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Characterization of self-adjoint domains for regular even order C-symmetric differential operators

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Abstract. Let C be a skew-diagonal constant matrix satisfying $C^{-1} = -C = C^*$. We characterize the self-adjoint domains for regular even order C-symmetric differential operators with two-point boundary conditions. The previously known characterizations are a special case of this one.

Keywords: C-symmetric, differential operators, boundary conditions, self-adjoint domains.

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1 Introduction

Consider the differential equation

$$My = \lambda wy \quad \text{on } J = (a,b),\ -\infty \le a < b \le \infty \tag{1.1}$$

with boundary conditions

$$AY(a) + BY(b) = 0, \qquad A, B \in M_n(\mathbb{C}), \tag{1.2}$$

where $M_n(\mathbb{C})$ denotes the set of $n \times n$ matrices of complex numbers. (This notation is standard and should not conflict with the notation M for differential expressions.)

In this paper, for regular endpoints a, b, any $n = 2k$, $k > 1$, and any skew-diagonal constant matrix C which satisfies

$$C^{-1} = -C = C^*, \tag{1.3}$$

we generate symmetric differential expressions $M = M_Q$ and characterize the boundary conditions (1.2) which determine self-adjoint operators S in $L^2(J, w)$ satisfying $S_{\min} \subset S = S^* \subset S_{\max}$. Here the matrix $Q \in Z_n(J, \mathbb{C})$ is a C-symmetric matrix in the sense that

$$Q = -C^{-1}Q^*C \tag{1.4}$$

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and $M = M_Q$ is generated by Q.

Such a characterization is well known [17] when

$$C = E = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n. \tag{1.5}$$

We prove the following theorem:

Theorem 1.1. *Let $Q \in Z_n(J, \mathbb{C})$, $n = 2k$, $k = 1, 2, 3, \dots$, let $M = M_Q$, let w be a weight function. Suppose a, b are regular endpoints. Assume that C satisfies (1.4) and Q satisfies the C-symmetry condition:*

$$Q = -C^{-1}Q^*C.$$

Then the linear manifold $D(S)$ defined by

$$D(S) = \{y \in D_{\max};\ (1.2) \text{ holds}\} \tag{1.6}$$

is the domain of a self-adjoint extension S of $S_{\min}$ (or restriction of $S_{\max}$) if and only if

$$\operatorname{rank}(A : B) = n \quad \text{and} \quad ACA^* = BCB^*. \tag{1.7}$$

Proof. The proof will be given below. □

Remark 1.2. We find it remarkable that the self-adjoint boundary conditions are characterized by the same matrix C which generates the symmetric operators M.

The definitions of $Z_n(J, \mathbb{C})$, the quasi-derivatives $y^{[j]}$, $j = 0, \dots, n-1$, and M_Q will be given in Section 2, the proof of the theorem in Section 3 and examples of matrices C and C-self-adjoint boundary conditions are given in Section 4. See [17] for definitions of $S_{\min}$, $S_{\max}$, $D_{\min}$, $D_{\max}$, etc.

2 C-symmetric expressions

In this section, we develop a general form of the C-symmetric quasi-differential expression M with complex coefficient of any even order $n = 2k$, $k \geq 1$ on an interval $J = (a, b)$, $-\infty < a < b < \infty$.

Let

$$\begin{aligned} Z_n(J) := \Big\{ Q = (q_{r,s})_{r,s=1}^n :\ & Q \in M_n(L_{\mathrm{loc}}(J)); \\ & q_{r,r+1} \neq 0 \text{ a.e. } J,\ q_{r,r+1}^{-1} \in L_{\mathrm{loc}}(J),\ 1 \leq r \leq n-1; \\ & q_{r,s} = 0 \text{ a.e. } J,\ 2 \leq r+1 < s \leq n \\ & q_{r,s} \in L_{\mathrm{loc}}(J),\ s \neq r+1,\ 1 \leq r \leq n-1 \Big\}. \end{aligned}$$

For $Q \in Z_n(J)$, in [3] define the quasi-derivatives $y^{[r]}$ $(0 \leq r \leq n)$ below:

$$\begin{aligned} & V_0 := \{y : J \to \mathbb{C},\ y \text{ is measurable}\}, \qquad y^{[0]} := y\ (y \in V_0), \\ & V_r := \{y \in V_{r-1} : y^{[r-1]} \in (AC_{\mathrm{loc}}(J))\}, \\ & y^{[r]} = q_{r,r+1}^{-1} \left\{ y^{[r-1]'} - \textstyle\sum_{s=1}^r q_{r,s} y^{[s-1]} \right\} \qquad (y \in V_r,\ r = 1, 2, \dots, n), \end{aligned}$$

where $q_{n,n+1} = 1$. Finally we set

$$My = i^n y^{[n]}, \qquad y \in V_n,$$

these expressions $M = M_Q$ are generated by or associated with Q and for V_n we also use the notations $D(Q)$ and $V(M)$. Since the quasi-derivatives depends on Q, we sometimes write $y_Q^{[r]}$ instead of $y^{[r]}$, $r = 1, 2, \dots, n$.

Remark 2.1. If $Q \in Z_n(J)$ has the format

$$\begin{aligned} q_{r,r+1} &= 1, \qquad r = 1, 2, \dots, n-1, \\ q_{r,s} &= 0, \qquad 1 \le r \le n-1,\ s \ne r+1, \end{aligned} \tag{2.1}$$

then M_Q will reduce to an ordinary differential expression M with $y^{[r]} = y^{(r)}$, $r = 1, 2, \dots, n-1$, the quasi-derivatives and ordinary derivatives are equal for $r = 1, 2, \dots, n-1$, when $y \in D(Q)$, and moreover

$$M_Q y = i^n y^{[n]} = i^n \left\{ y^{(n)} - \textstyle\sum_{s=1}^{n} q_{n,s} y^{(s-1)} \right\}. \tag{2.2}$$

Hence, in this case, M_Q is merely an ordinary differential expression M, see (1.1), with $p_n(x) = i^n$ on J. And conversely every such differential expression can be rewritten in the form of a quasi-differential expression.

In [11, 17] the expression M is called a Lagrange symmetric (or just a symmetric) differential expression if the matrix Q satisfies

$$Q = -E_n^{-1} Q^* E_n, \tag{2.3}$$

where E_n is the symplectic matrix of order n given by (1.5). However, (2.3) is not generally satisfied by the companion-type matrices (2.1).

For the Lagrange symmetric M_Q, the Green's formula has the form

$$\int_{[\alpha,\beta]} \{My\bar{z} - y\overline{Mz}\}dx = [y,z](\beta) - [y,z](\alpha) \qquad (y,\ z \in D(Q))$$

for any compact sub-interval $[\alpha, \beta]$ of (a, b). Here the skew-symmetric sesquilinear form $[\cdot, \cdot]$ maps $D(Q) \times D(Q) \to \mathbb{C}$. The explicit form of $[\cdot, \cdot]$ is given by

$$[y,z](x) = i^n \sum_{r=1}^{n} (-1)^{r-1} y^{[n-r]}(x) \overline{z^{[r-1]}(x)} = (-1)^{k+1} Z^* E_n Y, \tag{2.4}$$

where $Z(x)$, $Y(x)$ are the column vector function

$$Y = (y^{[0]}(x)\ y^{[1]}(x)\ \cdots\ y^{[n-1]}(x))^T,\ Z = (z^{[0]}(x)\ z^{[1]}(x)\ \cdots\ z^{[n-1]}(x))^T,\ x \in [\alpha, \beta].$$

The expression $w^{-1} M_Q = \lambda y$, $\lambda \in \mathbb{R}$ defines or generates a linear operator S, once the domain $D(S)$ is suitably $S_{\min}$ with their respective domains $D_{\max}$ and $D_{\min}$. In general, the minimal operator $S_{\min}$ is a nonself-adjoint operator, otherwise $S_{\min} = S_{\min}^* = S_{\max}$. So if S is a self-adjoint operator on $D(S)$, then $S_{\min} \subset S = S^* \subset S_{\max}$, and

$$\int_J \{My\bar{z} - y\overline{Mz}\}dx = 0 \tag{2.5}$$

for all $y,\ z \in D_{\max}$.

The GKN (Glazeman–Krein–Naimark) Theorem [4] which characterizes all self-adjoint extensions of $T_{Q,0}$ in H.

Theorem 2.2 (GKN)**.** *Let d be the deficiency index of minimal operator S_{min}, then a linear submanifold $D(S) \subset D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ in $H = L^2(J, w)$ if and only if there exist functions $v_1, v_2, \ldots, v_d$ in $D_{\max}$ such that*

(i) $v_1, v_2, \cdots, v_d$ are linearly independent modulo D_{min}, i.e. no nontrivial linear combination of $v_1, v_2, \ldots, v_d$ is in D_{min}.

(ii) $[v_i, v_j](b) - [v_i, v_j](a) = 0,\ i,\ j = 1, 2, \cdots, d$;

(iii) $D(S) = \{y \in D_Q : [y, v_j](b) - [y, v_j](a) = 0,\ j = 1, 2, \cdots, d\}$.

The GKN characterization depends on the maximal domain functions $v_j, j = 1, \ldots, d$. These functions depend on the coefficients of the differential equation and this dependence is implicit and complicated.

When both endpoints of J are regular, this dependence can be eliminated and an explicit characterization can be given in terms of two-point boundary conditions involving only solutions and their quasi-derivatives at the endpoints. This has the form:

$$D(S) = \{y \in D_{\max} : AY(a) + BY(b) = 0\}, \tag{2.6}$$

where the complex $n \times n$ matrices A, B satisfy

$$\operatorname{rank}(A : B) = n, \tag{2.7}$$

and

$$AE_nA^* = BE_nB^*. \tag{2.8}$$

It is much more explicit than the GKN Theorem and it can lead to a canonical form for self-adjoint boundary conditions such as the well known form in the second order Sturm–Liouville case, see formulas (4.2.3), (4.2.4) and (4.2.7) in [20]. Through the long history of Sturm–Liouville problems, these canonical representations have led to a comprehensive understanding, both theoretically and numerically, of the dependence of the eigenvalues on the boundary conditions. In [10, 15] canonical representations for regular problems of $n = 4$ are known. We will also go on with these canonical forms in our subsequent papers.

Notice that (2.4) and (2.8) hold for the constant matrix E_n satisfying $E_n^{-1} = -E_n = E_n^*$, this paper considers these forms for every general regular skew-diagonal constant matrix $C = (c_{r,s})_{r,s=1}^n$ satisfying $C^{-1} = -C = C^*$. Thus we have the following definition.

Definition 2.3. Let $Q \in Z_n(J)$. Define

$$\begin{aligned} &y^{[0]} := y, \quad y \in V_0, \\ &y_Q^{[r]} = q_{r,r+1}^{-1}\left\{y_Q^{[r-1]'} - \sum_{s=1}^{r} q_{r,s}y_Q^{[s-1]}\right\}, \quad y \in V_r,\ r = 1, \ldots, n, \end{aligned} \tag{2.9}$$

where $q_{n,n+1} := c_{n,1}$.

We set

$$My = M_Qy = i^n y^{[n]}, \tag{2.10}$$

with the domain $D(M_Q)$, which we usually write as $D(Q)$. The expression $M = M_Q$ is called the quasi-differential expression generated by or associated with Q. Suppose that

$$Q = Q^+ = -C_n^{-1}Q^*C_n, \tag{2.11}$$

i.e.,

$$q_{r,s} = c_{r,n+1-r}\overline{q}_{n+1-s,n+1-r}c_{n+1-s,s}, \tag{2.12}$$

then Q is said to be a C-symmetric matrix. In this case M_Q is called a C-symmetric quasi-differential expression. Note that $Q^{++} = Q$, $M_Q^{++} = M_Q$, where $M_Q^{+} := M_{Q^+}$, we call Q^+ the C-adjoint matrix of Q and M_Q^+ the C-adjoint expression of M_Q.

It is of special interest to note that if $C_n = E_n$, then

$$Q = -E_n^{-1}Q^*E_n,$$

and the expression $M = M_Q$ is reduced to the Lagrange symmetric differential expression.

Remark 2.4. What we really need to emphasize is that the constant matrix C_n is not only a skew-diagonal matrix satisfying

$$C_n^{-1} = -C_n = C_n^*, \tag{2.13}$$

but plays a key role in the construction of symmetric quasi-differential expressions as well as in the self-adjoint domain characterization for C-symmetric differential operators. In addition, the C-symmetric condition on the matrix Q means that Q is invariant under the composition of the following three operators: "flips" about the secondary diagonal, conjugation, multiplying $q_{r,s}$ by $(-1)^{r+s+1}$ (i.e., changing the sign of $q_{r,s}$ if $r+s$ is even).

Remark 2.5. The operator $M : D(Q) \longrightarrow L_{\mathrm{loc}}(J)$ is linear.

From Definition 2.3 we have the symmetric condition

$$Q = -C_n^{-1}Q^*C_n.$$

Set

$$C_n = \begin{pmatrix} 0_{k\times k} & C_{12} \\ C_{21} & 0_{k\times k} \end{pmatrix}, \qquad C_{21}, C_{12} \in M_k(\mathbb{C}).$$

Then

$$C_{21} = -C_{12}^*,\ C_{12}^{-1} = C_{12}^*,$$

i.e.,

$$C_n = \begin{pmatrix} 0_{k\times k} & C_{12} \\ -C_{12}^* & 0_{k\times k} \end{pmatrix} \tag{2.14}$$

and C_{12} is a skew-diagonal unitary matrix, that is,

$$\begin{aligned} c_{r,s}\overline{c}_{r,s} &= 1, \quad \text{for } r+s = n+1,\ 1 \le r \le k, \\ c_{r,s} &= 0, \quad \text{otherwise}. \end{aligned} \tag{2.15}$$

Set

$$c_{r,n-r+1} = e^{i\theta_r}, \qquad -\pi < \theta_r \le \pi,\ r = 1, 2, \ldots, k,$$

Thus C_n can be rewritten as

$$C_n = \text{skew-diagonal}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_k}, -e^{-i\theta_k}, \ldots, -e^{-i\theta_2}, -e^{-i\theta_1}). \tag{2.16}$$

Let

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \in Z_n(J),$$

$Q_{ij} \in M_k(\mathbb{C}),\ i,\ j = 1, 2$, then

$$Q^+ = \begin{pmatrix} -C_{12}Q_{22}^*C_{12}^* & C_{12}Q_{12}^*C_{12} \\ C_{12}^*Q_{21}^*C_{12}^* & -C_{12}^*Q_{11}^*C_{12} \end{pmatrix}.$$

From $Q = Q^+$, we have the C-symmetric matrix

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & -C_{12}^*Q_{11}^*C_{12} \end{pmatrix}, \tag{2.17}$$

where $Q_{12} = C_{12}Q_{12}^*C_{12},\ Q_{21} = C_{12}^*Q_{21}^*C_{12}^*$, i.e., $C_{12}^*Q_{12},\ C_{12}Q_{21}$ are symmetric matrices.

By direct calculation, the C-symmetric matrices $Q \in Z_n(J)$ have the form

$$\begin{pmatrix} q_{11} & q_{12} & 0 & \cdots & \cdots & 0 \\ q_{21} & q_{22} & q_{23} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{n-2,1} & q_{n-2,2} & \cdots & \cdots & -\bar{c}_{3,n-2}c_{2,n-1}\bar{q}_{23} & 0 \\ q_{n-1,1} & q_{n-1,2} & \cdots & \cdots & -\bar{q}_{22} & -\bar{c}_{2,n-1}c_{1,n}\bar{q}_{12} \\ q_{n,1} & \bar{c}_{1,n}\bar{c}_{2,n-1}\bar{q}_{n-1,1} & \cdots & \cdots & -\bar{c}_{1,n}c_{2,n-1}\bar{q}_{21} & -\bar{q}_{11} \end{pmatrix}, \tag{2.18}$$

where $q_{n,1} = \bar{c}_{1,n}^2\bar{q}_{n,1},\ q_{n-1,2} = \bar{c}_{2,n-1}^2\bar{q}_{n-1,2},\ \cdots,\ q_{k+1,k} = \bar{c}_{k,k+1}^2\bar{q}_{k+1,k},\ q_{k,k+1} = c_{k,k+1}^2\bar{q}_{k,k+1}$.

The self-adjoint operators S in the Hilbert space $L^2(J, w)$ generated by the equation

$$My = M_Q y = \lambda w y \quad \text{on } J,$$

where Q has the form (2.18). Then S satisfy

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{2.19}$$

So it is clear that these operators S differ from each other only by their domains. These domains $D(S)$ are characterized by Theorem 1.1 and the proof is given in next section.

3 Characterization of self-adjoint domains

In this section, we prove the main results in this paper: characterization of self-adjoint domains for general regular even order C-symmetric quasi-differential operators. Our starting point for this characterization is the Lagrange identity which plays a critical important role in the characterization of self-adjoint domains.

To prove Lagrange identity, we use the following two lemmas.

Lemma 3.1. *Let $Q_n,\ P_n \in Z_n(J)$. Let $F,\ G$ be $n \times 1$ function matrices on J. If $Y' = Q_nY + F$ and $Z' = P_nZ + G$ and the constant matrix $C_n \in M_n(\mathbb{C})$ satisfies*

$$C_n^* = -C_n = C_n^{-1}.$$

Then

$$(Z^*C_nY)' = Z^*(P_n^*C_n + C_nQ_n)Y + Z^*C_nF + G^*C_nY, \tag{3.1}$$

where

$$Y = \left(y^{[0]}\ y^{[1]}\ \cdots\ y^{[n-1]}\right)^T, \qquad Z = \left(z^{[0]}\ z^{[1]}\ \cdots\ z^{[n-1]}\right)^T.$$

Proof. From the differentiation of function matrix, we have

$$\begin{aligned}(Z^*C_nY)' &= (Z^*)'C_nY + Z^*C_n'Y + Z^*C_nY' \\ &= (Z')^*C_nY + Z^*C_nY' \\ &= (P_nZ+G)^*C_nY + Z^*C_n(Q_nY+F) \\ &= (Z^*P_n^* + G^*)C_nY + Z^*C_nQ_nY + Z^*C_nF \\ &= Z^*(P_n^*C_n + C_nQ_n)Y + G^*C_nY + Z^*C_nF.\end{aligned}$$

This completes the proof. □

Lemma 3.2. *Assume $Q_n \in Z_n(J)$ and $P_n = -C_n^{-1}Q_n^*C_n$, then $P_n \in Z_n(J)$ and if $Y' = Q_nY + F$ and $Z' = P_nZ + G$ on J, where F, G be $n\times 1$ function matrices on J. Then*

$$(Z^*C_nY)' = Z^*C_nF + G^*C_nY. \tag{3.2}$$

Proof. Let $Q_n = (q_{r,s})_{r,s=1}^n \in Z_n(J)$ and $P_n = (p_{r,s})_{r,s=1}^n = -C_n^{-1}Q_n^*C_n$, then we have

$$p_{r,s} = \sum_{l=1}^{n}(\sum_{j=1}^{n} c_{r,j}\bar{q}_{l,j})c_{l,s} = c_{r,n-r+1}\bar{q}_{n-s+1,n-r+1}c_{n-s+1,s}, \qquad r,s = 1,2,\cdots,n.$$

So for $1 \le r \le n-1$,

$$p_{r,r+1} = c_{r,n-r+1}\bar{q}_{n-r,n-r+1}c_{n-r,r+1}$$

is invertible a.e. on J.

Since for $2 \le r+1 < s \le n$, $r+1-s = (n-s-1)+1-(n-r+1) < 0$, $q_{n-s-1,n-r+1} = 0$, then

$$p_{r,s} = c_{r,n-r+1}\bar{q}_{n-s+1,n-r+1}c_{n-s+1,s} = 0.$$

This concludes that $P_n \in Z_n(J)$.

From C_n satisfy (2.13), and $C_nP_n = -Q_n^*C_n = -(C_n^*Q_n)^*$, we have $C_nQ_n = -(C_n^*Q_n) = (C_nP_n)^* = -P_n^*C_n$. Hence from (3.1) in Lemma 3.1, (3.2) is established. □

We obtain a new general version of the Lagrange identity as follows.

Theorem 3.3 (Lagrange identity)**.** *Let $Q \in Z_n(J)$, and $P = -C_n^{-1}Q^*C_n$, C_n is defined by* (2.14) *(or* (2.16)*). Then $P \in Z_n(J)$ and for any $y \in D(Q)$ and $z \in D(P)$, we have*

$$\bar{z}M_Qy - y\overline{M_Pz} = [y,z]', \qquad [y,z] = \widetilde{Z}^*C_n\widetilde{Y}, \tag{3.3}$$

and

$$\widetilde{Z}^*C_n\widetilde{Y} = \sum_{r=0}^{n-1} c_{n-r,r+1}\,\overline{z_P^{[n-r-1]}}\,y_Q^{[r]} = \sum_{r=1}^{k}\left\{c_{r,n-r+1}\,\overline{z_P^{[r-1]}}\,y_Q^{[n-r]} - \bar{c}_{r,n-r+1}\,\overline{z_P^{[n-r]}}\,y_Q^{[r-1]}\right\}, \tag{3.4}$$

where $\widetilde{Y} = (y^{[0]}\ y^{[1]}\ \cdots\ y^{[n-1]})^T$, $\widetilde{Z} = (z^{[0]}\ z^{[1]}\ \cdots\ z^{[n-1]})^T$ are generated by Q and P respectively.

Proof. Set $f = -\bar{c}_{1,n}y_Q^{[n]}$, $g = -\bar{c}_{1,n}z_P^{[n]}$, then we have

$$\widetilde{Y}' = Q\widetilde{Y} + F, \qquad \widetilde{Z}' = P\widetilde{Z} + G,$$

where

$$F = (0\ \ldots\ 0\ f)^T, \qquad G = (0\ \ldots\ 0\ g)^T.$$

So from the Lemma 3.2, we have

$$\begin{aligned}(\widetilde{Z}^*C_n\widetilde{Y})' &= \widetilde{Z}^*_P C_n F + G^* C_n \widetilde{Y}_Q \\ &= c_{1n}\overline{z^{[0]}}f - \overline{c}_{1n}\overline{g}y^{[0]} \\ &= -\overline{z^{[0]}}y^{[n]}_Q + \overline{z^{[n]}_P}y^{[0]} \\ &= -(-i)^n\{\overline{z^{[0]}}M_Q y - y^{[0]}\overline{M_P z}\}.\end{aligned}$$

After integrating both sides of the above equation on any subinterval $[\alpha, \beta] \subset J$, we get

$$[y, z]_\alpha^\beta = \int_\alpha^\beta \overline{z} M_Q y dx - \int_\alpha^\beta y\overline{M_P z} dx = (-1)^{k+1}\widetilde{Z}^*C_n\widetilde{Y}\mid_\alpha^\beta .$$

Hence from the arbitrariness of $\alpha,\ \beta \in J$ we have

$$\overline{z}M_Q y - y\overline{M_P z} = [y, z]',$$

and

$$[y, z] = (-1)^{k+1}\widetilde{Z}^*C_n\widetilde{Y}.$$

By calculation (3.4) is also established. This completes the proof. □

Remark 3.4.

(1) If in (2.16) for odd number in $1 \le j \le k$, we set $\theta_j = \pi$ and for even number in $1 \le j \le k$, $\theta_j = 0$, then $C_n = E_n$ and we have the classical Lagrange identity in the references [12, 17, 21] below:

Assume $Q \in Z_n(J)$, and $P = -E_n^{-1}Q^*E_n$, then $P \in Z_n(J)$ and for any $y \in D(Q)$ and $z \in D(P)$, we have

$$\overline{z}M_Q y - y\overline{M_P z} = [y, z]',$$

and

$$[y, z] = (-1)^k \sum_{r=0}^{n-1}(-1)^{n+1-r}\overline{z^{[n-r-1]}}y^{[r]} = (-1)^{k+1}Z^*E_nY. \tag{3.5}$$

(2) If we set $\theta_j = 0,\ j = 1, 2, 3, \ldots, k$ in (2.16), then $C_n = -F_n$, and we have the another classical type of Lagrange identity in the Naimark book [14] as follows:

Let $Q \in Z_n(J)$, and $P = -F_n^{-1}Q^*F_n$, then $P \in Z_n(J)$ and for any $y \in D(Q)$ and $z \in D(P)$, we have

$$\overline{z}M_Q y - y\overline{M_P z} = [y, z]',$$

and

$$[y, z] = (-1)^k \sum_{r=1}^{k}\{y^{[r-1]}\overline{z^{[n-r]}} - y^{[n-r]}\overline{z^{[r-1]}}\} = (-1)^k\widehat{Z}^*F_n\widehat{Y}, \tag{3.6}$$

where

$$F_n = \begin{pmatrix} 0_{k\times k} & -J_k \\ J_k & 0_{k\times k}\end{pmatrix}, \qquad J_k = (\delta_{r,k+1-s})_{r,s=1}^k. \tag{3.7}$$

Theorem 1.1 characterizes all self-adjoint realizations of the operators generated by differential equation

$$My = \lambda wy, \text{ on } J = (a,b), \qquad -\infty < a < b < \infty, \tag{3.8}$$

where M is C-symmetric quasi-differential expression.

Let (3.8) has the two-point boundary condition

$$A\widetilde{Y}(a) + B\widetilde{Y}(b) = 0, \qquad \widetilde{Y} = (y^{[0]}\ y^{[1]}\ \cdots\ y^{[n-1]})^T, \tag{3.9}$$

in the Hilbert space $H = L^2(J,w)$. Then according to Lemma 3.1, Lemma 3.2 and Theorem 3.3 we have the following proof of Theorem 1.1.

Proof. From Theorem 3.3 we have

$$\int_a^b \bar{z}Mydx - \int_a^b \overline{Mz}ydx = [y,z]_a^b = \widetilde{Z}^*(b)C_n\widetilde{Y}(b) - \widetilde{Z}^*(a)C_n\widetilde{Y}(a) = 0,$$

then

$$\widetilde{D}(S) = \left\{ y \in D_{\max} : A\widetilde{Y}(a) + B\widetilde{Y}(b) = 0 \right\}$$

is a self-adjoint domain if and only if

$$AC_nA^* = BC_nB^*.$$

Thus Theorem 1.1 is established. □

Remark 3.5. If $A, B \in M_n(\mathbb{R})$, then the condition (1.7) reduces to $\det(A) = \det(B)$. However, not all the real self-adjoint boundary conditions are generated in this way.

Remark 3.6.

(1) In [4,6] and [17,21] Everitt and Zettl et al. define a formally self-adjoint differential equation M_Q by

$$Q = Q^+ = -E_n^{-1}Q^*E_n, \qquad Q \in Z_n(J),$$

where constant $n \times n$ matrix E_n is defined by (1.5). E_n is a skew-diagonal matrix satisfying $E_n^{-1} = -E_n = E_n^*$, i.e., it is a special case of C_n. Then S is a self-adjoint extension of minimal operator generated by M_Q if and only if

$$D(S) = \{ y \in D_{\max} : AY(a) + BY(b) = 0,\ A,\ B \in M_n(\mathbb{C}) \}, \tag{3.10}$$

where

$$\operatorname{rank}(A : B) = n, \qquad AE_nA^* = BE_nB^*. \tag{3.11}$$

(2) In [14, Chapter V] the formally self-adjoint differential expressions are generated by the matrices

$$\widehat{Q} = -F_n^{-1}\widehat{Q}^*F_n,\ \widehat{Q} \in Z_n(J). \tag{3.12}$$

Notice that F_n is a constant skew-diagonal matrix and satisfy $F_n^{-1} = -F_n = F_n^*$, it is a special case of C_n. Let $M = M_{\widehat{Q}}$ is generated by (3.12), then the domain defined by

$$D(\widehat{S}) = \left\{ y \in D_{\max} : A\widehat{Y}(a) + B\widehat{Y}(b) = 0,\ A,\ B \in M_n(\mathbb{C}) \right\}, \tag{3.13}$$

is a self-adjoint domain, i.e.,

$$\widehat{S}_{\min} \subset \widehat{S} = \widehat{S}^* \subset \widehat{S}_{\max}$$

if and only if

$$\operatorname{rank}(A : B) = n, \qquad AF_nA^* = BF_nB^*. \tag{3.14}$$

(3) Theorem 1.1 unifies and generalizes the statement of (1)–(2). Furthermore the different characterizations of self-adjoint domains among (1.6), (3.10) and (3.13) are caused by the use of different definition of the quasi-derivatives. In fact, the self-adjoint characterization of C-symmetric differential operators are generalization of previously known characterizations [4–6, 8, 13, 14, 17, 18, 21].

Remark 3.7. In general, the matrices which determine symmetric differential expressions are not unique, two different matrices may determine the same quasi-symmetric differential expressions. Frentzen [9] extended the Shin–Zettl set of matrices $Z_n(J)$ and Everitt and Race [6] studied the relationship between the matrices in this extended set which generate the same symmetric expressions. Theorem 1.1 shows that, given any constant skew-symmetric matrix C satisfying

$$C^{-1} = -C = C^*,$$

the matrix

$$Q = -C^{-1}Q^*C$$

is C-symmetric. And, remarkably, this same matrix C determines all self-adjoint boundary conditions, i.e., $S_{\min}$ and $S_{\max}$ denote the minimal and maximal operators determined by Q, respectively, then all self-adjoint extensions of $S_{\min}$ (or equivalently self-adjoint restrictions of $S_{\max}$), i.e. all operators S in $L^2(J, w)$ satisfying

$$S_{\min} \subset S = S^* \subset S_{\max}$$

are determined by the boundary conditions (1.6), (1.7). In addition to the examples $C = E_n$, $C = F_n$, the general generator of the symplectic group

$$C = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where I is the identity matrix of order k, is another example. See also the example

$$C = \begin{pmatrix} 0 & 0 & 0 & e^{i\theta_1} \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & -e^{-i\theta_2} & 0 & 0 \\ -e^{-i\theta_1} & 0 & 0 & 0 \end{pmatrix}$$

below.

4 Examples

In order to get a better understanding about our main results in this section we give some simple examples for the special case $n = 2, 4, 6$.

Example 4.1. Let $C_2 = \left(\begin{smallmatrix} 0 & c_{12} \\ c_{21} & 0 \end{smallmatrix}\right) \in M_2(\mathbb{C})$ satisfy

$$C_2^{-1} = -C_2 = C_2^*,$$

then

$$C_2 = \begin{pmatrix} 0 & c_{12} \\ -\bar{c}_{12} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix}, \qquad -\pi < \theta \leq \pi. \tag{4.1}$$

Now, let $Q \in Z_2(J)$ satisfy

$$Q = Q^+ := -C_2^{-1}Q^*C_2. \tag{4.2}$$

Then

$$Q^+ = \begin{pmatrix} -\bar{q}_{22} & c_{12}^2\bar{q}_{12} \\ \bar{c}_{12}^2\bar{q}_{21} & -\bar{q}_{11} \end{pmatrix},$$

and we have a second order C-symmetric matrix

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & -\bar{q}_{11} \end{pmatrix}, \tag{4.3}$$

where $q_{12} = c_{12}^2\bar{q}_{12}$, $q_{21} = \bar{c}_{12}^2\bar{q}_{21}$.

The C-symmetric quasi-derivatives generated by (4.3) are:

$$\begin{aligned} y^{[0]} &= y,\ y^{[1]} = \frac{1}{q_{12}}\{(y^{[0]})' - q_{11}y\}, \\ y^{[2]} &= -c_{12}\{(y^{[1]})' - q_{21}y^{[0]} + \bar{q}_{11}y^{[1]}\} = -e^{i\theta}\{(y^{[1]})' - q_{21}y^{[0]} + \bar{q}_{11}y^{[1]}\}, \end{aligned} \tag{4.4}$$

and $M = M_Q$ is given by

$$My = i^2y^{[2]} = e^{i\theta}\left\{\left[\frac{1}{q_{12}}(y' - q_{11}y)\right]' - q_{21}y + \frac{\bar{q}_{11}}{q_{12}}(y' - q_{11}y)\right\}. \tag{4.5}$$

Let $Q \in Z_2(J)$, $P = -C_2^{-1}Q^*C_2$, then we obtain a new version of Lagrange identity for the second order case:

$$\bar{z}M_Qy - y\overline{M_Pz} = [y,z]', \qquad y \in D(Q),\ z \in D(P), \tag{4.6}$$

where

$$[y,z] = Z^*C_2Y = e^{i\theta}\overline{z^{[1]}}y^{[0]} - e^{-i\theta}\overline{z^{[0]}}y^{[1]}, \qquad -\pi < \theta \le \pi.$$

Let

$$My = \lambda wy, \quad \text{on } J = (a,b), \tag{4.7}$$

in Hilbert space $L^2(J,w)$, where M is defined by (4.5), it has the following boundary conditions

$$\widetilde{A}\left(y^{[0]}(a), y^{[1]}(a)\right)^T + \widetilde{B}\left(y^{[0]}(b), y^{[1]}(b)\right)^T = 0, \qquad \widetilde{A}, \widetilde{B} \in M_2(\mathbb{C}),$$

where $y^{[0]}$, $y^{[1]}$ are defined by (4.4).

Define

$$D(S) = \left\{y \in D_{\max} : \widetilde{A}Y(a) + \widetilde{B}Y(b) = 0,\ Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \end{pmatrix}\right\}, \tag{4.8}$$

and S is generated by (4.7) satisfying $S_{\min} \subset S \subset S_{\max}$, then $D(S)$ is a self-adjoint domain for the second-order C-symmetric differential operators if and only if

$$\widetilde{A}C_2\widetilde{A}^* = \widetilde{B}C_2\widetilde{B}^*, \qquad \operatorname{rank}(\widetilde{A} : \widetilde{B}) = 2. \tag{4.9}$$

Remark 4.2. If $\theta = \pi$, i.e., $C_2 = E_2$, then (4.3) is reduced to the Lagrange symmetric matrix

$$Q = \begin{pmatrix} q_{11} & r_1 \\ r_2 & -\bar{q}_{11} \end{pmatrix}, \tag{4.10}$$

where r_1, r_2 are real-valued functions. $S_{\min}$, $S_{\max}$ are determined by (4.10) and S is a self-adjoint extension of $S_{\min}$ if and only if the domain

$$\widetilde{D}(S) = \left\{ y \in D_{\max} : \widetilde{A}\widetilde{Y}(a) + \widetilde{B}\widetilde{Y}(b) = 0,\ \widetilde{A},\ \widetilde{B} \in M_2(\mathbb{C}) \right\} \tag{4.11}$$

satisfy

$$\operatorname{rank}(\widetilde{A} : \widetilde{B}) = 2, \quad \text{and} \quad \widetilde{A}E_2\widetilde{A}^* = \widetilde{B}E_2\widetilde{B}^*, \tag{4.12}$$

i.e., the well-known characterization (4.12) is a special case of (4.9).

Example 4.3. Let $Q \in Z_4(J)$ be C-symmetric, then from Definition 2.3 we get

$$Q = Q^+ = -C_4^{-1}Q^*C_4, \tag{4.13}$$

where C_4 has the form

$$C_4 = \begin{pmatrix} 0 & 0 & 0 & c_{14} \\ 0 & 0 & c_{23} & 0 \\ 0 & -\bar{c}_{23} & 0 & 0 \\ -\bar{c}_{14} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & e^{i\theta_1} \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & -e^{-i\theta_2} & 0 & 0 \\ -e^{-i\theta_1} & 0 & 0 & 0 \end{pmatrix}.$$

From (4.13) we have

$$Q^+ = \begin{pmatrix} -\bar{q}_{44} & -c_{14}\bar{c}_{23}\bar{q}_{34} & 0 & 0 \\ -\bar{c}_{14}c_{23}\bar{q}_{43} & -\bar{q}_{33} & c_{23}^2\bar{q}_{23} & 0 \\ \bar{c}_{14}\bar{c}_{23}\bar{q}_{42} & \bar{c}_{23}^2\bar{q}_{32} & -\bar{q}_{22} & -c_{14}\bar{c}_{23}\bar{q}_{12} \\ \bar{c}_{14}^2\bar{q}_{41} & \bar{c}_{14}\bar{c}_{23}\bar{q}_{31} & -\bar{c}_{14}c_{23}\bar{q}_{21} & -\bar{q}_{11} \end{pmatrix},$$

and it follows that

$$Q = \begin{pmatrix} q_{11} & q_{12} & 0 & 0 \\ q_{21} & q_{22} & q_{23} & 0 \\ q_{31} & q_{32} & -\bar{q}_{22} & -c_{14}\bar{c}_{23}\bar{q}_{12} \\ q_{41} & \bar{c}_{14}\bar{c}_{23}\bar{q}_{31} & -\bar{c}_{14}c_{23}\bar{q}_{21} & -\bar{q}_{11} \end{pmatrix}, \tag{4.14}$$

where $q_{23} = c_{23}^2\bar{q}_{23}$, $q_{32} = \bar{c}_{23}^2\bar{q}_{32}$, $q_{41} = \bar{c}_{14}^2\bar{q}_{41}$.

Thus the quasi-derivatives associated with the C-symmetric matrix Q are

$$\begin{aligned} y^{[0]} &= y,\ y^{[1]} = \frac{1}{q_{12}}\{(y^{[0]})' - q_{11}y\}, \\ y^{[2]} &= \frac{1}{q_{23}}\{(y^{[1]})' - q_{21}y^{[0]} - q_{22}y^{[1]}\}, \\ y^{[3]} &= -\frac{1}{c_{14}\bar{c}_{23}\bar{q}_{12}}\{(y^{[2]})' - q_{31}y^{[0]} - q_{32}y^{[1]} + \bar{q}_{22}y^{[2]}\}, \\ y^{[4]} &= -c_{14}\{(y^{[3]})' - q_{41}y^{[0]} - \bar{c}_{14}\bar{c}_{23}\bar{q}_{31}y^{[1]} + \bar{c}_{14}c_{23}\bar{q}_{21}y^{[2]} + \bar{q}_{11}y^{[3]}\}_{14}). \end{aligned} \tag{4.15}$$

So the fourth order C-symmetric quasi-differential expressions be given by

$$My = i^4y^{[4]} = -c_{14}\{(y^{[3]})' - q_{41}y^{[0]} - \bar{c}_{14}\bar{c}_{23}\bar{q}_{31}y^{[1]} + \bar{c}_{14}c_{23}\bar{q}_{21}y^{[2]} + \bar{q}_{11}y^{[3]}\}. \tag{4.16}$$

Set

$$My = \lambda wy, \tag{4.17}$$

where M is defined by (4.17). Then all self-adjoint extension S of minimal operator generated by (4.17) are characterized as follows:

$$\widetilde{D}(S) = \left\{ y \in D_{\max} : A\widetilde{Y}(a) + B\widetilde{Y}(b) = 0 \right\}, \tag{4.18}$$

where A, B satisfy

$$\operatorname{rank}(A : B) = 4, \quad AC_4A^* = BC_4B^*, \quad A, B \in M_4(\mathbb{C}), \tag{4.19}$$

and the quasi-derivatives in $\widetilde{Y}$ are defined by (4.15).

Remark 4.4. Note that $q_{11} = q_{21} = q_{22} = q_{31} = 0$ and $q_{12} = 1$ in (4.16) yields

$$My = c_{23}[(q_{23}^{-1}y'')' - q_{32}y']' + c_{14}q_{41}y. \tag{4.20}$$

Moreover,

(1) if $\theta_1 = \pi$, $\theta_2 = 0$, i.e., $c_{14} = -1, c_{23} = 1$ in (4.20), then it is reduced to the real Lagrange symmetric differential expression [21]

$$My = [(q_{23}^{-1}y'')' - q_{32}y']' - q_{41}y, \tag{4.21}$$

where q_{23}^{-1}, q_{32}, q_{41} are reals.

For this Lagrange symmetric differential expression we have characterization of self-adjoint domains

$$D(S) = \left\{ y \in D_{\max} : AY(a) + BY(b) = 0,\ Y = \begin{pmatrix} y \\ y' \\ \frac{1}{q_{23}}y'' \\ (\frac{1}{q_{23}}y'')' - q_{32}y' \end{pmatrix} \right\}, \tag{4.22}$$

where

$$\operatorname{rank}(A : B) = 4,\ AE_4A^* = BE_4B^*,\ A,\ B \in M_4(\mathbb{C}).$$

(2) If $\theta_1 = \theta_2 = 0$ in (4.20), then it is reduced to the modified Naimark form [14]

$$My = [(q_{23}^{-1}y'')' - q_{32}y']' + q_{41}y, \tag{4.23}$$

where q_{23}^{-1}, q_{32}, q_{41} are reals.

For this differential expression (4.23) we have the characterization of self-adjoint domains

$$\widehat{D}(S) = \left\{ y \in D_{\max} : A\widehat{Y}(a) + B\widehat{Y}(b) = 0,\ \widehat{Y} = \begin{pmatrix} y \\ y' \\ \frac{1}{q_{23}}y'' \\ q_{32}y' - \left(\frac{1}{q_{23}}y''\right)' \end{pmatrix} \right\}, \tag{4.24}$$

where

$$\operatorname{rank}(A : B) = 4, \quad AF_4A^* = BF_4B^*, \quad A, B \in M_4(\mathbb{C}).$$

Example 4.5. $n = 6$. Let $Q = (q_{r,s})_{r,s=1}^{6} \in Z_6(J)$ is C-symmetric, where

$$C = C_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & c_{16} \\ 0 & 0 & 0 & 0 & c_{25} & 0 \\ 0 & 0 & 0 & c_{34} & 0 & 0 \\ 0 & 0 & -\bar{c}_{34} & 0 & 0 & 0 \\ 0 & -\bar{c}_{25} & 0 & 0 & 0 & 0 \\ -\bar{c}_{16} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.25}$$

Then we obtain

$$Q = \begin{pmatrix} q_{11} & q_{12} & 0 & 0 & 0 & 0 \\ q_{21} & q_{22} & q_{23} & 0 & 0 & 0 \\ q_{31} & q_{32} & q_{33} & q_{34} & 0 & 0 \\ q_{41} & q_{42} & q_{43} & -\bar{q}_{33} & -c_{25}\bar{c}_{34}\bar{q}_{23} & 0 \\ q_{51} & q_{52} & \bar{c}_{25}\bar{c}_{34}\bar{q}_{42} & -c_{34}\bar{c}_{25}\bar{q}_{32} & -\bar{q}_{22} & -c_{16}\bar{c}_{25}\bar{q}_{12} \\ q_{61} & \bar{c}_{16}\bar{c}_{25}\bar{q}_{51} & \bar{c}_{16}\bar{c}_{34}\bar{q}_{41} & -c_{34}\bar{c}_{16}\bar{q}_{31} & -c_{25}\bar{c}_{16}\bar{q}_{21} & -\bar{q}_{11} \end{pmatrix}, \tag{4.26}$$

where $q_{34} = c_{34}^2\bar{q}_{34}$, $q_{43} = \bar{c}_{34}^2\bar{q}_{43}$, $q_{52} = \bar{c}_{25}^2\bar{q}_{52}$, $q_{61} = \bar{c}_{16}^2\bar{q}_{61}$.

Then we have the C-symmetric quasi-derivatives below:

$$\begin{aligned} y^{[0]} &= y,\ y^{[1]} = \frac{1}{q_{12}}\{(y^{[0]})' - q_{11}y\}, \\ y^{[2]} &= \frac{1}{q_{23}}\{(y^{[1]})' - q_{21}y^{[0]} - q_{22}y^{[1]}\}, \\ y^{[3]} &= \frac{1}{q_{34}}\{(y^{[2]})' - q_{31}y^{[0]} - q_{32}y^{[1]} - q_{33}y^{[2]}\}, \\ y^{[4]} &= -\frac{1}{c_{25}\bar{c}_{34}\bar{q}_{23}}\{(y^{[3]})' - q_{41}y^{[0]} - q_{42}y^{[1]} - q_{43}y^{[2]} + \bar{q}_{33}y^{[3]}\}, \\ y^{[5]} &= -\frac{1}{c_{16}\bar{c}_{25}\bar{q}_{12}}\{(y^{[4]})' - q_{51}y^{[0]} - q_{52}y^{[1]} - \bar{c}_{25}\bar{c}_{34}\bar{q}_{42}y^{[2]} + c_{34}\bar{c}_{25}\bar{q}_{32}y^{[3]} + \bar{q}_{22}y^{[4]}\}, \end{aligned} \tag{4.27}$$

and $My = M_Q y$ is given by

$$My = c_{16}(y^{[5]})' - c_{16}q_{61}y - \bar{c}_{25}\bar{q}_{51}y^{[1]} - \bar{c}_{34}\bar{q}_{41}y^{[2]} + c_{34}\bar{q}_{31}y^{[3]} + c_{25}\bar{q}_{21}y^{[4]} + \bar{q}_{11}y^{[5]}. \tag{4.28}$$

Set

$$My = \lambda w y, \tag{4.29}$$

where M is defined by (4.28). Then all self-adjoint extension S of minimal operator generated by (4.29) are characterized as follows:

$$\widetilde{D}(S) = \left\{ y \in D_{\max} : A\widetilde{Y}(a) + B\widetilde{Y}(b) = 0,\ A,\ B \in M_6(\mathbb{C}) \right\}, \tag{4.30}$$

where A, B satisfy

$$\operatorname{rank}(A : B) = 6,\ AC_6A^* = BC_6B^*,$$

and $\widetilde{Y}$ are defined by (4.27).

Note that $q_{11} = q_{21} = q_{22} = q_{31} = q_{32} = q_{33} = q_{41} = q_{42} = q_{51} = 0$ and $q_{12} = q_{23} = 1$ in (4.28) yields

$$My = \{c_{34}[(q_{34}^{-1}y''')' - q_{43}y'']' + c_{25}q_{52}y'\}' - c_{16}q_{61}y. \tag{4.31}$$

Furthermore we observe that $\theta_1 = \theta_3 = \pi$ and $\theta_2 = 0$ in (4.31) yields the Lagrange symmetric expression

$$My = \{[(q_{34}^{-1}y''')' - q_{43}y'']' - q_{52}y'\}' - q_{61}y, \tag{4.32}$$

where q_{34}^{-1}, q_{43}, q_{52}, q_{61} are real-valued functions.

For this Lagrange symmetric differential expression we have characterization of self-adjoint domains:

$$D(S) = \{y \in D_{\max} : AY(a) + BY(b) = 0,\ A,\ B \in M_6(\mathbb{C})\}, \tag{4.33}$$

where

$$\operatorname{rank}(A:B) = 6, \qquad AE_6A^* = BE_6B^*, \qquad Y = \begin{pmatrix} y \\ y' \\ y'' \\ \frac{1}{q_{34}}y''' \\ (\frac{1}{q_{34}}y''')' - q_{43}y'' \\ \{[q_{43}y'' - (\frac{1}{q_{34}}y''')']' - q_{52}y'\} \end{pmatrix}.$$

If $\theta_1 = \theta_2 = \theta_3 = 0$ in (4.31), then it is reduced to the real modified Naimark form

$$My = \{[(-q_{34}^{-1}y''')' + q_{43}y'']' - q_{52}y'\}' + q_{61}y, \tag{4.34}$$

where q_{34}^{-1}, q_{43}, q_{52}, q_{61} are real-valued functions.

For this special expressions (4.34), we have the characterization of self-adjoint domains:

$$\widehat{D}(S) = \left\{y \in D_{\max} : A\widehat{Y}(a) + B\widehat{Y}(b) = 0,\ A,\ B \in M_6(\mathbb{C})\right\}, \tag{4.35}$$

where

$$\operatorname{rank}(A:B) = 6, \qquad AF_6A^* = BF_6B^*, \qquad \widehat{Y} = \begin{pmatrix} y \\ y' \\ y'' \\ \frac{1}{q_{34}}y''' \\ q_{43}y'' - (\frac{1}{q_{34}}y''')' \\ q_{52}y' - [q_{43}y'' - (\frac{1}{q_{34}}y''')']' \end{pmatrix}.$$

Remark 4.6. (1) For $n = 4$ and $n = 6$, (4.21) and (4.32) are generated by the following matrix form [21]:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & q_{23} & 0 \\ 0 & q_{32} & 0 & 1 \\ q_{41} & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{34} & 0 & 0 \\ 0 & 0 & q_{43} & 0 & 1 & 0 \\ 0 & q_{52} & 0 & 0 & 0 & 1 \\ q_{61} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. However, (4.23) and (4.34) are generated by the G-N type matrix function [14]:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & q_{23} & 0 \\ 0 & q_{32} & 0 & -1 \\ q_{41} & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{34} & 0 & 0 \\ 0 & 0 & q_{43} & 0 & -1 & 0 \\ 0 & q_{52} & 0 & 0 & 0 & -1 \\ q_{51} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively.

(2) For $n = 4$, (4.18) contains the characterization (4.22) and (4.24). For $n = 6$, (4.30) contains the characterization (4.33) and (4.35).

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