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On a two-dimensional solvable system of difference equations

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Abstract. Here we solve the following system of difference equations

$$x_{n+1} = \frac{y_n y_{n-2}}{bx_{n-1} + ay_{n-2}}, \quad y_{n+1} = \frac{x_n x_{n-2}}{dy_{n-1} + cx_{n-2}}, \quad n \in \mathbb{N}_0,$$

where parameters a, b, c, d and initial values $x_{-j}, y_{-j}, j = \overline{0,2}$, are complex numbers, and give a representation of its general solution in terms of two specially chosen solutions to two homogeneous linear difference equations with constant coefficients associated to the system. As some applications of the representation formula for the general solution we obtain solutions to four very special cases of the system recently presented in the literature and proved by induction, without any theoretical explanation how they can be obtained in a constructive way. Our procedure presented here gives some theoretical explanations not only how the general solutions to the special cases are obtained, but how is obtained general solution to the general system.

Keywords: system of difference equations, general solution, representation of solutions.

2010 Mathematics Subject Classification: 39A20.

1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the sets of natural, integer, real and complex numbers, respectively, and $\mathbb{N}_l = \{n \in \mathbb{Z} : n \geq l\}$, where $l \in \mathbb{Z}$. Let $k, l \in \mathbb{Z}$, $k \leq l$, then instead of writing $k \leq j \leq l$, we will use the notation $j = \overline{k,l}$.

Finding closed-form formulas for solutions to difference equations has been studied for more than three centuries. The first results in the topic were essentially given by de Moivre

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(see, e.g., [24]) and systematized and extended later by Euler [10]. Further important results were given by Lagrange [15] and Laplace [16]. Presentations of some of these results and some results obtained later can be found, e.g., in [7,9,11,13,14,17–20,23,25,34]. Examples of some problems where closed-form formulas of solutions to the equations are applied can be found, e.g., in [5,11,13,14,17,21–23,34,35,43,44].

Having found methods for solving linear difference equations with constant coefficients experts looked for solvable nonlinear ones. One of the basic examples of such equations is the bilinear difference equation

$$z_{n+1} = \frac{\alpha z_n + \beta}{\gamma z_n + \delta}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta, z_0 \in \mathbb{R}$ (or $\in \mathbb{C}$). For some methods for solving equation (1.1) consult, e.g., [1,2,7,8,14,17,22,34]. For some results on the long-term behavior of its solutions see, e.g., [2,5,7,9].

There have been some activities in solvability theory and related topics in the last few decades (see, e.g., [6,12,28,29,32,33,36–53] and the references therein). This is caused, among other things, by use of computers and systems for symbolic computation. Although they are useful, there are some frequent problems by using them only, especially connected to getting essentially known results, and/or getting wrong formulas, which is also caused by not giving any theory behind the formulas presented in such papers (we have explained some of such cases in [40,47–49,53], see also [36] and some references therein).

Our first explanation of such a problem appeared in 2004, when we solved the following equation

$$z_n = \frac{z_{n-2}}{\alpha + \beta z_{n-2} z_{n-1}}, \quad n \in \mathbb{N},$$

by a constructive method, explaining a closed-form formula for the case $\alpha = \beta = 1$ previously presented in the literature. In [33,36,37] some extensions of the equation have been investigated later. The main point is that the previous equation is easily transformed to a solvable difference equation. After that we employed and developed successfully the method, e.g., in [6,38,39,47–49]. For some combinations of the method with other ones see, e.g., the following representative papers: [41,42,45,46,50–52].

In the last few decades Papaschinopoulos and Schinas have popularized the area of concrete systems of difference equations [26–32], which motivated us to work also in the field (see, e.g., [6,38–42,46–48,50–53] and the references therein).

There has been also some recent interest in representation of solutions to difference equations and systems in terms of specially chosen sequences, for example, in terms of Fibonacci sequences (for some basics on the sequence see, e.g., [3,14,54]). Many papers present such results, but in the majority cases the results are essentially known. For some representative papers in the area see [40] and [53], where you can find some citations which have such results.

The following four systems of difference equations

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{x_n x_{n-2}}{\pm y_{n-1} \pm x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

have been studied in recent paper [4], where some closed-form formulas for their solutions are given in terms of the initial values x_{-j}, y_{-j}, $j = \overline{0,2}$, and some subsequences of the Fibonacci sequence. The closed-form formulas are only given and proved by induction. There are no theoretical explanations for the formulas.

A natural problem is to explain what is behind all the formulas given in [4]. Since it is expected that the solvability is the main cause for this, we can try to use some of the ideas from our previous investigations, especially on rational difference equations and systems (e.g., the ones in [6,36–39,47–49]).

Here we consider the following extension of the systems in (1.2)

$$x_{n+1} = \frac{y_n y_{n-2}}{bx_{n-1} + ay_{n-2}}, \quad y_{n+1} = \frac{x_n x_{n-2}}{dy_{n-1} + cx_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where parameters a, b, c, d and initial values x_{-j}, y_{-j}, $j = \overline{0,2}$, are complex numbers.

Our aim is to show that system (1.3) is solvable by getting its closed-form formulas in an elegant constructive way, and to show that all the closed-form formulas obtained in [4] easily follow from the ones in our present paper.

2 Main results

Assume that $x_{n_0} = 0$ for some $n_0 \geq -2$. Then from the second equation in (1.3) it follows that $y_{n_0+1} = 0$, and consequently $dy_{n_0+1} + cx_{n_0} = 0$, from which it follows that y_{n_0+3} is not defined. Now, assume that $y_{n_1} = 0$ for some $n_1 \geq -2$. Then from the first equation in (1.3) it follows that $x_{n_1+1} = 0$, and consequently $bx_{n_1+1} + ay_{n_1} = 0$, from which it follows that x_{n_1+3} is not defined. This means that the set

$$\bigcup_{j=0}^{2} \{(x_{-j}, y_{-j}) \in \mathbb{C}^2 : x_{-j} = 0 \text{ or } y_{-j} = 0\},$$

is a subset of the domain of undefinable solutions to system (1.3).

Hence, from now on we will assume that

$$x_n \neq 0 \neq y_n, \quad n \geq -2. \tag{2.1}$$

Now we use some related ideas to those in [6,36–39,47–49]. Assume that $(x_n, y_n)_{n \geq -2}$ is a well-defined solution to system (1.3). Then from (1.3) we have

$$\frac{y_n}{x_{n+1}} = b\frac{x_{n-1}}{y_{n-2}} + a, \quad \frac{x_n}{y_{n+1}} = d\frac{y_{n-1}}{x_{n-2}} + c, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Let

$$u_{n+1} = \frac{y_n}{x_{n+1}}, \tag{2.3}$$

$$v_{n+1} = \frac{x_n}{y_{n+1}}, \tag{2.4}$$

for $n \geq -2$.

Then system (2.2) can be written as

$$u_{n+1} = \frac{b}{u_{n-1}} + a, \quad v_{n+1} = \frac{d}{v_{n-1}} + c, \quad n \in \mathbb{N}_0. \tag{2.5}$$

Let

$$u_m^{(j)} = u_{2m+j}, \quad v_m^{(j)} = v_{2m+j}, \tag{2.6}$$

for $m \geq -1$, $j = 1, 2$.

Then, from (2.5) we see that $(u_m^{(j)})_{m\geq -1}$, $j = 1, 2$, are two solutions to the following difference equation

$$z_m = \frac{b}{z_{m-1}} + a, \quad m \in \mathbb{N}_0, \tag{2.7}$$

whereas $(v_m^{(j)})_{m\geq -1}$, $j = 1, 2$, are two solutions to the following difference equation

$$\widehat{z}_m = \frac{d}{\widehat{z}_{m-1}} + c, \quad m \in \mathbb{N}_0. \tag{2.8}$$

Equations (2.7) and (2.8) are bilinear, so, solvable ones.

Let

$$z_m = \frac{w_{m+1}}{w_m}, \quad m \geq -1, \tag{2.9}$$

where

$$w_{-1} = 1 \quad \text{and} \quad w_0 = z_{-1}.$$

Then equation (2.7) becomes

$$w_{m+1} = aw_m + bw_{m-1}, \quad m \in \mathbb{N}_0. \tag{2.10}$$

Let $(s_m)_{m\geq -1}$ be the solution to equation (2.10) such that

$$s_{-1} = 0, \quad s_0 = 1. \tag{2.11}$$

Let λ_1 and λ_2 be the zeros of the characteristic polynomial $P_2(\lambda) = \lambda^2 - a\lambda - b$. Then general solution to equation (2.10) can be written in the following form [40]

$$w_m = bw_{-1}s_{m-1} + w_0 s_m, \quad m \geq -1, \tag{2.12}$$

(here for $m = -1$ is involved the term s_{-2}, which is calculated by using the following relation $s_{m-1} = (s_{m+1} - as_m)/b$ for $m = -1$).

From (2.9) and (2.12) it follows that

$$z_m = \frac{bw_{-1}s_m + w_0 s_{m+1}}{bw_{-1}s_{m-1} + w_0 s_m} = \frac{bs_m + z_{-1}s_{m+1}}{bs_{m-1} + z_{-1}s_m}, \quad m \geq -1. \tag{2.13}$$

Hence

$$u_m^{(j)} = \frac{bs_m + u_{-1}^{(j)} s_{m+1}}{bs_{m-1} + u_{-1}^{(j)} s_m}, \quad m \geq -1,$$

for $j = 1, 2$, that is,

$$u_{2m+j} = \frac{bs_m + u_{j-2}s_{m+1}}{bs_{m-1} + u_{j-2}s_m}, \quad m \geq -1, \tag{2.14}$$

for $j = 1, 2$.

Using (2.14) in (2.3), we obtain

$$\begin{aligned} x_{2m+1} = \frac{y_{2m}}{u_{2m+1}} &= y_{2m}\frac{bs_{m-1} + u_{-1}s_m}{bs_m + u_{-1}s_{m+1}} \\ &= y_{2m}\frac{bx_{-1}s_{m-1} + y_{-2}s_m}{bx_{-1}s_m + y_{-2}s_{m+1}}, \end{aligned} \tag{2.15}$$

and

$$x_{2m} = \frac{y_{2m-1}}{u_{2m}} = y_{2m-1}\frac{bs_{m-2} + u_0 s_{m-1}}{bs_{m-1} + u_0 s_m}$$
$$= y_{2m-1}\frac{bx_0 s_{m-2} + y_{-1}s_{m-1}}{bx_0 s_{m-1} + y_{-1}s_m}, \tag{2.16}$$

for $m \in \mathbb{N}_0$.

Let

$$\widehat{z}_m = \frac{\widehat{w}_{m+1}}{\widehat{w}_m}, \quad m \geq -1, \tag{2.17}$$

where

$$\widehat{w}_{-1} = 1 \quad \text{and} \quad \widehat{w}_0 = \widehat{z}_{-1}.$$

Then equation (2.8) becomes

$$\widehat{w}_{m+1} = c\widehat{w}_m + d\widehat{w}_{m-1}, \quad m \in \mathbb{N}_0. \tag{2.18}$$

Let $(\widehat{s}_m)_{m\geq -1}$ be the solution to equation (2.18) such that

$$\widehat{s}_{-1} = 0, \quad \widehat{s}_0 = 1. \tag{2.19}$$

Let $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ be the zeros of the characteristic polynomial $\widehat{P}_2(\lambda) = \lambda^2 - c\lambda - d$. Then general solution to equation (2.18) can be written in the following form

$$\widehat{w}_m = d\widehat{w}_{-1}\widehat{s}_{m-1} + \widehat{w}_0\widehat{s}_m, \quad m \geq -1. \tag{2.20}$$

From (2.17) and (2.20) it follows that

$$\widehat{z}_m = \frac{d\widehat{w}_{-1}\widehat{s}_m + \widehat{w}_0\widehat{s}_{m+1}}{d\widehat{w}_{-1}\widehat{s}_{m-1} + \widehat{w}_0\widehat{s}_m} = \frac{d\widehat{s}_m + \widehat{z}_{-1}\widehat{s}_{m+1}}{d\widehat{s}_{m-1} + \widehat{z}_{-1}\widehat{s}_m}, \quad m \geq -1. \tag{2.21}$$

From (2.6) and (2.21) it follows that

$$v_m^{(j)} = \frac{d\widehat{s}_m + v_{-1}^{(j)}\widehat{s}_{m+1}}{d\widehat{s}_{m-1} + v_{-1}^{(j)}\widehat{s}_m}, \quad m \geq -1,$$

for $j = 1, 2$, that is,

$$v_{2m+j} = \frac{d\widehat{s}_m + v_{j-2}\widehat{s}_{m+1}}{d\widehat{s}_{m-1} + v_{j-2}\widehat{s}_m}, \quad m \geq -1. \tag{2.22}$$

for $j = 1, 2$.

Using (2.22) in (2.4), we obtain

$$y_{2m+1} = \frac{x_{2m}}{v_{2m+1}} = x_{2m}\frac{d\widehat{s}_{m-1} + v_{-1}\widehat{s}_m}{d\widehat{s}_m + v_{-1}\widehat{s}_{m+1}}$$
$$= x_{2m}\frac{dy_{-1}\widehat{s}_{m-1} + x_{-2}\widehat{s}_m}{dy_{-1}\widehat{s}_m + x_{-2}\widehat{s}_{m+1}}, \tag{2.23}$$

and

$$y_{2m} = \frac{x_{2m-1}}{v_{2m}} = x_{2m-1}\frac{d\widehat{s}_{m-2} + v_0\widehat{s}_{m-1}}{d\widehat{s}_{m-1} + v_0\widehat{s}_m}$$
$$= x_{2m-1}\frac{dy_0\widehat{s}_{m-2} + x_{-1}\widehat{s}_{m-1}}{dy_0\widehat{s}_{m-1} + x_{-1}\widehat{s}_m}, \tag{2.24}$$

for $m \in \mathbb{N}_0$.

From (2.15), (2.16), (2.23) and (2.24), we have

$$x_{2m+1} = y_{2m}\frac{bx_{-1}s_{m-1} + y_{-2}s_m}{bx_{-1}s_m + y_{-2}s_{m+1}}$$
$$= x_{2m-1}\frac{dy_0\widehat{s}_{m-2} + x_{-1}\widehat{s}_{m-1}}{dy_0\widehat{s}_{m-1} + x_{-1}\widehat{s}_m}\frac{bx_{-1}s_{m-1} + y_{-2}s_m}{bx_{-1}s_m + y_{-2}s_{m+1}}, \tag{2.25}$$

$$x_{2m} = y_{2m-1}\frac{bx_0s_{m-2} + y_{-1}s_{m-1}}{bx_0s_{m-1} + y_{-1}s_m}$$
$$= x_{2m-2}\frac{dy_{-1}\widehat{s}_{m-2} + x_{-2}\widehat{s}_{m-1}}{dy_{-1}\widehat{s}_{m-1} + x_{-2}\widehat{s}_m}\frac{bx_0s_{m-2} + y_{-1}s_{m-1}}{bx_0s_{m-1} + y_{-1}s_m}, \tag{2.26}$$

$$y_{2m+1} = x_{2m}\frac{dy_{-1}\widehat{s}_{m-1} + x_{-2}\widehat{s}_m}{dy_{-1}\widehat{s}_m + x_{-2}\widehat{s}_{m+1}}$$
$$= y_{2m-1}\frac{dy_{-1}\widehat{s}_{m-1} + x_{-2}\widehat{s}_m}{dy_{-1}\widehat{s}_m + x_{-2}\widehat{s}_{m+1}}\frac{bx_0s_{m-2} + y_{-1}s_{m-1}}{bx_0s_{m-1} + y_{-1}s_m}, \tag{2.27}$$

$$y_{2m} = x_{2m-1}\frac{dy_0\widehat{s}_{m-2} + x_{-1}\widehat{s}_{m-1}}{dy_0\widehat{s}_{m-1} + x_{-1}\widehat{s}_m}$$
$$= y_{2m-2}\frac{dy_0\widehat{s}_{m-2} + x_{-1}\widehat{s}_{m-1}}{dy_0\widehat{s}_{m-1} + x_{-1}\widehat{s}_m}\frac{bx_{-1}s_{m-2} + y_{-2}s_{m-1}}{bx_{-1}s_{m-1} + y_{-2}s_m}, \tag{2.28}$$

for $m \in \mathbb{N}_0$.

Multiplying the equalities which are obtained from (2.25), (2.26), (2.27) and (2.28) from 1 to m, respectively, it follows that

$$x_{2m+1} = x_1\prod_{j=1}^{m}\frac{dy_0\widehat{s}_{j-2} + x_{-1}\widehat{s}_{j-1}}{dy_0\widehat{s}_{j-1} + x_{-1}\widehat{s}_j}\frac{bx_{-1}s_{j-1} + y_{-2}s_j}{bx_{-1}s_j + y_{-2}s_{j+1}}, \tag{2.29}$$

$$x_{2m} = x_0\prod_{j=1}^{m}\frac{dy_{-1}\widehat{s}_{j-2} + x_{-2}\widehat{s}_{j-1}}{dy_{-1}\widehat{s}_{j-1} + x_{-2}\widehat{s}_j}\frac{bx_0s_{j-2} + y_{-1}s_{j-1}}{bx_0s_{j-1} + y_{-1}s_j}, \tag{2.30}$$

$$y_{2m+1} = y_1\prod_{j=1}^{m}\frac{dy_{-1}\widehat{s}_{j-1} + x_{-2}\widehat{s}_j}{dy_{-1}\widehat{s}_j + x_{-2}\widehat{s}_{j+1}}\frac{bx_0s_{j-2} + y_{-1}s_{j-1}}{bx_0s_{j-1} + y_{-1}s_j}, \tag{2.31}$$

$$y_{2m} = y_0\prod_{j=1}^{m}\frac{dy_0\widehat{s}_{j-2} + x_{-1}\widehat{s}_{j-1}}{dy_0\widehat{s}_{j-1} + x_{-1}\widehat{s}_j}\frac{bx_{-1}s_{j-2} + y_{-2}s_{j-1}}{bx_{-1}s_{j-1} + y_{-2}s_j}, \tag{2.32}$$

for $m \in \mathbb{N}_0$.

From (2.29), since

$$x_1 = \frac{y_0y_{-2}}{bx_{-1} + ay_{-2}},$$

$$s_1 = as_0 + bs_{-1} = a, \tag{2.33}$$

and after some calculations we have

$$\begin{aligned} x_{2m+1} &= \frac{y_0 y_{-2}}{bx_{-1}+ay_{-2}} \frac{dy_0\widehat{s}_{-1}+x_{-1}\widehat{s}_0}{dy_0\widehat{s}_{m-1}+x_{-1}\widehat{s}_m} \frac{bx_{-1}s_0+y_{-2}s_1}{bx_{-1}s_m+y_{-2}s_{m+1}} \\ &= \frac{x_{-1}y_{-2}y_0}{(dy_0\widehat{s}_{m-1}+x_{-1}\widehat{s}_m)(bx_{-1}s_m+y_{-2}s_{m+1})}. \end{aligned}$$

From (2.30), (2.33) and after some calculations we have

$$\begin{aligned} x_{2m} &= x_0 \frac{dy_{-1}\widehat{s}_{-1}+x_{-2}\widehat{s}_0}{dy_{-1}\widehat{s}_{m-1}+x_{-2}\widehat{s}_m} \frac{bx_0 s_{-1}+y_{-1}s_0}{bx_0 s_{m-1}+y_{-1}s_m} \\ &= \frac{y_{-1}x_{-2}x_0}{(dy_{-1}\widehat{s}_{m-1}+x_{-2}\widehat{s}_m)(bx_0 s_{m-1}+y_{-1}s_m)}. \end{aligned}$$

From (2.31), since

$$y_1 = \frac{x_0 x_{-2}}{dy_{-1}+cx_{-2}},$$

$$\widehat{s}_1 = c\widehat{s}_0 + d\widehat{s}_{-1} = c, \tag{2.34}$$

and after some calculations we have

$$\begin{aligned} y_{2m+1} &= \frac{x_{-2}x_0}{dy_{-1}+cx_{-2}} \frac{dy_{-1}\widehat{s}_0+x_{-2}\widehat{s}_1}{dy_{-1}\widehat{s}_m+x_{-2}\widehat{s}_{m+1}} \frac{bx_0 s_{-1}+y_{-1}s_0}{bx_0 s_{m-1}+y_{-1}s_m} \\ &= \frac{y_{-1}x_{-2}x_0}{(dy_{-1}\widehat{s}_m+x_{-2}\widehat{s}_{m+1})(bx_0 s_{m-1}+y_{-1}s_m)}. \end{aligned}$$

From (2.32), (2.34) and after some calculations we have

$$\begin{aligned} y_{2m} &= y_0 \frac{dy_0\widehat{s}_{-1}+x_{-1}\widehat{s}_0}{dy_0\widehat{s}_{m-1}+x_{-1}\widehat{s}_m} \frac{bx_{-1}s_{-1}+y_{-2}s_0}{bx_{-1}s_{m-1}+y_{-2}s_m} \\ &= \frac{x_{-1}y_{-2}y_0}{(dy_0\widehat{s}_{m-1}+x_{-1}\widehat{s}_m)(bx_{-1}s_{m-1}+y_{-2}s_m)}. \end{aligned}$$

From the above consideration we see that the following result holds.

Theorem 2.1. *Consider system (1.3). Let s_n be the solution to equation (2.10) satisfying initial conditions (2.11), and $\widehat{s}_n$ be the solution to equation (2.18) satisfying initial conditions (2.19). Then, for every well-defined solution $(x_n, y_n)_{n\geq -2}$ to the system the following representation formulas hold*

$$x_{2n-1} = \frac{x_{-1}y_{-2}y_0}{(dy_0\widehat{s}_{n-2}+x_{-1}\widehat{s}_{n-1})(bx_{-1}s_{n-1}+y_{-2}s_n)}, \tag{2.35}$$

$$x_{2n} = \frac{y_{-1}x_{-2}x_0}{(dy_{-1}\widehat{s}_{n-1}+x_{-2}\widehat{s}_n)(bx_0 s_{n-1}+y_{-1}s_n)}, \tag{2.36}$$

$$y_{2n-1} = \frac{y_{-1}x_{-2}x_0}{(dy_{-1}\widehat{s}_{n-1}+x_{-2}\widehat{s}_n)(bx_0 s_{n-2}+y_{-1}s_{n-1})}, \tag{2.37}$$

$$y_{2n} = \frac{x_{-1}y_{-2}y_0}{(dy_0\widehat{s}_{n-1}+x_{-1}\widehat{s}_n)(bx_{-1}s_{n-1}+y_{-2}s_n)}, \tag{2.38}$$

for $n \in \mathbb{N}_0$.

3 Some applications

As some applications we show how are obtained closed-form formulas for solutions to the systems in (1.2), which were presented in [4].

First result proved in [4] is the following.

Corollary 3.1. *Let* $(x_n, y_n)_{n\geq -2}$ *be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{x_n x_{n-2}}{y_{n-1} + x_{n-2}}, \quad n \in \mathbb{N}_0. \tag{3.1}$$

Then

$$x_{2n-1} = \frac{x_{-1}y_{-2}y_0}{(y_0 f_{n-2} + x_{-1} f_{n-1})(x_{-1} f_{n-1} + y_{-2} f_n)}, \tag{3.2}$$

$$x_{2n} = \frac{x_0 x_{-2} y_{-1}}{(y_{-1} f_{n-1} + x_{-2} f_n)(x_0 f_{n-1} + y_{-1} f_n)}, \tag{3.3}$$

$$y_{2n-1} = \frac{x_0 x_{-2} y_{-1}}{(y_{-1} f_{n-1} + x_{-2} f_n)(x_0 f_{n-2} + y_{-1} f_{n-1})}, \tag{3.4}$$

$$y_{2n} = \frac{x_{-1}y_{-2}y_0}{(y_0 f_{n-1} + x_{-1} f_n)(x_{-1} f_{n-1} + y_{-2} f_n)}, \tag{3.5}$$

for $n \in \mathbb{N}_0$, *where* $(f_n)_{n\geq -1}$ *is the solution to the following difference equation*

$$f_{n+1} = f_n + f_{n-1}, \quad n \in \mathbb{N}_0, \tag{3.6}$$

satisfying the initial conditions $f_{-1} = 0$ *and* $f_0 = 1$.

Proof. System (3.1) is obtained from system (1.3) with $a = b = c = d = 1$. For these values of parameters a, b, c, d equations (2.10) and (2.18) are the same. Namely, they both are

$$w_{n+1} = w_n + w_{n-1}, \quad n \in \mathbb{N}_0. \tag{3.7}$$

Hence the sequences $(s_n)_{n\geq -1}$ and $(\widehat{s}_n)_{n\geq -1}$ satisfying conditions (2.11) and (2.19) respectively, are the same and we have

$$s_n = \widehat{s}_n = f_n, \quad n \geq -1. \tag{3.8}$$

By using (3.8) in formulas (2.35)–(2.38), formulas (3.2)–(3.5) follow. □

The following corollary is Theorem 3 in [4].

Corollary 3.2. *Let* $(x_n, y_n)_{n\geq -2}$ *be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{x_n x_{n-2}}{y_{n-1} - x_{n-2}}, \quad n \in \mathbb{N}_0. \tag{3.9}$$

Then

$$x_{2n-1} = \frac{(-1)^n x_{-1}y_{-2}y_0}{(y_0 f_{n-2} - x_{-1} f_{n-1})(x_{-1} f_{n-1} + y_{-2} f_n)}, \tag{3.10}$$

$$x_{2n} = \frac{(-1)^{n+1} x_0 x_{-2} y_{-1}}{(y_{-1} f_{n-1} - x_{-2} f_n)(x_0 f_{n-1} + y_{-1} f_n)}, \tag{3.11}$$

$$y_{2n-1} = \frac{(-1)^{n+1} x_0 x_{-2} y_{-1}}{(y_{-1} f_{n-1} - x_{-2} f_n)(x_0 f_{n-2} + y_{-1} f_{n-1})}, \tag{3.12}$$

$$y_{2n} = \frac{(-1)^{n+1} x_{-1}y_{-2}y_0}{(y_0 f_{n-1} - x_{-1} f_n)(x_{-1} f_{n-1} + y_{-2} f_n)}, \tag{3.13}$$

for $n \in \mathbb{N}_0$.

Proof. System (3.9) is obtained from system (1.3) with $a = b = -c = d = 1$. For these values of parameters a, b, c, d equation (2.10) becomes (3.7), whereas equation (2.18) becomes

$$\widehat{w}_{n+1} = -\widehat{w}_n + \widehat{w}_{n-1}, \tag{3.14}$$

for $n \in \mathbb{N}_0$.

From (2.11) and (3.7) we have

$$s_n = f_n, \quad n \geq -1. \tag{3.15}$$

Let

$$\widehat{w}_n = (-1)^n \widetilde{w}_n, \quad n \geq -1. \tag{3.16}$$

Employing (3.16) in (3.14) we obtain

$$\widetilde{w}_{n+1} = \widetilde{w}_n + \widetilde{w}_{n-1}, \quad n \in \mathbb{N}_0. \tag{3.17}$$

From (3.16) we have

$$\widetilde{s}_{-1} = 0 \quad \text{and} \quad \widetilde{s}_0 = 1. \tag{3.18}$$

From this and since $\widetilde{s}_n$ is a solution to equation (3.17) we have

$$\widetilde{s}_n = f_n, \quad n \geq -1, \tag{3.19}$$

from which along with (3.16) it follows that

$$\widehat{s}_n = (-1)^n f_n, \tag{3.20}$$

for $n \geq -1$.

By using (3.15) and (3.20) in formulas (2.35)–(2.38), after some simple calculations are obtained formulas (3.10)–(3.13). □

The following corollary is Theorem 4 in [4].

Corollary 3.3. *Let $(x_n, y_n)_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{x_n x_{n-2}}{-y_{n-1} + x_{n-2}}, \quad n \in \mathbb{N}_0. \tag{3.21}$$

Then

$$x_{6n-2} = \frac{(-1)^n x_{-2} x_0}{x_0 f_{3n-2} + y_{-1} f_{3n-1}}, \tag{3.22}$$

$$x_{6n-1} = \frac{(-1)^n x_{-1} y_{-2}}{x_{-1} f_{3n-1} + y_{-2} f_{3n}}, \tag{3.23}$$

$$x_{6n} = \frac{(-1)^n x_0 y_{-1}}{x_0 f_{3n-1} + y_{-1} f_{3n}}, \tag{3.24}$$

$$x_{6n+1} = \frac{(-1)^n y_0 y_{-2}}{x_{-1} f_{3n} + y_{-2} f_{3n+1}}, \tag{3.25}$$

$$x_{6n+2} = \frac{(-1)^n x_0 x_{-2} y_{-1}}{(x_{-2} - y_{-1})(x_0 f_{3n} + y_{-1} f_{3n+1})}, \tag{3.26}$$

$$x_{6n+3} = \frac{(-1)^n x_{-1} y_0 y_{-2}}{(x_{-1} - y_0)(x_{-1} f_{3n+1} + y_{-2} f_{3n+2})}, \tag{3.27}$$

$$y_{6n-2} = \frac{(-1)^n x_{-1} y_{-2}}{x_{-1} f_{3n-2} + y_{-2} f_{3n-1}}, \tag{3.28}$$

$$y_{6n-1} = \frac{(-1)^n x_0 y_{-1}}{x_0 f_{3n-2} + y_{-1} f_{3n-1}}, \tag{3.29}$$

$$y_{6n} = \frac{(-1)^n y_0 y_{-2}}{x_{-1} f_{3n-1} + y_{-2} f_{3n}}, \tag{3.30}$$

$$y_{6n+1} = \frac{(-1)^n x_0 x_{-2} y_{-1}}{(x_{-2} - y_{-1})(x_0 f_{3n-1} + y_{-1} f_{3n})}, \tag{3.31}$$

$$y_{6n+2} = \frac{(-1)^n x_{-1} y_0 y_{-2}}{(x_{-1} - y_0)(x_{-1} f_{3n} + y_{-2} f_{3n+1})}, \tag{3.32}$$

$$y_{6n+3} = \frac{(-1)^{n+1} x_0 x_{-2}}{x_0 f_{3n} + y_{-1} f_{3n+1}}, \tag{3.33}$$

for $n \in \mathbb{N}_0$.

Proof. System (3.21) is obtained from system (1.3) with $a = b = c = -d = 1$. For these values of parameters a, b, c, d equation (2.10) becomes equation (3.7), whereas equation (2.18) becomes

$$\widehat{w}_{n+1} = \widehat{w}_n - \widehat{w}_{n-1}, \quad n \in \mathbb{N}_0. \tag{3.34}$$

From (2.11) and (3.7) we have that (3.15) holds.

The solution $\widehat{s}_n$ to equation (3.34) satisfying the initial conditions in (2.19) is equal to

$$\widehat{s}_n = \frac{\widehat{\lambda}_1^{n+1} - \widehat{\lambda}_2^{n+1}}{\lambda_1 - \lambda_2}, \quad n \geq -1,$$

where

$$\lambda_{1,2} = \cos\frac{\pi}{3} \pm i \sin\frac{\pi}{3},$$

from which by some calculation it follows that

$$\widehat{s}_n = \frac{2}{\sqrt{3}} \sin\frac{(n+1)\pi}{3}, \quad n \geq -1. \tag{3.35}$$

Formula (3.35) shows that the sequence $\widehat{s}_n$ is six periodic. Namely, we have

$$\widehat{s}_{6m-1} = \widehat{s}_{6m+2} = 0, \tag{3.36}$$
$$\widehat{s}_{6m} = s_{6m+1} = 1, \tag{3.37}$$
$$\widehat{s}_{6m+3} = s_{6m+4} = -1, \tag{3.38}$$

for $m \geq -1$ (in fact, (3.36)–(3.38) hold for every $m \in \mathbb{Z}$).

Equalities (3.36)–(3.38) can be written as follows

$$\widehat{s}_{3m-1} = 0, \tag{3.39}$$
$$\widehat{s}_{3m} = (-1)^m, \tag{3.40}$$
$$\widehat{s}_{3m+1} = (-1)^m, \tag{3.41}$$

for $m \geq -1$.

Using equalities (3.39)–(3.41) in formulas (2.35)–(2.38), after some calculations we have

$$\begin{aligned}
x_{6n-2} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n-2} + x_{-2}\widehat{s}_{3n-1})(x_0 s_{3n-2} + y_{-1}s_{3n-1})} \\
&= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n-2})(x_0 f_{3n-2} + y_{-1}f_{3n-1})} \\
&= \frac{(-1)^n x_{-2}x_0}{x_0 f_{3n-2} + y_{-1}f_{3n-1}}, \\
x_{6n-1} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-2} + x_{-1}\widehat{s}_{3n-1})(x_{-1}s_{3n-1} + y_{-2}s_{3n})} \\
&= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-2})(x_{-1}f_{3n-1} + y_{-2}f_{3n})} \\
&= \frac{(-1)^n x_{-1}y_{-2}}{x_{-1}f_{3n-1} + y_{-2}f_{3n}}, \\
x_{6n} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n-1} + x_{-2}\widehat{s}_{3n})(x_0 s_{3n-1} + y_{-1}s_{3n})} \\
&= \frac{y_{-1}x_{-2}x_0}{(x_{-2}\widehat{s}_{3n})(x_0 f_{3n-1} + y_{-1}f_{3n})} \\
&= \frac{(-1)^n x_0 y_{-1}}{x_0 f_{3n-1} + y_{-1}f_{3n}}, \\
x_{6n+1} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-1} + x_{-1}\widehat{s}_{3n})(x_{-1}s_{3n} + y_{-2}s_{3n+1})} \\
&= \frac{x_{-1}y_{-2}y_0}{(x_{-1}\widehat{s}_{3n})(x_{-1}f_{3n} + y_{-2}f_{3n+1})} \\
&= \frac{(-1)^n y_0 y_{-2}}{x_{-1}f_{3n} + y_{-2}f_{3n+1}}, \\
x_{6n+2} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n} + x_{-2}\widehat{s}_{3n+1})(x_0 s_{3n} + y_{-1}s_{3n+1})} \\
&= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}(-1)^n + x_{-2}(-1)^n)(x_0 f_{3n} + y_{-1}f_{3n+1})} \\
&= \frac{(-1)^n x_0 x_{-2} y_{-1}}{(x_{-2} - y_{-1})(x_0 f_{3n} + y_{-1}f_{3n+1})},
\end{aligned}$$

$$\begin{aligned}
x_{6n+3} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n} + x_{-1}\widehat{s}_{3n+1})(x_{-1}s_{3n+1} + y_{-2}s_{3n+2})}\\
&= \frac{x_{-1}y_{-2}y_0}{(-y_0(-1)^n + x_{-1}(-1)^n)(x_{-1}f_{3n+1} + y_{-2}f_{3n+2})}\\
&= \frac{(-1)^n x_{-1}y_0y_{-2}}{(x_{-1} - y_0)(x_{-1}f_{3n+1} + y_{-2}f_{3n+2})},\\
y_{6n-2} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-2} + x_{-1}\widehat{s}_{3n-1})(x_{-1}s_{3n-2} + y_{-2}s_{3n-1})}\\
&= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-2})(x_{-1}f_{3n-2} + y_{-2}f_{3n-1})}\\
&= \frac{(-1)^n x_{-1}y_{-2}}{x_{-1}f_{3n-2} + y_{-2}f_{3n-1}},\\
y_{6n-1} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n-1} + x_{-2}\widehat{s}_{3n})(x_0s_{3n-2} + y_{-1}s_{3n-1})}\\
&= \frac{y_{-1}x_{-2}x_0}{(x_{-2}\widehat{s}_{3n})(x_0f_{3n-2} + y_{-1}f_{3n-1})}\\
&= \frac{(-1)^n x_0y_{-1}}{x_0f_{3n-2} + y_{-1}f_{3n-1}},\\
y_{6n} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-1} + x_{-1}\widehat{s}_{3n})(x_{-1}s_{3n-1} + y_{-2}s_{3n})}\\
&= \frac{x_{-1}y_{-2}y_0}{(x_{-1}\widehat{s}_{3n})(x_{-1}f_{3n-1} + y_{-2}f_{3n})}\\
&= \frac{(-1)^n y_0y_{-2}}{x_{-1}f_{3n-1} + y_{-2}f_{3n}},\\
y_{6n+1} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n} + x_{-2}\widehat{s}_{3n+1})(x_0s_{3n-1} + y_{-1}s_{3n})}\\
&= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}(-1)^n + x_{-2}(-1)^n)(x_0f_{3n-1} + y_{-1}f_{3n})}\\
&= \frac{(-1)^n x_0x_{-2}y_{-1}}{(x_{-2} - y_{-1})(x_0f_{3n-1} + y_{-1}f_{3n})},\\
y_{6n+2} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n} + x_{-1}\widehat{s}_{3n+1})(x_{-1}s_{3n} + y_{-2}s_{3n+1})}\\
&= \frac{x_{-1}y_{-2}y_0}{(-y_0(-1)^n + x_{-1}(-1)^n)(x_{-1}f_{3n} + y_{-2}f_{3n+1})}\\
&= \frac{(-1)^n x_{-1}y_0y_{-2}}{(x_{-1} - y_0)(x_{-1}f_{3n} + y_{-2}f_{3n+1})},\\
y_{6n+3} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n+1} + x_{-2}\widehat{s}_{3n+2})(x_0s_{3n} + y_{-1}s_{3n+1})}\\
&= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n+1})(x_0f_{3n} + y_{-1}f_{3n+1})}\\
&= \frac{(-1)^{n+1} x_0x_{-2}}{x_0f_{3n} + y_{-1}f_{3n+1}},
\end{aligned}$$

for $n \in \mathbb{N}_0$, as claimed. □

The following corollary is Theorem 5 in [4].

Corollary 3.4. *Let $(x_n, y_n)_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_ny_{n-2}}{x_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{x_nx_{n-2}}{-y_{n-1} - x_{n-2}}, \quad n \in \mathbb{N}_0. \tag{3.42}$$

Then

$$x_{6n-2} = \frac{x_{-2}x_0}{x_0 f_{3n-2} + y_{-1} f_{3n-1}}, \tag{3.43}$$

$$x_{6n-1} = \frac{x_{-1}y_{-2}}{x_{-1} f_{3n-1} + y_{-2} f_{3n}}, \tag{3.44}$$

$$x_{6n} = \frac{x_0 y_{-1}}{x_0 f_{3n-1} + y_{-1} f_{3n}}, \tag{3.45}$$

$$x_{6n+1} = \frac{y_0 y_{-2}}{x_{-1} f_{3n} + y_{-2} f_{3n+1}}, \tag{3.46}$$

$$x_{6n+2} = \frac{-x_0 x_{-2} y_{-1}}{(x_{-2} + y_{-1})(x_0 f_{3n} + y_{-1} f_{3n+1})}, \tag{3.47}$$

$$x_{6n+3} = \frac{-x_{-1} y_0 y_{-2}}{(x_{-1} + y_0)(x_{-1} f_{3n+1} + y_{-2} f_{3n+2})}, \tag{3.48}$$

$$y_{6n-2} = \frac{x_{-1}y_{-2}}{x_{-1} f_{3n-2} + y_{-2} f_{3n-1}}, \tag{3.49}$$

$$y_{6n-1} = \frac{x_0 y_{-1}}{x_0 f_{3n-2} + y_{-1} f_{3n-1}}, \tag{3.50}$$

$$y_{6n} = \frac{y_0 y_{-2}}{x_{-1} f_{3n-1} + y_{-2} f_{3n}}, \tag{3.51}$$

$$y_{6n+1} = \frac{-x_0 x_{-2} y_{-1}}{(x_{-2} + y_{-1})(x_0 f_{3n-1} + y_{-1} f_{3n})}, \tag{3.52}$$

$$y_{6n+2} = \frac{-x_{-1} y_0 y_{-2}}{(x_{-1} + y_0)(x_{-1} f_{3n} + y_{-2} f_{3n+1})}, \tag{3.53}$$

$$y_{6n+3} = \frac{x_0 x_{-2}}{x_0 f_{3n} + y_{-1} f_{3n+1}}, \tag{3.54}$$

for $n \in \mathbb{N}_0$.

Proof. System (3.42) is obtained from system (1.3) with $a = b = -c = -d = 1$. For these values of parameters a, b, c, d equation (2.10) becomes equation (3.7), whereas equation (2.18) becomes

$$\widehat{w}_{n+1} = -\widehat{w}_n - \widehat{w}_{n-1}, \quad n \in \mathbb{N}_0. \tag{3.55}$$

From (2.11) and (3.7) we have that (3.15) holds.

The solution $\widehat{s}_n$ to equation (3.55) satisfying initial conditions (2.19) is equal to

$$\widehat{s}_n = \frac{\widehat{\lambda}_1^{n+1} - \widehat{\lambda}_2^{n+1}}{\lambda_1 - \lambda_2}, \quad n \geq -1,$$

where

$$\lambda_{1,2} = \cos\frac{2\pi}{3} \pm i \sin\frac{2\pi}{3},$$

from which by some calculation it follows that

$$\widehat{s}_n = \frac{2}{\sqrt{3}} \sin\frac{2(n+1)\pi}{3}, \quad n \geq -1. \tag{3.56}$$

Formula (3.56) shows that the sequence $\widehat{s}_n$ is three periodic. Namely, we have

$$\widehat{s}_{3m} = 1, \tag{3.57}$$

$$\widehat{s}_{3m+1} = -1, \tag{3.58}$$

$$\widehat{s}_{3m+2} = 0, \tag{3.59}$$

for $m \geq -1$ (in fact, (3.57)–(3.59) hold for every $m \in \mathbb{Z}$).

Using equalities (3.57)–(3.59) in formulas (2.35)–(2.38), after some calculations we have

$$\begin{aligned}
x_{6n-2} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n-2} + x_{-2}\widehat{s}_{3n-1})(x_0 s_{3n-2} + y_{-1}s_{3n-1})} \\
&= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n-2})(x_0 f_{3n-2} + y_{-1}f_{3n-1})} \\
&= \frac{x_{-2}x_0}{x_0 f_{3n-2} + y_{-1}f_{3n-1}}, \\
x_{6n-1} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-2} + x_{-1}\widehat{s}_{3n-1})(x_{-1}s_{3n-1} + y_{-2}s_{3n})} \\
&= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-2})(x_{-1}f_{3n-1} + y_{-2}f_{3n})} \\
&= \frac{x_{-1}y_{-2}}{x_{-1}f_{3n-1} + y_{-2}f_{3n}}, \\
x_{6n} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n-1} + x_{-2}\widehat{s}_{3n})(x_0 s_{3n-1} + y_{-1}s_{3n})} \\
&= \frac{y_{-1}x_{-2}x_0}{(x_{-2}\widehat{s}_{3n})(x_0 f_{3n-1} + y_{-1}f_{3n})} \\
&= \frac{x_0 y_{-1}}{x_0 f_{3n-1} + y_{-1}f_{3n}}, \\
x_{6n+1} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-1} + x_{-1}\widehat{s}_{3n})(x_{-1}s_{3n} + y_{-2}s_{3n+1})} \\
&= \frac{x_{-1}y_{-2}y_0}{(x_{-1}\widehat{s}_{3n})(x_{-1}f_{3n} + y_{-2}f_{3n+1})} \\
&= \frac{y_0 y_{-2}}{x_{-1}f_{3n} + y_{-2}f_{3n+1}}, \\
x_{6n+2} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n} + x_{-2}\widehat{s}_{3n+1})(x_0 s_{3n} + y_{-1}s_{3n+1})} \\
&= \frac{y_{-1}x_{-2}x_0}{(-y_{-1} + x_{-2}(-1))(x_0 f_{3n} + y_{-1}f_{3n+1})} \\
&= \frac{-x_0 x_{-2}y_{-1}}{(x_{-2} + y_{-1})(x_0 f_{3n} + y_{-1}f_{3n+1})}, \\
x_{6n+3} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n} + x_{-1}\widehat{s}_{3n+1})(x_{-1}s_{3n+1} + y_{-2}s_{3n+2})} \\
&= \frac{x_{-1}y_{-2}y_0}{(-y_0 + x_{-1}(-1))(x_{-1}f_{3n+1} + y_{-2}f_{3n+2})} \\
&= \frac{-x_{-1}y_0 y_{-2}}{(x_{-1} + y_0)(x_{-1}f_{3n+1} + y_{-2}f_{3n+2})}, \\
y_{6n-2} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-2} + x_{-1}\widehat{s}_{3n-1})(x_{-1}s_{3n-2} + y_{-2}s_{3n-1})} \\
&= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-2})(x_{-1}f_{3n-2} + y_{-2}f_{3n-1})} \\
&= \frac{x_{-1}y_{-2}}{x_{-1}f_{3n-2} + y_{-2}f_{3n-1}},
\end{aligned}$$

$$\begin{aligned}
y_{6n-1} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n-1}+x_{-2}\widehat{s}_{3n})(x_0s_{3n-2}+y_{-1}s_{3n-1})}\\
&= \frac{y_{-1}x_{-2}x_0}{(x_{-2}\widehat{s}_{3n})(x_0f_{3n-2}+y_{-1}f_{3n-1})}\\
&= \frac{x_0y_{-1}}{x_0f_{3n-2}+y_{-1}f_{3n-1}},\\
y_{6n} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n-1}+x_{-1}\widehat{s}_{3n})(x_{-1}s_{3n-1}+y_{-2}s_{3n})}\\
&= \frac{x_{-1}y_{-2}y_0}{(x_{-1}\widehat{s}_{3n})(x_{-1}f_{3n-1}+y_{-2}f_{3n})}\\
&= \frac{y_0y_{-2}}{x_{-1}f_{3n-1}+y_{-2}f_{3n}},\\
y_{6n+1} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n}+x_{-2}\widehat{s}_{3n+1})(x_0s_{3n-1}+y_{-1}s_{3n})}\\
&= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}+x_{-2}(-1))(x_0f_{3n-1}+y_{-1}f_{3n})}\\
&= \frac{-x_0x_{-2}y_{-1}}{(x_{-2}+y_{-1})(x_0f_{3n-1}+y_{-1}f_{3n})},\\
y_{6n+2} &= \frac{x_{-1}y_{-2}y_0}{(-y_0\widehat{s}_{3n}+x_{-1}\widehat{s}_{3n+1})(x_{-1}s_{3n}+y_{-2}s_{3n+1})}\\
&= \frac{x_{-1}y_{-2}y_0}{(-y_0+x_{-1}(-1))(x_{-1}f_{3n}+y_{-2}f_{3n+1})}\\
&= \frac{-x_{-1}y_0y_{-2}}{(x_{-1}+y_0)(x_{-1}f_{3n}+y_{-2}f_{3n+1})},\\
y_{6n+3} &= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n+1}+x_{-2}\widehat{s}_{3n+2})(x_0s_{3n}+y_{-1}s_{3n+1})}\\
&= \frac{y_{-1}x_{-2}x_0}{(-y_{-1}\widehat{s}_{3n+1})(x_0f_{3n}+y_{-1}f_{3n+1})}\\
&= \frac{x_0x_{-2}}{x_0f_{3n}+y_{-1}f_{3n+1}},
\end{aligned}$$

for $n \in \mathbb{N}_0$, as claimed. □

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