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A note on the uniqueness of strong solution to the incompressible Navier–Stokes equations with damping

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Abstract. We study the Cauchy problem of the 3D incompressible Navier–Stokes equations with nonlinear damping term $\alpha|\mathbf{u}|^{\beta-1}\mathbf{u}$ ($\alpha > 0$ and $\beta \geq 1$). In [*J. Math. Anal. Appl.* **377**(2011), 414–419], Zhang et al. obtained global strong solution for $\beta > 3$ and the solution is unique provided that $3 < \beta \leq 5$. In this note, we aim at deriving the uniqueness of global strong solution for any $\beta > 3$.

Keywords: incompressible Navier–Stokes equations, strong solution, uniqueness, damping.

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1 Introduction

We are concerned with the following incompressible Navier–Stokes equations with damping in $\mathbb{R}^3$:

$$\begin{cases} \mathbf{u}_t - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \alpha|\mathbf{u}|^{\beta-1}\mathbf{u} + \nabla P = \mathbf{0}, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \\ \lim\limits_{|x|\to\infty} |\mathbf{u}(t,x)| = 0, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u^1(t,x), u^2(t,x), u^3(t,x))$ is the velocity field, $P(t,x)$ is a scalar pressure. $t \geq 0$ is the time, $x \in \mathbb{R}^3$ is the spatial coordinate. In the damping term, $\alpha > 0$ and $\beta \geq 1$ are two constants. The prescribed function $\mathbf{u}_0(x)$ is the initial velocity field with $\operatorname{div}\mathbf{u}_0 = 0$, while the constant $\mu > 0$ represents the viscosity coefficient of the flow.

When there is no damping term $\alpha|\mathbf{u}|^{\beta-1}\mathbf{u}$, the system (1.1) is reduced to the classical incompressible Navier–Stokes equations, which has been attracted quite a lot of attention, refer to [2–6,8] and references therein. The model (1.1) comes from porous media flow, friction

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effects, or some dissipative mechanisms, mainly as a limiting system from compressible flows (see [1] for the physical background). The system (1.1) was studied firstly by Cai and Jiu [1], they showed the existence of a global weak solution for any $\beta \geq 1$ and global strong solutions for $\beta \geq \frac{7}{2}$. Moreover, the uniqueness is shown for any $\frac{7}{2} \leq \beta \leq 5$. In [7], Zhang et al. proved for $\beta > 3$ and $\mathbf{u}_0 \in H^1 \cap L^{\beta+1}$ that the system (1.1) has a global strong solution and the strong solution is unique when $3 < \beta \leq 5$. Later, Zhou [10] improved the results in [1,7]. He obtained that the strong solution exists globally for $\beta \geq 3$ and $\mathbf{u}_0 \in H^1$. Moreover, regularity criteria for (1.1) is also established for $1 \leq \beta < 3$ as follows: if $\mathbf{u}(t,x)$ satisfies

$$\mathbf{u} \in L^s(0,T;L^\gamma) \quad \text{with } \frac{2}{s} + \frac{3}{\gamma} \leq 1,\ 3 < \gamma < \infty, \tag{1.2}$$

or

$$\nabla\mathbf{u} \in L^{\tilde{s}}(0,T;L^{\tilde{\gamma}}) \quad \text{with } \frac{2}{\tilde{s}} + \frac{3}{\tilde{\gamma}} \leq 1,\ 3 < \tilde{\gamma} < \infty, \tag{1.3}$$

then the solution remains smooth on $[0,T]$. Recently, Zhong [9] showed the global unique strong solution for any $\beta \geq 1$ provided that the viscosity constant μ is sufficiently large or $\|\mathbf{u}_0\|_{L^2}\|\nabla\mathbf{u}_0\|_{L^2}$ is small enough.

Now we define precisely what we mean by strong solutions to the system (1.1).

Definition 1.1 (Strong solutions)**.** A pair $(\mathbf{u},P)$ is called a strong solution to (1.1) in $\mathbb{R}^3 \times (0,T)$ if (1.1) holds almost everywhere in $\mathbb{R}^3 \times (0,T)$ and

$$\mathbf{u} \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)) \cap L^\infty(0,T;L^{\beta+1}(\mathbb{R}^3)).$$

The aim of this paper is to show the uniqueness of global strong solution. Our main result reads as follows.

Theorem 1.2. *Assume that $\beta > 3$ and $\mathbf{u}_0 \in H^1(\mathbb{R}^3) \cap L^{\beta+1}(\mathbb{R}^3)$ with $\operatorname{div}\mathbf{u}_0 = 0$. Then there exists a unique global strong solution $(\mathbf{u},P)$ to the system (1.1).*

Remark 1.3. It should be noted that the uniqueness of global strong solutions was shown in [1] for $\frac{7}{2} \leq \beta \leq 5$, while the authors [7] extended the uniqueness of global strong solutions for $3 < \beta \leq 5$. Thus, our theorem improves the uniqueness results in [1,7].

2 Proof of Theorem 1.2

Throughout this section, we denote

$$\int \cdot\, dx = \int_{\mathbb{R}^3} \cdot\, dx.$$

Since the global existence of strong solutions for $\beta > 3$ has been obtained in [7, Theorem 3.1], we only need to show the uniqueness for $\beta > 3$. To this end, let $(\mathbf{u},P)$ and $(\bar{\mathbf{u}},\bar{P})$ be two strong solutions to the system (1.1) on $\mathbb{R}^3 \times (0,T)$ with the same initial data, and denote

$$\mathbf{U} \triangleq \mathbf{u} - \bar{\mathbf{u}}, \qquad \pi \triangleq P - \bar{P}.$$

Subtracting $(1.1)_1$ satisfied by $(\mathbf{u},P)$ and $(\bar{\mathbf{u}},\bar{P})$ gives

$$\mathbf{U}_t - \mu\Delta\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{u} + \bar{\mathbf{u}}\cdot\nabla\mathbf{U} + \alpha(|\mathbf{u}|^{\beta-1}\mathbf{u} - |\bar{\mathbf{u}}|^{\beta-1}\bar{\mathbf{u}}) + \nabla\pi = \mathbf{0}. \tag{2.1}$$

Multiplying (2.1) by $\mathbf{U}$ and integrating the resulting equation by parts yield that

$$\frac{1}{2}\frac{d}{dt}\int |\mathbf{U}|^2 dx + \mu \int |\nabla \mathbf{U}|^2 dx + \alpha \int (|\mathbf{u}|^{\beta-1}\mathbf{u} - |\bar{\mathbf{u}}|^{\beta-1}\bar{\mathbf{u}}) \cdot \mathbf{U} dx$$
$$= -\int \mathbf{U} \cdot \nabla \mathbf{u} \cdot \mathbf{U} dx - \int \bar{\mathbf{u}} \cdot \nabla \mathbf{U} \cdot \mathbf{U} dx \triangleq I_1 + I_2. \tag{2.2}$$

It follows from the Hölder, Gagliardo–Nirenberg, and Young inequalities that

$$\begin{aligned} |I_1| &\le \|\mathbf{U}\|_{L^4}^2 \|\nabla \mathbf{u}\|_{L^2} \\ &\le C \|\mathbf{U}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{U}\|_{L^2}^{\frac{3}{2}} \|\nabla \mathbf{u}\|_{L^2} \\ &\le \frac{\mu}{2} \|\nabla \mathbf{U}\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{L^2}^4 \|\mathbf{U}\|_{L^2}^2. \end{aligned} \tag{2.3}$$

By divergence theorem and $\operatorname{div} \bar{\mathbf{u}} = 0$, one has

$$I_2 = -\int \bar{u}^i \partial_i U^j U^j dx = \int \bar{u}^i \partial_i U^j U^j dx,$$

which gives

$$I_2 = 0. \tag{2.4}$$

Applying Hölder's inequality, we obtain that for any $\beta > 3$,

$$\begin{aligned} &\int (|\mathbf{u}|^{\beta-1}\mathbf{u} - |\bar{\mathbf{u}}|^{\beta-1}\bar{\mathbf{u}}) \cdot \mathbf{U} dx \\ &= \int (|\mathbf{u}|^{\beta-1}\mathbf{u} - |\bar{\mathbf{u}}|^{\beta-1}\bar{\mathbf{u}}) \cdot (\mathbf{u} - \bar{\mathbf{u}}) dx \\ &= \int |\mathbf{u}|^{\beta+1} dx - \int |\bar{\mathbf{u}}|^{\beta-1}\bar{\mathbf{u}} \cdot \mathbf{u} dx - \int |\mathbf{u}|^{\beta-1}\mathbf{u} \cdot \bar{\mathbf{u}} dx + \int |\bar{\mathbf{u}}|^{\beta+1} dx \\ &\ge \|\mathbf{u}\|_{L^{\beta+1}}^{\beta+1} - \|\bar{\mathbf{u}}\|_{L^{\beta+1}}^{\beta} \|\mathbf{u}\|_{L^{\beta+1}} - \|\mathbf{u}\|_{L^{\beta+1}}^{\beta} \|\bar{\mathbf{u}}\|_{L^{\beta+1}} + \|\bar{\mathbf{u}}\|_{L^{\beta+1}}^{\beta+1} \\ &= \left(\|\mathbf{u}\|_{L^{\beta+1}}^{\beta} - \|\bar{\mathbf{u}}\|_{L^{\beta+1}}^{\beta} \right) (\|\mathbf{u}\|_{L^{\beta+1}} - \|\bar{\mathbf{u}}\|_{L^{\beta+1}}) \ge 0. \end{aligned} \tag{2.5}$$

Substituting (2.3)–(2.5) into (2.2) and noting $\alpha > 0$, we get

$$\frac{d}{dt} \|\mathbf{U}\|_{L^2}^2 \le C \|\nabla \mathbf{u}\|_{L^2}^4 \|\mathbf{U}\|_{L^2}^2.$$

Thus, Gronwall's inequality leads to

$$\|\mathbf{U}\|_{L^2}^2 \le \mathbf{U}_0 \exp\left(C \int_0^T \|\nabla \mathbf{u}\|_{L^2}^4 dt \right),$$

which combined with $\mathbf{u} \in L^\infty(0, T; H^1(\mathbb{R}^3))$ (since $\mathbf{u}$ is a strong solution of (1.1)) and $\mathbf{u}_0 = \bar{\mathbf{u}}_0$ (i.e., $\mathbf{U}_0 = \mathbf{0}$) implies $\mathbf{U}(x, t) = \mathbf{0}$ for almost everywhere $(x, t) \in \mathbb{R}^3 \times (0, T)$. This finishes the proof of Theorem 1.2. □

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