Electronic Journal of Qualitative Theory of Differential Equations
2020, No. 5, 1–16; https://doi.org/10.14232/ejqtde.2020.1.5 www.math.u-szeged.hu/ejqtde/

Fractional integral inequalities and global solutions of fractional differential equations

Tao Zhu ✉

Department of Mathematics and Physics, Nanjing Institute of Technology, Nanjing, 211100, P. R. China

Received 1 May 2019, appeared 20 January 2020
Communicated Jeff R. L. Webb

Abstract. New fractional integral inequalities are established, which generalize some famous inequalities. Then we apply these new fractional integral inequalities to study global existence results for fractional differential equations.

Keywords: fractional integral inequalities, fixed point theorem, fractional differential equations, global solutions.

2010 Mathematics Subject Classification: 34A12, 39B62, 34A08.

1 Introduction

In [9, p. 190], Henry obtained the following result about weakly singular Gronwall type inequality.

Theorem 1.1. *Let a, b, α, β be nonnegative constants with $\alpha < 1$, $\beta < 1$. Suppose that $u \in L^1[0, T]$ satisfies*

$$u(t) \leq at^{-\alpha} + b\int_0^t (t-s)^{-\beta} u(s)ds, \quad a.e.\ t \in (0, T]. \tag{1.1}$$

Then there is a constant $C(b, \beta, T)$ such that

$$u(t) \leq \frac{at^{-\alpha}}{1-\alpha} C(b, \beta, T), \quad a.e.\ t \in (0, T]. \tag{1.2}$$

One version of a doubly singular case of Henry is the following, cf. [9, p. 189].

Theorem 1.2. *Suppose $\beta > 0$, $\gamma > 0$, $\beta + \gamma > 1$ and $a \geq 0$, $b \geq 0$, u is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \leq t < T$, and u satisfies*

$$u(t) \leq a + b\int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s)ds, \quad a.e.\ t \in [0, T). \tag{1.3}$$

Then

$$u(t) \leq aE_{\beta,\gamma}\left(b\Gamma(\beta)^{\frac{1}{\beta+\gamma-1}} t\right), \tag{1.4}$$

where $E_{\beta,\gamma}(z)$ is given by an infinite series related to the two-parameter Mittag-Leffler function.

✉ Email: zhutaoyzu@sina.cn

Since fractional integral inequality is a well-known tool in the study of fractional differential equations and evolution equations, Henry's work was followed by many scholars (for example, see [6,12–14,19,21–23]). Recently, by the Hölder inequality and a method introduced by Medved' [13,14], Zhu [22] considered the following inequality

Theorem 1.3. *Let $0 < T \leq \infty$, $\beta > 0$, $a(t)$, $b(t)$ and $l(t)$ be continuous, nonnegative functions on $[0, T)$, and $u(t)$ be a continuous, nonnegative function on $[0, T)$ with*

$$u(t) \leq a(t) + \frac{b(t)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)u(s)ds, \tag{1.5}$$

then

$$u(t) \leq \left(A(t) + B(t) \int_0^t L(s)A(s) \exp\left(\int_s^t L(\tau)B(\tau)d\tau \right) ds \right)^{\frac{1}{p}}, \tag{1.6}$$

where

$$A(t) = 2^{p-1}a^p(t), \qquad B(t) = 2^{p-1} \left(\frac{b(t)}{\Gamma(\beta)(q(\beta-1)+1)^{\frac{1}{q}}} t^{\beta-1+\frac{1}{q}} \right)^p, \qquad L(t) = l^p(t),$$

and $p, q \in (0, \infty)$ such that $\frac{1}{q} + \beta > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$.

By a reduction to the classical Gronwall inequality, Webb [19] studied the following Gronwall type inequality with a double singularity.

Theorem 1.4. *Let $a, b \geq 0$ and $c > 0$ be constants. Let $0 < \alpha, \beta, \gamma < 1$ with $\alpha + \gamma < 1$ and $\beta + \gamma < 1$. Suppose that $u(t)t^\alpha \in L^\infty_+[0, T]$ and u satisfies*

$$u(t) \leq at^{-\alpha} + b + c\int_0^t (t-s)^{-\beta} s^{-\gamma} u(s)ds, \quad a.e.\ t \in (0, T]. \tag{1.7}$$

Then we have, for a.e. $t \in (0, T]$,

$$\begin{aligned} u(t) &\leq at^{-\alpha} + acB_1 t^{-\alpha+1-\beta-\gamma} + ac^2 B_1 B_2 t^{-\alpha+2(1-\beta-\gamma)} + \dots \\ &\quad + (b + ac^m B_1 B_2 \dots B_m t^{-\alpha+m(1-\beta-\gamma)}) \exp\left(\frac{ct_{r_1}^{-\beta}}{1-\beta-\gamma} t^{1-\gamma} \right), \end{aligned} \tag{1.8}$$

where m is the smallest positive integer such that $m(1-\beta-\gamma) - \alpha \geq 0$, $r_1 = \frac{\beta}{1-\gamma}$, and for $n \in \mathbb{N}$, $B_n = B(1-\beta, 1-\alpha-\gamma+(n-1)(1-\beta-\gamma))$. In particular, there is an explicit constant $C(b, c, \beta, \gamma, T)$ such that $u(t) \leq at^{-\alpha}C$ for a.e. $t \in (0, T]$.

In this paper, we study the following fractional integral inequalities

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s)u(s)ds, \qquad t \in [0, +\infty), \tag{1.9}$$

where $\gamma \geq 0$ and $\beta \in (0, 1)$, and

$$u(t) \leq at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s)u(s)ds, \qquad t \in (0, +\infty), \tag{1.10}$$

where $a, b \geq 0$, $\alpha > \delta \geq 0$ and $\beta \in (0, 1)$. The special cases $b(t) \equiv C$ or $\gamma = 0$ of the inequality (1.9) are proved in Medved' [13, Theorem 2 and Theorem 3] and Zhu [22, Theorem 2.4 and

Theorem 2.6]. Medved' also studied the inequality (1.9) in [13, Theorem 4] and obtained two different results with exponential functions for different β and γ. The conclusion of Theorem 4 in [13] has a more complicated appearance. Webb [19] obtained several results of inequality (1.10) for the special case $l(t) = t^{-\gamma}$ by reducing the inequality (1.10) to the classical Gronwall inequality. In this paper, we study the inequality (1.9) under the hypothesis $\beta \in (0,1)$ and $\gamma \geq 0$. The proof is more simple than Theorem 4 in [13]. We present a new method to study a integral inequality which was first studied by Willett [20]. By this integral inequality, we study the inequality (1.9) for the special cases $b(t) = t^{1-\beta}$ and $\gamma = 1-\beta$. The conclusion and the method of proof seem to be new in this case. We also obtain some results of the inequality (1.10) and examples show our results are improvements on [19].

Fractional differential equations (FDEs) have been of great interest in the past three decades. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications in various sciences. Recently, many researchers began to investigate the existence of solutions of nonlinear fractional differential equations (for example, see [4–6,8,11, 12,18,19,21–24] and references therein). In this paper, we continue to investigate the existence and uniqueness of global solutions of the following initial value problem

$$\begin{cases} D_r^{\beta} x(t) = f(t, x(t)) \qquad t \in (0, +\infty), \qquad \beta \in (0,1), \\ \lim_{t\to 0^+} t^{1-\beta} x(t) = x_0, \end{cases} \tag{1.11}$$

where D_r^{β} is the Riemann–Liouville fractional derivative. It should be pointed out that such global existence results are fundamental in the theory of fractional differential equations and crucial in stability analysis of fractional differential equations.

The existence and uniqueness of global solutions of the fractional differential equation (1.11) have been studied by many scholars. For example, under the assumption that f satisfies an inequality of the form

$$|f(t,x)| \leq p(t)\omega\left(\frac{|x|}{1+t^2}\right) + q(t),$$

Kou et al. [11] proved the global existence of solutions of fractional differential equation (1.11) in a special Banach space

$$E = \left\{ x(t) | x(t) \in C_{1-\beta}(0, +\infty), \lim_{t\to+\infty} \frac{t^{1-\beta}x(t)}{1+t^2} = 0 \right\}.$$

Trif [18] investigated the global existence of solutions to initial value problems for nonlinear fractional differential equation (1.11) by constructing a special locally convex space which is metrizable and complete. Webb [19] proved the existence results of equation (1.11) under the assumption that nonnegative function f satisfies $f(t,x) = t^{-\gamma} g(t,x)$, where $g(t,x) \leq M(1+x)$, $M > 0$ and $0 \leq \gamma < \beta$. Unlike all the previous papers, by new fractional inequality (1.9) and fixed point theorem, we present the existence and uniqueness results of the fractional differential equation (1.11). Our result includes the main result of [18, Theorem 4.2]. Finally, examples are given to illustrate the applicability of our results and can not be solved by Theorem 4.2 in [18].

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

Let $\beta \in (0,1)$, denote $C_\beta(0,T] = \{x : (0,T] \to \mathbb{R}$ and $x(t) = t^{-\beta}y(t)$ for some $y \in C[0,T]\}$. Let $\|x\|_\beta = \sup_{0<t\leq T} t^\beta |x(t)|$, then $C_\beta(0,T]$ endowed with the norm $\|\cdot\|_\beta$ is a Banach space. We denote $C_\beta(0,+\infty) = \{x : (0,+\infty) \to \mathbb{R}$ and $x(t) = t^{-\beta}y(t)$ for some $y \in C[0,+\infty)\}$. $L^p_{Loc}[0,+\infty)$ $(p \geq 1)$ is the space of all real valued functions which are Lebesgue integrable over every bounded subinterval of $[0,+\infty)$.

Definition 2.1. The Riemann–Liouville fractional integral of order $\beta \in (0,1)$ of a function $f \in L^1[0,T]$ is defined by

$$(I^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^{1-\beta}} ds.$$

Definition 2.2. The Riemann–Liouville fractional derivative of order $\beta \in (0,1)$ of a function f where $I^{1-\beta} f$ is absolutely continuous (AC) is defined by

$$(D_r^\beta f)(t) = \frac{d}{dt}(I^{1-\beta} f)(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\beta} ds.$$

Remark 2.3. If $f \in L^1[0,T]$, then the integral $(I^\beta f)(t)$ exists for almost every $t \in [0,T]$ and $I^\beta f \in L^1[0,T]$. If $f \in AC[0,T]$, then $D_r^\beta f$ exists almost everywhere in $[0,T]$. If $f \in I^\beta(L^1) = \{f : f = I^\beta g, g \in L^1[0,T]\}$, then $I^{1-\beta} f \in AC[0,T]$. For more details about fractional calculus, we refer the reader to the texts [7,10,16,17].

Theorem 2.4 ([3])**.** *Let $f(t,x)$ be a function that is continuous on the set*

$$\mathbf{B} = \left\{(t,x) \in \mathbb{R}^2 : 0 < t \leq T, x \in I\right\},$$

where $I \subseteq \mathbb{R}$ denotes an unbounded interval. Suppose a function $x : (0,T] \to I$ is continuous and that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,T]$. Then $x(t)$ satisfies the initial value problem (1.11) *on $(0,T]$ if and only if it satisfies the Volterra integral equation*

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s)) ds \tag{2.1}$$

on $(0,T]$.

Remark 2.5. f is absolutely integrable on $(0,T]$ if f is Riemann integrable on every closed interval $[\eta,T]$, where $\eta \in (0,T]$, and $\lim_{\eta\to 0^+} \int_\eta^T |f(t)|dt$ exists and is finite. From Proposition 2.1 in [3], if $f \in L^1[0,T]$ is continuous on $(0,T]$, then f is absolutely integrable on $(0,T]$.

Lemma 2.6 ([2,17])**.** *Suppose $\rho \in L^q[0,1]$. Then*

$$\int_0^t (t-s)^{\beta-1} \rho(s) ds$$

is continuous on $[0,1]$, where $\beta \in (0,1)$ and $q > \frac{1}{\beta}$.

Theorem 2.7 ([1])**.** *Let E be a Banach space, C a closed, convex subset of E and $0 \in C$. Let $F : C \to C$ be a continuous and completely continuous map, and let the set $\{x \in E : x = \lambda F x$ for some $\lambda \in (0,1)\}$ be bounded. Then F has at least one fixed point in E.*

3 Fractional integral inequalities

In this section, we are now to prove some results concerning fractional integral inequalities (1.9) and (1.10), which can be used to study the global existence of solutions of fractional differential equation (1.11).

Theorem 3.1. *Let $\beta \in (0,1)$ and $\gamma \geq 0$, $a(t)$ and $b(t)$ be nonnegative and continuous functions on $[0,+\infty)$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $t^{-\gamma}l(t) \in L^q_{Loc}[0,+\infty)$ $(q > \frac{1}{\beta})$, and $u(t)$ be a continuous, nonnegative function on $[0,+\infty)$ with*

$$u(t) \leq a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)u(s)ds. \tag{3.1}$$

Then

$$u(t) \leq \left(A(t) + B(t)\int_0^t L(s)A(s)\exp\left(\int_s^t L(\tau)B(\tau)d\tau\right)ds\right)^{\frac{1}{q}}, \qquad t \in [0,+\infty), \tag{3.2}$$

where $A(t) = 2^{q-1}a^q(t)$, $B(t) = \frac{2^{q-1}b^q(t)t^{q\beta-q+\frac{q}{p}}}{(p\beta-p+1)^{\frac{q}{p}}}$, $L(t) = t^{-q\gamma}l^q(t)$ and $p \in (1,+\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $q > \frac{1}{\beta}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\beta - 1 + \frac{1}{p} > 0$. From the inequality (3.1) and using the Hölder inequality, we have

$$\begin{aligned}
u(t) &\leq a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)u(s)ds \\
&\leq a(t) + b(t)\left(\int_0^t (t-s)^{p(\beta-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t (s^{-\gamma}l(s)u(s))^q ds\right)^{\frac{1}{q}} \\
&= a(t) + \frac{b(t)t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}\left(\int_0^t (s^{-\gamma}l(s)u(s))^q ds\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.3}$$

Then

$$u^q(t) \leq 2^{q-1}a^q(t) + \frac{2^{q-1}b^q(t)t^{q\beta-q+\frac{q}{p}}}{(p\beta-p+1)^{\frac{q}{p}}}\int_0^t s^{-q\gamma}l^q(s)u^q(s)ds.$$

Let $w(t) = u^q(t)$, $A(t) = 2^{q-1}a^q(t)$, $B(t) = \frac{2^{q-1}b^q(t)t^{q\beta-q+\frac{q}{p}}}{(p\beta-p+1)^{\frac{q}{p}}}$ and $L(t) = t^{-q\gamma}l^q(t)$, then

$$w(t) \leq A(t) + B(t)\int_0^t L(s)w(s)ds.$$

By the Gronwall–Beesack inequality [15, p. 356], we obtain

$$w(t) \leq A(t) + B(t)\int_0^t L(s)A(s)\exp(\int_s^t L(\tau)B(\tau)d\tau)ds.$$

Thus, we obtain the inequality (3.2) and complete the proof. □

Theorem 3.2. *Let $a,b \geq 0$, $\alpha > \delta \geq 0$ and $\beta \in (0,1)$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $t^{-\alpha}l(t) \in L^q_{Loc}[0,+\infty)$ $(q > \frac{1}{\beta})$. Suppose that $t^\alpha u(t)$ is a continuous, nonnegative function on $[0,+\infty)$ and $u(t)$ satisfies the inequality*

$$u(t) \leq at^{-\alpha} + bt^{-\delta}\int_0^t (t-s)^{\beta-1}l(s)u(s)ds, \qquad t \in (0,+\infty). \tag{3.4}$$

Then

$$u(t) \le t^{-\alpha}\left(2^{q-1}a^q + 2^{q-1}a^q B(t)\int_0^t L(s)\exp\left(\int_s^t L(\tau)B(\tau)d\tau\right)ds\right)^{\frac{1}{q}}, \qquad t\in(0,+\infty), \quad (3.5)$$

where $B(t) = \frac{2^{q-1}b^q t^{q\alpha-q\delta+q\beta-q+\frac{q}{p}}}{(p\beta-p+1)^{\frac{q}{p}}}$, $L(t) = t^{-q\alpha}l^q(t)$ *and* $p\in(1,+\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$ *.*

Proof. Let $v(t) = t^\alpha u(t)$, so that $v(t)$ satisfies the inequality

$$v(t) \le a + bt^{\alpha-\delta}\int_0^t (t-s)^{\beta-1}s^{-\alpha}l(s)v(s)ds, \qquad t\in[0,+\infty). \quad (3.6)$$

By Theorem 3.1, we obtain the inequality (3.5) and complete the proof. □

Lemma 3.3 ([20])**.** *Let* $1\le p<\infty$, $a(t)$ *and* $b(t)$ *be nonnegative continuous on* $[0,\infty)$, $l(t)$ *be a non-negative and continuous function on* $(0,+\infty)$ *and* $l(t)\in L^1_{Loc}[0,+\infty)$*. Suppose* $u(t)$ *is a nonnegative continuous function on* $[0,+\infty)$ *with*

$$u(t) \le a(t) + b(t)\left(\int_0^t l(s)u^p(s)ds\right)^{\frac{1}{p}}, \qquad t\in[0,\infty). \quad (3.7)$$

Then

$$u(t) \le a(t) + b(t)\frac{\left(\int_0^t l(s)e(s)a^p(s)ds\right)^{\frac{1}{p}}}{1-[1-e(t)]^{\frac{1}{p}}},$$

where $e(t) = \exp(-\int_0^t l(s)b^p(s)ds)$*.*

Theorem 3.4. *Let* $a,b\ge 0$, $\alpha>\delta\ge 0$ *and* $\beta\in(0,1)$, $l(t)$ *be a nonnegative and continuous function on* $(0,+\infty)$ *and* $t^{-\alpha}l(t)\in L^q_{Loc}[0,+\infty)$ $(q>\frac{1}{\beta})$*. Suppose that* $t^\alpha u(t)$ *is a continuous, nonnegative function on* $[0,+\infty)$ *and* $u(t)$ *satisfies the inequality*

$$u(t) \le at^{-\alpha} + bt^{-\delta}\int_0^t (t-s)^{\beta-1}l(s)u(s)ds, \qquad t\in(0,+\infty). \quad (3.8)$$

Then

$$u(t) \le at^{-\alpha} + at^{-\alpha}B(t)\frac{\left(\int_0^t L(s)e(s)ds\right)^{\frac{1}{q}}}{1-[1-e(t)]^{\frac{1}{q}}}, \qquad t\in(0,+\infty), \quad (3.9)$$

where $B(t) = \frac{bt^{\alpha-\delta+\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}$, $L(t) = t^{-q\alpha}l^q(t)$, $e(t) = \exp(-\int_0^t L(s)B^q(s)ds)$, *and* $p\in(1,+\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$ *.*

Proof. Let $v(t) = t^\alpha u(t)$ and using the Hölder inequality, we have

$$\begin{aligned} v(t) &\le a + bt^{\alpha-\delta}\left(\int_0^t (t-s)^{p(\beta-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t (s^{-\alpha}l(s)v(s))^q ds\right)^{\frac{1}{q}} \\ &= a + \frac{bt^{\alpha-\delta+\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}\left(\int_0^t s^{-q\alpha}l^q(s)v^q(s)ds\right)^{\frac{1}{q}}. \end{aligned} \quad (3.10)$$

By Lemma 3.3, we get

$$v(t) \le a + aB(t)\frac{\left(\int_0^t L(s)e(s)ds\right)^{\frac{1}{q}}}{1-[1-e(t)]^{\frac{1}{q}}},$$

where $B(t) = \frac{bt^{\alpha-\delta+\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}$, $L(t) = t^{-q\alpha}l^q(t)$ and $e(t) = \exp(-\int_0^t L(s)B^q(s)ds)$. Then we obtain the inequality (3.9) and complete the proof. □

In [20], Willett studied the inequality (3.7) by using the Minkowski inequality. Now, we use a new method to study the inequality (3.7).

Lemma 3.5. *Let $1 \le p < \infty$, $a(t)$ and $b(t)$ be continuous and nonnegative functions on $[0,\infty)$, nonnegative function $l(t) \in L^p_{Loc}[0,+\infty)$, and $u(t)$ be a continuous and nonnegative function with*

$$u(t) \le a(t) + b(t)\left(\int_0^t l^p(s)u^p(s)ds\right)^{\frac{1}{p}}, \qquad t \in [0,\infty). \tag{3.11}$$

Then

$$u(t) \le a(t) + b(t)\left(A(t)\exp(\int_0^t L(s)ds)\right)^{\frac{1}{p}}, \qquad t \in [0,\infty), \tag{3.12}$$

where $A(t) = \int_0^t 2^{p-1}l^p(s)a^p(s)ds$ and $L(t) = 2^{p-1}l^p(t)b^p(t)$.

Proof. From (3.11), we know

$$l(t)u(t) \le l(t)a(t) + l(t)b(t)\left(\int_0^t l^p(s)u^p(s)ds\right)^{\frac{1}{p}}$$

and

$$\begin{aligned}\int_0^t l^p(s)u^p(s)ds &\le \int_0^t \left(l(s)a(s) + l(s)b(s)\left(\int_0^s l^p(\tau)u^p(\tau)d\tau\right)^{\frac{1}{p}}\right)^p ds \\ &\le \int_0^t 2^{p-1}l^p(s)a^p(s) + 2^{p-1}l^p(s)b^p(s)\int_0^s l^p(\tau)u^p(\tau)d\tau ds.\end{aligned} \tag{3.13}$$

Let $w(t) = \int_0^t l^p(s)u^p(s)ds$, $A(t) = \int_0^t 2^{p-1}l^p(s)a^p(s)ds$ and $L(t) = 2^{p-1}l^p(t)b^p(t)$, then

$$w(t) \le A(t) + \int_0^t L(s)w(s)ds.$$

Since $A(t)$ is a nondecreasing function and using Gronwall integral inequality, thus we obtain

$$w(t) \le A(t)\exp\left(\int_0^t L(s)ds\right).$$

Thus, we obtain the inequality (3.12) and complete the proof. □

If we replace $b(t)$ by $t^{1-\beta}$ and γ by $1-\beta$ in Theorem 3.1, and using Lemma 3.5, we can obtain the following conclusions under the hypotheses $l(t) \in L^q_{Loc}[0,+\infty)$.

Theorem 3.6. *Let $\beta \in (0,1)$, $a(t)$ be a nonnegative and continuous function on $[0,+\infty)$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $l(t) \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$, and $u(t)$ be a continuous, nonnegative function on $[0,+\infty)$ with*

$$u(t) \leq a(t) + t^{1-\beta} \int_0^t (t-s)^{\beta-1} s^{\beta-1} l(s) u(s) ds. \tag{3.14}$$

Then

$$u(t) \leq a(t) + b(t) \left(A(t) \exp\left(\int_0^t L(s) ds \right) \right)^{\frac{1}{q}}, \qquad t \in [0,\infty), \tag{3.15}$$

where $b(t) = \frac{2^{\frac{1}{p}} t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}$, $A(t) = \int_0^t 2^{q-1} l^q(s) a^q(s) ds$, $L(t) = 2^{q-1} l^q(t) b^q(t)$ and $p \in (1,+\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $q > \frac{1}{\beta}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $1 < p < \frac{1}{1-\beta}$. From the inequality (3.14) we have

$$\begin{aligned}
u(t) &\leq a(t) + \int_0^t \left(\frac{t}{(t-s)s} \right)^{1-\beta} l(s) u(s) ds \\
&= a(t) + \int_0^t \left(\frac{1}{t-s} + \frac{1}{s} \right)^{1-\beta} l(s) u(s) ds \\
&\leq a(t) + \left(\int_0^t \left(\frac{1}{t-s} + \frac{1}{s} \right)^{p(1-\beta)} ds \right)^{\frac{1}{p}} \left(\int_0^t (l(s)u(s))^q ds \right)^{\frac{1}{q}} \\
&\leq a(t) + \left(\int_0^t (t-s)^{p(\beta-1)} + s^{p(\beta-1)} ds \right)^{\frac{1}{p}} \left(\int_0^t (l(s)u(s))^q ds \right)^{\frac{1}{q}} \\
&= a(t) + \frac{2^{\frac{1}{p}} t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}} \left(\int_0^t l^q(s) u^q(s) ds \right)^{\frac{1}{q}}.
\end{aligned} \tag{3.16}$$

Let $b(t) = \frac{2^{\frac{1}{p}} t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}$. Then by Lemma 3.5, we obtain the inequality (3.15). □

Corollary 3.7. *Let $\beta \in (0,1)$ and $u_0 > 0$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $l(t) \in L^q_{Loc}[0,+\infty)$ $(q > \frac{1}{\beta})$, and nonnegative function $u(t) \in C_{1-\beta}(0,+\infty)$ with*

$$u(t) \leq u_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s) u(s) ds, \qquad t \in (0,+\infty). \tag{3.17}$$

Then

$$u(t) \leq u_0 t^{\beta-1} + t^{\beta-1} b(t) \left(A(t) \exp\left(\int_0^t L(s) ds \right) \right)^{\frac{1}{q}}, \qquad t \in (0,+\infty), \tag{3.18}$$

where $b(t) = \frac{2^{\frac{1}{p}} t^{\beta-1+\frac{1}{p}}}{\Gamma(\beta)(p\beta-p+1)^{\frac{1}{p}}}$, $A(t) = \int_0^t 2^{q-1} u_0^q l^q(s) ds$, $L(t) = 2^{q-1} l^q(t) b^q(t)$ and $p \in (1,+\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $u(t) \in C_{1-\beta}(0,+\infty)$, then $v(t) = t^{1-\beta} u(t) \in C[0,+\infty)$ and

$$v(t) \leq u_0 + \frac{t^{1-\beta}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\beta-1} l(s) v(s) ds.$$

By Theorem 3.6, we obtain the inequality (3.18) and complete the proof. □

Remark 3.8. Medved' studied the inequality (1.9) in [13, Theorem 4] for different β and γ. If $\beta > \frac{1}{2}$ and $\gamma > 1 - \frac{1}{2p}$ $(p > 1)$, then Medved' obtained the bound of the inequality (1.9). If $\beta = \frac{1}{m+1}$ and $\gamma > 1 - \frac{1}{kq}$ $(m \geq 1,\ k > 1$ and $q = m+2)$, then Medved' obtained another bound. In Theorem 3.1, we study the inequality (1.9) under the hypothesis $\beta \in (0,1)$ and $\gamma \geq 0$. The proof of the inequality (1.9) is more simple than Theorem 4 in [13]. Lemma 3.5 and Theorem 3.6 we now discuss seem to be new. For the special $b(t)$ and γ, the hypothesis in Theorem 3.6 is weaker than that in Theorem 3.1.

Example 3.9. Suppose that $t^{\frac{1}{2}}u(t)$ is a continuous, nonnegative function on $[0,+\infty)$ and $u(t)$ satisfies the inequality

$$u(t) \leq t^{-\frac{1}{2}} + t^{-\frac{1}{3}} \int_0^t (t-s)^{-\frac{1}{3}} \frac{\sqrt[6]{s}}{\sqrt{1+s^2}} u(s)ds, \qquad t \in (0,+\infty). \tag{3.19}$$

Let $p = q = 2$, by Theorem 3.2, then we have

$$u(t) \leq t^{-\frac{1}{2}} (2 + 12 t^{\frac{2}{3}} \exp(6 \arctan t) \int_0^t \frac{s^{\frac{-2}{3}}}{1+s^2} \exp(-6 \arctan s) ds)^{\frac{1}{2}}.$$

We know

$$\begin{aligned} \int_0^t \frac{s^{\frac{-2}{3}}}{1+s^2} \exp(-6 \arctan s) ds &\leq \int_0^{+\infty} \frac{s^{\frac{-2}{3}}}{1+s^2} ds \\ &= \frac{1}{2} \int_0^1 (1-u)^{\frac{-5}{6}} u^{\frac{-1}{6}} du \\ &= \pi, \end{aligned} \tag{3.20}$$

where $u = \frac{1}{1+s^2}$. Then we obtain

$$u(t) \leq t^{-\frac{1}{2}} \left(2 + 12\pi \exp(3\pi) t^{\frac{2}{3}}\right)^{\frac{1}{2}}, \qquad t \in (0,+\infty).$$

Example 3.10. Suppose that $t^{\frac{1}{3}}u(t)$ is a continuous, nonnegative function on $[0,+\infty)$ and $u(t)$ satisfies the inequality

$$u(t) \leq t^{-\frac{1}{3}} + \int_0^t (t-s)^{-\frac{1}{3}} s^{-\frac{1}{3}} u(s) ds, \qquad t \in (0,+\infty). \tag{3.21}$$

Let $v(t) = t^{\frac{1}{3}}u(t)$, then

$$v(t) \leq 1 + t^{\frac{1}{3}} \int_0^t (t-s)^{-\frac{1}{3}} s^{-\frac{2}{3}} v(s) ds, \qquad t \in [0,+\infty).$$

Let $p = \frac{8}{3}$ and $q = \frac{8}{5}$, by Theorem 3.6, we have

$$\begin{aligned} v(t) &\leq 1 + 18^{\frac{3}{8}} t^{\frac{1}{24}} \left(\frac{15}{7} 2^{\frac{3}{5}} t^{\frac{7}{15}} \exp\left(\frac{15}{8} 36^{\frac{3}{5}} t^{\frac{8}{15}} \right) \right)^{\frac{5}{8}} \\ &= 1 + 36^{\frac{3}{8}} \left(\frac{15}{7} \right)^{\frac{5}{8}} t^{\frac{1}{3}} \exp\left(\frac{75}{64} 36^{\frac{3}{5}} t^{\frac{8}{15}} \right) \\ &\leq 1 + 7 t^{\frac{1}{3}} \exp(11 t^{\frac{8}{15}}) \end{aligned} \tag{3.22}$$

and
$$u(t) \le t^{\frac{-1}{3}} + 7\exp(11t^{\frac{8}{15}}), \qquad t \in (0, +\infty).$$
We know $t^{\frac{-2}{3}} \notin L^q_{Loc}[0, +\infty)$ $(q > \frac{3}{2})$. Thus, Theorem 3.2 can not be applied to Example 3.10.

Using Theorem 3.9 in [19], we know
$$u(t) \le t^{\frac{-1}{3}} + B_1 \exp(6B_0 t^{\frac{2}{3}}), \qquad t \in (0, +\infty),$$
where $B_0 = B(\frac{2}{3}, \frac{2}{3})$ and $B_1 = B(\frac{2}{3}, \frac{1}{3})$ ($B(p,q) = \int_0^1 (1-s)^{p-1} s^{q-1} ds$ is the Beta function). Due to $\frac{8}{15} < \frac{2}{3}$, this indicates that our results are improvements on [19] as $t \to \infty$. Theorem 3.9 of [19] can also be applied to the inequality (1.10) when $l(t) = t^{-\gamma}$.

4 Global solutions of fractional differential equations

In this section, we give the existence and uniqueness results of the initial value problem (1.11).

Lemma 4.1. *Suppose $f : (0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exist nonnegative functions $l(t), k(t)$ with $t^{\beta-1} l(t) \in C(0, T] \bigcap L^q[0, T]$ and $k(t) \in C(0, T] \bigcap L^q[0, T]$ $(q > \frac{1}{\beta}, \beta \in (0, 1))$ such that*
$$|f(t, x)| \le l(t)|x| + k(t)$$
for all $(t, x) \in (0, T] \times \mathbb{R}$. Then the following Volterra integral equation
$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \tag{4.1}$$
has at least one solution in $C_{1-\beta}(0, T]$.

Proof. Let $G : C_{1-\beta}(0, T] \to C_{1-\beta}(0, T]$ be the operator defined by
$$Gx(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \tag{4.2}$$
for all $x \in C_{1-\beta}(0, T]$.

Step 1: we show that the operator G is continuous. To see this let $x_n \to x$ in $C_{1-\beta}(0, T]$ and we will show that $Gx_n \to Gx$ in $C_{1-\beta}(0, T]$. Now $x_n \to x$ implies that there exists $r > 0$ such that $\|x_n\|_{1-\beta} \le r$ and $\|x\|_{1-\beta} \le r$. For each $s \in (0, T]$, we have
$$f(s, x_n(s)) \to f(s, x(s)).$$
Using the assumption of f, we get
$$(t-s)^{\beta-1} |f(s, x_n(s)) - f(s, x(s))| \le 2(t-s)^{\beta-1} (r s^{\beta-1} l(s) + k(s)).$$
Since $t^{\beta-1} l(t) \in C(0, T] \bigcap L^q[0, T]$ and $k(t) \in C(0, T] \bigcap L^q[0, T]$, using the Hölder inequality, then we know the function
$$s \to 2r(t-s)^{\beta-1} s^{\beta-1} l(s) + 2(t-s)^{\beta-1} k(s)$$
is integrable for $s \in [0, t]$. By means of the Lebesgue dominated convergence theorem yields
$$t^{1-\beta} \left| \int_0^t (t-s)^{\beta-1} [f(s, x_n(s)) - f(s, x(s))] ds \right| \to 0$$

as $n \to +\infty$. Therefore $t^{1-\beta}Gx_n(t) \to t^{1-\beta}Gx(t)$ pointwise on $[0,T]$ as $n \to +\infty$. If we show the convergence is uniform then of course G is continuous. Let $t_1, t_2 \in [0,T]$ with $t_2 < t_1$. Then

$$\begin{aligned} &\left|t_1^{1-\beta}Gx(t_1) - t_2^{1-\beta}Gx(t_2)\right| \\ &\quad \le \left|\frac{t_1^{1-\beta} - t_2^{1-\beta}}{\Gamma(\beta)}\right| \left|\int_0^{t_2}(t_2-s)^{\beta-1}f(s,x(s))ds\right| \\ &\quad + \frac{t_1^{1-\beta}}{\Gamma(\beta)}\left|\int_0^{t_1}(t_1-s)^{\beta-1}f(s,x(s))ds - \int_0^{t_2}(t_2-s)^{\beta-1}f(s,x(s))ds\right|. \end{aligned} \tag{4.3}$$

Since

$$|f(t,x(t))| \le l(t)|x(t)| + k(t) \le t^{\beta-1}l(t)t^{1-\beta}|x(t)| + k(t),$$

from the assumptions of f, we know $f(t,x(t)) \in L^q[0,T]$ $(q > \frac{1}{\beta})$ when $x(t) \in C_{1-\beta}(0,T]$. From Lemma 2.6, we obtain

$$\int_0^t (t-s)^{\beta-1}f(s,x(s))ds$$

is continuous on $[0,T]$. As $t_1 \to t_2$, the right-hand side of the above inequality (4.3) tends to zero. Now (4.3) together with the fact that $t^{1-\beta}Gx_n(t) \to t^{1-\beta}Gx(t)$ pointwise on $[0,T]$ implies that the convergence is uniform. Consequently $G : C_{1-\beta}(0,T] \to C_{1-\beta}(0,T]$ is continuous.

Step 2: Next we claim that the operator G is completely continuous. To see this let $\Omega \in C_{1-\beta}(0,T]$ be bounded and $\|x\|_{1-\beta} \le M$ for each $x \in \Omega$, we will show that $t^{1-\beta}G(\Omega)$ is uniformly bounded and equicontinuous on $[0,T]$. The equicontinuity of $t^{1-\beta}G(\Omega)$ on $[0,T]$ follows essentially the same reasoning as that used to prove (4.3). Also $t^{1-\beta}G(\Omega)$ is uniformly bounded. Since for $t \in [0,T]$, we have

$$\begin{aligned} |t^{1-\beta}Gx(t)| &\le |x_0| + \frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\beta-1}l(s)s^{1-\beta}|x(s)|ds + \frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}k(s)ds \\ &\le |x_0| + \frac{t^{1-\beta}}{\Gamma(\beta)}\left(\int_0^t (t-s)^{p(\beta-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t (Ms^{\beta-1}l(s))^q ds\right)^{\frac{1}{q}} \\ &\quad + \frac{t^{1-\beta}}{\Gamma(\beta)}\left(\int_0^t (t-s)^{p(\beta-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t k^q(s)ds\right)^{\frac{1}{q}} \\ &\le |x_0| + \frac{t^{\frac{1}{p}}}{\Gamma(\beta)(p(\beta-1)+1)^{\frac{1}{p}}}\left[\left(\int_0^t (Ms^{\beta-1}l(s))^q ds\right)^{\frac{1}{q}} + \left(\int_0^t k^q(s)ds\right)^{\frac{1}{q}}\right], \end{aligned} \tag{4.4}$$

then

$$\|Gx\|_{1-\beta} \le |x_0| + \frac{T^{\frac{1}{p}}}{\Gamma(\beta)(p(\beta-1)+1)^{\frac{1}{p}}}\left[\left(\int_0^T (Ms^{\beta-1}l(s))^q ds\right)^{\frac{1}{q}} + \left(\int_0^T k^q(s)ds\right)^{\frac{1}{q}}\right].$$

Consequently $G : C_{1-\beta}(0,T] \to C_{1-\beta}(0,T]$ is completely continuous.

Step 3: If $x \in C_{1-\beta}(0,T]$ is any solution of

$$x(t) = \lambda\left(x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s,x(s))ds\right), \qquad t \in (0,T]$$

for $\lambda \in (0,1)$. Let $v(t) = t^{1-\beta}x(t) \in C[0,T]$, then

$$\begin{aligned} |v(t)| &\leq |x_0| + \left|\frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, s^{\beta-1}v(s))ds\right| \\ &\leq |x_0| + \frac{t^{\frac{1}{p}}}{\Gamma(\beta)(p(\beta-1)+1)^{\frac{1}{p}}}\left(\int_0^t k^q(s)ds\right)^{\frac{1}{q}} + \frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\beta-1}l(s)|v(s)|ds. \end{aligned} \tag{4.5}$$

Consequently, by Theorem 3.1, we can get

$$|v(t)| \leq \left(A(t) + B(t)\int_0^t L(s)A(s)\exp\left(\int_s^t L(\tau)B(\tau)d\tau\right)ds\right)^{\frac{1}{q}}, \qquad t \in [0,T],$$

where

$$\begin{aligned} A(t) &= 2^{q-1}\left(|x_0| + \frac{t^{\frac{1}{p}}}{\Gamma(\beta)\,(p(\beta-1)+1)^{\frac{1}{p}}}\left(\int_0^t k^q(s)ds\right)^{\frac{1}{q}}\right)^q, \\ B(t) &= \frac{2^{q-1}t^{\frac{q}{p}}}{\Gamma^q(\beta)(p\beta - p + 1)^{\frac{q}{p}}}, \\ L(t) &= t^{q(\beta-1)}l^q(t) \end{aligned}$$

and $p \in (1,+\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we get

$$\|v\| = \|x\|_{1-\beta} \leq \left(A(T) + B(T)\int_0^T L(s)A(s)\exp\left(\int_s^T L(\tau)B(\tau)d\tau\right)ds\right)^{\frac{1}{q}}.$$

Finally, by applying fixed point Theorem 2.7, the operator G has a fixed point $x(t) \in C_{1-\beta}(0,T]$, which is the solution of the integral equation (4.1). □

Lemma 4.2. *Let f be as in Lemma 4.1. A function $x \in C_{1-\beta}(0,T]$ is a solution of fractional differential equation (1.11) if and only if it is a solution of the Volterra integral equation*

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, x(s))ds, \qquad t \in (0,T]. \tag{4.6}$$

Proof. Since $x \in C_{1-\beta}(0,T]$ and

$$|f(t,x(t))| \leq l(t)|x(t)| + k(t) = t^{\beta-1}l(t)t^{1-\beta}|x(t)| + k(t)$$

with $t^{\beta-1}l(t) \in C(0,T]\bigcap L^q[0,T]$ and $k(t) \in C(0,T]\bigcap L^q[0,T]$, then we have $x \in C(0,T]\bigcap L^1[0,T]$ and $f(t,x(t)) \in C(0,T]\bigcap L^1[0,T]$. By virtue of Theorem 2.4, then we complete the proof. □

Theorem 4.3. *Suppose $f : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exist nonnegative functions $l(t), k(t)$ with $t^{\beta-1}l(t) \in C(0,+\infty)\bigcap L^q_{Loc}[0,+\infty)$ and $k(t) \in C(0,+\infty)\bigcap L^q_{Loc}[0,+\infty)$ $(q > \frac{1}{\beta}, \beta \in (0,1))$ such that*

$$|f(t,x)| \leq l(t)|x| + k(t)$$

for all $(t,x) \in (0,+\infty) \times \mathbb{R}$. Then the initial value problem (1.11) has at least one continuous solution in $C_{1-\beta}(0,+\infty)$.

Proof. From Lemma 4.1 and Lemma 4.2, We know the equation (1.11) has at least one solution in $C_{1-\beta}(0,T]$. Since T can be chosen arbitrarily constant, then the equation (1.11) has at least one global solution on $(0,+\infty)$. Thus, we complete the proof of Theorem 4.3. $\square$

Theorem 4.4. *If $f:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and*

$$|f(t,x)-f(t,y)|\le l(t)|x-y|$$

for all $x,y\in\mathbb{R}$ and $t\in(0,+\infty)$, where $t^{\beta-1}l(t)\in C(0,+\infty)\bigcap L^q_{Loc}[0,+\infty)$ and $|f(t,0)|\in L^q_{Loc}[0,+\infty)$ $(q>\frac{1}{\beta})$. Then equation (1.11) has a unique solution on $(0,+\infty)$.

Proof. We know

$$|f(t,x)|\le|f(t,x)-f(t,0)|+|f(t,0)|\le l(t)|x|+|f(t,0)|.$$

By Theorem 4.3, we suppose $x_1(t)$, $x_2(t)$ are two global solutions of equation (1.11). Then

$$\begin{aligned}|x_1(t)-x_2(t)|&=\left|\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}(f(s,x_1(s))-f(s,x_2(s)))ds\right|\\&\le\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}l(s)|x_1(s)-x_2(s)|ds\\&=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}s^{\beta-1}l(s)s^{1-\beta}|x_1(s)-x_2(s)|ds.\end{aligned}\tag{4.7}$$

Let $u(t)=t^{1-\beta}|x_1(t)-x_2(t)|$, then

$$u(t)\le\frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}s^{\beta-1}l(s)u(s)ds.$$

By Theorem 3.1, we can get $x_1(t)=x_2(t)$. Thus the proof is complete. $\square$

Remark 4.5. In [18], Trif proved that the equation (1.11) has a unique solution when continuous function $f(t,x)=p(t)x+q(t)$ for all $(t,x)\in(0,+\infty)\times\mathbb{R}$, where $p\in C_\alpha(0,+\infty)$ and $q\in C_{1-\beta}(0,+\infty)$ with $0\le\alpha<\beta$. Then under the above conclusions, Trif presented the existence result when $f(t,x)\le p(t)x+q(t)$, where $p\in C_\alpha(0,+\infty)$ and $q\in C_{1-\beta}(0,+\infty)$ with $0\le\alpha<\beta$ and $2\beta-\alpha>1$. In fact, if $p\in C_\alpha(0,+\infty)$ and $q\in C_{1-\beta}(0,+\infty)$, let $1+\alpha-\beta<\frac{1}{q}<\beta$, then $t^{\beta-1}p(t)\in C(0,+\infty)\bigcap L^q_{Loc}[0,+\infty)$ and $q(t)\in C(0,+\infty)\bigcap L^q_{Loc}[0,+\infty)$. Thus, our result includes the main result of [18, Theorem 4.2]. Theorem 4.11 of [19] also states a global existence result of the equation (1.11) but with only a sketch of the proof.

Example 4.6.

$$\begin{cases}D_r^{\frac{3}{4}}x(t)=(t^{\frac{-1}{4}}+1)\sqrt{x(t)}+t^{\frac{-1}{2}},\\\lim_{t\to0^+}t^{\frac{3}{4}}x(t)=1.\end{cases}\tag{4.8}$$

We know

$$(t^{\frac{-1}{4}}+1)\sqrt{x(t)}+t^{\frac{-1}{2}}\le\frac{t^{\frac{-1}{4}}+1}{2}|x(t)|+\frac{t^{\frac{-1}{4}}+1}{2}+t^{\frac{-1}{2}}.\tag{4.9}$$

Let $q=\frac{5}{3}$, then $\frac{t^{\frac{-1}{4}}(t^{\frac{-1}{4}}+1)}{2}\in C(0,+\infty)\bigcap L^{\frac{5}{3}}_{Loc}[0,+\infty)$ and $\frac{t^{\frac{-1}{4}}+1}{2}+t^{\frac{-1}{2}}\in C(0,+\infty)\bigcap L^{\frac{5}{3}}_{Loc}[0,+\infty)$. From Theorem 4.3, equation (4.8) has at least one global solution on $(0,+\infty)$.

A global solution is proved in [18] under the following hypothesis $f(t,x)\le p(t)x+q(t)$, where $p\in C_\alpha(0,+\infty)$ and $q\in C_{1-\beta}(0,+\infty)$ with $0\le\alpha<\beta$ and $2\beta-\alpha>1$. From (4.9), we know $\frac{t^{\frac{-1}{4}}+1}{2}+t^{\frac{-1}{2}}\notin C_{\frac{1}{4}}(0,+\infty)$.

Example 4.7.

$$\begin{cases} D_r^{\frac{3}{4}}x(t) = t^{\frac{-1}{3}}\frac{1+x^2(t)}{1+x(t)} + t^{\frac{-1}{2}}, \\ \lim_{t\to 0^+} t^{\frac{3}{4}}x(t) = 1. \end{cases} \tag{4.10}$$

We know

$$\left|\frac{1+x^2}{1+x} - \frac{1+y^2}{1+y}\right| \le |x-y|,$$

where $x, y \in [0,+\infty)$. Since $t^{\frac{-7}{12}} \in C(0,+\infty) \bigcap L^q_{Loc}[0,+\infty)$ and $t^{\frac{-1}{3}} + t^{\frac{-1}{2}} \in C(0,+\infty) \bigcap L^q_{Loc}[0,+\infty)$ $(q > \frac{4}{3})$, then from Theorem 4.4, equation (4.10) has a unique global solution on $(0,+\infty)$.

Acknowledgements

The research was supported by Scientific Research Foundation of Nanjing Institute of Technology (No: CKJB201508). I thank the referee for pointing out some useful references and for valuable comments which led to improvements of the paper.

References

[1] R. P. Agarwal, M. Meehan, D. O'Regan, *Fixed point theory and applications*, Cambridge University Press, Cambridge, 2001. https://doi.org/10.1017/CBO978051154300; MR1825411

[2] R. P. Agarwal, D. O'Regan, S. Staněk, Positive solutions for mixed problems of singular fractional differential equations, *Math. Nachr.* **285**(2012), No. 1, 27–41. https://doi.org/10.1002/mana.201000043; Zbl 1232.26005

[3] L. C. Becker, T. A. Burton, I. K. Purnaras, Complementary equations: a fractional differential equation and a Volterra integral equation, *Electron. J. Qual. Theory. Differ. Equ.* **2015**, No. 12, 1–24. https://doi.org/10.14232/ejqtde.2015.1.12; Zbl 1349.34005

[4] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204**(1996), No. 2, 609–625. https://doi.org/10.1006/jmaa.1996.0456; Zbl 0881.34005

[5] Z. Denton, J. D. Ramírez, Existence of minimal and maximal solutions to RL fractional integro-differential initial value problems, *Opuscula. Math.* **37**(2017), No. 5, 705–724. https://doi.org/10.7494/OpMath.2017.37.5.705; Zbl 06963036

[6] Z. Denton, A. S. Vatsala, Fractional integral inequalities and applications, *Comput. Math. Appl.* **59**(2010), No. 3, 1087–1094. https://doi.org/10.1016/j.camwa.2009.05.012; Zbl 1189.26044

[7] K. Diethelm, *The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type*, Lecture Notes in Mathematics, Vol. 2004, Springer-Verlag, Berlin, 2010. https://doi.org/10.1007/978-3-642-14574-2; Zbl 1215.34001

[8] P. W. Eloe, T. Masthay, Initial value problems for Caputo fractional differential equations, *J. Fract. Calc. Appl.* **9**(2018), No. 2, 178–195. MR3772633

[9] D. Henry, *Geometric theory of semilinear parabolic equations*, Springer-Verlag, Berlin, Heidelberg, New York, 1981. https://doi.org/10.1007/BFb0089647; Zbl 0456.35001

[10] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier, Amsterdam, 2006. Zbl 1092.45003

[11] C. Kou, H. Zhou, Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis, *Nonlinear. Anal.* **74**(2011), No. 17, 5975–5986. https://doi.org/10.1016/j.na.2011.05.074; Zbl 1235.34022

[12] Q. Ma, J. Pečarić, Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential equations and integral equations, *J. Math. Anal. Appl.* **341**(2008), No. 2, 894–905. https://doi.org/10.1016/j.jmaa.2007.10.036; Zbl 1142.26015

[13] M. Medveď, A new approach to an analysis of Henry type integral inequalities and their Bihari type versions, *J. Math. Anal. Appl.* **214**(1997), No. 2, 349–366. https://doi.org/10.1006/jmaa.1997.5532; Zbl 0893.26006

[14] M. Medveď, Integral inequalities and global solutions of semilinear evolution equations, *J. Math. Anal. Appl.* **267**(2002), No. 2, 634–650. https://doi.org/10.1006/jmaa.2001.7798; Zbl 1028.34055

[15] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, 1991. https://doi.org/10.1007/978-94-011-3562-7; Zbl 0744.26011

[16] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, San Diego, 1999. Zbl 0924.34008

[17] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993. Zbl 0818.26003

[18] T. Trif, Existence of solutions to initial value problems for nonlinear fractional differential equations on the semi-axis, *Frac. Calc. Appl. Anal.* **16**(2013), No. 3, 595–612. https://doi.org/10.2478/s13540-013-0038-3; Zbl 1312.34026

[19] J. R. L. Webb, Weakly singular Gronwall inequalities and applications to fractional differential equations, *J. Math. Anal. Appl.* **471**(2019), No. 1–2, 692–711. https://doi.org/10.1016/j.jmaa.2018.11.004; Zbl 1404.26022

[20] D. Willett, Nonlinear vector integral equations as contraction mappings, *Arch. Ration. Mech. Anal.* **15**(1964), No. 1, 79–86. https://doi.org/10.1007/BF00257405; MR159200; Zbl 0161.31902

[21] H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.* **328**(2007), No. 2, 1075–1081. https://doi.org/10.1016/j.jmaa.2006.05.061; Zbl 1120.26003

[22] T. Zhu, New Henry–Grownall integral inequalities and their applications to fractional differential equations, *Bull. Braz. Math. Soc.* **49**(2018), No. 3, 647–657. https://doi.org/10.1007/s00574-018-0074-z; Zbl 1403.26026

[23] T. Zhu, Existence and uniqueness of positive solutions for fractional differential equations, *Bound. Value. Probl.* **2019**, No. 22, 1–11. https://doi.org/10.1186/s13661-019-1141-0

[24] T. Zhu, C. Zhong, C. Song, Existence results for nonlinear fractional differential equations in $C[0, T)$, *J. Appl. Math. Comput.* **57**(2018), No. 1–2, 57–68. https://doi.org/10.1007/s12190-017-1094-3; Zbl 1395.34013