Existence of solution for two classes of Schrödinger equations in $\mathbb{R}^N$ with magnetic field and zero mass

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Abstract. In this paper, we consider the existence of a nontrivial solution for the following Schrödinger equations with a magnetic potential $A$

$$-\Delta_A u = K(x)f(|u|^2)u, \quad \text{in } \mathbb{R}^N$$

where $N \geq 3$, $K$ is a nonnegative function verifying two kinds of conditions and $f$ is continuous with subcritical growth. We discuss the above equation with $K$ asymptotically periodic and $K \in L^r$.

Keywords: Schrödinger equation, magnetic field, zero mass, periodic condition, asymptotically periodic condition.

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1 Introduction

In this paper, we consider the existence of a nontrivial solution for the following equation

$$-\Delta_A u = K(x)f(|u|^2)u, \quad \text{in } \mathbb{R}^N.$$  \hfill (1.1)

where $N \geq 3$, $K : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative function and $f : \mathbb{R} \to \mathbb{R}$ is continuous with subcritical growth.

Problem (1.1) is motivated by the following nonlinear Schrödinger equation

$$\left(\frac{\hbar}{i} \nabla - A(x)\right)^2 \psi = K(x)f(|\psi|^2)\psi,$$

where $N \geq 3$, $\hbar$ is the Planck constant and $A$ is a magnetic potential of a given magnetic field $B = \text{curl} A$, and the nonlinear term $f$ is a nonlinear coupling and $K$ is nonnegative. The function $A : \mathbb{R}^N \to \mathbb{R}^N$ denotes a magnetic potential and the Schrödinger operator is defined by

$$-\Delta_A \psi = -\Delta \psi + |A|^2 \psi - 2i A \nabla \psi - i \text{div} A, \quad \text{in } \mathbb{R}^N.$$ 

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This class of problem with the nonlinearity \( f \) verifying the condition \( f'(0) = 0 \) is known as zero mass.

In recent years, much attention has been paid to the nonlinear Schrödinger equations, we may refer to \([6, 13, 23, 25–29]\). In particular, we notice that the existence of solutions for the problems with zero mass and without magnetic field, namely, \( A \equiv 0 \) and \( f'(0) = 0 \). In \([5]\), Alves and Souto investigated the following problem

\[
- \Delta u = K(x)f(u), \quad x \in \mathbb{R}^N,
\]

where \( f \) is a continuous function with quasicritical growth and \( K \) is nonnegative function. Using the variational method and some technical lemmas, the authors gave the existence of positive solution for problem (1.2).

In \([20]\), Li, Li and Shi considered a nonlinear Kirchhoff type problem

\[
- \left( a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 \right) \Delta u = K(x)f(u), \quad x \in \mathbb{R}^N,
\]

where \( N \geq 3 \), \( a \) is a positive constant, \( \lambda \geq 0 \) is a parameter and \( K \) is a potential function. The authors used an a priori estimate and a Pohozaev type identity in the case with constant coefficient nonlinearity. And in the problem with the variable-coefficient, a cut-off functional and Pohozaev type identity were used to find Palais–Smale sequences.

In \([1]\), Alves studied a quasilinear equation given by

\[
- \Delta u + V(x)u - k\Delta (u^2)u = K(x)f(u), \quad x \in \mathbb{R}^N,
\]

where \( N \geq 1 \), \( k \in \mathbb{R} \), \( V : \mathbb{R}^N \to \mathbb{R} \) is the potential, and \( f : \mathbb{R} \to \mathbb{R} \) and \( K : \mathbb{R}^N \to \mathbb{R} \) are continuous. The variational methods were used to establish a Berestycki–Lions type result. For further results about the elliptic equations with zero mass, we may refer to \([4, 7, 8, 19, 24]\).

Inspired by \([1, 5, 20]\), we would like to consider Schrödinger equations in \( \mathbb{R}^N \) with magnetic field and zero mass.

Due to the appearance of the magnetic field, the problem cannot be changed into a pure real-valued problem, hence we should deal with a complex-valued directly, which causes more new difficulties in employing the methods and some estimates. Thus there are a few results for the Schrödinger equations with magnetic field than ones for that without the magnetic field. In \([18]\), Ji and Yin showed the existence of nontrivial solutions for the following Schrödinger equation

\[
- \Delta_A u + V(x)u = f(|u|^2)u, \quad u \in \mathbb{R}^N,
\]

where \( N \geq 3 \), \( f \) has subcritical growth, and the potential \( V \) is nonnegative. The solution is obtained by the variational method combined with penalization technique of del Pino and Felmer \([17]\) and Moser iteration.

In \([15]\), Chabrowski and Szulkin discussed the semilinear Schrödinger equation

\[
- \Delta_A u + V(x)u = Q(x)|u|^{2^* - 2}u, \quad u \in H^1_{A,V,+}(\mathbb{R}^N),
\]

where \( V \) changes sign. The authors considered the problem by a min-max type argument based on a topological linking. For the more results involving the magnetic Schrödinger equations, we see \([2, 3, 9, 11, 12, 16, 25]\) and the references therein.

In this paper, we consider problem (1.1) with the different function \( K \). First of all, we assume the potential \( A \) verifying
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(A) $A \in L^2_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N)$.

In the first case, we propose the following assumptions for function $K$:

(K1) there exist $k_0 > 0$ such that

\[ K(x) \geq k_0, \quad \text{for } \forall x \in \mathbb{R}^N, \]

(K2) there exist a positive continuous periodic function $K_p : \mathbb{R}^N \to \mathbb{R}$

\[ K_p(x + y) = K_p(x), \quad \forall x \in \mathbb{R}^N \text{ and } \forall y \in \mathbb{Z}^N, \]

such that

\[ |K(x) - K_p(x)| \to 0 \quad \text{as } |x| \to +\infty. \]

(K3) $K_p$ is defined in (K2) such that

\[ K(x) \geq K_p(x), \quad \forall x \in \mathbb{R}^N. \]

In addition, we assume that function $f$ satisfies:

(f1) there holds

\[ \lim_{t \to 0^+} \frac{f(t)}{t^{\frac{2}{2^*}}} = \lim_{t \to +\infty} \frac{f(t)}{t^{\frac{2}{2^*}}} = 0, \]

where $2^* = \frac{2N}{N-2}$ and $N \geq 3$.

(f2) function $F$ is defined by $F(t) = \int_0^t f(s)ds$, and

\[ \frac{F(t)}{t} \to \infty \quad \text{as } t \to +\infty, \]

(f3) function $H(t) = tf(t) - F(t)$ is increasing in $t$ and $H(0) = 0$.

Now we are in a position to state the first result.

**Theorem 1.1.** Assume that (A), (K1)–(K3) and (f1)–(f3) hold. Then, problem (1.1) has a nontrivial solution.

In the second case, we involve that $K$ is positive almost everywhere:

(K4) the Lebesgue measure of $\{ x \in \mathbb{R}^N : K(x) \leq 0 \}$ is zero.

Then, we state the second result as follows.

**Theorem 1.2.** Assume that $K \in L^\infty(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, for some $r \geq 1$, satisfies (K4), and (A), (f1)–(f3) hold. Then, problem (1.1) has a ground state solution.

**Remark 1.3.** In fact, we consider the second case under a weaker condition than $K \in L^r(\mathbb{R}^N)$. We only require to suppose that for all $R > 0$ and any sequence of Borel sets $\{E_n\}$ of $\mathbb{R}^N$ such that $|E_n| \leq R$, for every $n$, we have

\[ \lim_{R \to +\infty} \int_{E_n \cap B^c_R(0)} K(x)dx = 0, \quad \text{uniformly in } n \in \mathbb{N}. \quad (1.3) \]

The paper is organized as follows. In the next section, we state the functional setting and give some preliminary lemmas. In Section 3, when $K$ verifies the periodic condition, we study problem (1.1) and establish the existence of a ground state solution. In Section 4, we give the existence of a nontrivial solution for asymptotically periodic problem, proving Theorem 1.1. In the last section we consider problem (1.1) with condition (K4) and we prove Theorem 1.2.
2 Preliminaries

In this section, we outline the variational framework for problem (1.1) and give some preliminary lemmas. We write

\[ \Delta_A u := (\nabla + iA)^2 u \]

and

\[ \nabla_A u := (\nabla + iA)u. \]

Let \( N \geq 3 \) and \( 2^* = 2N/(N-2) \). We denote \( D^{1,2}_A(\mathbb{R}^N) \) the Hilbert space with the scalar product

\[ \langle u, v \rangle_A = \text{Re} \int_{\mathbb{R}^N} (\nabla u + iA(x)u)(\nabla v + iA(x)v)dx, \]

and the norm induced by the product \( \langle \cdot, \cdot \rangle_A \) is

\[ \|u\|_A = \left( \int_{\mathbb{R}^N} |\nabla_A u|^2 dx \right)^{1/2}. \]

Let \( C^\infty_0(\mathbb{R}^N, \mathbb{C}) \) be dense in \( D^{1,2}_A(\mathbb{R}^N) \) with respect to the norm \( \|u\|_A \). It is easy to know that \( D^{1,2}_A(\mathbb{R}^N) \) is continuous for \( N \geq 3 \).

3 A periodic problem

In the section, we will discuss the existence of a ground state solution for the following equation

\[ \begin{cases} -\Delta_A u = K_p(x)|u|^2 u, & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}_A(\mathbb{R}^N, \mathbb{C}), \end{cases} \]

where \( K_p : \mathbb{R}^N \to \mathbb{R} \) is a continuous function verifying the following hypotheses

(K5) for all \( x \in \mathbb{R}^N \) and \( y \in \mathbb{Z}^N \),

\[ K_p(x + y) = K_p(x), \]

(K6) there is a positive constant \( k_1 \geq 0 \) such that

\[ K_p(x) \geq k_1, \quad \forall x \in \mathbb{R}^N. \]

In this section, the main result is the following.
Theorem 3.1. Assume that (A), (K5)–(K6) and (f1)–(f3) hold. Then, problem (3.1) has a nontrivial solution.

We denote by $I : D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ the energy functional for the problem (3.1), which is defined by

$$ I(u) = \frac{1}{2} \|u\|_A^2 - \frac{1}{2} \int_{\mathbb{R}^N} K_p(x) |u|^2 \, dx, $$

(3.2)

with derivative, for $\forall u, v \in D^{1,2}_A(\mathbb{R}^N, \mathbb{C})$,

$$ I'(u)v = \text{Re} \int_{\mathbb{R}^N} \nabla_A u \nabla_A \bar{v} \, dx - \text{Re} \int_{\mathbb{R}^N} K_p(x) f(|u|^2) u \bar{v} \, dx. $$

(3.3)

The weak solution for (3.1) are the critical points of $I$. Furthermore, we can use (f1)–(f3) to check that functional $I$ satisfies the geometry of the mountain pass. There is a sequence $(u_n) \subset D^{1,2}_A(\mathbb{R}^N, \mathbb{C})$ such that

$$ I(u_n) \to c $$

(3.4)

and

$$ (1 + \|u_n\|_A) \|I'(u_n)\| \to 0, $$

(3.5)

where $c$ is the mountain pass level given by

$$ c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) $$

with

$$ \Gamma = \left\{ \gamma \in C([0,1], D^{1,2}_A(\mathbb{R}^N, \mathbb{C})) : \gamma(0) = 0 \text{ and } I(\gamma(1)) \leq 0 \right\}. $$

This sequence is called as Cerami sequence for $I$ at level $c$, see [14].

Notice that from (f3) one obtains $H(s) \geq 0$ for every $s \in \mathbb{R}$. Then, we have the next estimates: by (f1), for $\forall \varepsilon > 0$, there exist a $\tau = \tau(\varepsilon)$ and $c_\varepsilon > 0$ such that

$$ |s^2 f(s^2)| \leq \varepsilon |s|^{2^*} + c_\varepsilon |s|^p \chi_{\{|s| \geq \tau\}}(s) $$

(3.6)

and, by (f3),

$$ |F(s^2)| \leq \varepsilon |s|^{2^*} + c_\varepsilon |s|^p \chi_{\{|s| \geq \tau\}}(s) $$

(3.7)

where $\chi$ is the characteristic function to the set $T = \{ t \in \mathbb{R}^N : \|t\| \geq \tau \}$.

In the proof of Theorem 3.1, we announce a lemma which resembles a classical result in [22].

Lemma 3.2. Let $(u_n)$ be a bounded sequence in $D^{1,2}_A(\mathbb{R}^N, \mathbb{C})$. Then either

(i) there are $R, \eta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that $\int_{B_R(y_n)} |u_n|^2 \geq \eta$, for all $n$,

or

(ii) $\int_{\mathbb{R}^N} |\tilde{u}_n|^q \to 0$, where $\tilde{u}_n = u_n \chi_{\{|s| \geq \tau\}}$, $\forall q \in (2, 2^*)$ and $\tau > 0$.

Proof. If (i) does not happen, going if necessary to a subsequence, we have

$$ \lim_{n \to +\infty} \sup_{y \in \mathbb{R}} \int_{B_R(y)} |u_n|^2 = 0. $$
Let \( \psi : C \to \mathbb{R} \) be a smooth function such that
\[
0 \leq \psi(s) \leq 1, \quad \psi(s) = 0 \text{ for } |s| < \frac{\tau}{2} \text{ and } \psi(s) = 1 \text{ for } |s| \geq \tau,
\]
it is easy to check that the sequence \( \tilde{u}_n = \psi(u_n)u_n \) belongs to \( D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) \) and satisfies
\[
\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_k(y)} |\tilde{u}_n|^2 = 0.
\]

Hence, by [22],
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\tilde{u}_n|^p = 0, \quad \forall q \in (2, 2^*),
\]
from where it follows that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\tilde{u}_n|^p = 0, \quad \forall q \in (2, 2^*) \text{ and } \tau > 0,
\]
finishing the proof. \( \square \)

The next lemma is used to prove that the Cerami sequence is bounded in \( D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) \).

**Lemma 3.3.** There is a positive constant \( M > 0 \) such that \( I(tu_n) \leq M \) for every \( t \in [0, 1] \) and \( n \in \mathbb{N} \).

**Proof.** Let \( t_n \in [0, 1] \) be such that \( I(t_n u_n) = \max_{t \geq 0} I(tu_n) \). If either \( t_n = 0 \) or \( t_n = 1 \), we are done. Thereby, we can assume that \( t_n \in (0, 1) \), and so \( I'(t_n u_n) t_n u_n = 0 \). From this
\[
2I(t_n u_n) = 2I(t_n u_n) - I'(t_n u_n) t_n u_n = \int_{\mathbb{R}^N} K_p(x) H(|t_n u_n|^2).
\]
Once that \( K_p \) is positive, it follows that (f3)
\[
2I(t_n u_n) \leq \int_{\mathbb{R}^N} K_p(x) H(|u_n|^2) = 2I(u_n) - I'(u_n) u_n = 2I(u_n) + o_n(1).
\]
Since \( (I(u_n)) \) converges to \( c \), so \( I(tu_n) \) is bounded. \( \square \)

**Lemma 3.4.** The sequence \( (u_n) \) is bounded in \( D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) \).

**Proof.** Suppose by contradiction that \( \|u\|_A \to \infty \) and set \( w_n = \frac{u_n}{\|u_n\|_A} \). Since \( \|w_n\|_A = 1 \), there exists \( w \in D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) \) such that \( w_n \to w \) in \( D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) \). Next, we will show that \( w = 0 \). First of all, notice that
\[
o_n(1) + 1 = \int_{\mathbb{R}^N} \frac{K_p(x) F(|u_n|^2)}{\|u_n\|_A^2} = \int_{\mathbb{R}^N} \frac{K_p(x) F(|u_n|^2)}{|u_n|^2} |w_n|^2.
\]
By (f2), for each \( M > 0 \), there is \( \xi > 0 \) such that
\[
\frac{F(s^2)}{s^2} \geq M, \quad \text{for } |s| \geq \xi,
\]
hence
\[
o_n(1) + 1 \geq \int_{\Omega \cap \{|u_n| \geq \xi\}} \frac{K_p(x) F(|u_n|^2)}{|u_n|^2} |w_n|^2 \geq M_k \int_{\Omega \cap \{|u_n| \geq \xi\}} |w_n|^2,
\]
where \( \Omega = \{ x \in \mathbb{R}^N : w(x) \neq 0 \} \). By Fatou’s Lemma
\[
1 \geq M_k \int_{\Omega} |w|^2 dx.
\]
Therefore $|\Omega| = 0$, showing that $w = 0$.

Notice that for each $C > 0$, one has $\frac{C}{\|u_n\|_A} \in [0, 1]$ for $n$ sufficiently large. Thus

$$I(t_n u_n) \geq I\left(\frac{C}{\|u\|_A} u_n\right) = I(Cw_n) = \frac{C^2}{2} - \frac{1}{2} \int_{\mathbb{R}^N} K_p(x) F(C^2|w_n|^2).$$

We claim that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} K_p(x) F(C^2|w_n|^2) = 0. \quad (3.8)$$

We postpone for minutes the proof of (3.8). But if it were true, we would get

$$\lim_{n \to +\infty} I(t_n u_n) \geq \frac{C^2}{2}, \quad \text{for every } C > 0,$$

which is a contradiction with Lemma 3.3, since $(I(t_n u_n)) \leq M$.

We prove (3.8) by using Lemma 3.2, which gives two alternatives: either

$$\int_{B_N(y_n)} |w_n|^2 \geq \eta \quad \text{for some } \eta > 0 \text{ and } (y_n) \in \mathbb{Z}^N,$$

or

$$\int_{\mathbb{R}^N} |\tilde{w}_n|^p dx \to 0, \quad \text{where } \tilde{w}_n = w_n \chi_{\{|u_n| \geq \tau\}}, \quad p \in (2, 2^*) \text{ and } \tau > 0.$$

By showing the boundedness of $(u_n)$, we will prove that the first alternative does not hold. If the first alternative occurs, we define $\tilde{u}_n = u_n(x + y_n)$ and $\tilde{w}_n = \frac{\tilde{u}_n}{\|\tilde{u}_n\|_A}$. These two sequences satisfy

$$I(\tilde{u}_n) \to c, \quad \left(1 + \|\tilde{u}_n\|_A\right) \|I'(\tilde{u}_n)\| \to 0 \quad \text{and} \quad \tilde{w}_n \to \tilde{w} \neq 0,$$

which is a contraction compared to what we have written in the beginning of this proof. Hence, the second alternative holds and

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\tilde{w}_n|^p dx = 0.$$

Then

$$|K_p(x) F(C^2|w_n|^2)| \leq \|K_p\|_\infty |F(C^2|w_n|^2)| \leq \|K_p\|_\infty \left[\varepsilon C^2 \|w_n\|^2 + c \varepsilon C^p |w_n|^p \chi_{\{|w_n| \geq \delta\}}\right],$$

from where it follows

$$|K_p(x) F(C^2|w_n|^2)| \leq \|K_p\|_\infty \left[\varepsilon C^2 \|w_n\|^2 + c \varepsilon C^p |w_n|^p\right].$$

Consequently

$$\int_{\mathbb{R}^N} |K_p(x) F(C^2|w_n|^2)| dx \leq \|K_p\|_\infty \left[\varepsilon C^2 \int_{\mathbb{R}^N} |w_n|^2 dx + c \varepsilon C^p \int_{\mathbb{R}^N} |w_n|^p dx\right],$$

showing that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} |K_p(x) F(C^2|w_n|^2)| dx = 0,$$

and the proof is finished. \qed
Proof of Theorem 3.1. Since \((u_n)\) is bounded, by applying Lemma 3.2, we have two alternatives, either

(i) there are \(R, \eta > 0\) and \((y_n) \subset \mathbb{R}^N\) such that \(\int_{B_R(y_n)} |u_n|^2 \geq \eta\) for all \(n\), or

\[
\int_{\mathbb{R}^N} |\tilde{u}_n|^q \to 0, \quad \text{where} \quad \tilde{u}_n = u_n \chi_{\{|s| \geq \tau\}}, \quad q \in (2, 2^*) \quad \text{and} \quad \tau > 0.
\]

Notice that (ii) does not occur. Otherwise, the inequality

\[
\int_{\mathbb{R}^N} |K_p(x)f(|u_n|^2)|u_n|^2| \leq \|K_p\|_{\infty} \left[ \varepsilon \int_{\mathbb{R}^N} |u_n|^2 + c_\varepsilon \int_{\mathbb{R}^N} |u_n|^p \right]
\]

leads to

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} |K_p(x)f(|u_n|^2)|u_n|^2| = 0,
\]

and so

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} K_p(x)f(|u_n|^2)|u_n|^2| = 0.
\]

The fact that \(I'(u_n)u_n = o_n(1)\) imply that \(\|u_n\|_A \to 0\), constituting a contradiction. Since alternative (i) is true and \(K_p\) is periodic, the sequence \(\tilde{u}_n(x) = u_n(x + y_n)\) is a Cerami sequence for \(I\) at level \(c\), namely,

\[
I(\tilde{u}_n) \to c, \quad \left(1 + \|\tilde{u}_n\|_A\right)\|I'(u_n)\| \to 0 \quad \text{and} \quad \tilde{u}_n \rightharpoonup \tilde{u} \text{ in } D^{1,2}_A(\mathbb{R}^N, \mathbb{C}).
\]

A direct computation indicates that \(I'(\tilde{u}) = 0\), and \(\tilde{u}\) is a nontrivial weak solution for problem (3.1). Then, we will prove that \(\tilde{u}\) is a ground state solution for (3.1). We will check that \(I(\tilde{u})\) accords with the mountain pass level. By Fatou’s Lemma,

\[
2c = \liminf_{n \to +\infty} I(\tilde{u}_n) = \liminf_{n \to +\infty} \left(2I(\tilde{u}_n) - I'(\tilde{u}_n)\tilde{u}_n\right) = \liminf_{n \to +\infty} \int_{\mathbb{R}^N} K_p(x)H(|\tilde{u}_n|^2) \geq \int_{\mathbb{R}^N} K_p(x)H(|\tilde{u}|^2).
\]

Since

\[
2I(\tilde{u}) = 2I(\tilde{u}) - I'(\tilde{u})\tilde{u} = \int_{\mathbb{R}^N} K_p(x)H(|\tilde{u}|^2)dx,
\]

we can conclude that \(I(\tilde{u}) \leq c\). But then, the condition (f3) leads to

\[
c = \inf \left\{ I(u) : u \in D^{1,2}_A(\mathbb{R}^N) \setminus \{0\} \text{ and } I'(u)u = 0 \right\}.
\]

It follows that \(I'(\tilde{u}) \geq c\), and so \(I'(\tilde{u}) = c\). \(\square\)

4 The proof of Theorem 1.1

In the section, we will discuss the existence of a nontrivial solution for problem (1.1), thus showing Theorem 1.1. Therefore, we need to prove Lemmas 4.1 and 4.2 below. Hence, we will presume that the condition (A), (K1)–(K3) and (f1)–(f3) hold.

We recall that \(u \in D^{1,2}_A(\mathbb{R}^N, \mathbb{C})\) is a weak solution of problem (1.1), if

\[
\text{Re} \int_{\mathbb{R}^N} \nabla_A u \nabla_A \overline{u} dx = \text{Re} \int_{\mathbb{R}^N} K(x)F(|u|^2)u \overline{u} dx,
\]

where \(K(x)F(|u|^2)u \overline{u} dx\).
for all \( v \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C}) \).

The Energy functional associated to (1.1) is

\[
J(u) = \frac{1}{2} \|u\|_A^2 - \frac{1}{2} \int_{\mathbb{R}^N} K(x) F(|u|^2) dx, \quad \forall u \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})
\]

with derivative

\[
J'(u)v = \text{Re} \int_{\mathbb{R}^N} \nabla_A u \nabla_A \bar{v} dx - \text{Re} \int_{\mathbb{R}^N} K(x) f(|u|^2) u \bar{v} dx, \quad \forall u, v \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C}).
\]

As in the proof of the periodic case, one observes that \( J \) satisfying the geometry of the mountain pass. Therefore, there is a sequence \((v_n) \subset D_A^{1,2}(\mathbb{R}^N, \mathbb{C})\) verifying

\[
J(v_n) \to d \quad \text{and} \quad \left( 1 + \|v_n\|_A \right) \|J'(v_n)\| \to 0,
\]

where \( d \) denotes the mountain pass level correlative of \( J \).

Since \( I(u) = c \), by property (K3), one obtains \( d \leq c \). With loss of generality, we can assume that \( K \neq K_p \), consequently

\[
d \leq \max_{t \geq 0} J(tu) = J(t_0 u) < I(t_0 u) \leq I(u) = c.
\]

**Lemma 4.1.** The sequence \((u_n)\) is bounded in \( D_A^{1,2}(\mathbb{R}^N, \mathbb{C}) \).

**Proof.** Let \( t_n \in [0,1] \) be such that \( J(t_n v_n) = \max_{t \geq 0} J(tv_n) \). If either \( t_n = 0 \) or \( t_n = 1 \), we are done. Thereby, we can assume \( t_n \in (0,1) \), and so \( J'(t_n v_n) t_n v_n = 0 \). From this

\[
2J(t_n v_n) = 2J(t_n v_n) - J'(t_n v_n) t_n v_n = \int_{\mathbb{R}^N} K(x) H(t_n^2 |v_n|^2).
\]

Since \( K \) is a nonnegative function, from (f3),

\[
2J(t_n v_n) \leq \int_{\mathbb{R}^N} K(x) H(|v_n|^2) = 2J(v_n) - J'(v_n) v_n = 2J(v_n) + o_n(1).
\]

Since \((J(v_n))\) is convergent, so it is bounded.

Suppose by contradiction that \( \|v_n\|_A \to \infty \). Proving as in Lemma 3.4, the sequence \( w_n = \frac{v_n}{\|v_n\|_A} \) weakly converges to 0 in \( D_A^{1,2}(\mathbb{R}^N, \mathbb{C}) \). Since \( \|w_n\|_A = 1 \), by applying Lemma 3.2, we have two alternatives, either

1. \((i)\) there are \( R, \eta > 0 \) and \( (y_n) \subset \mathbb{R}^N \) such that \( \int_{B_R(y_n)} |w_n|^2 \geq \eta \), for all \( n \),

or

2. \((ii)\)

\[
\int_{\mathbb{R}^N} |\hat{w}_n|^q \to 0, \quad \text{where} \quad \hat{w}_n = w_n \chi_{\{|s| \geq \tau\}}, \quad \forall q \in (2, 2^*) \text{ and } \tau > 0.
\]

If that (i) occurred, we could define the functions \( \hat{v}_n(x) = v_n(x + y_n) \) and \( \hat{w}_n(x) = \frac{s_n(x)}{\|s_n\|_A} \).

These two sequences satisfy

\[
J(\hat{v}_n) \to d, \quad \left( 1 + \|\hat{v}_n\|_A \right) \|J'(\hat{v}_n)\| \to 0 \quad \text{and} \quad \hat{w}_n \to \hat{w} \neq 0,
\]

which contradicts \( w_n \to 0 \).
Suppose that (ii) is true. As in the proof of Lemma 3.4

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x) F(C^2|w_n|^2) = 0 \quad (4.5)
\]

for each \( C > 0 \), and one has \( \frac{C}{\|v_n\|_A} \in [0, 1] \) for \( n \) sufficiently large. There is a constant \( M > 0 \) such that \( J(tv_n) \leq M \) for every \( t \in [0, 1] \) and \( n \in \mathbb{N} \). Thus

\[
J(t_n v_n) \geq J\left( \frac{C}{\|v_n\|_A}v_n \right) = J(Cw_n) = \frac{C^2}{2} - \frac{1}{2} \int_{\mathbb{R}^N} K(x) F(C^2|w_n|^2).
\]

By (4.5), one would get

\[
\lim_{n \to +\infty} J(t_n v_n) \geq \frac{C^2}{2}, \quad \text{for every } C > 0,
\]

which constitutes a contradiction, since \( (J(t_n v_n)) \) is bounded. Consequently, the sequence \( (v_n) \) is bounded.

From the preceding lemma, since the Hilbert space \( D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) \) is reflexive, there exists \( \bar{v} \in D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) \) and a subsequence of \( (v_n) \), still denoted by \( (v_n) \), such that \( v_n \rightharpoonup \bar{v} \) in \( D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) \).

**Lemma 4.2.** The weak limit \( \bar{v} \) of \( (v_n) \) is nontrivial.

**Proof.** Suppose by contradiction that \( \bar{v} \equiv 0 \). Since

\[
\int_{B_R} |K(x) - K_p(x)||F(|v_n|^2)|dx \leq \epsilon \int_{B_R} |K(x) - K_p(x)||v_n|^2 dx + \int_{B_R} |K(x) - K_p(x)||v_n|^p dx,
\]

as consequence of \( \bar{v} \equiv 0 \), it follows that

\[
\int_{B_R} |K(x) - K_p(x)||F(|v_n|^2)|dx \to 0 \quad \text{as } n \to +\infty. \quad (4.6)
\]

On the other hand, from (K2), given \( \epsilon > 0 \) there exists \( R = R(\epsilon) \) such that

\[
|K(x) - K_p(x)| < \epsilon, \quad \text{for all } |x| > R.
\]

Thus

\[
\int_{B_R^c} |K(x) - K_p(x)||F(|v_n|^2)|dx \leq \epsilon M \quad (4.7)
\]

where

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} |F(|v_n|^2)|dx = M.
\]

From (4.6) and (4.7)

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |K(x) - K_p(x)||F(|v_n|^2)|dx = 0, \quad (4.8)
\]

and

\[
|J(v_n) - I(v_n)| \to 0 \quad \text{as } n \to +\infty.
\]

A similar argument shows that

\[
|J'(v_n)v_n - I'(v_n)v_n| \to 0 \quad \text{as } n \to +\infty.
\]
Consequently, 
\[ I(v_n) = d + o_n(1) \quad \text{and} \quad I'(v_n)v_n = o_n(1). \] (4.9)

Let \( s_n \) be positive number verifying
\[ I'(s_nv_n)v_n = 0. \] (4.10)

We claim that \( (s_n) \) converges to 1 as \( n \to +\infty \). We begin proving that
\[ \limsup_{n \to +\infty} s_n \leq 1. \] (4.11)

Suppose by contradiction that, going if necessary to a subsequence, \( s_n \geq 1 + \delta \) for all \( n \in \mathbb{N} \), for some \( \delta > 0 \). From (4.9),
\[ \|v_n\|_A^2 = \int_{\mathbb{R}^N} K_p(x)f(|v_n|^2)|v_n|^2dx + o_n(1). \]

On the other hand, from (4.10),
\[ s_n\|v_n\|_A^2 = \int_{\mathbb{R}^N} K_p(x)f(s_n^2|v_n|^2)s_n|v_n|^2dx. \]

Consequently
\[ \int_{\mathbb{R}^N} K_p(x)\left[f(s_n^2|v_n|^2) - f(|v_n|^2)\right]|v_n|^2dx = o_n(1), \]
and from (f3) combined with (K1)–(K3),
\[ \int_{\mathbb{R}^N} \left[f(s_n^2|v_n|^2) - f(|v_n|^2)\right]|v_n|^2dx = o_n(1). \] (4.12)

Since \( (v_n) \) is bounded, by Lemma 3.2 again, we have two alternatives, either

(i) there are \( R, \eta > 0 \) and \( (y_n) \subset \mathbb{R}^N \) such that \( \int_{B_R(y_n)} |v_n|^2 \geq \eta \), for all \( n \),

or

(ii) \( \int_{\mathbb{R}^N} |\delta_n|^q \to 0 \), where \( \delta_n = v_n\chi_{\{|s| \geq \tau\}} \), \( \forall q \in (2, 2^*) \) and \( \tau > 0 \).

In case (ii), we derive
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^N} f(|v_n|^2)|v_n|^2dx = 0, \]
which implies \( v_n \to 0 \) in \( D^1_A(\mathbb{R}^N, \mathbb{C}) \) that is impossible.

Let \( (y_n) \) be given by (i), and define \( \bar{\delta}_n(x) = v_n(x + y_n) \). Since
\[ \int_{B_R(0)} |\bar{\delta}_n|^2dx \geq \eta > 0, \]
there exists \( \bar{\delta} \neq 0 \) in \( n D^1_A(\mathbb{R}^N, \mathbb{C}) \) such that \( (v_n) \) is weakly convergent to \( \bar{\delta} \) in \( D^1_A(\mathbb{R}^N, \mathbb{C}) \). From (4.12) and (f3), Fatou’s Lemma yields,
\[ 0 < \int_{\mathbb{R}^N} \left[f((1 + \delta)^2|\bar{\delta}_n|^2) - f(|\bar{\delta}_n|^2)\right]|\bar{\delta}_n|^2dx = 0, \]
which is impossible. Hence

\[ \limsup_{n \to +\infty} s_n \leq 1. \]

From this, \((s_n)\) is bounded. Without loss of generality, we can assume that

\[ \lim_{n \to +\infty} s_n = s_0 \leq 1. \]

If \(s_0 < 1\), we have that \(s_n < 1\) for \(n\) large enough. Hence, by Fatou’s Lemma

\[ 0 < \int_{\mathbb{R}^N} f(|\tilde{u}_n|^2) - f(s_0^2|\tilde{u}_n|^2) |\tilde{u}_n|^2 dx = 0, \quad \text{when } s_0 > 0, \]

and

\[ 0 < \int_{\mathbb{R}^N} f(|\tilde{u}_n|^2) |\tilde{u}_n|^2 dx = 0, \quad \text{when } s_0 = 0, \]

which are impossible. Therefore,

\[ \lim_{n \to +\infty} s_n = 1. \quad (4.13) \]

As a consequence of (4.13),

\[ \int_{\mathbb{R}^N} K_p(x) F(s_n^2|v_n|^2) dx - \int_{\mathbb{R}^N} K_p(x) F(|v_n|^2) dx = o_n(1) \]

and

\[ (s_n^2 - 1) \|v_n\|_A^2 = o_n(1), \]

leading to

\[ I(s_n v_n) - I(v_n) = o_n(1). \]

Then, by (4.9)

\[ c \leq I(s_n v_n) = I(v_n) + o_n(1) = d + o_n(1). \]

Taking \(n \to +\infty\), we find \(c \leq d\), which obtain a contradiction, because, by (4.4), \(d < c\). This contradiction comes from the assumption that \(v \equiv 0\). \(\square\)

5 The proof of Theorem 1.2

In this section, we mean to prove Theorem 1.2. As the proof in the preceding section, we can prove that the functional \(I\) satisfies the geometry of the mountain pass and there is a Cerami sequence \((u_n) \in D^{1,2}_A(\mathbb{R}^N, \mathbb{C})\) satisfying (3.4) and (3.5). Finally, we have proved Lemma 3.3. In order to check that \((u_n)\) is bounded in \(D^{1,2}_A(\mathbb{R}^N, \mathbb{C})\), we should show that the (3.8) holds and proceed as in the proof of Lemma 3.4.

Let \(\Omega, \xi, w, M\) be defined as in the proof of Lemma 3.4. Notice that \(|\Omega| = 0\), since

\[ o_n(1) + 1 \geq \int_{\mathbb{R}^N} \frac{K(x) F(|u_n|^2)}{|u_n|^2} |w_n|^2 \]

implies that

\[ 1 \geq M \int_{\Omega} K(x) |w|^2, \]

and from (K4), we have \(w = 0\).
Let us prove the limit (3.8). From (f1), for each \( \varepsilon > 0 \), we have \( \delta > 0 \) and \( C_\varepsilon > 0 \) such that
\[
|s^2 f(s^2)| < \varepsilon |s|^2 + C_\varepsilon \chi_{\{|s| \geq \delta\}}, \quad \text{for all } s \in \mathbb{R}^N,
\] (5.1)
and
\[
|F(s^2)| < \varepsilon |s|^2 + C_\varepsilon \chi_{\{|s| \geq \delta\}}, \quad \text{for all } s \in \mathbb{R}^N.
\] (5.2)

By Sobolev embedding and (2.1), there exists \( \hat{S} > 0 \) such that
\[
\int_{\mathbb{R}^N} |v|^2 \, dx \leq \hat{S} \left( \int_{\mathbb{R}^N} |\nabla_A v|^2 \, dx \right)^{\frac{2}{2}},
\]
for all \( v \in D_{A}^{1,2}(\mathbb{R}^N, \mathbb{C}) \). Observe that \( \Delta_n = \{ x \in \mathbb{R}^N : |Cw_n(x)| \geq \delta \} \) is such that
\[
\int_{\Delta_n} |w_n|^2 \, dx \leq \hat{S}.
\]

This implies, besides (5.2), that
\[
\int_{|x| \geq R} K(x)F(|Cw_n|^2) \, dx \leq \varepsilon C^2 \|K\|_\infty \int_{B_R(0)} |w_n|^2 \, dx + C \int_{B_R(0) \cap \Delta_n} K(x) \, dx,
\]
and from (1.3)
\[
\lim_{R \to +\infty} \int_{|x| \geq R} K(x)F(|Cw_n|^2) \, dx \leq \varepsilon \hat{S} C^2 \|K\|_\infty, \quad \text{uniformly in } n.
\]

On the other hand, for any \( R > 0 \), from (f1) and Strauss’ compactness lemma (see [10])
\[
\lim_{n \to +\infty} \int_{|x| \leq R} K(x)F(|Cw_n|^2) \, dx = 0,
\]
which shows that (3.8) holds and \( (u_n) \) is bounded in \( D_{A}^{1,2}(\mathbb{R}^N, \mathbb{C}) \).

To prove Theorem 1.2, it is important to show that \( (u_n) \) converges in \( D_{A}^{1,2}(\mathbb{R}^N, \mathbb{C}) \). In this way we can see that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x)F(|u_n|^2) |u_n|^2 \, dx = \int_{\mathbb{R}^N} K(x)F(|u|^2) |u|^2 \, dx.
\] (5.3)

To verify (5.3), consider \( E_n = \{ x \in \mathbb{R}^N : |u_n(x)| \geq \delta \} \) which satisfies \( \sup_{n \in \mathbb{N}} |E_n| < \infty \). From (5.1)
\[
\int_{|x| \geq R} K(x)F(|u_n|^2) |u_n|^2 \, dx \leq \varepsilon \|K\|_\infty \int_{B_R(0)} |u_n|^2 \, dx + C \int_{B_R(0) \cap E_n} K(x) \, dx
\]
and from (1.3)
\[
\limsup_{R \to +\infty} \int_{|x| \geq R} K(x)F(|u_n|^2) |u_n|^2 \, dx \leq \varepsilon \hat{S} \|K\|_\infty, \quad \text{uniformly in } n.
\]

Again, from (f1) and Strauss’ compactness lemma
\[
\lim_{n \to +\infty} \int_{|x| \leq R} K(x)F(|u_n|^2) |u_n|^2 \, dx = \int_{|x| \leq R} K(x)F(|u|^2) |u|^2 \, dx,
\]
for all \( r > 0 \) fixed, and it shows that (5.3) holds. Since \( l'(u_n)u_n \to 0 \), (5.3) implies that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla_A u_n|^2 \, dx = \int_{\mathbb{R}^N} K(x)F(|u|^2) |u|^2 \, dx = \int_{\mathbb{R}^N} |\nabla_A u|^2 \, dx
\]
finishing the proof of Theorem 1.2.
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References


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