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On the first eigenvalue for a $(p(x), q(x))$-Laplacian elliptic system

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Abstract. In this article, we deal with the first eigenvalue for a nonlinear gradient type elliptic system involving variable exponents growth conditions. Positivity, boundedness and regularity of associated eigenfunctions for auxiliaries systems are established.

Keywords: $p(x)$-Laplacian, variable exponents, weak solution, eigenvalue, regularity, boundedness.

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1 Introduction and setting of the problem

In the present paper, we focus to find a non zero first eigenvalue for the system of quasilinear elliptic equations

$$(P) \quad \begin{cases} -\Delta_{p(x)} u = \lambda c(x)(\alpha(x)+1)|u|^{\alpha(x)-1} u |v|^{\beta(x)+1} & \text{in } \Omega \\ -\Delta_{q(x)} v = \lambda c(x)(\beta(x)+1)|u|^{\alpha(x)+1} v |v|^{\beta(x)-1} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$. For any function p, $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is usually named the $p(x)$-Laplacian.

During the last decade, the interest for partial differential equations involving the $p(x)$-Laplacian operator is increasing. When the exponent variable function $p(\cdot)$ is reduced to be a constant, $\Delta_{p(x)} u$ becomes the well-known p-Laplacian operator $\Delta_p u$. The $p(x)$-Laplacian operator possesses more complicated nonlinearity than the p-Laplacian. Consequently, the problems arising from the $p(x)$-Laplacian operator cannot always be transposed to the results achieved with the p-Laplacian. The process of resolving these problems is often very complicated and needs a mathematical tool (Lebesgue and Sobolev spaces with variable exponents,

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see for instance [5] and its abundant reference). Among them, find the first eigenvalue of $p(x)$-Laplacian Dirichlet presents more singular phenomenon which do not appear in the constant case. This singularity appears for instance by solving the Dirichlet eigenvalue $(\mathcal{D}_\lambda)$: $-\Delta_{p(x)}u = \lambda f(x,u)$ in the Sobolev space $W_0^{1,p(x)}(\Omega)$ where Ω is an open bounded domain with smooth boundary. For more inquiries on this topic we refer, for instance, to [2], where the authors show that the Dirichlet parameter problem admits a nontrivial weak solution provided λ is in a well estimate interval of parameters. In the constant exponent case the function $p(\cdot)$ is constant (see for instance [12] for $p(x) = 2$) a lower bound of the parameter λ depends on the first eigenvalue of the Laplacian Dirichlet problem, while it is zero in the variable exponent case.

More precisely, it is well known that the first eigenvalue for the $p(x)$-Laplacian Dirichlet problem may be equal to zero (for more details, see [10]). In [10], the authors consider that Ω is a bounded domain and p is a continuous function from $\overline{\Omega}$ to $]1,+\infty[$. They gave some geometrical conditions which insure that the first eigenvalue is equal to 0. Otherwise, in one dimensional space, the monotonicity assumption on the function p is a necessary and sufficient condition which guarantees that the first eigenvalue is strictly positive. Here, it should be noted that the monotinicity condition on p prevents the existence of strictly local minimum or maximum in Ω, with which the first eigenvalue does not exist (see [10, Theorem 3.1]). The same conclusion is obtained in higher dimensional case under a monotonicity assumption required for a suitable function depending on p.

The fact that the first eigenvalue is zero, has been observed earlier by [8]. Indeed, the authors illustrate this phenomena by taking $\Omega = (-2,2)$ and $p(x) = 3\chi_{[0,1]}(x) + (4-|x|)\chi_{[1,2]}(x)$. In this condition, the Rayleigh quotient

$$\mu_1 = \inf_{u\in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}}{\int_\Omega |u|^{p(x)}}$$

is equal to zero [22]. The main reason comes from the fact that the well-known Poincaré inequality is not always fulfilled. In some particular situations, Maeda in [18] establishes a version of Poincaré inequality. In this paper, the author also discusses others versions given in [13] by Fu.

Further works established suitable conditions drawing to a non zero first eigenvalue (one can see for instance [11,19]).

Compared to a single equation, elliptic systems have their own peculiarity with respect to the first eigenvalue. First of all, when $p(x)$ and $q(x)$ are constant on Ω, in [4], the following elliptic Dirichlet system is considered

$$\begin{cases} -\Delta_p u = \lambda u|u|^{\alpha-1}|v|^{\beta+1} & \text{in } \Omega \\ -\Delta_q v = \lambda |u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

More precisely, the author establishes the existence of the first eigenvalue $\lambda_{pq} > 0$ associated to a positive and unique eigenfunction (u^*, v^*). In this study Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and the constant exponents $-1 < \alpha, \beta$ and $1 < p, q < N$ obey the following conditions

$$C_{p,q}^{\alpha,\beta} : \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1 \quad \text{and} \quad (\alpha+1)\frac{N-p}{Np} + (\beta+1)\frac{N-q}{Nq} < 1. \tag{1.3}$$

Furthermore, this result has been extended by Kandilakis et al. [15] for the system under mixed boundary conditions

$$\begin{cases} \Delta_p u + \lambda a(x)|u|^{p-2}u + \lambda b(x)u|u|^{\alpha-1}|v|^{\beta+1} = 0 & \text{in } \Omega, \\ \Delta_q v + \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v = 0 & \text{on } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \nu + c_1(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega \\ |\nabla v|^{q-2}\nabla v \cdot \nu + c_2(x)|v|^{q-2}v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is an unbounded domain in $\mathbb{R}^N$ with non compact and smooth boundary $\partial\Omega$, the constant exponents $0 < \alpha, \beta$ and $1 < p, q < N$ are as follows

$$\tfrac{\alpha+1}{p} + \tfrac{\beta+1}{q} = 1 \quad \text{and} \quad (\alpha+1)\tfrac{N-p}{Np} < q, \quad (\beta+1)\tfrac{N-q}{Nq} < p. \tag{1.5}$$

Inspired by [4], Khalil et al. [16] showed that the first eigenvalue λ_{pq} of (1.2) is simple and moreover they established stability (continuity) for the function $(p, q) \longmapsto \lambda_{pq}$.

Motivated by the aforementioned papers, in our work we establish the existence of one-parameter family of nontrivial solutions $((\hat{u}_R, \hat{v}_R), \lambda_R^*)$ for all $R > 0$ for problem (1.1). In addition, we show that the corresponding eigenfunction $(\hat{u}_R, \hat{v}_R)$ is positive in Ω, bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$ and belongs to $C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$ for certain $\gamma \in (0, 1)$ if $p, q \in C^1(\overline{\Omega}) \cap C^{0,\theta}(\overline{\Omega})$. Moreover, by means of geometrical conditions on the domain Ω, we prove that the infimum of the eigenvalues of (1.1) is positive.

To the best of our knowledge, it is the first time that the positive infimum eigenvalue for systems involving $p(x)$-Laplacian operator is studied. However, we point out that in this paper, the existence of an eigenfunction corresponding to the infimum of the eigenvalues of (1.1) is not established and therefore, this issue still remains an open problem.

The rest of the paper is organized as follows. Section 2 contains hypotheses, some auxiliary and useful results involving variable exponents Lebesgue–Sobolev spaces and our main results. Sections 3 and 4 present the proof of our main results.

2 Hypotheses – main results and some auxiliary results

Let $L^{p(x)}(\Omega)$ be the generalized Lebesgue space that consists of all measurable real-valued functions u satisfying

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx < +\infty.$$

$L^{p(x)}(\Omega)$ is endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf\left\{\tau > 0 : \rho_{p(x)}\left(\frac{u}{\tau}\right) \leq 1\right\}.$$

Throughout the paper, to simplify, we will use the notation $\|u\|_{p(x)}$ instead of $\|u\|_{L^{p(x)}(\Omega)}$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$

The classical norm associated is $\|u\|_{W_0^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \||\nabla u|\|_{p(x)}$.

$W_0^{1,p(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W_0^{1,p(x)}(\Omega)}}$ denotes the closure of $C_0^\infty(\Omega)$ respect with the norm of $W^{1,p(x)}(\Omega)$. The norm on $W_0^{1,p(x)}(\Omega)$ is denoted as $\|u\|_{W_0^{1,p(x)}(\Omega)}$ and it is well known that

$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}$. This norm makes $W_0^{1,p(x)}(\Omega)$ a Banach space and the following embedding

$$W_0^{1,p(x)} \hookrightarrow L^{r(x)}(\Omega) \tag{2.1}$$

is compact with $1 < r(x) < \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$.

In the sequel, we will also use the simplified notation $\|u\|_{1,p(x)}$ instead of $\|u\|_{W_0^{1,p(x)}(\Omega)}$.

2.1 Hypotheses

(H.1) Ω is an bounded open of $\mathbb{R}^N$, its boundary $\partial\Omega$ of class $C^{2,\delta}$, for certain $0 < \delta < 1$,

(H.2) $c : \Omega \longrightarrow \mathbb{R}_+$ and $c \in L^\infty(\Omega)$,

(H.3) $\alpha, \beta : \overline{\Omega} \to]1, +\infty[$ are two continuous functions satisfying

$$1 < \alpha^- = \inf_{x\in\Omega} \alpha(x) \leq \alpha^+ = \sup_{x\in\Omega} \alpha(x) < \infty, 1 < \beta^- = \inf_{x\in\Omega} \beta(x) \leq \beta^+ = \sup_{x\in\Omega} \beta(x) < \infty$$

and

$$\frac{\alpha(x)+1}{p(x)} + \frac{\beta(x)+1}{q(x)} = 1,$$

(H.4) p and q are two variable exponents of class $C^1(\overline{\Omega})$ satisfying

$$p(x) < \frac{Np(x)}{N-p(x)}, \qquad q(x) < \frac{Nq(x)}{N-q(x)}, \quad \text{for all } x \in \overline{\Omega}.$$

with

$$1 < p^- = \inf_{x\in\Omega} p(x) \leq p^+ = \sup_{x\in\Omega} p(x) < \infty,$$
$$1 < q^- = \inf_{x\in\Omega} q(x) \leq q^+ = \sup_{x\in\Omega} q(x) < \infty.$$

2.2 Main results

Throughout this paper, the notation $X_0^{p(x),q(x)}(\Omega)$ designates the product space $W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$.

Define on $X_0^{p(x),q(x)}(\Omega)$ the functionals $\mathcal{A}$ and $\mathcal{B}$ are given by:

$$\mathcal{A}(z,w) = \int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} \, dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} \, dx, \tag{2.2}$$

$$\mathcal{B}(z,w) = \int_\Omega c(x)|z|^{\alpha(x)+1}|w|^{\beta(x)+1} dx, \tag{2.3}$$

and denote by $\|(z,w)\| = \|z\|_{1,p(x)} + \|w\|_{1,q(x)}$. The same reasoning exploited in [9] implies that $\mathcal{A}$ and $\mathcal{B}$ are of class $C^1(X_0^{p(x),q(x)}(\Omega), \mathbb{R})$. The Fréchet derivatives of $\mathcal{A}$ and $\mathcal{B}$ at (z,w) in $X_0^{p(x),q(x)}(\Omega)$ are given by

$$\mathcal{A}'(z,w) \cdot (\varphi, \psi) = \int_\Omega |\nabla z|^{p(x)-2} \nabla z \cdot \nabla \varphi \, dx + \int_\Omega |\nabla w|^{q(x)-2} \nabla w \cdot \nabla \psi \, dx \tag{2.4}$$

and

$$\mathcal{B}'(z,w)\cdot(\varphi,\psi) = \int_\Omega c(x)(\alpha(x)+1)|z|^{\alpha(x)-1}z|w|^{\beta(x)+1}\varphi + \int_\Omega c(x)(\beta(x)+1)|z|^{\alpha(x)+1}|w|^{\beta(x)-1}w\psi\, dx, \tag{2.5}$$

where $(\varphi,\psi) \in X_0^{p(x),q(x)}(\Omega)$.

Let $R > 0$ be fixed, we set

$$\mathcal{X}_R = \{(z,w) \in X_0^{p(x),q(x)}(\Omega);\ \mathcal{B}(z,w) = R\}.$$

It is obvious to notice that the set $\mathcal{X}_R$ is not empty. Indeed, let $(z_0, w_0) \in X_0^{p(x),q(x)}(\Omega)$ such that $\mathcal{B}(z_0, w_0) = b_0 > 0$, if $b_0 = R$ we are done. Otherwise, for $z_R = (R/b_0)^{1/p(x)}z_0$ and $w_R = (R/b_0)^{1/q(x)}w_0$, it is easy to note that $\mathcal{B}(z_R, w_R) = R$.

Now, define the Rayleigh quotients

$$\lambda_R^* = \inf_{(z,w)\in\mathcal{X}_R} \frac{\mathcal{A}(z,w)}{\mathcal{B}(z,w)}, \tag{2.6}$$

$$\lambda_{p(x),q(x)}^* = \inf_{(z,w)\in X_0^{1,p(x),q(x)}(\Omega)\setminus\{0\}} \frac{\mathcal{A}(z,w)}{\mathcal{B}(z,w)} \tag{2.7}$$

and

$$\lambda_{*R} = \inf_{(z,w)\in\mathcal{X}_R} \frac{\mathcal{A}(z,w)}{\int_\Omega c(x)(\alpha(x)+\beta(x)+2)|z|^{\alpha(x)+1}|w|^{\beta(x)+1}dx}. \tag{2.8}$$

Remark 2.1. The constant λ_R^* in (2.6) can be written as follows

$$R\lambda_R^* = \inf_{\{(z,w)\in\mathcal{X}_R\}} \mathcal{A}(z,w). \tag{2.9}$$

Our first main result provides the existence of a one-parameter family of solutions for the system (1.1).

Theorem 2.2. *Assume that* (H.1)–(H.4) *hold. Then, the system* (1.1) *has a one-parameter family of nontrivial solutions* $((\hat{u}_R, \hat{v}_R), \lambda_R^*)$ *for all* $R \in (0, +\infty)$. *Moreover, if one of the following conditions holds:*

(a.1) *There are two vectors* $l_1,\ l_2 \in \mathbb{R}^N\setminus\{0\}$ *such that for all* $x \in \Omega$, $f(t_1) = p(x + t_1 l_1)$ *and* $g(t_2) = q(x + t_2 l_2)$ *are monotone for* $t_i \in I_{i,x} = \{t_i;\ x + t_i l_i \in \Omega\}$, $i = 1, 2$.

(a.2) *There are* x_1 *and* $x_2 \notin \overline{\Omega}$ *such that for all* $w_1, w_2 \in \mathbb{R}\setminus\{0\}$ *with* $\|w_1\|, \|w_2\| = 1$, *the functions* $f(t_1) = p(x_0 + t_1 w_1)$ *and* $g(t_2) = p(x_2 + t_2 w_2)$ *are monotone for* $t_i \in I_{x_i,w_i} = \{t_i \in \mathbb{R};\ x_i + t_i w_i \in \Omega\}$, $i = 1, 2$.

Then, $\lambda_{p(x),q(x)}^* = \inf_{R>0} \lambda_R^* > 0$ *is the positive infimum eigenvalue of the problem* (1.1).

A second main result treats positivity, boundedness and regularity properties for a solution of the problem (1.1).

Theorem 2.3. *Let R be a fixed and strictly positive real. Assume that* (H.3) *holds.*

Then, $(\hat{u}_R, \hat{v}_R)$ *the nontrivial solution of problem* (1.1) *is positive and bounded in* $L^\infty(\Omega) \times L^\infty(\Omega)$. *Moreover, if* $p, q \in C^1(\overline{\Omega}) \cap C^{0,\gamma}(\overline{\Omega})$ *for certain* $\gamma \in (0,1)$ *then* $(\hat{u}_R, \hat{v}_R)$ *belongs to* $C^{1,\delta}(\overline{\Omega}) \times C^{1,\delta}(\overline{\Omega})$, $\delta \in (0,1)$.

The proof of Theorem 2.2 will be done in Section 3 while in Section 4 we will present the proof of Theorem 2.3.

2.3 Some preliminaries lemmas

Lemma 2.4 ([5], [8, Theorems 1.2 and 1.3]).

(*i*) *For any* $u \in L^{p(x)}(\Omega)$ *we have*

$$\|u\|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p^+} \quad \text{if} \;\; \|u\|_{p(x)} > 1,$$
$$\|u\|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p^-} \quad \text{if} \;\; \|u\|_{p(x)} \le 1.$$

(*ii*) *For* $u \in L^{p(x)}(\Omega)\backslash\{0\}$ *we have*

$$\|u\|_{p(x)} = a \quad \text{if and only if} \quad \rho_{p(x)}\left(\frac{u}{a}\right) = 1. \tag{2.10}$$

Lemma 2.5 ([5, Theorem 8.2.4]). *Under assumption* (H.4)*, for every* $u \in W_0^{1,p(\cdot)}(\Omega)$ *it holds*

$$\|u\|_{p(\cdot)} \le C_{N,p}\|\nabla u\|_{p(\cdot)}, \tag{2.11}$$

with a constant $C_{N,p} > 0$.

Recall that if there exists a constant $L > 0$ and an exponent $\theta \in (0,1)$ such as

$$|p(x_1) - p(x_2)| \le L|x_1 - x_2|^\theta \quad \text{for all } x_1, x_2 \in \overline{\Omega},$$

then the function p is said to be Hölder continuous on $\overline{\Omega}$ and we observe that p is a function of class $C^{0,\theta}(\overline{\Omega})$.

For a later use, we have the next result.

Lemma 2.6. *For* $s \in (0,1)$ *it holds*

$$\sum\nolimits_{n=1}^{r}(n-1)s^{n-1} \le \frac{s}{(s-1)^2}.$$

Proof. The proof is immediate and it is left to the reader. □

3 Proof of Theorem 2.2

Taking account of the assumption (H.3), we note that the system (1.1) is arising from a nonlinear eigenvalue type problem. Solvability of general class of nonlinear eigenvalues problems of type $\mathcal{A}'(x) = \lambda\mathcal{B}'(x)$ has been treated by M. S. Berger in [1]. We recall this main tool.

Theorem 3.1 ([1]). *Suppose that the* C^1 *functionals* $\mathcal{A}$ *and* $\mathcal{B}$ *defined on the reflexive Banach space* X *have the following properties:*

1. $\mathcal{A}$ *is weakly lower semicontinuous and coercive on* $\{x : \mathcal{B}(x) \le \text{const.}, x \in X\}$;
2. $\mathcal{B}$ *is continuous with respect to weak sequential convergence and* $\mathcal{B}'(x) = 0$ *only at* $x = 0$.

Then the equation $\mathcal{A}'(x) = \lambda\mathcal{B}'(x)$ *has a one-parameter family of nontrivial solutions* (x_R, λ_R) *for all* R *in the range of* $\mathcal{B}(x)$ *such that* $\mathcal{B}(x_R) = R$; *and* x_R *is characterized as the minimum of* $\mathcal{A}(x)$ *over the set* $\{\mathcal{B}(x) = R\}$.

Remark 3.2. In the statement (2) of Theorem 3.1, the condition "$\mathcal{B}'(x) = 0$ only at $x = 0$" may be replaced by "$\mathcal{B}(x) = 0$ only at $x = 0$". Indeed, in the proof of Theorem 3.1, assume that the minimizing problem $\inf_{\{\mathcal{B}(x)=R\}} \mathcal{A}(x)$ is attained at $x_R \in X$ then because $\mathcal{A}$ and $\mathcal{B}$ are differentiable there exists (λ_1, λ_2) (λ_1 and λ_2 are not both zero) a pair of Lagrange multipliers such that

$$\lambda_1 \mathcal{A}'(x_R) + \lambda_2 \mathcal{B}'(x_R) = 0.$$

$\lambda_1 \neq 0$ since otherwise $\mathcal{B}'(x_R) = 0$ implies $x_R = 0$.

In particular, this is true if we assume that there exists $\gamma > 0$ such that

$$(\mathcal{B}'(x), x) \geq \gamma \mathcal{B}(x) \quad \text{for all } x \in X.$$

3.1 Properties of $\mathcal{A}$ and $\mathcal{B}$

Lemma 3.3.

(i) *$\mathcal{A}(z, w)$ is coercive on $X_0^{p(x),q(x)}(\Omega)$.*

(ii) *$\mathcal{B}$ is a weakly continuous functional, namely, $(z_n, w_n) \rightharpoonup (z, w)$ (weak convergence) implies $\mathcal{B}(z_n, w_n) \to \mathcal{B}(z, w)$.*

(iii) *Let (z, w) be in $X_0^{p(x),q(x)}(\Omega)$. Assume that $\mathcal{B}'(z, w) = 0$ in $X^{-1,p'(x),q'(x)}(\Omega)$ then $\mathcal{B}(z, w) = 0$.*

Proof. (i) For any $(z, w) \in X_0^{p(x),q(x)}(\Omega)$ with $\|z\|_{1,p(x)}, \|w\|_{1,q(x)} > 1$, using Lemma 2.4 we have

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)}\, dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)}\, dx \\
&\quad \geq \frac{1}{p^+} \int_\Omega |\nabla z|^{p(x)}\, dx + \frac{1}{q^+} \int_\Omega |\nabla w|^{q(x)}\, dx \\
&\quad \geq \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\} (\|z\|_{1,p(x)}^{p^-} + \|w\|_{1,q(x)}^{q^-}) \\
&\quad \geq 2^{-\min\{p^-,q^-\}} \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\} (\|z\|_{1,p(x)} + \|w\|_{1,q(x)})^{\min\{p^-,q^-\}}.
\end{aligned}$$

Since $\min\{p^-, q^-\} > 1$ (see (H.4)) the above inequality implies that

$$\mathcal{A}(z, w) \to +\infty \quad \text{as} \quad \|(z, w)\| \to +\infty.$$

(ii) Let $(z_n, w_n) \rightharpoonup (z, w)$ in $X_0^{p(x),q(x)}(\Omega)$. By the first part in (H.4) and (2.1) the embeddings $W_0^{1,p(x)} \hookrightarrow L^{p(x)}(\Omega)$ and $W_0^{1,q(x)} \hookrightarrow L^{q(x)}(\Omega)$ are both compact, so we get

$$(z_n, w_n) \to (z, w) \quad \text{in } L^{p(x)}(\Omega) \times L^{q(x)}(\Omega). \tag{3.1}$$

Using (H.3) and the definition of $\mathcal{B}$, we have

$$\begin{aligned}
&|\mathcal{B}(z_n, w_n) - \mathcal{B}(z, w)| \\
&\quad \leq \|c\|_\infty \left[\int_\Omega |z|^{\alpha(x)+1} \left(|w|^{\beta(x)+1} - |w_n|^{\beta(x)+1}\right) dx + \int_\Omega |w_n|^{\beta(x)+1} \left(|z|^{\alpha(x)+1} - |z_n|^{\alpha(x)+1}\right) dx\right] \\
&\quad \leq 2^{\max(\alpha^+,\beta^+)} \|c\|_\infty \left[\int_\Omega |z|^{\alpha(x)+1} |w - w_n|^{\beta(x)+1}\, dx + \int_\Omega |\, w_n|^{\alpha(x)+1} |z - z_n|^{\alpha(x)+1}\, dx\right].
\end{aligned}$$

By Hölder's inequality one has

$$\int_\Omega |z|^{\alpha(x)+1} |w-w_n|^{\beta(x)+1}\,dx \le C_{\alpha,\beta,p,q} \left\| |z|^{\alpha(x)+1} \right\|_{\frac{p(x)}{\alpha(x)+1}} \left\| |w_n-w|^{\alpha(x)+1} \right\|_{\frac{q(x)}{\beta(x)+1}}$$

where $C_{\alpha,\beta,p,q} > 0$ is a constant. Observe that

$$\left\| |w-w_n|^{\beta(x)+1} \right\|^{q^+}_{\frac{q(x)}{\beta(x)+1}} \le \int_\Omega (|w-w_n|^{\beta(x)+1})^{\frac{q(x)}{\beta(x)+1}}dx = \rho_{q(\cdot)}(w-w_n)$$

and also

$$\rho_{q(\cdot)}(w-w_n) \le \left\| |w-w_n|^{\beta(x)+1} \right\|^{q^-}_{q(x)}.$$

Then it follows that

$$\left\| |w-w_n|^{\beta(x)+1} \right\|_{\frac{q(x)}{\beta(x)+1}} \le \rho_{q(\cdot)}(w-w_n)^{1/q^+} \le \left\| |w-w_n|^{\beta(x)+1} \right\|^{q^-/q^+}_{q(x)}.$$

Therefore, the strong convergence in (3.1) ensures that

$$\left\| |w-w_n|^{\beta(x)+1} \right\|_{\frac{q(x)}{\beta(x)+1}} \to 0 \quad \text{as } n \to +\infty.$$

A quite similar argument provides

$$\left\| |z-z_n|^{\alpha(x)+1} \right\|_{\frac{p(x)}{\alpha(x)+1}} \to 0 \quad \text{as } n \to +\infty.$$

(iii) Clearly, let us notice that for any $(z,w) \in X_0^{p(x),q(x)}(\Omega)$, doing $\varphi = z/p(x)$ and $\psi = w/q(x)$ in (2.5), we get the following identity

$$\mathcal{B}'(z,w)\cdot(z/p(x), w/q(x)) = \mathcal{B}(z,w).$$

Then the statement *(iii)* follows. This concludes the proof of the lemma. □

3.2 A priori bound for $\mathcal{A}$

Lemma 3.4. *Let R a fixed and strictly positive real number. There exists a constant $\mathcal{K}(R) > 0$ depending on R such that*

$$\mathcal{A}(z,w) \ge \mathcal{K}(R) > 0, \qquad \forall (z,w) \in \mathcal{X}_R. \tag{3.2}$$

Proof. First, observe from Lemma 2.5 that if $\|\nabla z\|_{L^{p(x)}(\Omega)} < 1$, we have

$$\left\| \tfrac{z}{C_{N,p}} \right\|_{p(x)} < 1.$$

Then if follows that

$$\rho_{p(x)}(\tfrac{z}{C_{N,p}}) \le \left\| \tfrac{z}{C_{N,p}} \right\|^{p^-}_{p(x)}, \tag{3.3}$$

which combined with Lemma 2.5 leads to

$$\int_\Omega \frac{|z|^{p(x)}}{C_{N,p}^{p(x)}}dx \le \|\nabla z\|^{p^-}_{p(x)}.$$

Hence it holds

$$\int_\Omega |z|^{p(x)}dx \leq K_{N,p}\|\nabla z\|_{p(x)}^{p^-} \leq K_{N,p}\|\nabla z\|_{p(x)}^{p^-/p^+}, \tag{3.4}$$

where

$$K_{N,p} = \begin{cases} C_{N,p}^{p^+} & \text{if } C_{N,p} > 1 \\ C_{N,p}^{p^-} & \text{if } C_{N,p} < 1. \end{cases}$$

A quite similar argument shows that

$$\int_\Omega |w|^{q(x)}dx \leq K_{N,q}\|\nabla w\|_{q(x)}^{q^-/q^+}, \tag{3.5}$$

where

$$K_{N,q} = \begin{cases} C_{N,q}^{q^+} & \text{if } C_{N,q} > 1 \\ C_{N,q}^{q^-} & \text{if } C_{N,q} < 1. \end{cases}$$

For every $(z, w) \in X_0^{p(x),q(x)}(\Omega)$, Young's inequality and (H.3) implies

$$\begin{aligned}\int_\Omega c(x)|z|^{\alpha(x)+1}|w|^{\beta(x)+1}dx &\leq \|c\|_\infty \int_\Omega \left[\frac{\alpha(x)+1}{p(x)}|z|^{p(x)} + \frac{\beta(x)+1}{q(x)}|w|^{q(x)}\right] dx \\ &\leq \|c\|_\infty \left(\int_\Omega |z|^{p(x)}dx + \int_\Omega |w|^{q(x)}dx\right).\end{aligned} \tag{3.6}$$

Assume that $(z, w) \in \mathcal{X}_R$ is such as

$$\max\left(\|\nabla z\|_{p(\cdot)}, \|\nabla w\|_{q(\cdot)}\right) < 1. \tag{3.7}$$

Bearing in mind (H.3), (H.4) and (i) of Lemma 2.4, we have

$$\max\left\{\int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}dx,\ \int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}dx\right\} < 1. \tag{3.8}$$

Then, from (3.4)–(3.8), it follows that

$$R \leq K_1\left(\int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}dx\right)^{p^-/p^+} + K_2\left(\int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}dx\right)^{q^-/q^+}. \tag{3.9}$$

From the hypothesis (H.4) on p^-, p^+, q^- and q^+, it follows that

$$\begin{aligned}R^{\frac{p^+q^+}{p^-q^-}} \leq 2^{\frac{p^+q^+}{p^-q^-}-1}\Bigg[&K_1^{\frac{p^+q^+}{p^-q^-}}\left(\int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}dx\right)^{q^+/q^-} \\ &+ K_2^{p^+q^+/p^-q^-}\left(\int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}dx\right)^{p^+/p^-}\Bigg].\end{aligned} \tag{3.10}$$

Or again

$$R^{\frac{p^+q^+}{p^-q^-}} \leq (2K_3)^{\frac{p^+q^+}{p^-q^-}}\left[\int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}dx\right] \tag{3.11}$$

where

$$K_1 = K_{N,p}\left(p^+\right)^{p^-/p^+}\|c\|_\infty, \qquad K_2 = K_{N,q}\left(q^+\right)^{q^-/q^+}\|c\|_\infty$$

and $K_3 = K_1 + K_2$. Thus, from (3.11), we conclude that

$$\mathcal{A}(z,w) \geq \left(\frac{R}{2K_3}\right)^{\frac{q^+p^+}{q^-p^-}}. \tag{3.12}$$

Now, we deal with the case when $(z,w) \in \mathcal{X}_R$ is such as

$$\max\left(\|\nabla z\|_{p(\cdot)}, \|\nabla w\|_{q(\cdot)}\right) \geq 1.$$

This implies that

$$\max\left(\int_\Omega |\nabla z|^{p(x)}dx, \int_\Omega |\nabla w|^{q(x)}dx\right) \geq 1.$$

If $\int_\Omega |\nabla z|^{p(x)}dx \geq 1$, we have

$$p^+\int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}dx \geq \int_\Omega |\nabla z|^{p(x)}dx \geq 1,$$

which in turn yields

$$\mathcal{A}(z,w) = \int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}dx > \frac{1}{p^+}. \tag{3.13}$$

Now for $\int_\Omega |\nabla w|^{q(x)}dx \geq 1$ a quite similar argument provides

$$\mathcal{A}(z,w) > \frac{1}{q^+}. \tag{3.14}$$

We notice that if $\max\left(\|\nabla z\|_{p(\cdot)}, \|\nabla w\|_{q(\cdot)}\right) \geq 1$, from (3.13) and (3.14), it is clear that

$$\mathcal{A}(z,w) > \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\}. \tag{3.15}$$

Thus, according to (3.12) and (3.15), for all $(z,w) \in \mathcal{X}_R$, one has

$$\mathcal{A}(z,w) \geq \min\left\{\left(\frac{R}{2K_3}\right)^{\frac{q^+p^+}{q^-p^-}}, \frac{1}{p^+}, \frac{1}{q^+}\right\} > 0. \tag{3.16}$$

Consequently, there exists a constant $\mathcal{K}(R) > 0$ depending on R such that (3.2) holds. □

3.3 Proof of (2.6)

We begin with a proposition.

Proposition 3.5. *Assume that* (H.3) *holds. Then, for* $R > 0$,

(i) $0 < \frac{\lambda_R^*}{(\alpha^+ + \beta^+ + 2)} < \lambda_{*R} < \lambda_R^*$.

(ii) *Any* $\lambda < \lambda_{*R}$ *is not an eigenvalue of problem* (1.1).

(iii) *There exists* $(\hat{u}_R, \hat{v}_R) \in \mathcal{X}_R$ *such that* λ_R^* *is a corresponding eigenvalue for the system* (1.1).

Proof. **(i)**. First let us show that $0 < \frac{\lambda_R^*}{(\alpha^+ + \beta^+ + 2)} \leq \lambda_{*R} \leq \lambda_R^*$. Obviously, for all $(z, w) \in \mathcal{X}_R$, we have

$$\frac{\mathcal{A}(z,w)}{(\alpha^+ + \beta^+ + 2)R} \leq \frac{\mathcal{A}(z,w)}{\int_\Omega c(x)(\alpha(x) + \beta(x) + 2)|z|^{\alpha(x)+1}|w|^{\beta(x)+1}dx} \leq \frac{\mathcal{A}(z,w)}{R}.$$

From (2.6) and (2.8), it follows that $\frac{\lambda_R^*}{(\alpha^+ + \beta^+ + 2)} \leq \lambda_{*R} \leq \lambda_R^*$. Now suppose that $\lambda_{*R} = 0$. Then $\lambda_R^* = 0$ and in virtue of Lemma 3.4 and Remark 2.1 this is a contradiction. Hence $\lambda_{*R} > 0$.

(ii). Next, we show that λ cannot be an eigenvalue for $\lambda < \lambda_{*R}$. Indeed, suppose by contradiction that λ is an eigenvalue of problem (1.1). Then there exists $(u, v) \in X_0^{p(x),q(x)}(\Omega) - \{(0, 0)\}$ such as

$$\begin{aligned} \int_\Omega |\nabla u|^{p(x)}\, dx &= \lambda \int_\Omega c(x)(\alpha(x) + 1)|u|^{\alpha(x)+1}|v|^{\beta(x)+1} \\ \int_\Omega |\nabla v|^{q(x)}\, dx &= \lambda \int_\Omega c(x)(\beta(x) + 1)|u|^{\alpha(x)+1}|v|^{\beta(x)+1}. \end{aligned} \tag{3.17}$$

On the basis of (H.3), (H.4), (2.8) and (3.17), we get

$$\begin{aligned} \lambda_{*R} &\int_\Omega c(x)(\alpha(x) + \beta(x) + 2)|u|^{\alpha(x)+1}|v|^{\beta(x)+1}dx \\ &\leq \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla v|^{q(x)} \right) dx \\ &\leq \int_\Omega |\nabla u|^{p(x)}\, dx + \int_\Omega |\nabla v|^{q(x)}\, dx \\ &= \lambda \int_\Omega c(x)(\alpha(x) + \beta(x) + 2)|u|^{\alpha(x)+1}|v|^{\beta(x)+1}dx \\ &< \lambda_{*R} \int_\Omega c(x)(\alpha(x) + \beta(x) + 2)|u|^{\alpha(x)+1}|v|^{\beta(x)+1}dx, \end{aligned}$$

which is not possible and the conclusion follows.

(iii). Now, we claim that the infimum in (2.9) is achieved at an element of $\mathcal{X}_R$. Indeed, thanks to Lemma 3.3, $\mathcal{B}$ is weakly continuous on $X_0^{p(x),q(x)}(\Omega)$, then the nonempty set $\mathcal{X}_R$ is weakly closed. So, since $\mathcal{A}$ is weakly lower semicontinuous, we conclude that there exists an element of $\mathcal{X}_R$ which we denote $(\hat{u}, \hat{v}_R)$ such that (2.9) is feasible. Since $(\hat{u}_R, \hat{v}_R) \neq 0$, we also have $\mathcal{B}'(\hat{u}_R, \hat{v}_R) \neq 0$ otherwise it implies $\mathcal{B}(\hat{u}_R, \hat{v}_R) = 0$, which contradicts $(\hat{u}_R, \hat{v}_R) \in \mathcal{X}_R$. So, owing to Lagrange multiplier method (see e.g. [1, Theorem 6.3.2, p. 325] or [6, Theorem 6.3.2, p. 402]), there exists $\lambda_R \in \mathbb{R}$ such that

$$\mathcal{A}'(\hat{u}_R, \hat{v}_R) \cdot (\varphi, \psi) = \lambda_R \mathcal{B}'(\hat{u}_R, \hat{v}_R) \cdot (\varphi, \psi), \qquad \forall (\varphi, \psi) \in X_0^{p(x),q(x)}(\Omega) \tag{3.18}$$

where $\mathcal{A}'$ and $\mathcal{B}'$ are defined as in (2.4) and (2.5) respectively.

In the sequel, we show that λ_R is equal to λ_R^*. To this end, let us denote by Ω^+ and Ω^- the sets defined as follows

$$\Omega^+ = \{x \in \Omega;\ |\nabla \hat{u}_R|^{p(x)} - \lambda_R(\alpha(x) + 1)c(x)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1} \geq 0\}$$

and

$$\Omega_- = \{x \in \Omega;\ |\nabla \hat{u}_R|^{p(x)} - \lambda_R(\alpha(x) + 1)c(x)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1} < 0\}.$$

By taking $\varphi = \hat{u}_R\, 1_{\Omega^+}$ and $\psi = 0$ in (3.18) one has

$$\int_{\Omega^+} \left(|\nabla \hat{u}_R|^{p(x)} - \lambda_R c(x)(\alpha(x) + 1)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1} \right) dx = 0 \tag{3.19}$$

and likewise, by choosing $\varphi = \hat{u}_R\, 1_{\Omega^-}$ and $\psi = 0$ in (3.18) we get

$$\int_{\Omega^-} \left(|\nabla \hat{u}_R|^{p(x)} - \lambda_R c(x)(\alpha(x)+1)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1} \right) dx = 0. \tag{3.20}$$

We claim that

$$\int_\Omega \frac{1}{p(x)} |\nabla \hat{u}_R|^{p(x)} dx = \lambda_R \int_\Omega c(x) \frac{\alpha(x)+1}{p(x)} |\hat{u}_R|^{\alpha(x)+1} |\hat{v}|^{\beta(x)+1} dx. \tag{3.21}$$

Indeed, on account of (H.4), (3.19) and (3.20) we have

$$\begin{aligned}
&\left| \int_\Omega \frac{|\nabla \hat{u}_R|^{p(x)}}{p(x)} dx - \lambda_R \int_\Omega \frac{\alpha(x)+1}{p(x)} c(x) |\hat{u}_R|^{\alpha(x)+1} |\hat{v}|^{\beta(x)+1} dx \right| \\
&\quad \leq \int_\Omega p(x) \left| \frac{|\nabla \hat{u}_R|^{p(x)}}{p(x)} - \lambda_R \frac{\alpha(x)+1}{p(x)} c(x) |\hat{u}_R|^{\alpha(x)+1} |\hat{v}_R|^{\beta(x)+1} \right| dx \\
&\quad = \int_\Omega \left| |\nabla \hat{u}_R|^{p(x)} - \lambda_R (\alpha(x)+1) c(x) |\hat{u}_R|^{\alpha(x)+1} |\hat{v}_R|^{\beta(x)+1} \right| dx \\
&\quad \leq \int_{\Omega^+} \left(|\nabla \hat{u}_R|^{p(x)} - \lambda_R (\alpha(x)+1) c(x) |\hat{u}_R|^{\alpha(x)+1} |\hat{v}_R|^{\beta(x)+1} \right) dx \\
&\qquad - \int_{\Omega_-} \left(|\nabla \hat{u}_R|^{p(x)} - \lambda_R (\alpha(x)+1) c(x) |\hat{u}_R|^{\alpha(x)+1} |\hat{v}_R|^{\beta(x)+1} \right) dx = 0,
\end{aligned}$$

showing that (3.21) holds. In the same manner we can prove that

$$\int_\Omega \frac{1}{q(x)} |\nabla \hat{v}_R|^{q(x)} dx = \lambda_R \int_\Omega c(x) \frac{\beta(x)+1}{q(x)} |\hat{u}_R|^{\alpha(x)+1} |\hat{v}_R|^{\beta(x)+1} dx. \tag{3.22}$$

Adding together (3.21) and (3.22), on account of (H.3) and (3.14), we achieve that

$$\mathcal{A}(\hat{u}_R, \hat{v}_R) = R\lambda_R.$$

Then, bearing in mind (3.15) it turns out that $\lambda_R = \lambda_R^*$, showing that λ_R^* is at least one eigenvalue of (1.1).

Therefore, combining this last point with the characterization (3.18), we get

$$\mathcal{A}'(\hat{u}_R, \hat{v}_R) \cdot (\varphi, 0) = \lambda_R^* \mathcal{B}'(\hat{u}_R, \hat{v}_R) \cdot (\varphi, 0), \qquad \forall \varphi \in W_0^{1,q(x)}(\Omega)$$

and

$$\mathcal{A}'(\hat{u}_R, \hat{v}_R) \cdot (0, \psi) = \lambda_R^* \mathcal{B}'(\hat{u}_R, \hat{v}_R) \cdot (0, \psi), \qquad \forall \psi \in W_0^{1,q(x)}(\Omega).$$

In other words, it means that $((\hat{u}_R, \hat{v}_R), \lambda_R^*)$ is a solution of the system (1.1). □

3.4 Proof of Theorem 2.2

Employing again the statement of Lemma 3.3, we can apply Theorem 3.1 due to [1]. Then the system (1.1) has a one-parameter family of nontrivial solutions $((\hat{u}_R, \hat{v}_R), \lambda_R)$ for all $R > 0$. Moreover, from **(iii)** of Proposition 3.5, $\lambda_R = \lambda_R^*$.

It remains to prove that $\lambda^*_{p(x),q(x)} = \inf_{R>0} \lambda_R^* > 0$. From (3.6), for $(z, w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\}$, one has

$$\begin{aligned}
&\frac{1}{\|c\|_\infty} \cdot \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega |z|^{p(x)} dx + \int_\Omega |w|^{q(x)} dx} \\
&\qquad \leq \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx}.
\end{aligned} \tag{3.23}$$

Recall that under assumption (a.1) or (a.2), the authors in [10] proved that the first eignevalues

$$\begin{cases} \lambda^*_{p(x)} = \inf_{z\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla z|^{p(x)}dx}{\int_\Omega |z|^{p(x)}dx} \\ \lambda^*_{q(x)} = \inf_{z\in W_0^{1,q(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla z|^{q(x)}dx}{\int_\Omega |z|^{q(x)}dx}, \end{cases} \tag{3.24}$$

are strictly positive. Hence, combining with (3.23) it follows that

$$\begin{aligned} \min\left\{\frac{\lambda^*_{p(x)}}{p^+\|c\|_\infty}, \frac{\lambda^*_{q(x)}}{q^+\|c\|_\infty}\right\} &= \min\left\{\frac{\lambda^*_{p(x)}}{p^+\|c\|_\infty}, \frac{\lambda^*_{q(x)}}{q^+\|c\|_\infty}\right\} \cdot \frac{\int_\Omega |z|^{p(x)}\,dx + \int_\Omega |w|^{q(x)}\,dx}{\int_\Omega |z|^{p(x)}\,dx + \int_\Omega |w|^{q(x)}\,dx} \\ &\le \frac{\frac{\lambda^*_{p(x)}}{p^+\|c\|_\infty}\int_\Omega |z|^{p(x)}\,dx + \frac{\lambda^*_{q(x)}}{q^+\|c\|_\infty}\int_\Omega |w|^{q(x)}\,dx}{\int_\Omega |z|^{p(x)}\,dx + \int_\Omega |w|^{q(x)}\,dx} \\ &\le \frac{\int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}\,dx}{\int_\Omega c(x)|z|^{\alpha(x)+1}|w|^{\beta(x)+1}dx}. \end{aligned}$$

Then

$$\begin{aligned} 0 &< \min\left\{\frac{\lambda^*_{p(x)}}{p^+\|c\|_\infty}, \frac{\lambda^*_{q(x)}}{q^+\|c\|_\infty}\right\} \\ &\le \inf_{(z,w)\in X_0^{1,p(x),q(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}\,dx}{\int_\Omega c(x)|z|^{\alpha(x)+1}|w|^{\beta(x)+1}dx}. \end{aligned} \tag{3.25}$$

On the other hand, since

$$\bigcup_{R>0} \mathcal{X}_R \subset \left\{(z,w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\}\right\},$$

one gets

$$\begin{aligned} &\inf_{(z,w)\in X_0^{1,p(x),q(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}\,dx}{\int_\Omega c(x)|z|^{\alpha(x)+1}|w|^{\beta(x)+1}dx} \\ &\le \inf_{\{\mathcal{B}(z,w)=R\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla z|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}\,dx}{\int_\Omega c(x)|z|^{\alpha(x)+1}|w|^{\beta(x)+1}dx}. \end{aligned} \tag{3.26}$$

Thus, gathering (3.25) and (3.26) together we infer that

$$0 < \min\left\{\frac{\lambda^*_{p(x)}}{p^+\|c\|_\infty}, \frac{\lambda^*_{q(x)}}{q^+\|c\|_\infty}\right\} \le \lambda^*_{p(x),q(x)} \le \inf_{R>0} \lambda^*_R.$$

Next, let us prove that $\lambda^*_{p(x),q(x)} \ge \inf_{R>0} \lambda^*_R$. To this end, let a constant $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that $\lambda^*_{R_\varepsilon} < \inf_{R>0} \lambda^*_R + \varepsilon$. This implies that

$$\lambda^*_{R_\varepsilon} < \lambda^*_R + \varepsilon \quad \text{for all } R > 0 \text{ and } \varepsilon > 0. \tag{3.27}$$

Now, let $(z,w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\}$ such that $\mathcal{B}(z,w) > 0$ and set $R_{(z,w)} = \mathcal{B}(z,w)$. According to (iii) in Propostion 3.5, the constant

$$\lambda^*_{R_{(z,w)}} = \inf_{\{\mathcal{B}(z,w)=R_{(z,w)}\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x)|z|^{\alpha(x)+1}|w|^{\beta(x)+1} dx}$$

exists and then

$$\lambda^*_{R_{(z,w)}} \leq \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x)|z|^{\alpha(x)+1}|w|^{\beta(x)+1} dx}.$$

At this point, combining with (3.27) yields

$$\lambda^*_{R_\varepsilon} < \lambda^*_{R_{(z,w)}} + \varepsilon \leq \frac{\mathcal{A}(z,w)}{\mathcal{B}(z,w)} + \varepsilon \quad \text{for all } \varepsilon > 0,$$

which, it turn, leads to

$$\lambda^*_{R_\varepsilon} < \lambda^*_{R_{(z,w)}} + \varepsilon \leq \inf_{(z,w)\in X_0^{1,p(x),q(x)}(\Omega)\setminus\{0\}} \frac{\mathcal{A}(z,w)}{\mathcal{B}(z,w)} + \varepsilon \quad \text{for all } \varepsilon > 0.$$

This is equivalent to $\lambda^*_{R_\varepsilon} \leq \lambda^*_{p(x),q(x)} + \varepsilon$. Consequently,

$$\inf_{R>0} \lambda^*_R \leq \lambda^*_{R_\varepsilon} \leq \lambda^*_{p(x),q(x)} + \varepsilon \leq \inf_{R>0} \lambda^*_R + \varepsilon \quad \text{for all } \varepsilon > 0.$$

Finally, passing to the limit as $\varepsilon \to 0$ implies that $\lambda^*_{p(x),q(x)} = \inf_{R>0} \lambda^*_R$. This ends the proof of Theorem 2.2. □

4 Proof of Theorem 2.3

Let $(\hat{u}_R, \hat{v}_R) \in X_0^{p(x),q(x)}(\Omega)$ be a solution of problem (1.1) corresponding to the positive infimum eigenvalue λ^*_R and let $d > 0$ be a constant such as

$$d = \frac{\hat{d}}{\max\{p^+, q^+\}}, \tag{4.1}$$

where

$$1 < \max\{p^+, q^+\} < \hat{d} \leq \max\{p^+, q^+\} \cdot \min\left\{\frac{\pi_p^-}{p^-}, \frac{\pi_p^+}{p^+}, \frac{\pi_q^-}{q^-}, \frac{\pi_q^+}{q^+}\right\} \tag{4.2}$$

and

$$\pi_p(x) = \frac{Np(x)}{N - p(x)}, \qquad \pi_p^- = \inf_{x\in\Omega} \pi_p(x) \quad \text{and} \quad \pi_p^+ = \sup_{x\in\Omega} \pi_p(x). \tag{4.3}$$

In this section, the goal consists in proving that $(\hat{u}_R, \hat{v}_R)$ is bounded in Ω. Notice that from the above section, we have

$$\begin{cases} \int_\Omega |\nabla \hat{u}_R|^{p(x)-2} \nabla \hat{u}_R \nabla \varphi dx = \lambda^*_R \int_\Omega c(x)(\alpha(x)+1)\hat{u}_R|\hat{u}_R|^{\alpha(x)-1}|\hat{v}_R|^{\beta(x)+1}\varphi dx \\ \int_\Omega |\nabla \hat{v}_R|^{q(x)-2} \nabla \hat{v}_R \nabla \psi dx = \lambda^*_R \int_\Omega c(x)(\beta(x)+1)|\hat{u}_R|^{\alpha(x)+1}\hat{v}_R|\hat{v}_R|^{\beta(x)-1}\psi dx. \end{cases} \tag{4.4}$$

Remark 4.1. Since $p(x) \leq p^+$ in Ω, the embeddings $C_c^\infty(\Omega) \subset C^1(\overline{\Omega}) \subset W_0^{1,p^+}(\Omega) \subset W_0^{1,p(x)}(\Omega)$ hold. Moreover, $C_c^\infty(\Omega)$ is dense in $W_0^{1,p(x)}(\Omega)$ with respect the norm on $W^{1,p(x)}(\Omega)$, we may assume that $\hat{u}_R \in C^1(\overline{\Omega})$ (see, e.g., [8]). The same argument enable us to assume that $\hat{v}_R \in C^1(\overline{\Omega})$.

For a better reading, we divide the proof of Theorem 2.3 in several lemmas.

Lemma 4.2. *Assume that the hypotheses* (H.1)–(H.4) *hold. Then, for any fixed k in $\mathbb{N}$, there exist $x_k, y_k \in \Omega$ such that the following estimates hold:*

$$\int_\Omega \hat{u}_R^{1+p(x)(d^k-1)} dx \le \max\{1, |\Omega|\} \max\Big\{ \|\hat{u}_R\|_{p(x)d^k}^{p(x_k)d^k}, \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k} \Big\}, \tag{4.5}$$

$$\int_\Omega \hat{u}_R |\hat{u}_R|^{\alpha(x)-1} |\hat{v}_R|^{\beta(x)+1} |\hat{u}_R|^{1+p(x)(d^k-1)} dx \le 2 \max\left\{ \|\hat{u}_R\|_{p(x)d^k}^{p(x_k)d^k}, \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k} \right\}, \tag{4.6}$$

where $|\Omega|$ denotes the Lebesgue measure of a set Ω in $\mathbb{R}^N$.

Proof. Before starting the proof, let us note that

$$\begin{aligned} \frac{\alpha(x)+1+p(x)(d^k-1)}{p(x)d^k} + \frac{\beta(x)+1}{q(x)d^k} &= \left[\frac{\alpha(x)+1}{p(x)} + \frac{\beta(x)+1}{q(x)} \right] \frac{1}{d^k} + \frac{d^k-1}{d^k} \\ &= \frac{1}{d^k} + \frac{d^k-1}{d^k} = 1, \end{aligned} \tag{4.7}$$

where d is chosen as in (4.1). Let us prove (4.5). Since $\hat{u}_R \in L^{p(x)d^k}(\Omega)$ and $p(x)d^k > p(x)d^k - p(x) + 1 > 0$ then $\hat{u}_R \in L^{\frac{p(x)d^k}{1+p(x)(d^k-1)}}(\Omega)$. Therefore, by Hölder's inequality and the mean value theorem, there exist x_k and $t_k \in \Omega$ such as

$$\begin{aligned} \int_\Omega \hat{u}_R^{1+p(x)(d^k-1)} dx &\le \|1_\Omega\|_{d^k p'(x)} \|\hat{u}_R\|_{d^k p(x)}^{d^k p(x_k)} = \|1_\Omega\|_{d^k p'(x)}^{\frac{d^k p'(t_k)}{d^k p'(t_k)}} \|\hat{u}_R\|_{d^k p(x)}^{d^k p(x_k)} \\ &= |\Omega|^{\frac{1}{d^k p'(t_k)}} \|\hat{u}_R\|_{d^k p(x)}^{d^k p(x_k)} \le \max\{1, |\Omega|\} \|\hat{u}_R\|_{d^k p(x)}^{d^k p(x_k)}. \end{aligned}$$

This shows that the inequality (4.5) holds true. Here p' and p are conjugate variable exponents functions.

Next, we show (4.6). By (4.7) and Young's inequality, we get

$$\begin{aligned} \left| \int_\Omega |\hat{u}_R|^{\alpha(x)+1+p(x)(d^k-1)} |\hat{v}_R|^{\beta(x)+1} dx \right| &\le \int_\Omega |\hat{u}_R|^{\alpha(x)+1+p(x)(d^k-1)} |\hat{v}_R|^{\beta(x)+1} dx \\ &\le \int_\Omega \frac{\alpha(x)+1+p(x)(d^k-1)}{p(x)d^k} |\hat{u}_R|^{p(x)d^k} dx + \int_\Omega \frac{\beta(x)+1}{q(x)d^k} |\hat{v}_R|^{q(x)d^k} dx \\ &\le \int_\Omega |\hat{u}_R|^{p(x)d^k} dx + \int_\Omega |\hat{v}_R|^{q(x)d^k} dx. \end{aligned} \tag{4.8}$$

Observe from (3.13) that

$$\int_\Omega \Big| \frac{\hat{u}_R}{\|\hat{u}_R\|_{p(x)d^k}} \Big|^{p(x)d^k} dx = 1.$$

Using the mean value theorem, there exists $x_k \in \Omega$ such that

$$\int_\Omega |\hat{u}_R|^{p(x)d^k} dx = \|\hat{u}_R\|_{p(x)d^k}^{p(x_k)d^k}. \tag{4.9}$$

Similarly, we can find $y_k \in \Omega$ such that

$$\int_\Omega |\hat{v}_R|^{q(x)d^k} dx = \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k}. \tag{4.10}$$

Then, combining (4.8), (4.9) and (4.10), the inequality (4.6) holds true, ending the proof of Lemma 4.2. □

Using Lemma 4.2, we can prove the next result.

Lemma 4.3. *Assume that the hypotheses* (H.1)–(H.4) *hold. Let* $(\hat{u}_R, \hat{v}_R) \in X_0^{p(x),q(x)}(\Omega)$ *be a solution of problem* (1.1)*. Then,*

$$(\hat{u}_R, \hat{v}_R) \in L^{p(x)d^k}(\Omega) \times L^{q(x)d^k}(\Omega), \qquad \forall k \in \mathbb{N}.$$

Proof. We employ a recursive reasoning. Since $(\hat{u}_R, \hat{v}_R) \in X_0^{p(x),q(x)}(\Omega)$, it is obvious that $(\hat{u}_R, \hat{v}_R) \in L^{p(x)}(\Omega) \times L^{q(x)}(\Omega)$. So, (4.5) remains true for $k = 0$.

Assume that the conjecture "$(\hat{u}_R, \hat{v}_R) \in L^{p(x)d^l}(\Omega) \times L^{q(x)d^l}(\Omega)$" holds at every level $l \leq k$ and we claim that

$$(\hat{u}_R, \hat{v}_R) \in L^{p(x)d^{k+1}}(\Omega) \times L^{q(x)d^{k+1}}(\Omega). \tag{4.11}$$

To do it, we insert $\varphi = \hat{u}_R^{1+p(x)(d^k-1)}$ in (4.4) we get

$$\begin{aligned}&\int_\Omega |\nabla \hat{u}_R|^{p(x)-2} \nabla \hat{u}_R \nabla ({\hat{u}_R}^{1+p(x)(d^k-1)})dx \\ &\quad = \lambda_R^* \int_\Omega c(x)(\alpha(x)+1)\hat{u}_R|\hat{u}_R|^{\alpha(x)-1}|\hat{v}_R|^{\beta(x)+1}\hat{u}_R^{1+p(x)(d^k-1)}dx.\end{aligned} \tag{4.12}$$

Observe that

$$\begin{aligned}&\int_\Omega |\nabla \hat{u}_R|^{p(x)-2} \nabla \hat{u}_R \nabla \left(\hat{u}_R^{1+p(x)(d^k-1)}\right) dx \\ &\quad = \int_\Omega (d^k-1)\nabla p \nabla \hat{u}_R\, |\nabla \hat{u}_R|^{p(x)-2}\, \hat{u}_R^{1+p(x)(d^k-1)} \ln \hat{u}_R \; dx \\ &\qquad + \int_\Omega \left[1+p(x)(d^k-1)\right] |\nabla \hat{u}_R|^{p(x)}\, \hat{u}_R^{p(x)(d^k-1)} dx\end{aligned} \tag{4.13}$$

and

$$|\nabla \hat{u}_R|^{p(x)}\, \hat{u}_R^{p(x)(d^k-1)} = \frac{1}{d^{kp(x)}} |\nabla (\hat{u}_R)^{d^k}|^{p(x)}. \tag{4.14}$$

Then on the one hand

$$\begin{aligned}\int_\Omega \frac{1+p(x)(d^k-1)}{d^{kp(x)}} |\nabla \, (\hat{u}_R)^{d^k}\, |^{p(x)} dx &\geq \int_\Omega \frac{d^k}{d^{kp(x)}} |\nabla \, (\hat{u}_R)^{d^k}\, |^{p(x)} dx \\ &\geq \frac{1}{d^{k(p^+-1)}} \int_\Omega |\nabla \, (\hat{u}_R)^{d^k}\, |^{p(x)} dx,\end{aligned} \tag{4.15}$$

on the other hand, since $\hat{u}_R$ is assumed of class $C^1(\overline{\Omega})$ and taking $\sup_{x\in\Omega} |\nabla p| = M_p < +\infty$, we have

$$\textstyle\int_\Omega (d^k-1)\, |\nabla p|\, |\nabla \hat{u}_R|^{p(x)-1}\, \hat{u}_R^{1+p(x)(d^k-1)}\, |\ln \hat{u}|\; dx \leq \hat{C} M_p \int_\Omega \hat{u}_R^{1+p(x)(d^k-1)}\, dx, \tag{4.16}$$

with some constant $\hat{C} > 0$. Hence, gathering (4.12), (4.13), (4.15) and (4.16) together, one has

$$\begin{aligned}\int_\Omega |\nabla \, (\hat{u}_R)^{d^k}\, |^{p(x)} dx &\leq d^{k(p^+-1)} \int_\Omega \left[1+p(x)(d^k-1)\right] |\nabla \hat{u}_R|^{p(x)}\, \hat{u}_R^{p(x)(d^k-1)} dx \\ &\leq d^{k(p^+-1)} \int_\Omega (d^k-1)\, |\nabla \hat{u}_R|^{p(x)-1}\, |\nabla p|\, \hat{u}_R^{1+p(x)(d^k-1)}\, |\ln \hat{u}_R| \; dx \\ &\quad + \lambda_R^* \|c\|_\infty (\alpha^+ +1) d^{k(p^+-1)} \int_\Omega |\hat{u}_R|^{\alpha(x)+1+p(x)(d^k-1)} |\hat{v}|^{\beta(x)+1} dx \\ &\leq \hat{C}_p d^{k(p^+-1)} \left[\int_\Omega \hat{u}_R^{1+p(x)(d^k-1)} dx \right. \\ &\qquad \left. + \lambda_R^* \|c\|_\infty (\alpha^+ +1) \int_\Omega |\hat{u}_R|^{\alpha(x)+1+p(x)(d^k-1)} |\hat{v}_R|^{\beta(x)+1} dx \right],\end{aligned} \tag{4.17}$$

where $\hat{C}_p = \max\{1, \hat{C}M_p\}$.

Thanks to the use of the hypothesis (H.3), the embeddings $L^{\pi_p(x)}(\Omega) \hookrightarrow L^{dp(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\pi_p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{dp(x)}(\Omega)$ are continuous and thus, for any $z \in W_0^{1,p(x)}(\Omega)$. We can conclude that there exists a constant $K > 0$ so that

$$\|z\|_{p(x)d} \leq K\|z\|_{1,p(x)}. \tag{4.18}$$

From (3.13) and through the mean value theorem observe that there exists $\xi_k \in \Omega$ such that

$$\begin{aligned} 1 &= \int_\Omega \left| \frac{|\hat{u}_R|}{\|\hat{u}_R\|_{p(x)d^{k+1}}} \right|^{p(x)d^{k+1}} dx \\ &= \int_\Omega \left| \frac{|\hat{u}_R|^{d^k}}{\||\hat{u}_R|^{d^k}\|_{p(x)d}} \right|^{p(x)d} \times \left(\frac{\||\hat{u}_R|^{d^k}\|_{p(x)d}}{\||\hat{u}_R|\|^{d^k}_{p(x)d^{k+1}}} \right)^{p(x)d} dx = \left(\frac{\||\hat{u}_R|^{d^k}\|_{p(x)d}}{\||\hat{u}_R|\|^{d^k}_{p(x)d^{k+1}}} \right)^{p(\xi_k)d}, \end{aligned}$$

which leads to

$$\||\hat{u}_R|^{d^k}\|_{p(x)d} = \|\hat{u}_R\|^{d^k}_{p(x)d^{k+1}}. \tag{4.19}$$

Recalling from (2.10) that for every $z \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$

$$\int_\Omega \left| \frac{|\nabla z|}{\|z\|_{1,p(x)}} \right|^{p(x)} dx = 1. \tag{4.20}$$

Applying (4.18) and (4.20) to $z = \hat{u}_R^{d^k}$, besides the mean value theorem and (4.19), there exists $x_k \in \Omega$ such that

$$\begin{aligned} K^{p(x_k)} \int_\Omega |\nabla(\hat{u}_R)^{d^k}|^{p(x)} dx &= K^{p(x_k)} \|\hat{u}_R^{d^k}\|^{p(x_k)}_{1,p(x)} = K^{p(x_k)} \|\hat{u}_R^{d^k}\|^{p(x_k)}_{1,p(x)} \\ &\geq \|\hat{u}_R^{d^k}\|^{p(x_k)}_{dp(x)} = \|\hat{u}_R\|^{p(x_k)d^k}_{p(x)d^{k+1}} = \left(\|\hat{u}_R\|^{p(x_k)d^{k+1}}_{p(x)d^{k+1}} \right)^{\frac{1}{d}}. \end{aligned} \tag{4.21}$$

Combining (4.17), (4.21) with Lemma 4.2, we get the following estimate

$$\|\hat{u}_R\|^{p(x_k)d^{k+1}}_{p(x)d^{k+1}} \leq C_1 d^{kd(p^+-1)} \left(\max\left\{ \|\hat{u}_R\|^{p(x_k)d^k}_{p(x)d^k}, \|\hat{v}_R\|^{q(y_k)d^k}_{q(x)d^k} \right\} \right)^d, \tag{4.22}$$

Acting also in (4.4) with $\psi = \hat{v}^{1+q(x)(d^k-1)}$ and repeating the argument above, we obtain

$$\|\hat{v}_R\|^{q(x_k)d^{k+1}}_{q(x)d^{k+1}} \leq C_2 d^{kd(q^+-1)} \left(\max\left\{ \|\hat{v}_R\|^{p(x_k)d^k}_{p(x)d^k}, \|\hat{v}_R\|^{q(y_k)d^k}_{q(x)d^k} \right\} \right)^d, \tag{4.23}$$

where C_1 and C_2 are two strictly positive constants.

So, it derives

$$\begin{aligned} \max\left\{ \|\hat{u}_R\|^{p(x_{k+1})d^{k+1}}_{p(x)d^{k+1}}, \|\hat{v}_R\|^{q(y_{k+1})d^{k+1}}_{q(x)d^{k+1}} \right\} &\leq C_3 d^{k\hat{d}} \left(\max\left\{ \|\hat{v}_R\|^{p(x_k)d^k}_{p(x)d^k}, \|\hat{v}_R\|^{q(y_k)d^k}_{q(x)d^k} \right\} \right)^d \\ &\leq C_3 d^{k\hat{d}} \left(\max\left\{ \|\hat{v}_R\|^{p(x_k)d^k}_{p(x)d^k}, \|\hat{v}_R\|^{q(y_k)d^k}_{q(x)d^k} \right\} \right)^d, \end{aligned} \tag{4.24}$$

where $\hat{d}$ satisfies (4.2) and $C_3 = \max\{C_1, C_2\}$.

Before continuing, we distinguish the cases where $\|\hat{u}_R\|_{p(x)d^{k+1}}$, $\|\hat{v}_R\|_{q(x)d^{k+1}}$, $\|\hat{u}_R\|_{p(x)d^k}$ and $\|\hat{v}_R\|_{p(x)d^k}$ are each either less than one or either greater than one. Using (H.4) and (4.1) we obtain

$$\ln\Big(\max\Big\{\|\hat{u}_R\|_{p(x)d^{k+1}}^{d^{k+1}}, \|\hat{v}_R\|_{q(x)d^{k+1}}^{d^{k+1}}\Big\}\Big) \leq \ln(C_3 d^{k\hat{d}}) + \hat{d}\ln\Big(\max\Big\{\|\hat{u}_R\|_{p(x)d^k}^{d^k}, \|\hat{v}_R\|_{q(x)d^k}^{d^k}\Big\}\Big). \quad (4.25)$$

Now set

$$E_k = \max\Big\{\ln\|\hat{u}_R\|_{p(x)d^k}^{d^k}, \ln\|\hat{v}_R\|_{q(x)d^k}^{d^k}\Big\} \quad \text{and} \quad \rho_k = ak + b, \quad (4.26)$$

with

$$a = \ln d^{\hat{d}} \qquad b = \ln C_3. \quad (4.27)$$

Then the recursive rule (4.25) becomes

$$E_{k+1} \leq \rho_k + \hat{d}E_k, \quad (4.28)$$

which in turn gives

$$E_{k+1} \leq E\hat{d}^k, \quad (4.29)$$

where

$$E = E_1 + \frac{b}{\hat{d}-1} + \frac{a\hat{d}}{(\hat{d}-1)^2}. \quad (4.30)$$

Indeed, using (4.28), (4.26) and Lemma 2.6, we get

$$\begin{aligned} E_{k+1} &\leq \rho_k + \hat{d}E_k \leq E_{k+1} \leq \rho_k + \hat{d}\rho_{k-1} + \hat{d}^2 E_{k-1} \\ &\leq \rho_k + \hat{d}\rho_{k-1} + \hat{d}^2\rho_{k-2} + \hat{d}^3 E_{k-2} \\ &\vdots \\ &\leq \sum_{i=0}^{k-1} \hat{d}^i \rho_{k-i} + \hat{d}^k E_1 = \hat{d}^k\left(a\sum_{i=1}^{k}\frac{i}{\hat{d}^i} + b\sum_{i=1}^{k}\frac{1}{\hat{d}^i} + E_1\right) \\ &\leq \hat{d}^k\left(\frac{a\hat{d}}{(\hat{d}-1)^2} + \frac{b}{\hat{d}-1} + E_1\right) = \hat{d}^k E. \end{aligned} \quad (4.31)$$

Here Lemma 2.6 is applied choosing $s = 1/\hat{d} < 1$ and $r = k+1$. So on, according to (4.26) and (4.29), its follows that

$$\max\{\|\hat{u}_R\|_{p(x)d^k}, \|\hat{v}_R\|_{q(x)d^k}\} \leq e^{E\frac{\max\{p^+,q^+\}^{k-1}}{d}}. \quad (4.32)$$

We fix k in $\mathbb{N}$, then we conclude that the assertion (4.3) in Lemma 4.3 holds. The proof of Lemma 4.3 is complete. □

Now, let us end the proof of Theorem 2.3 by showing that $(\hat{u}_R, \hat{v}_R)$ is bounded in Ω.

Lemma 4.4. *Let $(\hat{u}_R, \hat{v}_R)$ be a solution of (1.1) corresponding to the eigenvalue λ_R^*. Assume that hypotheses (H.1)–(H.4) hold. Then, $\hat{u}_R$ and $\hat{v}_R$ are bounded in Ω.*

Proof. Argue by contradiction. It means that we suppose that for all $L > 0$, there exists $\Omega_L \subset \Omega$, $|\Omega_L| > 0$ such that for all $x \in \Omega_L$ we have $|\hat{u}_R(x)| > L$. Fix k and choose L large enough so that

$$\frac{p^-\ln L}{p^+ E\max\{p^+,q^+\}^{k+1}} > 1. \quad (4.33)$$

From Lemma 2.4 we get

$$L^{p^- d^{k+1}} |\Omega_L| \le \int_{\Omega_L} L^{p(x)d^{k+1}}\, dx \le \int_{\Omega_L} |\hat{u}_R|^{p(x)d^{k+1}}\, dx \le \int_{\Omega} |\hat{u}_R|^{p(x)d^{k+1}}\, dx \le \max\{\|\hat{u}_R\|^{p^+ d^{k+1}}_{p(x)d^{k+1}}, \|\hat{u}_R\|^{p^- d^{k+1}}_{p(x)d^{k+1}}\}.$$

By (4.26), (4.29), and (4.1) it follows that

$$d^{k+1} p^- \ln L + \ln |\Omega_L| \le p^+ E_{k+1} \le p^+ E \hat{d}^k$$

After using (4.33) and dividing by $\hat{d}^{k+1}$, we get

$$1 + \frac{\ln |\Omega_L|}{\hat{d}^{k+1}} < 1/\hat{d}. \tag{4.34}$$

We choose k sufficiently large in (4.34). This forces $\hat{d} < 1$, which contradicts (4.2). This proves Lemma 4.4. □

Next, we show that $\hat{u}_R$ and $\hat{v}_R$ are strictly positive in Ω.

Lemma 4.5. *Let $(\hat{u}_R, \hat{v}_R)$ be a solution of (1.1) corresponding to the eigenvalue λ^*_R. Then, the following assertions hold:*

1. *$\hat{u}_R > 0$ (resp. $\hat{v}_R > 0$) in Ω.*
2. *There exists $\delta \in (0, 1)$ such that $\hat{u}_R$ is of class $C^{1,\delta}(\overline{\Omega})$.*

Proof.

Step 1. $\hat{u}_R \ge 0$ (resp. $\hat{v}_R \ge 0$ in Ω)

First, observe that

$$|u| = \max(u, 0) + \min(u, 0) \in W^{1,p(x)}_0(\Omega)$$

and

$$|\nabla |u|| \le |\nabla \max(u, 0)| + |\nabla \min(u, 0)| \le |\nabla u|.$$

Then it turns out that

$$\mathcal{A}(|\hat{u}_R|, |\hat{v}_R|) \le \mathcal{A}(\hat{u}_R, \hat{v}_R) \text{ and } \mathcal{B}(|\hat{u}_R|, |\hat{v}_R|) = \mathcal{B}(\hat{u}_R, \hat{v}) = R.$$

Thereby (2.6) and (3.15), it follows that

$$\mathcal{A}(|\hat{u}_R|, |\hat{v}_R|) \le \mathcal{A}(\hat{u}_R, \hat{v}_R) = R\lambda^*_R \le \mathcal{A}(|\hat{u}_R|, |\hat{v}_R|),$$

which implies that $\mathcal{A}(|\hat{u}_R|, |\hat{v}_R|) = R\lambda^*_R$, showing that $(|\hat{u}_R|, |\hat{v}_R|)$ is a solution of (1.1). Therefore, we can assume that $\hat{u}_R, \hat{v}_R \ge 0$ in Ω.

Step 2. $\hat{u}_R > 0$ (resp. $\hat{v}_R > 0$) in Ω

Inspired by the ideas in [17], let $m > 0$ be a constant such that $h(\cdot) \in C^2(\overline{\partial\Omega}_{3m})$, with $\overline{\partial\Omega}_{3m} = \{x \in \overline{\Omega} : h(x) \le 3m\}$. Define the functions

$$\mathcal{U}(x) = \begin{cases} e^{\kappa h(x)} - 1 & \text{if } h(x) < \sigma_1 \\ e^{\kappa h(x)} - 1 + \kappa e^{\kappa \sigma_1} \int_{\sigma_1}^{h(x)} \left(\frac{2m-t}{2m-\sigma_1}\right)^{\frac{2}{p^- - 1}} dt & \text{if } \sigma_1 \le h(x) < 2\sigma_1 \\ e^{\kappa h(x)} - 1 + \kappa e^{\sigma_1} \int_{\sigma_1}^{2m} \left(\frac{2m-t}{2m-\sigma_1}\right)^{\frac{2}{p^- - 1}} dt & \text{if } 2\sigma_1 \le h(x) \end{cases}$$

and

$$\mathcal{V}(x) = \begin{cases} e^{\kappa h(x)} - 1 & \text{if } h(x) < \sigma_2, \\ e^{\kappa h(x)} - 1 + \kappa e^{\kappa\sigma_2} \int_{\sigma_2}^{h(x)} (\frac{2m-t}{2m-\sigma_2})^{\frac{2}{q^- - 1}} dt & \text{if } \sigma_2 \leq h(x) < 2\sigma_2, \\ e^{\kappa h(x)} - 1 + \kappa e^{\kappa\sigma_2} \int_{\sigma_2}^{2m} (\frac{2m-t}{2m-\sigma_2})^{\frac{2}{q^- - 1}} dt & \text{if } 2\sigma_2 \leq h(x), \end{cases}$$

where $(\sigma_1, \sigma_2) = (\frac{\ln 2}{\kappa p^+}, \frac{\ln 2}{\kappa q^+})$ and $\kappa > 0$ is a parameter. Similar calculations as in [17, pages 11 and 12] furnish

$$-\Delta_{p(x)}(\mu_1 \mathcal{U}) \leq \lambda_R^* c(x)(\alpha(x)+1)(\mu_1 \mathcal{U})^{\alpha(x)} \hat{v}_R^{\beta(x)+1} \quad \text{in } \Omega \tag{4.35}$$

and

$$-\Delta_{q(x)}(\mu_2 \mathcal{V}) \leq \lambda_R^* c(x)(\beta(x)+1) \hat{u}_R^{\alpha(x)+1} (\mu_2 \mathcal{V})^{\beta(x)} \quad \text{in } \Omega, \tag{4.36}$$

where $\mu_1 = \exp\big(\kappa \frac{1-p^-}{\max_{\overline{\Omega}} |\nabla p|+1}\big)$ and $\mu_2 = \exp\big(\kappa \frac{1-q^-}{\max_{\overline{\Omega}} |\nabla q|+1}\big)$, provided that $\kappa > 0$ is large enough.

Now, for any $(z, w) \in X_0^{p(x),q(x)}(\Omega)$, set

$$\mathcal{L}_p(z,w) = -\Delta_{p(x)} z - \lambda_R^* c(x)(\alpha(x)+1) z|z|^{\alpha(x)-1} |w|^{\beta(x)+1}$$

and

$$\mathcal{L}_q(z,w) = -\Delta_{q(x)} w - \lambda_R^* c(x)(\beta(x)+1) |z|^{\alpha(x)+1} w|w|^{\beta(x)-1},$$

(4.35) and (4.36) may be formulated respectively as follows

$$\mathcal{L}_p(\mu_1 \mathcal{U}, \hat{v}_R) \leq 0 \quad \text{and} \quad \mathcal{L}_q(\hat{u}_R, \mu_2 \mathcal{V}) \leq 0, \quad \text{in } \Omega.$$

Hence, from the above notation, we get

$$\mathcal{L}_p(\mu_1 \mathcal{U}, \hat{v}_R) \leq 0 \leq \mathcal{L}_p(\hat{u}_R, \hat{v}_R) \quad \text{in } \Omega$$

and

$$\mathcal{L}_q(\hat{u}_R, \mu_2 \mathcal{V}) \leq 0 \leq \mathcal{L}_q(\hat{u}_R, \hat{v}_R) \quad \text{in } \Omega.$$

Since $\mu_1 \mathcal{U} = \hat{u}_R = 0$ and $\mu_2 \mathcal{V} = \hat{v}_R = 0$ on $\partial\Omega$, we are allowed to apply [21, Lemma 2.3] and we deduce that

$$\hat{u}_R \geq \mu_1 \mathcal{U} > 0 \quad \text{and} \quad \hat{v}_R \geq \mu_2 \mathcal{V} > 0 \quad \text{in } \Omega.$$

Thereby the positivity of $(\hat{u}_R, \hat{v}_R)$ in Ω is proven.

To end the proof of Lemma 4.4, we claim a regularity property for $\hat{u}_R$ and $\hat{v}_R$.

Step 3. Regularity property

For $p, q \in C^1(\overline{\Omega}) \cap C^{0,\theta}(\overline{\Omega})$ for certain $\theta \in (0,1)$, owing to [7, Theorem 1.2] the solution $(\hat{u}_R, \hat{v}_R)$ belongs to $C^{1,\delta}(\overline{\Omega}) \times C^{1,\delta}(\overline{\Omega})$ for certain $\delta \in (0,1)$. This completes the proof. □

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