Global bifurcation and nodal solutions for homogeneous Kirchhoff type equations

Fang Liu¹, Hua Luo² and Guowei Dai*

¹School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, PR China
²School of Economics and Finance, Shanghai International Studies University, Shanghai, 201620, PR China

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Abstract. In this paper, we shall study unilateral global bifurcation phenomenon for the following homogeneous Kirchhoff type problem

\[- \left( \int_0^1 |u'|^2 \, dx \right) u'' = \lambda u^3 + h(x, u, \lambda) \quad \text{in} \ (0, 1),
\]

\[u(0) = u(1) = 0.\]

As application of bifurcation result, we shall determine the interval of \(\lambda\) in which there exist nodal solutions for the following homogeneous Kirchhoff type problem

\[- \left( \int_0^1 |u'|^2 \, dx \right) u'' = \lambda f(x, u) \quad \text{in} \ (0, 1),
\]

\[u(0) = u(1) = 0,\]

where \(f\) is asymptotically cubic at zero and infinity. To do this, we also establish a complete characterization of the spectrum of a homogeneous nonlocal eigenvalue problem.

Keywords: bifurcation, spectrum, nonlocal problem, nodal solution, regularity results.

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1 Introduction

Consider the following problem

\[- \left( \int_0^1 |u'|^2 \, dx \right) u'' = \lambda u^3 + h(x, u, \lambda) \quad \text{in} \ (0, 1),
\]

\[u(0) = u(1) = 0,\]  

(1.1)

where \(\lambda\) is a nonnegative parameter and \(h : (0, 1) \times \mathbb{R}^2 \to \mathbb{R}\) is a continuous function satisfying

\[\lim_{s \to 0} \frac{h(x, s, \lambda)}{s^3} = 0\]  

(1.2)

*Corresponding author. Email: daiguowei@dlut.edu.cn
uniformly for all \( x \in (0, 1) \) and \( \lambda \) on bounded sets.

The problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string [16]. Some important and interesting results can be found, for example, in [1, 4, 12, 13, 15, 19, 25]. Recently, there are many mathematicians studying the problem (1.1), see [5, 6, 8, 17, 20, 21, 22, 24, 26] and the references therein. A distinguishing feature of problem (1.1) is that the first equation contains a nonlocal coefficient \( \int_0^1 |u'|^2 \, dx \), and hence the equation is no longer a pointwise identity, which raises some essential difficulties to the study of this kind of problems. In particular, the bifurcation theory of [11, 23] does not work on it.

Based on Theorem 1.1, we study the existence of nodal solutions for the following problem

\[
\begin{align*}
- \left( \int_0^1 |u'|^2 \, dx \right) u'' &= \lambda f(x, u) \quad \text{in } (0, 1), \\
\ u(0) &= u(1) = 0.
\end{align*}
\]
We assume that $f$ satisfies the following conditions

(f1) $f : (0, 1) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x, s)s > 0$ for all $x \in (0, 1)$ and any $s \neq 0$.

(f2) there exist $f_0, f_\infty \in (0, +\infty)$ such that

$$\lim_{s \to 0^+} \frac{f(x, s)}{s^3} = f_0, \quad \lim_{s \to +\infty} \frac{f(x, s)}{s^3} = f_\infty$$

uniformly with respect to all $x \in (0, 1)$.

The last main theorem of this paper is the following result.

**Theorem 1.3.** Assume that $f$ satisfies (f1)–(f2). Then the pair $(\mu_k / f_0, 0)$ is a bifurcation point of (1.5) and there are two distinct unbounded continua in $\mathbb{R} \times H^1_0(0, 1), \mathcal{C}^{-}_k$ and $\mathcal{C}^{+}_k$, emanating from $(\mu_k / f_0, 0)$, such that $\mathcal{C}^{+}_k \subseteq \{ (\mu_k / f_0, 0) \} \cup \Phi_k^{\nu}$ and links $(\mu_k / f_0, 0)$ to $(\mu_k / f_\infty, \infty)$.

The rest of this paper is arranged as follows. In Section 2, we establish the spectrum of problem (1.4). In Section 3 and 4, we give the proofs of Theorem 1.1 and 1.3, respectively.

## 2 Spectrum of (1.4)

Let $X$ be the usual Sobolev space $H^1_0(0, 1)$ with the norm $\|u\| = \left( \int_0^1 |u'|^2 \, dx \right)^{1/2}$. For any $\alpha \in (0, 1]$, we use $C^\alpha[0, 1]$ to denote all the real functions such that

$$\|u\|_\alpha := \sup_{x, y \in [0,1], x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} < +\infty.$$

Firstly, we have the following regularity result.

**Proposition 2.1.** Any weak solution $u \in X$ of problem (1.4) is also a classical solution, i.e., $u \in C^2[0, 1]$ satisfying (1.4).

**Proof.** Let $u$ be a nontrivial weak solution of problem (1.4) and

$$f(x) = \frac{\lambda |u(x)|^pu(x)}{\|u\|^p}.$$

Note that

$$H^1_0(0, 1) = \{ u \in AC[0, 1] : u' \in L^2(0, 1) \text{ and } u(0) = u(1) = 0 \}.$$

Then it is obvious that $f \in L^2(0, 1)$, in fact continuous by the compact embedding $X \hookrightarrow C^{1/2}[0, 1]$. According to the definition of weak solution, we have

$$-\left( \int_0^1 |u'|^2 \, dx \right)^{\frac{p}{2}} u'' = \lambda |u|^pu$$

in the sense of distribution. It follows that

$$u'(x) = u'(0) - \int_0^x f(t) \, dt.$$
Thus, \( u(x) = \int_0^x u'(t) \, dt. \)

So, we have that
\[
 u(x) = \int_0^x \left( u'(0) - \int_0^\tau f(\tau) \, d\tau \right) \, dt = u'(0) x - \int_0^x \int_0^\tau f(\tau) \, d\tau \, dt.
\]

Then, in view of \( f \in C[0,1] \), we get that \( u \in C^2[0,1] \) and satisfies (1.4).

**Lemma 2.2.** If \((\lambda, u)\) is a solution of (1.4) and \( u \) has a double zero, then \( u \equiv 0 \).

**Proof.** Let \( u \) be a solution of (1.4) and \( x^* \in [0,1] \) be a double zero. If \( \|u\| = 0 \), the conclusion is obvious. Next, we assume that \( \|u\| \neq 0 \). We note that
\[
 u(x) = -\frac{\lambda}{\|u\|^p} \int_{x^*}^x \int_{x^*}^s |u|^p u \, ds \, d\tau.
\]

Firstly, we consider \( x \in [0,x^*] \). Then
\[
 |u(x)| = \left| -\frac{\lambda}{\|u\|^p} \int_{x^*}^x \int_{x^*}^s |u|^p u \, ds \, d\tau \right| \leq \frac{\lambda}{\|u\|^p} \int_{x^*}^x \int_{x^*}^s |u|^p u \, ds \, d\tau
\]
\[
 = \frac{\lambda}{\|u\|^p} (x - x^*) \int_{x^*}^x |u|^p u \, d\tau
\]
\[
 \leq \frac{\lambda}{\|u\|^p} \int_{x^*}^x |u|^p+1 \, d\tau \leq \frac{\lambda \|u\|^{p+1}}{\|u\|^p} \int_{x^*}^x |u| \, d\tau \leq \lambda \int_{x^*}^x |u| \, d\tau.
\]

By the Gronwall–Bellman inequality [7, Lemma 2.2], we get \( u \equiv 0 \) on \([0,x^*]\). Similarly, we can get \( u \equiv 0 \) on \([x^*,1]\) and the proof is completed. \( \square \)

**Lemma 2.3.** Each nontrivial solution \((\lambda, u)\) of (1.4) has a finite number of zeros.

**Proof.** Suppose, on the contrary, that \( u \) has a sequence zeros \( x_n \). Since \([0,1]\) is compact, up to a subsequence, there exists \( x_0 \in [0,1] \) such that \( \lim_{n \to +\infty} x_n = x_0 \). By the continuity of \( u \), we have that \( u(x_0) = \lim_{n \to +\infty} u(x_n) = 0 \). So, we have that
\[
 u'(x_0) = \lim_{n \to +\infty} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0.
\]

Thus, \( x_0 \) is a double zero of \( u \). By Lemma 2.2, we get that \( u \equiv 0 \), which is a contradiction. \( \square \)

Let \( J \) be a strict sub-interval of \( I \). Let \( \lambda_1(J) \) denote the first eigenvalue
\[
 \left\{ \begin{array}{ll}
 - \left( \int_0^1 |u'|^2 \, dx \right)^{p/2} u'' = \lambda |u|^p u & \text{in } J, \\
 u(x) = 0 & \text{on } \partial J,
\end{array} \right.
\]

where \( p \in [0,2] \).

**Lemma 2.4.** \( \lambda_1(I) \) verifies the strict monotonicity property with respect to the domain \( I \), i.e. if \( J \) is a strict subinterval of \( I \), then \( \lambda_1(I) < \lambda_1(J) \).
Proof. Let \( \varphi_1 \) with \( \| \varphi_1 \| = 1 \) be the eigenfunction of (1.4) on \( I \) corresponding to \( \lambda_1(I) \), and denote by \( \tilde{\varphi}_1 \) the extension by zero on \( J \). Then we have that

\[
\frac{1}{\lambda_1(J)} = \int_J |\varphi_1|^{p+2} \, dx = \int_I |\tilde{\varphi}_1|^{p+2} \, dx < \sup_{u \in X, \|u\| = 1} \int_0^1 |u|^{p+2} \, dx = \frac{1}{\lambda_1(I)}.
\]

The last strict inequality holds from the fact that \( \tilde{\varphi}_1 \) vanishes in \( I \setminus J \) so cannot be an eigenfunction corresponding to the principal eigenvalue \( \lambda_1(I) \).

Proof of Theorem 1.2. Let \( \varphi_1 \) be a positive eigenfunction corresponding to \( \lambda_1(p) \). It follows from the symmetry of (1.4) and Theorem 3.1 of [9] (or Theorem 2.4 of [18]) that \( \varphi_1(x) = \varphi_1(1-x) \) for \( x \in [0,1] \), i.e. \( \varphi_1 \) is even with respect to \( 1/2 \). For any \( k \geq 2 \), set

\[
\varphi_k(x) = \begin{cases} 
\varphi_1(kx), & x \in [0, \frac{1}{k}], \\
-\varphi_1(kx-1), & x \in \left[ \frac{1}{k}, \frac{2}{k} \right], \\
& \vdots \\
(-)^k \varphi_1(kx-k+1), & x \in \left[ \frac{k-1}{k}, 1 \right].
\end{cases}
\]

Then \( \varphi_k \) is an eigenfunction of (1.4) associated with the eigenvalue \( \lambda_k(p) = k^{p+2} \lambda_1(p) \). Clearly, the continuity of \( \lambda_k(p) \) implies that \( \lambda_k(p) \) is continuous with respect to \( p \).

On the other hand, let \( u = u(x) \) be an eigenfunction of (1.4) associated with some eigenvalue \( \lambda_* > \lambda_1(p) \). According to Theorem 3.1 of [9], \( u \) changes sign in \( (0,1) \). Lemmas 2.2 and 2.3 imply that \( u \in S_k \) for some \( k \geq 2 \). Without loss of generality, we may assume that \( u''(0) > 0 \). Let

\[
0 < \tau_1 < \tau_2 < \cdots < \tau_{k-1} < 1
\]

denote the zeros of \( u \) in \( (0,1) \). Without loss of generality, we may assume that \( \tau_1 \leq 1/k \).

Applying Lemma 2.4 on \( [0,1/k] \), we have that \( \lambda_* \geq \lambda_k \). By Lemma 2 of [2], there exist integers \( p \) and \( q \), \( 1 \leq p \leq k-1 \), \( 1 \leq q \leq k-1 \), such that

\[
\tau_p \leq \frac{1}{q+1} < \frac{1}{q} \leq \tau_{p+1}.
\]

Applying Lemma 2.4 on \( [\tau_p, \tau_{p+1}] \), we have that \( \lambda_* \leq \lambda_k \). So we have that \( \lambda_* = \lambda_k \). Furthermore, if \( \tau_1 < 1/k \), we have that \( \lambda_* > \lambda_k \); if \( \tau_1 > 1/k \), we have that \( \lambda_* < \lambda_k \). Thus we have \( \tau_1 = 1/k \) and \( u = c_1 \varphi_k(x) \) for \( x \in [0,1/k] \). Similarly, we can obtain that \( \tau_i = i/k \) and \( u = c_i \varphi_k(x) \) for \( x \in [(i-1)/k, i/k] \), \( 2 \leq i \leq k-1 \). Let us normalize \( u \) as \( u''(0) = \varphi_k'(0) \). It follows that \( c_1 = 1 \). Hence \( \varphi_k'(\frac{1}{k}) = c_2 \varphi_k'(\frac{1}{k}) \). So we have \( c_2 = 1 \). Similarly, one has \( c_i = 1 \) for all \( 3 \leq i \leq k-1 \). Therefore, we have that \( u(x) = \varphi_k(x) \), \( x \in [0,1] \).

3 Global bifurcation

Consider the following auxiliary problem

\[
\begin{cases} 
-\left( \int_0^1 |u'|^2 \, dx \right)^{p/2} u'' = f(x) & \text{in } (0,1), \\
u(0) = u(1) = 0
\end{cases}
\]

(3.1)

for any \( p \in [0,2] \) and a given \( f \in X^* \). We have shown in [9] that problem (3.1) has a unique weak solution. Let us denote by \( R_p(f) \) the unique weak solution of (3.1). Then \( R_p : X^* \to X \).
is a continuous operator. Since the embedding of \( X \hookrightarrow L^\infty(0,1) \) is compact, the restriction of \( R_p \) to \( L^1(0,1) \) is a completely continuous (i.e., continuous and compact) operator. From the obvious modification of Lemma 4.2 of [9], we can get the following compactness and continuity of the operator \( R_p \) with respect to \( p \) and \( f \).

**Lemma 3.1.** The operator \( R : [0,2] \times L^1(0,1) \rightarrow L^\infty(0,1) \) defined by \( R(p,f) = R_p(f) \) is completely continuous.

Now, we consider (1.4) again. Clearly, \( u \) is a weak solution of (1.4) if and only if \( u \in X \), \( \lambda \in [0, +\infty) \) satisfy

\[
    u = R_p(\lambda|u|^pu) = \lambda^{\frac{1}{p+1}} R_p(|u|^pu) := T^\lambda_p(u).
\]

For any \( u \in X \), we define

\[
    K_p(u) = |u|^pu.
\]

Then we see that \( K_p(u) \in L^1(0,1) \). We claim that \( K_p : X \hookrightarrow L^1(0,1) \) is continuous. Assume that \( u_n \rightarrow u \) in \( X \). Since embedding \( X \hookrightarrow C[0,1] \) is compact, we have \( u_n \rightarrow u \) in \( C[0,1] \). It follows that \( u_n(x) \rightarrow u(x) \) for any \( x \in [0,1] \). So, we have that \( K_p(u_n) \rightarrow K_p(u) \) in \( L^1(0,1) \).

Since \( R_p : L^1(0,1) \rightarrow X \) is a compact, we have that \( T^\lambda_p = \lambda^{\frac{1}{p+1}} R_p \circ K_p : X \rightarrow X \) is completely continuous. Thus the Leray–Schauder degree

\[
    \deg_X \left( I - T^\lambda_p, B_r(0), 0 \right)
\]

is well-defined for arbitrary \( r \)-ball \( B_r(0) \) and \( \lambda \neq \lambda_k(p) \). It is well known that

\[
    \deg_X \left( I - T^{\lambda_0}_p, B_r(0), 0 \right) = (-1)^\beta,
\]

where \( \beta \) is the number of eigenvalues of problem (1.4) with \( p = 0 \) less than \( \lambda \). As far as the general \( p \), we can compute it through the deformation along \( p \).

**Proposition 3.2.** Let \( r > 0 \) and \( \overline{p} \in [0,2] \). Then

\[
    \deg_X \left( I - T^{\lambda}_p, B_r(0), 0 \right) = \begin{cases} 
    1, & \text{if } \lambda \in (0, \lambda_1(\overline{p})) \\
    (-1)^k, & \text{if } \lambda \in (\lambda_k(\overline{p}), \lambda_{k+1}(\overline{p})) 
    \end{cases}
\]

**Proof.** If \( \lambda \in (0, \lambda_1(\overline{p})) \), the conclusion has done in [9]. So we only need to prove the case \( \lambda \in (\lambda_k(\overline{p}), \lambda_{k+1}(\overline{p})) \). Since \( p \to \lambda_k(p) \) is continuous, we can define a continuous function \( \chi : [0,2] \rightarrow \mathbb{R} \) such that \( \lambda_k(p) < \chi(p) < \lambda_{k+1}(p) \) and \( \lambda = \chi(\overline{p}) \). Set

\[
    d(p) = \deg_X \left( I - T^{\chi(p)}_p, B_r(0), 0 \right).
\]

We shall show that \( d(p) \) is constant in \([0,2]\).

Define \( S_p : L^\infty(0,1) \rightarrow X \) by \( S_p(u) = R_p(\chi(p)|u|^pu) \). We see that \( S_p(u) = \chi^{\frac{1}{p+1}}(p) R_p \circ K_p(u) \), where \( K_p(u) = |u|^pu \). By the definition of \( K_p \), we can easily verify that \( K_p : L^\infty(0,1) \rightarrow L^1(0,1) \) is continuous. Since \( R_p : L^1(0,1) \rightarrow X \) is a compact, we get that \( S_p : L^\infty(0,1) \rightarrow X \) is completely continuous. Also we have that \( T^{\chi(p)}_p = S_p \circ i \) where \( i : X \rightarrow L^\infty(0,1) \) is the usual inclusion. From Lemma 2.4 of [14], we obtain that

\[
    d(p) = \deg_{L^\infty} \left( I - i \circ S_p, \Omega_s, 0 \right) \quad \text{for } p \in [0,2],
\]
where $\Omega_s$ is any open bounded set in $L^\infty(0,1)$ containing 0. It is not difficult to verify that the operator $\varphi : [0,2] \times L^\infty(0,1) \to L^1(0,1)$ defined by $\varphi(p,u) = |u|^p u$ is continuous. This fact, the continuity of $\chi(p)$ and Lemma 3.1 imply that $(p,u) \mapsto R_p (\chi(p)|u|^p u) = (i \circ S_p)(u) : [0,2] \times L^\infty(0,1) \to L^\infty(0,1)$ is completely continuous. Since $\lambda_k(p) < \chi(p) < \lambda_{k+1}(p)$ for any $p \in [0,2]$, we have that $u - R_p (\chi(p)|u|^p u) \neq 0$ on $\partial \Omega_s$. The invariance of the Leray–Schauder degree under a compact homotopy follows that $d(p) \equiv \text{constant}$ for $p \in [0,2]$. So, $d(\overline{p}) = d(0) = (-1)^k$, as desired. 

In particular, we have the following corollary.

**Corollary 3.3.** Let $r > 0$. Then

$$\deg_X \left( I - T_2^\lambda, B_r(0), 0 \right) = \begin{cases} 1, & \text{if } \lambda \in (0, \mu_1), \\ (-1)^k, & \text{if } \lambda \in (\mu_k, \mu_{k+1}), \end{cases}$$

where $\mu_k$ is the $k$-th eigenvalue of (1.3).

Clearly, the pair $(\lambda, u)$ is a solution of (1.1) if and only if $(\lambda, u)$ satisfies

$$u = R_2 (\lambda u^3 + h(x, u, \lambda)) : = G_\lambda(u).$$

It is easy to see that $G_\lambda : X \to X$ is completely continuous and $G_\lambda(0) = 0$, $\forall \lambda \in [0, +\infty)$. $\mu_k$ is the $\lambda_k$. Let $X_0$ be any complement to span $\{ \varphi_k \}$ in $X$.

**Theorem 3.4.** The pair $(\mu_k, 0)$ is a bifurcation point of (1.1). Moreover, there are two distinct continua in $\mathbb{R} \times X$, $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$, consisting of the bifurcation branch $\mathcal{C}_k$ emanating from $(\mu_k, 0)$, which contain $\{(\mu_k, 0)\}$ and each of them satisfies one of the following non-excluding alternatives:

1. it is unbounded in $\mathbb{R} \times X$;
2. it contains a pair $(\mu_j, 0)$ with $j \neq k$;
3. it contains a point $(\lambda, y) \in \mathbb{R} \times (X_0 \setminus \{0\})$.

**Proof.** We use the abstract bifurcation result of [10] to prove this theorem. An operator $L$ defined on $X$ is called homogeneous if $L(cu) = cL(u)$ for any $c \in \mathbb{R}$ and $u \in X$. It is not difficult to verify that $L(\lambda) := T_2^\lambda : X \to X$ is homogeneous and completely continuous. Let $\tilde{h}(x, u, \lambda) = \max_{0 \leq s \leq u} |h(x,s,\lambda)|$ for all $x \in (0,1)$ and $\lambda$ on bounded sets, then $\tilde{h}$ is nondecreasing with respect to $u$ and

$$\lim_{u \to 0^+} \frac{\tilde{h}(x,u,\lambda)}{u^3} = 0. \tag{3.2}$$

Further it follows from (3.2) that

$$\frac{h(x, u, \lambda)}{|u|^3} \leq \frac{\tilde{h}(x,|u|,\lambda)}{|u|^3} \leq \frac{\tilde{h}(x,|u|_{\infty},\lambda)}{|u|_{\infty}^3} \to 0 \quad \text{as } |u| \to 0 \tag{3.3}$$

uniformly for $x \in (0,1)$ and $\lambda$ on bounded sets. Let

$$H(\lambda, u) = G_\lambda(u) - L(\lambda)u.$$

By (3.3), we can easily verify that $H : \mathbb{R} \times X \to X$ is completely continuous with $H = o(|u|)$ near $u = 0$ uniformly on bounded $\lambda$ intervals. Noting Corollary 3.3, the desired conclusions can be obtained by applying Theorem 1 of [10].
By an argument similar to that of Proposition 2.1, we can get the following regularity result.

**Proposition 3.5.** Any weak solution \( u \in X \) of problem (1.1) is also a classical solution, i.e., \( u \in C^2(0, 1) \cap C^1,\lambda(0, 1] \) satisfying (1.1) and \( u(0) = u(1) = 0 \).

**Lemma 3.6.** If \((\lambda, u)\) is a solution of (1.1) and \( u \) has a double zero, then \( u \equiv 0 \).

**Proof.** Let \( u \) be a solution of (1.1) and \( x^* \in [0, 1] \) be a double zero. If \( \|u\| = 0 \), the conclusion is done. Next, we assume that \( \|u\| \neq 0 \). We note that

\[
R \mid u(x) \mid \leq \int_{x_0}^{x} \left( \mid \lambda u^3 + h(x, u, \lambda) \mid \right) d\tau.
\]

In view of (1.2), for any \( \varepsilon > 0 \), there exists a constant \( \delta > 0 \) such that

\[
\mid h(x, s, \lambda) \mid \leq \varepsilon |s|
\]

uniformly with respect to all \( x \in (0, 1) \) and fixed \( \lambda \) when \( |s| \in [0, \delta] \). Hence,

\[
\mid u(x) \mid \leq \int_{x_0}^{x} \left( \mid \lambda \mid + \varepsilon + \max_{s \in [\delta, \|u\|_\infty]} \mid \frac{h(\tau, s, \lambda)}{s^3} \mid \right) |u(\tau)| d\tau.
\]

By the Gronwall–Bellman inequality [7], we get \( u \equiv 0 \) on \([0, x^*] \). Similarly, we can get \( u \equiv 0 \) on \([x^*, 1] \) and the proof is complete.

**Proof of Theorem 1.1.** Lemma 3.1 of [10] implies that there exists a bounded open neighborhood \( \mathcal{O}_k \) of \((\mu_k, 0)\) such that \( \mathcal{O}_k^\circ \subseteq \mathcal{O}_k \) or \( \mathcal{O}_k^\circ \subseteq \mathcal{O}_k^\circ \). Without loss of generality, we assume that \( \mathcal{O}_k^\circ \subseteq \mathcal{O}_k \).

Next, we show that \( \mathcal{O}_k^\circ \subseteq \mathcal{O}_k \). Suppose \( \mathcal{O}_k^\circ \not\subseteq \mathcal{O}_k \). Then there exists \((\mu, u) \in \mathcal{O}_k \cap (\mathbb{R} \times \partial \mathcal{S}_k)\) such that \( (\mu, u) \neq (\mu_k, 0) \) and \( (\lambda_n, u_n) \to (\mu, u) \) with \( (\lambda_n, u_n) \in \mathcal{O}_k \subseteq \mathcal{O}_k \cap (\mathbb{R} \times \partial \mathcal{S}_k)\). Since \( u \in \partial \mathcal{S}_k \), by Lemma 3.6, \( u \equiv 0 \). Let \( v_n := u_n/\|u_n\| \), then \( v_n \) should be a solution of the following problem

\[
v = R_2 \left( \lambda_n v^3 + \frac{h(x, u_n, \lambda_n)}{\|u_n(x)\|^3} \right).
\]

By (3.3), (3.4) and the compactness of \( R_2 \) we obtain that for some convenient subsequence \( v_n \to v_0 \neq 0 \) as \( n \to +\infty \). Now \( v_0 \) verifies the equation

\[
-\int_0^1 |v'|^2 \, dxv'' = \mu v^3
\]

and \( \|v_0\| = 1 \). Hence \( \mu = \mu_j \), for some \( j \neq k \). Hence \( v_0 \in S_j \) which is an open set in \( X \), and as a consequence for some \( n \) large enough, \( u_n \in S_j \) and this is a contradiction. Thus, we have that

\[
\mathcal{O}_k^\circ \subseteq \mathcal{O}_k \subseteq (\mathcal{O}_k \cap (\mathbb{R} \times \partial \mathcal{S}_k)).
\]
Furthermore, by an argument similar to the above, we can easily show that $\mathcal{C}_k \cap (\mathbb{R} \times \{0\}) = \{(\mu_k, 0)\}$. So Theorem 1 of [10] implies that $\mathcal{C}_k$ is unbounded.

We claim that both $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are unbounded. Introduce the following auxiliary problem

$$
\begin{cases}
- \left(\int_0^1 |u'|^2 \, dx \right) u'' = \lambda u^3 + \tilde{h}(x, u, \lambda) & \text{in } (0, 1), \\
u(0) = u(1) = 0,
\end{cases}
$$

where $\tilde{h}$ is defined by

$$
\tilde{h}(x, u, \lambda) = \begin{cases}
h(x, u, \lambda), & \text{if } u'(0) > 0, \\
-h(x, -u, \lambda), & \text{if } u'(0) < 0.
\end{cases}
$$

The previous argument shows that an unbounded continuum $\tilde{\mathcal{C}}_k$ bifurcates from $(\mu_k, 0)$ and can be split into $\tilde{\mathcal{C}}_k^+$ and $\tilde{\mathcal{C}}_k^-$ with $\tilde{\mathcal{C}}_k^\nu$ connected, $\tilde{\mathcal{C}}_k^\nu \subseteq \{(\mu_k, 0)\} \cup (\mathbb{R} \times \mathcal{S}_k)$. It is easy to see that $\tilde{\mathcal{C}}_k^- = -\tilde{\mathcal{C}}_k^+$. It follows that both $\tilde{\mathcal{C}}_k^+$ and $\tilde{\mathcal{C}}_k^-$ are unbounded. It is clear that $\tilde{\mathcal{C}}_k^+ \subseteq \mathcal{C}_k^+$. Therefore $\mathcal{C}_k^+$ must be unbounded. A symmetric argument shows that $\mathcal{C}_k^-$ is also unbounded. 

\section{Nodal solutions}

In this section, we apply Theorem 1.1 to study the existence of nodal solutions for (1.5).

**Proof of Theorem 1.3.** Let $g : (0, 1) \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x, s) = f_0 s^3 + g(x, s)$$

with

$$\lim_{s \to 0} \frac{g(x, s)}{s^3} = 0 \quad \text{uniformly with respect to all } x \in (0, 1). \quad (4.1)$$

From (4.1), we can see that $\lambda g$ satisfies the assumptions of (1.2). Now, using Theorem 1.1, we have that there are two distinct unbounded continua, $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ emanating from $(\mu_k / f_0, 0)$, such that

$$\mathcal{C}_k^\nu \subseteq \{(\mu_k / f_0, 0)\} \cup \mathcal{S}_k^\nu.$$

It is sufficient to show that $\mathcal{C}_k^\nu$ joins $(\mu_k / f_0, 0)$ to $(\mu_k / f_\infty, 0)$. Let $(\zeta_n, u_n) \in \mathcal{C}_k^\nu$ where $u_n \neq 0$ satisfies $|\zeta_n| + \|u_n\| \to +\infty$. Proposition 5.1 of [8] implies that $(0, 0)$ is the only solution of (1.5) for $\lambda = 0$, we have $\mathcal{C}_k^\nu \cap (\{0\} \times X) = \emptyset$. It follows that $\zeta_n > 0$ for all $n \in \mathbb{N}$.

Next we show that $u_n$ is one-signed in some interval $(\alpha, \beta) \subseteq (0, 1)$ with $\alpha < \beta$. Let

$$0 < \tau(1, n) < \tau(2, n) < \cdots < \tau(k-1, n) < 1$$

denote the zeros of $u_n$ in $(0, 1)$. Let $\tau(0, n) = 0$ and $\tau(k, n) = 1$. Then, after taking a subsequence if necessary,

$$\lim_{n \to +\infty} \tau(l, n) = \tau(l, \infty), \quad l \in \{0, 1, \ldots, k\}.$$

We claim that there exists $l_0 \in \{0, 1, \ldots, k\}$ such that

$$\tau(l_0, \infty) < \tau(l_0 + 1, \infty).$$
Otherwise, we have that
\[ 1 = \Sigma_{i=0}^{k-1} (\tau(l+1,n) - \tau(l,n)) = \Sigma_{i=0}^{k-1} (\tau(l+1,\infty) - \tau(l,\infty)) = 0. \]
This is a contradiction. Let \((\alpha, \beta) \subset (\tau(l_0,\infty), \tau(l_0+1,\infty))\) with \(\alpha < \beta\). For all \(n\) sufficiently large, we have \((\alpha, \beta) \subset (\tau(l_0,n), \tau(l_0+1,n))\). So \(u_n\) does not change its sign in \((\alpha, \beta)\).

We claim that there exists a constant \(M\) such that \(\xi_n \in (0, M]\) for \(n \in \mathbb{N}\) large enough. On the contrary, we suppose that \(\lim_{n \to +\infty} \xi_n = +\infty\). Since \((\xi_n, u_n) \in C^\alpha_\eta\), it follows that
\[
\|u_n\| \to +\infty \quad \text{as} \quad n \to +\infty.
\]
Let \(h : (0,1) \times \mathbb{R} \to \mathbb{R}\) be a continuous function such that
\[
f(x,s) = f_\infty s^3 + h(x,s)
\]
with
\[
\lim_{|s| \to +\infty} \frac{h(x,s)}{s^3} = 0, \quad \lim_{|s| \to 0} \frac{h(x,s)}{s^3} = f_0 - f_\infty \quad \text{uniformly with respect to all} \quad x \in (0,1).
\]
Then \((\xi_n, u_n)\) satisfies
\[
u_n = R_2 \left( \xi_n f_\infty u_n^3 + h(x,u_n) \right).
\]
Dividing the above equation by \(\|u_n\|\) and letting \(\bar{u}_n = u_n / \|u_n\|\), we get that
\[
\bar{u}_n = R_2 \left( \xi_n f_\infty \bar{u}_n^3 + \frac{h(x,u_n)}{\|u_n\|^3} \right).
\]
Let
\[
\tilde{h}(x,u) = \max_{0 \leq |s| \leq u} |h(x,s)| \quad \text{for any} \quad x \in (0,1),
\]
then \(\tilde{h}\) is nondecreasing with respect to \(u\). Define
\[
\bar{h}(x,u) = \max_{u/2 \leq |s| \leq u} |h(x,s)| \quad \text{for any} \quad x \in (0,1).
\]
Then we can see that
\[
\lim_{u \to +\infty} \frac{\bar{h}(x,u)}{u^3} = 0 \quad \text{and} \quad \tilde{h}(x,u) \leq \bar{h} \left( x, \frac{u}{2} \right) + \bar{h}(x,u).
\]
It follows that
\[
\limsup_{u \to +\infty} \frac{\bar{h}(x,u)}{u^3} \leq \limsup_{u \to +\infty} \frac{\bar{h} \left( x, \frac{u}{2} \right)}{u^3} = \limsup_{u/2 \to +\infty} \bar{h} \left( x, \frac{u}{2} \right)^3.
\]
So we have
\[
\lim_{u \to +\infty} \frac{\hat{h}(x, u)}{u^3} = 0.
\]
(4.2)

Further it follows from (4.2) that
\[
\frac{h(x, u_n)}{\|u_n\|^3} \leq \frac{\hat{h}(x, |u_n|)}{\|u_n\|^3} \leq c \frac{\hat{h}(x, \|u_n\|)}{c^3 \|u_n\|^3} \to 0 \quad \text{as} \quad n \to +\infty
\]
uniformly for \( x \in (0, 1) \).

By the compactness of \( R_2 \) we obtain that
\[
-\|u\|^2 u'' = \mu f \infty u^3,
\]
where \( \overline{\mu} = \lim_{n \to +\infty} \mu_n \) and \( \overline{\mu} = \lim_{n \to +\infty} \mu_n \), again choosing a subsequence and relabeling it if necessary. It follows from \( \|\mu\| = \lim_{n \to +\infty} \|\mu_n\| \). Since \( \|\mu_n\| = 1 \), we obtain that \( \|\mu\| = 1 \). It is clear that \( \mu \in \mathcal{C}^\nu_k \). Theorem 1.2 of [3] shows that \( \mu = \mu_k / f_\infty \). Therefore, \( \mathcal{C} \) joins \((\mu_k / f_\infty, 0)\) to \((\mu_k / f_\infty, \infty)\). \( \square \)

From Theorem 1.3, we can easily get the following corollary.

**Corollary 4.1.** Assume that \( f \) satisfies (f1)–(f2). Then for
\[
\lambda \in \left( \frac{\mu_k}{f_0}, \frac{\mu_k}{f_\infty} \right) \cup \left( \frac{\mu_k}{f_\infty}, \frac{\mu_k}{f_0} \right),
\]
problem (1.5) possesses at least two solutions \( u^+_k \) and \( u^-_k \) such that \( u^+_k \) has exactly \( k - 1 \) simple zeros in \( (0, 1) \) and is positive near 0, and \( u^-_k \) has exactly \( k - 1 \) simple zeros in \( (0, 1) \) and is negative near 0.

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**References**


