Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **31**, 1–12; https://doi.org/10.14232/ejqtde.2019.1.31 www.math.u-szeged.hu/ejqtde/

# Boundedness in a quasilinear two-species chemotaxis system with consumption of chemoattractant

**Jing Zhang**, **Xuegang Hu**✉, **Liangchen Wang** and **Li Qu**

School of Science, Chongqing University of Posts and Telecommunications,
No. 2 Chongwen Road, Chongqing, 400065, P.R. China

Received 18 March 2019, appeared 29 April 2019
Communicated by Maria Alessandra Ragusa

**Abstract.** This paper deals with a two-species chemotaxis system

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u) - \nabla\cdot(u\chi_1(w)\nabla w) + \mu_1 u(1-u-a_1 v), & x\in\Omega, \quad t>0,\\ v_t = \nabla\cdot(D_2(v)\nabla v) - \nabla\cdot(v\chi_2(w)\nabla w) + \mu_2 v(1-a_2 u - v), & x\in\Omega, \quad t>0,\\ w_t = \Delta w - (\alpha u + \beta v) w, & x\in\Omega, \quad t>0, \end{cases}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$; $\chi_i (i=1,2)$ are chemotactic functions satisfying $\chi_i'\geq 0$; the parameters $\mu_1,\mu_2>0, a_1,a_2>0$ and $\alpha,\beta>0$, the initial data $(u_0,v_0)\in(C^0(\overline{\Omega}))^2$ and $w_0\in W^{1,\infty}(\Omega)$ are non-negative. Based on the maximal Sobolev regularity, it is shown that this system possesses a unique global bounded classical solution provided that the logistic growth coefficients $\mu_1$ and $\mu_2$ are sufficiently large.

**Keywords:** two-species chemotaxis system, boundedness, maximal Sobolev regularity.

**2010 Mathematics Subject Classification:** 92C17, 35A09, 35M13, 35K55.

## 1 Introduction

This paper considers the following quasilinear chemotaxis system

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u) - \nabla\cdot(u\chi_1(w)\nabla w) + \mu_1 u(1-u-a_1 v), & x\in\Omega,\ t>0,\\ v_t = \nabla\cdot(D_2(v)\nabla v) - \nabla\cdot(v\chi_2(w)\nabla w) + \mu_2 v(1-a_2 u - v), & x\in\Omega,\ t>0,\\ w_t = \Delta w - (\alpha u + \beta v) w, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\nu$ denotes the outer normal vector to $\partial\Omega$, the constants $\mu_1,\mu_2,a_1,a_2,\alpha$ *and* $\beta$ are positive. We consider the

✉Corresponding author. Email: huxg@cqupt.edu.cn

initial data as follows

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) & \text{with } u_0 \geq 0 \text{ in } \Omega, \\ v_0 \in C^0(\overline{\Omega}) & \text{with } v_0 \geq 0 \text{ in } \Omega, \\ w_0 \in W^{1,\infty}(\Omega) & \text{with } w_0 \geq 0 \text{ in } \Omega. \end{cases} \tag{1.2}$$

The chemotactic sensitivity function $\chi_i(w)$ $(i = 1, 2)$ satisfy

$$\chi_i(w) > 0 \quad \text{and} \quad \chi_i'(w) \geq 0. \tag{1.3}$$

Furthermore, we assume that the diffusion function $D_i(s) \in C^2([0, \infty))$ $(i = 1, 2)$ as well as

$$D_i(s) \geq c_{D_i}(s+1)^{m-1} \quad \text{for all } s \geq 0, \tag{1.4}$$

where $c_{D_i} > 0$ and $m \in \mathbb{R}$. In model (1.1), $u = u(x,t)$ and $v = v(x,t)$ represent densities of two populations, respectively, and $w = w(x,t)$ denotes the concentration of oxygen.

System (1.1) is used in mathematical biology as a model to study the mechanism of two-species chemotaxis. The model describes the nonlinear diffusion of competing species which move towards the gradient of a substance called chemoattractant. Chemotaxis system plays a crucial role in cellular communication, for instance, in the governing of immune cells migration, in wound healing, in tumours growth or in the organization of embryonic cell positioning (see e.g. [3,5,38,40]).

The classical Keller–Segel model was proposed by Keller and Segel [14], and the existence of traveling wave solutions was proved under some conditions. Based on the Keller–Segel model, various chemotaxis models have attracted many authors to explore their mathematical properties, such as the boundedness, the stabilization of solutions and the blow-up of solutions [4,6,8,12,17,18,21,23,24,27,34–37,39,41].

A typical chemotaxis process is considered where the signal is degraded, but not produced by the cells. More precisely, the following oxygen consumption model is studied

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0, \end{cases} \tag{1.5}$$

where $u$ and $v$ represent the density of the bacteria and the concentration of oxygen, respectively. $D(u)$ denotes the diffusion function and $f(u)$ is the logistic source. The analysis of this model has attracted many interests and many results are presented. For instance, in the absence of the logistic source (i.e. $f(u) \equiv 0$), when $D(u) = 1$, the global bounded solutions have been shown by Tao [20] under the condition of $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)\chi}$. For arbitrarily large initial data, in three-dimensional case, the global bounded weak solutions and smoothness in $\Omega \times (T, +\infty)$ are proved with some $T > 0$ by Tao and Winkler [22]. Moreover, when $D(u)$ satisfies (1.4), Wang et al. prove that system (1.5) possesses a unique global bounded classical solution if $m > \frac{1}{2}$ in the case $n = 1$ or $m > 2 - \frac{2}{n}$ in the case $n \geq 2$ [32], the domain can be extended to $m > 2 - \frac{6}{n+4}$ in the case $n \geq 3$, but the solutions maybe unbounded in [31]. Furthermore, the global bounded solutions are proved [9,33] provided that $m > 2 - \frac{n+2}{2n}$, which improves the results in [31,32]. Recently, the diffusivity $D(u)$ exponential decay as $u \to \infty$ is studied in [16,26].

If the logistic source $f(u) = au - \mu u^\gamma$ with $\gamma > 1$ and $D(u) = \delta$ in system (1.5), the global bounded solution is studied if

$$\|v_0\|_{L^\infty(\Omega)} < \frac{1}{\chi}\sqrt{\frac{\delta}{2(n+1)}}\left[\pi - 2\arctan\frac{\delta - 1}{2}\sqrt{\frac{2(n+1)}{\delta}}\right]$$

in [2]. Similarly, Lankeit and Wang [15] prove this system has global bounded solutions if $\mu > c_1(n)\|\chi v_0\|_{L^\infty(\Omega)}^{\frac{1}{n}} + c_2(n)\|\chi v_0\|_{L^\infty(\Omega)}^{2n}$, where $c_1(n)$ and $c_2(n)$ are constants about $n$. The chemotaxis-consumption model (1.5) with nonlinear diffusion function and nontrivial source terms has also already been considered in [28,30].

To better discuss model (1.1), we need to mention the following two species chemotaxis(-Navier)–Stokes system with Lotka–Volterra competitive kinetics [25]

$$\begin{cases} u_t + V\cdot\nabla u = \Delta u - \nabla\cdot(u\chi_1(w)\nabla w) + \mu_1 u(1-u-a_1 v), & x\in\Omega, \quad t>0,\\ v_t + V\cdot\nabla v = \Delta v - \nabla\cdot(v\chi_2(w)\nabla w) + \mu_2 v(1-a_2 u - v), & x\in\Omega, \quad t>0,\\ w_t + V\cdot\nabla w = \Delta w - (\alpha u + \beta v)w, & x\in\Omega, \quad t>0,\\ V_t + \kappa(V\cdot\nabla V) = \Delta V - \nabla P + (\gamma u + \delta v)\nabla\phi, & x\in\Omega, \quad t>0,\\ \nabla\cdot V = 0, & x\in\Omega, \quad t>0, \end{cases} \tag{1.6}$$

which describes the evolution of two competing species that reacts on a chemoattractant in the environment of fulling the fluid. Here $u, v$ and $w$ are represented as model (1.1), and $V$ denotes the velocity field of the fluid belonging to an incompressible Navier–Stokes equation with pressure $P$. Moreover, $\phi$ is a potential function, and $\kappa$ is a constant concerning the strength of nonlinear fluid convection. Boundedness and asymptotic behavior of model (1.6) are researched in the case two-dimension and three-dimension [7, 11, 13]. When the fluid is stationary or the effect of fluid is absent, i.e. $V \equiv 0$, model (1.6) is ascribed to the fundamental chemotaxis model (1.1).

Motivated by the arguments in [19,29,30,37,41], in this paper, we extend their method and then obtain global boundedness of solution of model (1.1). Our main results are as follows.

**Theorem 1.1.** *Assume $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ is a bounded domain with smooth boundary, $\chi_i(w)$ $(i = 1,2)$ satisfy (1.3), and $D_1(u)$ and $D_2(v)$ satisfy (1.4). Moreover, assume that there exists $\mu_0 > 0$ such that $\min\{\mu_1, \mu_2\} > \mu_0$. Then for the initial data $(u_0, v_0, w_0)$ satisfies (1.2), system (1.1) possesses a unique classical solution $(u, v, w)$ which is uniformly bounded in the sense that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} < C \quad \text{for all } t > 0 \tag{1.7}$$

*with some constants $C > 0$.*

**Remark 1.2.** For $i = 1,2$, when $D_i(s) = d_i > 0$ is constant, if

$$0 < \|w_0\|_{L^\infty(\Omega)} \le \frac{1}{3(n+1)\|\chi_i\|_{L^\infty[0,\|w_0\|_{L^\infty(\Omega)}]}} \min\left\{\frac{2d_i}{d_i+1}, 1\right\},$$

model (1.1) has global bounded solutions in [29], but which is independent of $\mu_1$ and $\mu_2$. Theorem 1.1 gives a qualitative result, namely, if $\mu_i$ $(i = 1,2)$ are sufficiently large, model (1.1) has global bounded solutions, which improves above results in some sense.

The rest of this paper is organized as follows. In the next section, we show the local existence of a solution to model (1.1) and give some preliminary inequalities those are important for our proofs. In Section 3, we will give the complete proof of Theorem 1.1.

## 2 Preliminaries

In order to prove our result, we first give one result concerning local-in-time existence of a classical solution to system (1.1).

**Lemma 2.1.** *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary, $\mu_1, \mu_2 > 0$, $\alpha, \beta > 0$ and $a_1, a_2 > 0$. Moreover, assume that the initial data $(u_0, v_0, w_0)$ satisfies (1.2), $\chi_i(w)$ $(i = 1, 2)$ satisfy (1.3), and $D_1(u)$ and $D_2(v)$ satisfy (1.4). Then there exists $t \in (0, T_{\max})$ such that system (1.1) has a unique local-in-time non-negative triple solution*

$$u, v, w \in C(\overline{\Omega} \times (0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})). \tag{2.1}$$

*In addition, if $T_{\max} < \infty$, then*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}. \tag{2.2}$$

*Proof.* Let $U = (u, v, w) \in \mathbb{R}^n (n \geq 1)$. And (1.1) can be transformed to

$$\begin{cases} U_t = \nabla \cdot (A(U)\nabla U) + F(U), \\ \frac{\partial U}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ U(x, 0) = (u_0(x), v_0(x), w_0(x)), & x \in \Omega, \end{cases} \tag{2.3}$$

where

$$A(U) = \begin{pmatrix} D_1(u) & 0 & -\chi_1(w) \\ 0 & D_2(v) & -\chi_2(w) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad F(U) = \begin{pmatrix} \mu_1 u(1 - u - a_1 v) \\ \mu_2 v(1 - a_2 u - v) \\ -(\alpha u + \beta v)w \end{pmatrix}.$$

Since the eigenvalues of $A$ are positive, system (2.3) is normally parabolic. Applying Theorems 14.4, 14.6 and 15.5 of [1], (2.1) and (2.2) can be proved. And the initial data satisfies (1.2), the maximum principle ensures that $u$, $v$ and $w$ are non-negative in $\Omega \times (0, T_{\max})$. □

The following characteristic of the solution of the third equation in model (1.1) plays an essential role in the later proof.

**Lemma 2.2.** *Let $(u, v, w)$ be the solution of model (1.1), then we have*

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} \tag{2.4}$$

*for all $t \in (0, T_{\max})$.*

*Proof.* According to the third equation of model (1.1), and the non-negative $u, v, w$ and $\alpha, \beta > 0$, we claim result (2.4) upon an application of the maximum principle. □

Finally, we provide the result referred to as a variation of Maximal Sobolev regularity, which was proposed in Theorem 3.1 in [10] (see also Lemma 3.1 in [6], Lemma 2.2 in [37] and Lemma 2.2 in [30]).

**Lemma 2.3.** *Assume that $T \in (0, \infty)$, we mention the following homogeneous heat equations*

$$\begin{cases} y_t = \Delta y - f y, & x \in \Omega, \quad t \in (0, T), \\ \frac{\partial y}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T), \\ y(x, 0) = y_0(x), & x \in \Omega, \end{cases} \tag{2.5}$$

*where $y_0 \in W^{2,\theta}(\Omega)$ $(\theta > 1)$ is non-negative with $\frac{\partial y_0}{\partial \nu} = 0$ on $\partial\Omega$ and any functions $f \in L^\theta((0, T); L^\theta(\Omega))$ are non-negative, there exists a unique solution*

$$y \in W^{1,\theta}((0, T); L^\theta(\Omega)) \cap L^\theta((0, T); W^{2,\theta}(\Omega)),$$

*and*

$$\begin{aligned}&\int_0^T\int_\Omega y^\theta dxdt+\int_0^T\int_\Omega |y_t|^\theta dxdt+\int_0^T\int_\Omega |\Delta y|^\theta dxdt\\&\quad\le C_\theta\left(\int_0^T\int_\Omega (fy)^\theta dxdt+\int_\Omega y_0^\theta dx+\int_\Omega |\Delta y_0|^\theta dx\right),\end{aligned}\tag{2.6}$$

*with some constant $C_\theta > 0$. Moreover, for $s \in (0, T)$, $y(\cdot, s) \in W^{2,\theta}(\Omega)(\theta > 1)$ with $\frac{\partial y(\cdot,s)}{\partial \nu} = 0$ on $\partial\Omega$, then*

$$\begin{aligned}&\int_s^T\int_\Omega y^\theta dxdt+\int_s^T\int_\Omega |y_t|^\theta dxdt+\int_s^T\int_\Omega |\Delta y|^\theta dxdt\\&\quad\le C_\theta\left(\int_s^T\int_\Omega (fy)^\theta dxdt+\int_\Omega y^\theta(\cdot,s) dx+\int_\Omega |\Delta y(\cdot,s)|^\theta dx\right),\end{aligned}\tag{2.7}$$

*and*

$$\begin{aligned}\int_s^T\int_\Omega e^{\theta t}|\Delta y|^\theta dxdt\le{}& C_\theta\int_s^T\int_\Omega e^{\theta t}y^\theta|1-f|^\theta dxdt\\&+C_\theta\int_\Omega y^\theta(\cdot,s)dx+C_\theta\int_\Omega|\Delta y(\cdot,s)|^\theta dx.\end{aligned}\tag{2.8}$$

*Proof.* (2.6) and (2.7) are proved in [6]. Now we prove (2.8). Similar to Lemma 2.2 in [37], let $z(x, \tau) = e^\tau y(x, \tau)$, then we have

$$\begin{cases} z_\tau=\Delta z+e^\tau y(1-f), & \Omega\times(0,T),\\ \frac{\partial z}{\partial\nu}=0, & \partial\Omega\times(0,T),\\ z(x,0)=z_0(x), & x\in\Omega.\end{cases}$$

Using Theorem 3.1 in [10], we get

$$\int_0^T\int_\Omega|\Delta z|^\theta dxd\tau\le C_0\left(\int_0^T\int_\Omega e^{\theta\tau}y^\theta|1-f|^\theta dxd\tau+\int_\Omega y_0^\theta dx+\int_\Omega|\Delta y_0|^\theta dx\right),$$

which implies

$$\int_0^T\int_\Omega e^{\theta\tau}|\Delta y|^\theta dxd\tau\le C_0\left(\int_0^T\int_\Omega e^{\theta\tau}y^\theta|1-f|^\theta dxd\tau+\int_\Omega y_0^\theta dx+\int_\Omega|\Delta y_0|^\theta dx\right).$$

Hence, we replace $y(\tau)$ by $y(\tau + s)$. Then, the inequality (2.8) is obtained. □

## 3 Global boundedness

In this section, global boundedness of solutions is proved to model (1.1). Firstly, to prove Theorem 1.1, we make an estimate for $(u, v, w, \Delta w)$ when $s_0 \in (0, T_{\max})$ and $s_0 < 1$. According to Lemma 2.1, it shows that $u(\cdot, s_0), v(\cdot, s_0), w(\cdot, s_0) \in C^2(\overline{\Omega})$ with $\frac{\partial w(\cdot,s_0)}{\partial\nu} = 0$ on $\partial\Omega$. Subsequently, we pick $M_0 > 0$ such that

$$\begin{cases}\sup\limits_{0\le t\le s_0}\|u(\cdot,t)\|_{L^\infty(\Omega)}\le M_0, & \sup\limits_{0\le t\le s_0}\|v(\cdot,t)\|_{L^\infty(\Omega)}\le M_0,\\ \sup\limits_{0\le t\le s_0}\|w(\cdot,t)\|_{L^\infty(\Omega)}\le M_0, & \|\Delta w(\cdot,t)\|_{L^\infty(\Omega)}\le M_0.\end{cases}\tag{3.1}$$

Next, we prove boundedness in $t \in (s_0, T_{\max})$.

**Lemma 3.1.** *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary and $\chi_i(w)$ $(i = 1, 2)$ satisfy (1.3). For any $p > 1$ and $\eta > 0$, there exists $\mu_{p,\eta} > 0$ such that if $\min\{\mu_1, \mu_2\} > \mu_{p,\eta}$, then*

$$\|u(\cdot,t)\|_{L^p(\Omega)} + \|v(\cdot,t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (s_0, T_{\max}) \tag{3.2}$$

*where $C = C(p, |\Omega|, \mu_1, \mu_2, \eta, u_0, v_0, w_0) > 0$.*

*Proof.* By direct calculations, we obtain from the first and third equations in model (1.1) that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= \int_\Omega u^{p-1}[\nabla\cdot(D_1(u)\nabla u) - \nabla\cdot(u\chi_1(w)\nabla w) + \mu_1 u(1-u-a_1 v)] \\
&= -\int_\Omega (p-1)u^{p-2}D_1(u)|\nabla u|^2 + \int_\Omega (p-1)u^{p-1}\chi_1(w)\nabla u\cdot\nabla w \\
&\quad + \mu_1\int_\Omega u^p - \mu_1\int_\Omega u^{p+1} - \mu_1 a_1\int_\Omega u^p v \\
&\leq \frac{p-1}{p}\int_\Omega \nabla u^p\cdot\nabla\Phi_1(w) + \mu_1\int_\Omega u^p - \mu_1\int_\Omega u^{p+1} \\
&= -\frac{p-1}{p}\int_\Omega u^p\chi_1(w)\Delta w - \frac{p-1}{p}\int_\Omega u^p\chi_1'(w)|\nabla w|^2 + \mu_1\int_\Omega u^p - \mu_1\int_\Omega u^{p+1},
\end{aligned}$$

where $\Phi_i(w) = \int_1^w \chi_i(s)ds$ $(i = 1, 2)$, so we have $\nabla\Phi_i(w) = \chi_i(w)\nabla w$ and $\Delta\Phi_i(w) = \chi_i'(w)|\nabla w|^2 + \chi_i(w)\Delta w$. Thanks to $\chi_i'(w) \geq 0$ $(i = 1, 2)$, we arrive at

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \leq -\frac{p+1}{p}\int_\Omega u^p - \frac{p-1}{p}\int_\Omega u^p\chi_1(w)\Delta w + \left(\mu_1 + \frac{p+1}{p}\right)\int_\Omega u^p - \mu_1\int_\Omega u^{p+1} \tag{3.3}$$

for all $t \in (s_0, T_{\max})$. For any $\varepsilon > 0$, based on Young's inequality, we conclude

$$\left(\mu_1 + \frac{p+1}{p}\right)\int_\Omega u^p \leq \varepsilon\int_\Omega u^{p+1} + c_1|\Omega| \tag{3.4}$$

and

$$\begin{aligned}
-\frac{p-1}{p}\int_\Omega u^p\chi_1(w)\Delta w &\leq \int_\Omega u^p\chi_1(w)|\Delta w| \leq M_1\int_\Omega u^p|\Delta w| \\
&\leq \eta\int_\Omega u^{p+1} + c_2\eta^{-p}M_1^{p+1}\int_\Omega |\Delta w|^{p+1},
\end{aligned} \tag{3.5}$$

where $\chi_i(w) \leq M_i := \chi_i(\|w_0\|_{L^\infty(\Omega)})$ due to $\chi_i'(w) \geq 0$ $(i = 1, 2)$ and (2.4), and constants $c_1 = \frac{1}{p+1}(1+\frac{1}{p})^{-p}\varepsilon^{-p}(\mu_1 + \frac{p+1}{p})^{p+1} > 0$ and $c_2 = \sup_{p>1}\frac{1}{p+1}(1+\frac{1}{p})^{-p} < \infty$. Inserting (3.4) and (3.5) into (3.3), we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{p}\int_\Omega u^p\right) \leq &-(p+1)\left(\frac{1}{p}\int_\Omega u^p\right) - (\mu_1 - \varepsilon - \eta)\int_\Omega u^{p+1} \\
&+ c_2\eta^{-p}M_1^{p+1}\int_\Omega |\Delta w|^{p+1} + c_1|\Omega|.
\end{aligned} \tag{3.6}$$

Applying the variation-of-constants formula to the inequality (3.6), it shows that

$$\begin{aligned}
\frac{1}{p}\int_\Omega u^p(\cdot,t) \leq\ & e^{-(p+1)(t-s_0)}\frac{1}{p}\int_\Omega u^p(\cdot,s_0) - (\mu_1-\varepsilon-\eta)\int_{s_0}^t e^{-(p+1)(t-s)}\int_\Omega u^{p+1} \\
&+ c_2\eta^{-p}M_1^{p+1}\int_{s_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta w|^{p+1} + c_1|\Omega|\int_{s_0}^t e^{-(p+1)(t-s)} \\
\leq & -(\mu_1-\varepsilon-\eta)e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}u^{p+1} \\
&+ c_2\eta^{-p}M_1^{p+1}e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}|\Delta w|^{p+1} + c_3
\end{aligned} \tag{3.7}$$

for all $t \in (s_0, T_{\max})$, where $c_3 = c_1|\Omega|\frac{1}{p+1} + \frac{1}{p}\int_\Omega u^p(\cdot, s_0) > 0$. According to Lemma 2.3, there exists $C_p > 0$ such that

$$\begin{aligned}
&\int_{s_0}^t \int_\Omega e^{(p+1)s}|\Delta w|^{p+1} \\
&\quad \le C_p \int_{s_0}^t \int_\Omega e^{(p+1)s} w^{p+1}|1-(\alpha u+\beta v)|^{p+1} + C_p\int_\Omega w^{p+1}(\cdot, s_0) + C_p\int_\Omega |\Delta w(\cdot, s_0)|^{p+1} \\
&\quad \le C_p \int_{s_0}^t \int_\Omega e^{(p+1)s} w^{p+1}|(\alpha u+\beta v)+1|^{p+1} + C_p\int_\Omega w^{p+1}(\cdot, s_0) + C_p\int_\Omega |\Delta w(\cdot, s_0)|^{p+1}.
\end{aligned}$$

Thanks to the inequality $(a+b)^d \le 2^d(a^d+b^d)$ with $a, b \ge 0$ and $d \ge 1$, we have

$$\begin{aligned}
&c_2\eta^{-p}M_1^{p+1}e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}|\Delta w|^{p+1} \\
&\quad \le c_2\eta^{-p}M_1^{p+1}C_p e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}w^{p+1}2^{p+1}[1+(\alpha u+\beta v)^{p+1}] \\
&\qquad + c_2\eta^{-p}M_1^{p+1}C_p e^{-(p+1)t}\left(\int_\Omega w^{p+1}(\cdot, s_0) + \int_\Omega |\Delta w(\cdot, s_0)|^{p+1}\right) \\
&\quad \le c_2\eta^{-p}M_1^{p+1}C_p\int_{s_0}^t\int_\Omega e^{-(p+1)(t-s)}w^{p+1}[2^{p+1}+2^{2p+2}(\alpha u)^{p+1}+2^{2p+2}(\beta v)^{p+1}] \\
&\qquad + c_2\eta^{-p}M_1^{p+1}C_p e^{-(p+1)t}\|w(\cdot, s_0)\|^{p+1}_{W^{2,p+1}(\Omega)} \\
&\quad = c_2\eta^{-p}M_1^{p+1}e^{-(p+1)t}\left[c_4\int_{s_0}^t\int_\Omega e^{(p+1)s}u^{p+1} + c_5\int_{s_0}^t\int_\Omega e^{(p+1)s}v^{p+1}\right] \\
&\qquad + c_2\eta^{-p}M_1^{p+1}c(t),
\end{aligned} \tag{3.8}$$

where $c_4 = C_p 2^{2p+2}\alpha^{p+1}\|w_0\|^{p+1}_{L^\infty(\Omega)}$, $c_5 = C_p 2^{2p+2}\beta^{p+1}\|w_0\|^{p+1}_{L^\infty(\Omega)}$, and

$$\begin{aligned}
c(t) &= C_p e^{-(p+1)t}\|w(\cdot, s_0)\|^{p+1}_{W^{2,p+1}(\Omega)} + C_p 2^{p+1}\int_{s_0}^t\int_\Omega e^{-(p+1)(t-s)}w^{p+1} \\
&\le C_p\|w(\cdot, s_0)\|^{p+1}_{W^{2,p+1}(\Omega)} + C_p 2^{p+1}\|w_0\|^{p+1}_{L^\infty(\Omega)}|\Omega|\int_{s_0}^t e^{-(p+1)(t-s)}ds \\
&= C_p\|w(\cdot, s_0)\|^{p+1}_{W^{2,p+1}(\Omega)} + \frac{|\Omega|}{p+1}C_p 2^{p+1}\|w_0\|^{p+1}_{L^\infty(\Omega)} =: c_6.
\end{aligned}$$

Inserting (3.8) into (3.7), we obtain

$$\begin{aligned}
\frac{1}{p}\int_\Omega u^p(\cdot, t) \le &-(\mu_1-\varepsilon-\eta)e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}u^{p+1} \\
&+ c_2c_4\eta^{-p}M_1^{p+1}e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}u^{p+1} \\
&+ c_2c_5\eta^{-p}M_1^{p+1}e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}v^{p+1} + c_7
\end{aligned} \tag{3.9}$$

with some $c_7 > 0$. Similarly,

$$\begin{aligned}
\frac{1}{p}\int_\Omega v^p(\cdot, t) \le &-(\mu_2-\varepsilon-\eta)e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}v^{p+1} \\
&+ c_2c_5\eta^{-p}M_2^{p+1}e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}v^{p+1} \\
&+ c_2c_4\eta^{-p}M_2^{p+1}e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}u^{p+1} + c_8
\end{aligned} \tag{3.10}$$

with some $c_8 > 0$. Adding (3.9) and (3.10), we have

$$\begin{aligned}\frac{1}{p}&\left[\int_\Omega u^p(\cdot,t)+\int_\Omega v^p(\cdot,t)\right]\\ &\le -(\mu_1-\varepsilon-\eta-c_2c_4\eta^{-p}M_1^{p+1}-c_2c_4\eta^{-p}M_2^{p+1})e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}u^{p+1}\\ &\quad-(\mu_2-\varepsilon-\eta-c_2c_5\eta^{-p}M_2^{p+1}-c_2c_5\eta^{-p}M_1^{p+1})e^{-(p+1)t}\int_{s_0}^t\int_\Omega e^{(p+1)s}v^{p+1}+c_9\end{aligned}\tag{3.11}$$

with some $c_9 > 0$.

Let $\mu_{p,\eta} = \max\{\eta + c_2c_4\eta^{-p}M_1^{p+1} + c_2c_4\eta^{-p}M_2^{p+1}, \eta + c_2c_5\eta^{-p}M_2^{p+1} + c_2c_5\eta^{-p}M_1^{p+1}\}$, we can choose $\varepsilon \in (0, \min\{\mu_1,\mu_2\} - \mu_{p,\eta})$ such that

$$\mu_1-\varepsilon-\eta-c_2c_4\eta^{-p}M_1^{p+1}-c_2c_4\eta^{-p}M_2^{p+1}>0$$

and

$$\mu_2-\varepsilon-\eta-c_2c_5\eta^{-p}M_2^{p+1}-c_2c_5\eta^{-p}M_1^{p+1}>0.$$

Hence, using (3.11), we conclude

$$\frac{1}{p}\left[\int_\Omega u^p(\cdot,t)+\int_\Omega v^p(\cdot,t)\right]\le c_9$$

for all $t \in (s_0, T_{\max})$, with some constant $c_9 = c_9(\mu_1,\mu_2,\varepsilon,\eta,p,w(s_0))$. □

Now our main result can be easily obtained.

*Proof of Theorem 1.1.* Applying Moser-type iteration techniques, which can be found in Lemma A.1 in [21] (see also [10]). Firstly, we claim that there is a constant $p_0 > n$, such that if

$$\|u(\cdot,t)\|_{L^p(\Omega)}+\|v(\cdot,t)\|_{L^p(\Omega)}<\infty$$

for all $p \ge p_0$ and $t \in (s_0, T_{\max})$, then there exists $c_{10} > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}+\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le c_{10}\tag{3.12}$$

for all $t \in (s_0, T_{\max})$. Assume that $\mu_0$ satisfies

$$\inf_{\eta>0}\mu_{p_0,\eta}=\inf_{\eta>0}\left(\max\left\{\eta+c_2c_4'\eta^{-p_0}\left(M_1^{p_0+1}+M_2^{p_0+1}\right),\eta+c_2c_5'\eta^{-p_0}\left(M_2^{p_0+1}+M_1^{p_0+1}\right)\right\}\right)=\mu_0,$$

where $c_4' = C_{p_0}2^{2p_0+2}\alpha^{p_0+1}\|w_0\|^{p_0+1}_{L^\infty(\Omega)}$ and $c_5' = C_{p_0}2^{2p_0+2}\beta^{p_0+1}\|w_0\|^{p_0+1}_{L^\infty(\Omega)}$. According to $\min\{\mu_1,\mu_2\} > \mu_0$, we have $\min\{\mu_1,\mu_2\} > \mu_{p_0,\eta}$ for some $\eta > 0$. Hence, using Lemma 3.1 implies (3.12) is true for $t \in (s_0, T_{\max})$. Due to (3.1) and Lemma 2.2 we obtain that $u, v, w$ are bounded in $(0, T_{\max})$. Finally, in view of Lemma 2.1 we can complete the proof of Theorem 1.1. □

## Acknowledgements

The authors are very grateful to the anonymous reviewers for their carefully reading and valuable suggestions which greatly improved this work. The second author is supported by Chongqing Natural Science Foundation (Grant No. cstc2017jcyjXB0037); the third author is supported the NNSF of China (Grant No. 11601052) and the Basic and Advanced Research Project of Chongqing (Grant No. cstc2017jcyjAX0178).

## References


[1] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, Differential Operators and Nonlinear Analysis, Vol. 133, Vieweg+Teubner Verlag, Wiesbaden, 1993, pp. 9–126. https://doi.org/10.1007/978-3-663-11336-2_1

[2] K. Baghaei, A. Khelghati, Global existence and boundedness of classical solutions for a chemotaxis model with consumption of chemoattractant and logistic source, *Math. Methods Appl. Sci.* **40**(2016), No. 10, 3799–3807. https://doi.org/10.1002/mma.4264; Zbl 06761645

[3] A. Barbagallo, M. A. Ragusa, On Lagrange duality theory for dynamics vaccination games, *Ric. Mat.* **67**(2018), No. 2, 969–982. https://doi.org/10.1007/s11587-018-0414-8; Zbl 1403.49031

[4] N. Bellomo, M. Winkler, A degenerate chemotaxis system with flux limitation: Maximally extended solutions and absence of gradient blow-up, *Comm. Partial Differential Equations* **42**(2017), No. 10, 436–473. https://doi.org/10.1080/03605302.2016.1277237; Zbl 06717968

[5] C. Bianca, F. Pappalardo, S. Motta, M. A. Ragusa, Persistence analysis in a Kolmogorov-type model for cancer-immune system competition, *AIP Conference Proceedings* **1558**(2013), 1797–1800. https://doi.org/10.1063/1.4825874

[6] X. Cao, Boundedness in a quasilinear parabolic-parabolic Keller–Segel system with logistic source, *J. Math. Anal. Appl.* **412**(2014), No. 1, 181–188. https://doi.org/10.1016/j.jmaa.2013.10.061; Zbl 1364.35123

[7] X. Cao, S. Kurima, M. Mizukami, Global existence and asymptotic behavior of classical solutions for a 3D two-species chemotaxis-Stokes system with competitive kinetics, *Math. Methods Appl. Sci.* **41**(2018), No. 8, 3138-3154. https://doi.org/10.1002/mma.4807; Zbl 1393.35083

[8] T. Cieślak, C. Stinner, Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller–Segel system in higher dimensions, *J. Differential Equations* **252**(2012), No. 10, 5832–5851. https://doi.org/10.1016/j.jde.2012.01.045; Zbl 1252.35087

[9] L. Fan, H. Jin, Global existence and asymptotic behavior to a chemotaxis system with consumption of chemoattractant in higher dimensions, *J. Math. Phys.* **58**(2017), No. 1, 011503. https://doi.org/10.1063/1.4974245; Zbl 1355.76090

[10] M. Hieber, J. Prüss, Heat kernels and maximal $l^p - l^q$ estimate for parabolic evolution equations, *Comm. Partial Differential Equations* **22**(1997), No. 9–10, 1647–1669. https://doi.org/10.1080/03605309708821314; Zbl 0886.35030

[11] M. Hirata, S. Kurima, M. Mizukami, T. Yokota, Boundedness and stabilization in a two-dimensional two-species chemotaxis-Navier–Stokes system with competitive kinetics, *J. Differential Equations* **263**(2017), No. 1, 470–490. https://doi.org/10.1016/j.jde.2017.02.045; Zbl 1362.35049

[12] D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Differential Equations* **215**(2005), No. 1, 52–107. https://doi.org/10.1016/j.jde.2004.10.022; Zbl 1085.35065

[13] H. Jin, T. Xiang, Convergence rates of solutions for a two-species chemotaxis-Navier–Stokes system with competitive kinetics, *Discrete Contin. Dyn. Syst. Ser. B* **24**(2019), No. 4, 1919–1942. https://doi.org/10.3934/dcdsb.2018249

[14] E. F. Keller, L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.* **26**(1970), No. 3, 399–415. https://doi.org/10.1016/0022-5193(70)90092-5; Zbl 1170.92306

[15] J. Lankeit, Y. Wang, Global existence, boundedness and stabilization in a high-dimensional chemotaxis system with consumption, *Discrete Contin. Dyn. Syst.* **37**(2017), No. 12, 6099–6121. https://doi.org/10.3934/dcds.2017262; Zbl 1375.35575

[16] B. Liu, M. Dong, Global solutions in a quasilinear parabolic-parabolic chemotaxis system with decaying diffusivity and consumption of a chemoattractant, *J. Math. Anal. Appl.* **467**(2018), No. 1, 32–44. https://doi.org/10.1016/j.jmaa.2018.06.001; Zbl 1396.35030

[17] M. Mizukami, Boundedness and stabilization in a two-species chemotaxis-competition system of parabolic-parabolic-elliptic type, *Math. Methods Appl. Sci.* **41**(2018), No. 1, 234–249. https://doi.org/10.1002/mma.4607; Zbl 1387.35337

[18] M. Mizukami, T. Yokota, A unified method for boundedness in fully parabolic chemotaxis systems with signal-dependent sensitivity, *Math. Nachr.* **290**(2017), No. 16, 2648–2660. https://doi.org/10.1002/mana.201600399; Zbl 1379.35162

[19] C. Stinner, J.I. Tello, M. Winkler, Competitive exclusion in a two-species chemotaxis model, *J. Math. Biol.* **68**(2014), No. 7, 1607–1626. https://doi.org/10.1007/s00285-013-0681-7; Zbl 1319.92050

[20] Y. Tao, Boundedness in a chemotaxis model with oxygen consumption by bacteria, *J. Math. Anal. Appl.* **381**(2011), No. 2, 521–529. https://doi.org/10.1016/j.jmaa.2011.02.041; Zbl 1225.35118

[21] Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller–Segel system with subcritical sensitivity, *J. Differential Equations* **252**(2012), No. 1, 692–715. https://doi.org/10.1016/j.jde.2011.08.019; Zbl 1382.35127

[22] Y. Tao, M. Winkler, Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant, *J. Differential Equations* **252**(2012), No. 3, 2520–2543. https://doi.org/10.1016/j.jde.2011.07.010; Zbl 1268.35016

[23] J. I. Tello, M. Winkler, A chemotaxis system with logistic source, *Comm. Partial Differential Equations* **32**(2007), No. 6, 849–877. https://doi.org/10.1080/03605300701319003; Zbl 1121.37068

[24] J. I. Tello, M. Winkler, Stabilization in a two-species chemotaxis system with a logistic source, *Nonlinearity* **25**(2012), No. 5, 1413–1425. https://doi.org/10.1088/0951-7715/25/5/1413; Zbl 1260.92014

[25] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler, R.E. Goldstein, Bacterial swimming and oxygen transport near contact lines, *Proc. Natl. Acad. Sci. USA* **102**(2005), No. 7, 2277–2282. https://doi.org/10.1073/pnas.0406724102; Zbl 1277.35332

[26] L. Wang, X. Hu, P. Zheng, L. Li, Boundedness in a chemotaxis model with exponentially decaying diffusivity and consumption of chemoattractant, *Comput. Math. Appl.* **74**(2017), No. 10, 2444–2448. https://doi.org/10.1016/j.camwa.2017.07.023; Zbl 1396.92011

[27] L. Wang, Y. Li, C. Mu, Boundedness in a parabolic-parabolic quasilinear chemotaxis system with logistic source, *Discrete Contin. Dyn. Syst.* **34**(2014), No. 2, 789–802. https://doi.org/10.3934/dcds.2014.34.789; Zbl 1277.35215

[28] L. Wang, S. U.-D. Khan, S. U.-D. Khan, Boundedness in a chemotaxis system with consumption of chemoattractant and logistic source, *Electron. J. Differential Equations* **2013**, No. 209, 1–9. Zbl 1295.35255

[29] L. Wang, C. Mu, X. Hu, P. Zheng, Boundedness and asymptotic stability of solutions to a two-species chemotaxis system with consumption of chemoattractant, *J. Differential Equations* **264**(2018), No. 5, 3369–3401. https://doi.org/10.1016/j.jde.2017.11.019; Zbl 1380.35025

[30] L. Wang, C. Mu, X. Hu, P. Zheng, Boundedness in a quasilinear chemotaxis model with consumption of chemoattractant and logistic source, *Appl. Anal.* **97**(2018), No. 5, 756–774. https://doi.org/10.1080/00036811.2017.1286645; Zbl 1398.92037

[31] L. Wang, C. Mu, K. Lin, J. Zhao, Global existence to a higher-dimensional quasilinear chemotaxis system with consumption of chemoattractant, *Z. Angew. Math. Phys.* **66**(2015), No. 4, 1633–1648. https://doi.org/10.1007/s00033-014-0491-9; Zbl 1328.92019

[32] L. Wang, C. Mu, S. Zhou, Boundedness in a parabolic-parabolic chemotaxis system with nonlinear diffusion, *Z. Angew. Math. Phys.* **65**(2014), No. 6, 1137–1152. https://doi.org/10.1007/s00033-013-0375-4; Zbl 1305.92024

[33] Y. Wang, Z. Xiang, Global existence and boundedness in a higher-dimensional quasilinear chemotaxis system, *Z. Angew. Math. Phys.* **66**(2015), No. 6, 3159–3179. https://doi.org/10.1007/s00033-015-0557-3; Zbl 1330.35201

[34] M. Winkler, Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity, *Math. Nachr.* **283**(2010), No. 11, 1664–1673. https://doi.org/10.1002/mana.200810838; Zbl 1205.35037

[35] M. Winkler, Does a 'volume-filling effect' always prevent chemotactic collapse?, *Math. Methods Appl. Sci.* **33**(2010), No. 1, 12–24. https://doi.org/10.1002/mma.1146; Zbl 1182.35220

[36] M. Winkler, Stabilization in a two-dimensional chemotaxis-Navier–Stokes system, *Arch. Ration. Mech. Anal.* **211**(2014), No. 2, 455–487. https://doi.org/10.1007/s00205-013-0678-9; Zbl 1293.35220

[37] C. Yang, X. Cao, Z. Jiang, S. Zheng, Boundedness in a quasilinear fully parabolic Keller-Segel system of higher dimension with logistic source, *J. Math. Anal. Appl.* **430**(2015), No. 1, 585–591. https://doi.org/10.1016/j.jmaa.2015.04.093; Zbl 06445432

[38] W. Zhang, Z. Liu, L. Zhou, A free boundary problem for an attraction-repulsion chemotaxis system, *Bound. Value Probl.* **2018**, Paper No. 191, 38 pp. https://doi.org/10.1186/s13661-018-1105-9

[39] J. Zhao, C. Mu, L. Wang, Blow up and bounded solutions in a two-species chemotaxis system in two dimensional domains, *Acta. Appl. Math.* **153**(2018), No. 1, 197–220. https://doi.org/10.1007/s10440-017-0128-z; Zbl 1383.92020

[40] P. Zheng, X. Hu, L. Wang, Y. Tian, A note on the boundedness in a chemotaxis-growth system with nonlinear sensitivity, *Electron. J. Qual. Theory Differ. Equ.* **2018**, No. 7, 1–9. https://doi.org/10.14232/ejqtde.2018.1.7; Zbl 06931338

[41] S. Zhou, C. Yang, Boundedness in a quasilinear fully parabolic two-species chemotaxis system of higher dimension, *Bound. Value Probl.* **2017**, Paper No. 115, 8 pp. https://doi.org/10.1186/s13661-017-0846-1; Zbl 1377.92014