(\alpha+1)/\alpha}(t) \le 0, \tag{2.17}$$
2. using comparison with a second-order linear differential equation; by employing linearization techniques,
3. using comparison with a first-order linear delay differential equation; where the delay is essential, but the information about the second-order nature of the equation is lost.

Another method based on the weighted Hardy inequality was presented in [8]. Any of works [1,8,11,13–15,20,21,23,25], employing the methods (1) or (2) gives at best
$$q_0 > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$
for the Euler equation (2.14) with $r(t) = t^{\alpha+1}$ and $\tau(t) = kt$, $k \in (0,1]$, which is sharp only in the ordinary case (2.3). Here, it is easy to see that the influence of the delay is completely lost. Some improvement was recently made by present authors [10] under assumption that $\pi(\tau(t))/\pi(t) \ge \lambda > 1$, which yields
$$\lambda^{q_0} q_0 > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$
On the other hand, the method (3) employed in [7] requires
$$q_0^{1/\alpha} \ln\frac{1}{k} > \frac{1}{\mathrm{e}}.$$
for (2.14) with $r(t) = t^{\alpha+1}$ to be oscillatory.

The purpose of the paper is to obtain the best-possible single-condition oscillation criterion for (1.1), where both the above-mentioned factors jointly contribute. The ideas partly exploit the very recent ones from [6] for the linear equation
$$\big(r(t)y'(t)\big)' + q(t)y(\tau(t)) = 0. \tag{2.18}$$
Theorem A (See [6, Theorem 3.4])**.** *Assume that $\tau(t)$ is nondecreasing, $\tau(t) < t$,*
$$\int_{t_0}^{\infty} q(s)\pi(s)\mathrm{d}s = \infty, \tag{2.19}$$
and there exists a constant $\beta_0 > 0$ such that
$$q(t)\pi^2(t)r(t) \ge \beta_0$$
eventually. If there exists $n \in \mathbb{N}$, such that $\beta_n < 1$ for $n = 0,1,2,\dots,n-1$, and
$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s)\tau(s) > \frac{1-\beta_n}{e},$$
where
$$\beta_n := \frac{\beta_0 \lambda^{\beta_{n-1}}}{1-\beta_{n-1}}$$
for $n \in \mathbb{N}$ and λ satisfying
$$\frac{\pi(\tau(t))}{\pi(t)} \ge \lambda$$
eventually, then (2.18) is oscillatory.

Our newly obtained results (Theorems 3.1 and 3.4) can be regarded as a natural extension of the oscillation part of Proposition 2.1 to a half-linear delay differential equation. Their advantage over the known results is threefold: first of all, Theorem 3.1 involves the oscillation constant which is optimal for the most general Euler-type comparison equation (2.14), and hence unimprovable. Secondly, in contrast with related works [1,7,8,10,11,13–15,20,21,23,25], we relaxed the assumption that τ is nondecreasing. Thirdly, our results in a special case $\alpha = 1$ improve Theorem A in several ways:

1. we use the limit inferior of quantities $q(t)\pi^2(t)r(t)$ and $\pi(\tau(t))/\pi(t)$ in definitions of corresponding constants, which is less-restrictive to apply;
2. we show that the iteration process can be omitted in final criteria;
3. our results do not require $\tau(t) < t$ nor the monotonicity of τ, as we have already mentioned.

3 Main results

In this section, we state the main results of the paper.

Theorem 3.1. *Let*

$$\lambda_* := \liminf_{t\to\infty} \frac{\pi(\tau(t))}{\pi(t)} < \infty. \tag{3.1}$$

If

$$\liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) > \max\{c(m) := \alpha m^{\alpha}(1-m)\lambda_*^{-\alpha m} : 0 < m < 1\}, \tag{3.2}$$

then (1.1) *is oscillatory.*

Corollary 3.2. *By some computations, one has*

$$\max\{c(m) : 0 < m < 1\} = c(m_{\max}),$$

where

$$m_{\max} = \begin{cases} \dfrac{\alpha}{\alpha+1}, & \text{for } \lambda_* = 1 \\ \dfrac{-\sqrt{(\alpha r+\alpha+1)^2 - 4\alpha^2 r} + \alpha r + \alpha + 1}{2\alpha r}, & \text{for } \lambda_* \neq 1 \text{ and } r = \ln \lambda_*, \end{cases}$$

and $c(m)$ *is defined by* (3.2).

Remark 3.3. It is easy to verify that for $\tau(t) = t$, condition (3.2) reduces to (2.6). In view of (2.16), it is clear that condition (3.2) is sharp in the sense that the strict inequality cannot be replaced by the nonstrict one without affecting the validity of the theorem. Hence, Theorem 3.1 can be viewed as a sharp extension of Kneser oscillation criterion (2.6) to a delay half-linear equation.

For the remaining case when (3.1) is violated, we have the following result.

Theorem 3.4. *Let*

$$\lim_{t\to\infty} \frac{\pi(\tau(t))}{\pi(t)} = \infty. \tag{3.3}$$

If

$$\liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) > 0,$$

then (1.1) *is oscillatory.*

In the linear case $\alpha = 1$, it is possible to transfer the oscillation property from (1.1) to the canonical equation

$$\left(\tilde{r}(t)x'(t)\right)' + \tilde{q}(t)x(\tau(t)) = 0, \qquad t \geq t_0 > 0, \tag{3.4}$$

where $\tilde{r}$ and $\tilde{q}$ are continuous positive functions, and

$$R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{\tilde{r}(s)} \to \infty \quad \text{as } t \to \infty.$$

Theorem 3.5. *Let*

$$\delta_* := \liminf_{t\to\infty} \frac{R(t)}{R(\tau(t))} < \infty.$$

If

$$\liminf_{t\to\infty} \tilde{r}(t)\tilde{q}(t)R(t)R(\tau(t)) > \max\{m(1-m)\delta_*^{-m} : 0 < m < 1\},$$

then (3.4) *is oscillatory.*

Theorem 3.6. *Let*

$$\lim_{t\to\infty} \frac{R(t)}{R(\tau(t))} = \infty.$$

If

$$\liminf_{t\to\infty} \tilde{r}(t)\tilde{q}(t)R(t)R(\tau(t)) > 0,$$

then (3.4) *is oscillatory.*

4 Auxiliary lemmas and proofs of main results

Let us define

$$\beta_* := \liminf_{t\to\infty} \frac{1}{\alpha} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t). \tag{4.1}$$

The arguments in the proofs are based on the existence of positive β_*, which is also necessary for the validity of Theorems 3.1 and 3.4. Then, for arbitrary fixed $\beta \in (0, \beta_*)$ and $\lambda \in [1, \lambda_*)$, there is a $t_1 \geq t_0$, such that

$$\frac{1}{\alpha} q(t)r^{1/\alpha}(t)\pi^{\alpha+1}(t) \geq \beta \quad \text{and} \quad \frac{\pi(\tau(t))}{\pi(t)} \geq \lambda \quad \text{on } [t_1, \infty). \tag{4.2}$$

In the sequel, we assume that all functional inequalities hold eventually, that is, they are satisfied for all t large enough.

Lemma 4.1. *Let $\beta_* > 0$. If* (1.1) *has an eventually positive solution y, then*

(i) y is eventually decreasing with $\lim_{t\to\infty} y(t) = 0$;

(ii) y/π is eventually nondecreasing.

Proof. (i). By [9, Theorem 1], the conclusion applies if

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} \left(\int_{t_0}^{t} q(s)\mathrm{d}s\right)^{1/\alpha} \mathrm{d}t = \infty. \tag{4.3}$$

Indeed, by simple computations, we see that

$$\int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)} \left(\int_{t_1}^{u} q(s) \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u \geq \sqrt[\alpha]{\beta} \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)} \left(\int_{t_1}^{u} \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u$$
$$= \sqrt[\alpha]{\beta} \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)} \left(\frac{1}{\pi^{\alpha}(u)} - \frac{1}{\pi^{\alpha}(t_1)} \right)^{1/\alpha} \mathrm{d}u$$

with β defined by (4.2). Since $\pi^{-\alpha}(t) \to \infty$ as $t \to \infty$, for any $\ell \in (0,1)$ and t large enough, we have $\pi^{-\alpha}(t) - \pi^{-\alpha}(t_1) \geq \ell^{\alpha} \pi^{-\alpha}(t)$ and hence

$$\int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)} \left(\int_{t_1}^{u} q(s) \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u \geq \ell \sqrt[\alpha]{\beta} \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)\pi(u)} \mathrm{d}u = \ell \sqrt[\alpha]{\beta} \ln \frac{\pi(t_1)}{\pi(t)} \to \infty \quad \text{as } t \to \infty.$$

(ii). Using the fact that $r^{1/\alpha} y'$ is nondecreasing, we obtain

$$y(t) \geq -\int_{t}^{\infty} \frac{1}{r^{1/\alpha}(s)} r^{1/\alpha}(s) y'(s) \mathrm{d}s \geq -r^{1/\alpha}(t) y'(t) \pi(t),$$

i.e.

$$\left(\frac{y}{\pi} \right)' = \frac{r^{1/\alpha} y' \pi + y}{r^{1/\alpha} \pi^2} \geq 0.$$

The proof is complete. □

Remark 4.2. Compared to the original Lemma statement used in [9, Theorem 1], we replaced the integral condition (4.3) by the requirement of positive β_*. In Theorem A, condition (2.19) was used to arrive at the same conclusion.

To improve the (i)-part of Lemma 4.1, we define a sequence $\{\beta_n\}$ by

$$\begin{aligned} \beta_0 &= \sqrt[\alpha]{\beta_*} \\ \beta_n &= \frac{\beta_0 \lambda_*^{\beta_{n-1}}}{\sqrt[\alpha]{1 - \beta_{n-1}}}, \quad n \in \mathbb{N}. \end{aligned} \tag{4.4}$$

By induction, it is easy to show that if for any $n \in \mathbb{N}$, $\beta_i < 1$, $i = 0, 1, 2, \ldots, n$, then β_{n+1} exists and

$$\beta_{n+1} = \ell_n \beta_n > \beta_n, \tag{4.5}$$

where ℓ_n is defined by

$$\begin{aligned} \ell_0 &= \frac{\lambda_*^{\beta_0}}{\sqrt[\alpha]{1 - \beta_0}} \\ \ell_{n+1} &= \lambda_*^{\beta_0(\ell_n - 1)} \sqrt[\alpha]{\frac{1 - \beta_n}{1 - \ell_n \beta_n}}, \qquad n \in \mathbb{N}_0. \end{aligned}$$

Lemma 4.3. *Let $\beta_* > 0$ and $\lambda_* < \infty$. If (1.1) has an eventually positive solution y, then for any $n \in \mathbb{N}_0$ y/π^{β_n} is eventually decreasing.*

Proof. Let y be a positive solution of (1.1) on $[t_1, \infty)$ where $t_1 \geq t_0$ is such that $y(\tau(t)) > 0$ and (4.2) holds for $t \geq t_1$. Integrating (1.1) from t_1 to t, we have

$$-r(t) \left(y'(t) \right)^{\alpha} = -r(t_1) \left(y'(t_1) \right)^{\alpha} + \int_{t_1}^{t} q(s) y^{\alpha}(\tau(s)) \mathrm{d}s. \tag{4.6}$$

By (i) of Lemma 4.1, y is decreasing and so $y(\tau(t)) \geq y(t)$ for $t \geq t_1$. Therefore,

$$-r(t)\left(y'(t)\right)^\alpha \geq -r(t_1)\left(y'(t_1)\right)^\alpha + \int_{t_1}^t q(s)y^\alpha(s)\mathrm{d}s \geq -r(t_1)\left(y'(t_1)\right)^\alpha + y^\alpha(t)\int_{t_1}^t q(s)\mathrm{d}s.$$

Using (4.2) in the above inequality, we get

$$\begin{aligned}-r(t)\left(y'(t)\right)^\alpha &\geq -r(t_1)\left(y'(t_1)\right)^\alpha + \beta y^\alpha(t)\int_{t_1}^t \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s \\ &\geq -r(t_1)\left(y'(t_1)\right)^\alpha + \beta\frac{y^\alpha(t)}{\pi^\alpha(t)} - \beta\frac{y^\alpha(t)}{\pi^\alpha(t_1)}.\end{aligned} \tag{4.7}$$

From (i)-part of Lemma 4.1, we have that $\lim_{t\to\infty} y(t) = 0$. Hence, there is a $t_2 \in [t_1, \infty)$ such that

$$-r(t_1)\left(y'(t_1)\right)^\alpha - \beta\frac{y^\alpha(t)}{\pi^\alpha(t_1)} > 0, \qquad t \geq t_2.$$

Thus,

$$-r(t)\left(y'(t)\right)^\alpha > \beta\frac{y^\alpha(t)}{\pi^\alpha(t)} \tag{4.8}$$

or

$$-r^{1/\alpha}(t)y'(t)\pi(t) > \sqrt[\alpha]{\beta}y(t) = \varepsilon_0\beta_0 y(t),$$

where $\varepsilon_0 = \sqrt[\alpha]{\beta}/\beta_0$ is an arbitrary constant from $(0,1)$. Therefore,

$$\left(\frac{y}{\pi^{\sqrt[\alpha]{\beta}}}\right)' = \frac{r^{1/\alpha}y'\pi^{\sqrt[\alpha]{\beta}} + \sqrt[\alpha]{\beta}\pi^{\sqrt[\alpha]{\beta}-1}y}{r^{1/\alpha}\pi^{2\sqrt[\alpha]{\beta}}} = \frac{\pi^{\sqrt[\alpha]{\beta}-1}\left(\sqrt[\alpha]{\beta}y + \pi r^{1/\alpha}y'\right)}{r^{1/\alpha}\pi^{2\sqrt[\alpha]{\beta}}} \leq 0, \qquad t \geq t_2. \tag{4.9}$$

Integrating (1.1) from t_2 to t and using that $y/\pi^{\sqrt[\alpha]{\beta}}$ is decreasing, we have

$$\begin{aligned}-r(t)\left(y'(t)\right)^\alpha &\geq -r(t_2)\left(y'(t_2)\right)^\alpha + \left(\frac{y(\tau(t))}{\pi^{\sqrt[\alpha]{\beta}}(\tau(t))}\right)^\alpha \int_{t_2}^t q(s)\pi^{\alpha\sqrt[\alpha]{\beta}}(\tau(s))\mathrm{d}s \\ &\geq -r(t_2)\left(y'(t_2)\right)^\alpha + \left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^\alpha \int_{t_2}^t q(s)\left(\frac{\pi(\tau(s))}{\pi(s)}\right)^{\alpha\sqrt[\alpha]{\beta}} \pi^{\alpha\sqrt[\alpha]{\beta}}(s)\mathrm{d}s.\end{aligned}$$

By virtue of (4.2), we see that

$$\begin{aligned}-r(t)\left(y'(t)\right)^\alpha \geq&\ -r(t_2)\left(y'(t_2)\right)^\alpha + \beta\left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^\alpha \int_{t_2}^t \frac{\alpha\left(\frac{\pi(\tau(s))}{\pi(s)}\right)^{\alpha\sqrt[\alpha]{\beta}}}{r^{1/\alpha}(s)\pi^{\alpha+1-\alpha\sqrt[\alpha]{\beta}}(s)}\mathrm{d}s \\ \geq&\ -r(t_2)\left(y'(t_2)\right)^\alpha \\ &+ \frac{\beta}{1-\sqrt[\alpha]{\beta}}\lambda^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^\alpha \int_{t_2}^t \frac{\alpha(1-\sqrt[\alpha]{\beta})}{r^{1/\alpha}(s)\pi^{\alpha+1-\alpha\sqrt[\alpha]{\beta}}(s)}\mathrm{d}s \\ =&\ -r(t_2)\left(y'(t_2)\right)^\alpha \\ &+ \frac{\beta}{1-\sqrt[\alpha]{\beta}}\lambda^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^\alpha \left(\frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t)} - \frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t_2)}\right).\end{aligned} \tag{4.10}$$

Now, we claim that $\lim_{t\to\infty} y(t)/\pi^{\sqrt[\alpha]{\beta}}(t) = 0$. It suffices to show that there is $\varepsilon > 0$ such that $y/\pi^{\sqrt[\alpha]{\beta}+\varepsilon}$ is eventually decreasing. Since $\pi(t)$ tends to zero, there is a constant

$$\ell \in \left(\frac{\sqrt[\alpha]{1-\sqrt[\alpha]{\beta}}}{\lambda^{\sqrt[\alpha]{\beta}}}, 1 \right)$$

and a $t_3 \geq t_2$ such that

$$\frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t)} - \frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t_2)} > \ell^\alpha \frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t)}, \qquad t \geq t_3.$$

Using the above inequality in (4.10) yields

$$-r(t)\left(y'(t)\right)^\alpha \geq \frac{\ell^\alpha \beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} \left(\frac{y(t)}{\pi(t)}\right)^\alpha,$$

i.e.,

$$-r^{1/\alpha} y'(t) \geq \left(\sqrt[\alpha]{\beta} + \varepsilon\right) \frac{y(t)}{\pi(t)}, \tag{4.11}$$

where

$$\varepsilon = \sqrt[\alpha]{\beta} \left(\frac{\ell \lambda^{\sqrt[\alpha]{\beta}}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\beta}}} - 1 \right) > 0.$$

Thus, from (4.11),

$$\left(\frac{y}{\pi^{\sqrt[\alpha]{\beta}+\varepsilon}}\right)' \leq 0, \qquad t \geq t_3,$$

and hence the claim holds. Therefore, for $t_4 \in [t_3, \infty)$,

$$-r(t_2)\left(y'(t_2)\right)^\alpha - \frac{\beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} \left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^\alpha \frac{1}{\pi^{\alpha-\alpha\sqrt[\alpha]{\beta}}(t_2)} > 0, \qquad t \geq t_4.$$

Turning back to (4.10) and using the above inequality,

$$\begin{aligned}
-r(t)\left(y'(t)\right)^\alpha \geq & -r(t_2)\left(y'(t_2)\right)^\alpha + \frac{\beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} \left(\frac{y(t)}{\pi(t)}\right)^\alpha \\
& - \frac{\beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} \left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^\alpha \frac{1}{\pi^{\alpha-\alpha\sqrt[\alpha]{\beta}}(t_2)} \\
> & \frac{\beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} y^\alpha,
\end{aligned}$$

or

$$-r^{1/\alpha} y' \pi > \frac{\sqrt[\alpha]{\beta}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\beta}}} \lambda^{\sqrt[\alpha]{\beta}} y = \varepsilon_1 \beta_1 y, \qquad t \geq t_4,$$

where

$$\varepsilon_1 = \sqrt[\alpha]{\frac{\beta(1-\sqrt[\alpha]{\beta_*})}{\beta_*(1-\sqrt[\alpha]{\beta})}} \frac{\lambda^{\sqrt[\alpha]{\beta}}}{\lambda_*^{\sqrt[\alpha]{\beta_*}}}$$

is arbitrary constant from $(0,1)$ approaching 1 if $\beta \to \beta_*$ and $\lambda \to \lambda_*$. Hence,

$$\left(\frac{y}{\pi^{\varepsilon_1 \beta_1}}\right)' < 0, \qquad t \geq t_4.$$

By induction, one can show that for any $n \in \mathbb{N}_0$ and t large enough,

$$\left(\frac{y}{\pi^{\varepsilon_n \beta_n}}\right)' < 0,$$

where ε_n given by

$$\varepsilon_0 = \sqrt[\alpha]{\frac{\beta}{\beta_*}}$$
$$\varepsilon_{n+1} = \varepsilon_0 \sqrt[\alpha]{\frac{1-\beta_n}{1-\varepsilon_n\beta_n}\frac{\lambda^{\varepsilon_n\beta_n}}{\lambda_*^{\beta_n}}}, \quad n \in \mathbb{N}_0$$

is arbitrary constant from $(0,1)$ approaching 1 if $\beta \to \beta_*$ and $\lambda \to \lambda_*$. Finally, we claim that from any $n \in \mathbb{N}_0$, $y/\pi^{\varepsilon_{n+1}\beta_{n+1}}$ decreasing implies that that y/π^{β_n} is decreasing as well. To see this, we use that from (4.5) and the fact that ε_{n+1} is arbitrarily close to 1,

$$\varepsilon_{n+1}\beta_{n+1} > \beta_n.$$

Hence, for t large enough,

$$-r^{1/\alpha}y'\pi > \varepsilon_{n+1}\beta_{n+1}y > \beta_n y$$

and so for any $n \in \mathbb{N}_0$ and t large enough,

$$\left(\frac{y}{\pi^{\beta_n}}\right)' < 0.$$

The proof is complete. □

Now, we are prepared to give straightforward proofs of the main results.

Proof of Theorem 3.1. Assume that y is an eventually positive solution of (1.1). Lemmas 4.1 and 4.3 ensure that $(y/\pi)' \geq 0$ and $(y/\pi^{\beta_n})' < 0$ for any $n \in \mathbb{N}_0$ and t large enough. This is the case when

$$\beta_n < 1 \quad \text{for any } n \in \mathbb{N}_0.$$

Hence the sequence $\{\beta_n\}$ defined by (4.4) is increasing and bounded from above, and so there exists a finite limit

$$\lim_{n\to\infty} \beta_n = m,$$

where m is the smaller positive root of the equation

$$c(m) = \liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t). \tag{4.12}$$

However, by (3.2), equation (4.12) cannot have positive solutions. This contradiction concludes the proof. □

Proof of Theorem 3.4. Let y be a positive solution of (1.1) on $[t_1, \infty)$ where $t_1 \geq t_0$ is such that $y(\tau(t)) > 0$ for $t \geq t_1$. In view of (3.3), for any $M > 0$ there is sufficiently large t such that

$$\frac{\pi(\tau(t))}{\pi(t)} \geq M^{1/\sqrt[\alpha]{\beta}}. \tag{4.13}$$

Proceeding as in the proof of Lemma 4.3, we show that $y/\pi^{\sqrt[\alpha]{\beta}}$ is decreasing eventually, say for $t \geq t_2 \geq t_1$. Using this monotonicity in (4.6), we have

$$\begin{aligned}-r(t)\left(y'(t)\right)^\alpha &= -r(t_2)\left(y'(t_2)\right)^\alpha + \int_{t_2}^t q(s) y^\alpha(\tau(s)) \mathrm{d}s \\ &\geq -r(t_2)\left(y'(t_2)\right)^\alpha + M^\alpha \beta y^\alpha(t) \int_{t_2}^t \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} \mathrm{d}s > M^\alpha \left(\frac{y(t)}{\pi(t)}\right)^\alpha,\end{aligned}$$

from which we deduce that y/π^M is decreasing. Since M is arbitrary, we get a contradiction with (ii)-part of Lemma 4.1 upon which y/π is nondecreasing. The proof is complete. □

Proof of Theorem 3.5. It can be directly verified that the canonical equation (3.4) is equivalent to a noncanonical equation (1.1) with $\alpha = 1$,

$$\begin{aligned} r(t) &= \tilde{r}(t) R^2(t) \\ q(t) &= \tilde{q}(t) R(t) R(\tau(t)) \end{aligned}$$

and

$$y(t) = \frac{x(t)}{R(t)}.$$

Here,

$$\pi(t) = \int_t^\infty \frac{\mathrm{d}s}{\tilde{r}(s) R^2(s)} = \frac{1}{R(t)}.$$

Then the conclusion immediately follows from Theorem 3.1. □

Proof of Theorem 3.6. Using the equivalent noncanonical representation of (3.4) as in the proof of Theorem 3.5, the conclusion follows from Theorem 3.4. □

Acknowledgements

The work on this research has been supported by the internal grant project no. FEI-2020-69.

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