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On the existence of solutions for a boundary value problem on the half-line

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Abstract. In this note we consider Dirichlet boundary value problem on a half line. Using critical point theory we prove the existence of at least one nontrivial solution.

Keywords: Dirichlet boundary value problem, half line, critical point, mountain pass.

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1 Introduction

In this paper we are going to prove two existence results concerning boundary value problems on a half line using critical point theory approach. Problems on a half line received lately some attention but the main approach concerning the existence issue was by fixed point theorems and the method of lower and upper solutions. The results by critical point theory are less frequent due to the lack of the Poincaré inequality and also due to the fact that the space in which the solutions are obtained is not compactly embedded into the space of continuous functions.

Let $\lambda > 0$ be a numerical parameter and assume that $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $q : [0, +\infty) \to (0, +\infty)$ is a function with $q \in L^1(0, +\infty)$. In the space $H_0^1(0, +\infty)$ we consider the following Dirichlet problem

$$\begin{cases} -u''(t) + u(t) = \lambda q(t) f(t, u(t)), & t \in (0;\ +\infty), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{1.1}$$

Using some appropriate growth conditions upon the nonlinear term f, we investigate solutions to (1.1) as critical points to the Euler action functional $J : H_0^1(0, +\infty) \to \mathbb{R}$ given by

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$$J(u) = \frac{1}{2}\int_0^{+\infty} \left(u'(t)\right)^2 dt + \frac{1}{2}\int_0^{+\infty} u^2(t)dt - \lambda \int_0^{+\infty} q(t)F(t,u(t))dt \tag{1.2}$$

where as always

$$F(t,u) = \int_0^u f(t,s)ds.$$

Let $p : [0,+\infty) \to (0,+\infty)$ be a continuously differentiable and bounded function such that $M = 2\max\left(\|p\|_{L^2}, \|p'\|_{L^2}\right) < +\infty$. In order to have the term $\int_0^{+\infty} \lambda q(t)F(t,u(t))dt$ well defined we assume that

A *for any constant* $r > 0$ *there exists a nonnegative function* h_r *for which* $\frac{q}{p}h_r \in L^1(0,+\infty)$ *such that*

$$\sup_{|y|\le r}\left|f\left(t,\frac{y}{p(t)}\right)\right| \le h_r(t) \quad \text{for a.e. } t \in [0,+\infty).$$

The above assumption is due to the fact that the space $H_0^1(0,+\infty)$ is not compactly embedded into $C[0,+\infty)$ contrary to the case of bounded interval setting as we mentioned before. In order to overcome this problem we may take into account the embedding results contained in [6] and [7]. These will allow us to have the counterpart of a definition of L^1-Carathéodory function commonly applied in the case of bounded interval. In the literature, for example [2], the idea of L^2-Carathéodory function is used and the embedding into the space of bounded continuous functions is utilized.

As it is common with variational problems for O.D.E. (1.1) admits two types of solutions, namely a weak and a classical one. Function $u \in H_0^1(0,+\infty)$ is a weak solution of (1.1) if

$$\int_0^{+\infty} u'(t)v'(t)dt + \int_0^{+\infty} u(t)v(t)dt - \lambda\int_0^{+\infty} q(t)f(t,u(t))v(t)dt = 0, \quad \forall v \in H_0^1(0,+\infty). \tag{1.3}$$

Function $u \in H_0^1(0,+\infty)$ is a classical solution to (1.1) if both u and u' are locally absolutely continuous functions on $[0,+\infty)$,

$$-u''(t) + u(t) = \lambda q(t)f(t,u(t)), \quad \text{for a.e. } t \in [0,+\infty)$$

and the boundary conditions $u(0) = u(+\infty)$ are satisfied. We would like to recall, following [3], that any function $u \in H_0^1(0,+\infty)$ is locally absolutely continuous, i.e. absolutely continuous on any closed bounded interval contained in $[0,+\infty)$ however it is not in general absolutely continuous on the whole half line which makes the problem different from the classical bounded one.

We will look for solutions of (1.1) which are critical points to (1.2) and in order to obtain them we will apply two approaches. The first one is connected with the usage of the mountain pass geometry, see book [5] for some background. Such an approach requires that the problem under consideration satisfies some suitable geometric conditions pertaining to behaviour around 0 and also compactness condition in a form of a Palais–Smale condition.

For the second approach we will use some abstract critical point theorem derived in [8]. This result provides the existence of a critical point located in some set which need not be open and was applied already to some problems in bounded domains only. This approach does not require compactness pertaining to the usage of a Palais–Smale condition but on the other hand the nonlinear part of the equation must have enough monotonicity in order to yield that the corresponding term of the action functional, namely $\int_0^{+\infty} q(t)F(t,u(t))dt$, is convex.

Both methods use partially different assumptions while the common assumption concerns the issue of integrability of terms appearing in the action functional and the issue of connection between weak and classical solutions. Both approaches yield the existence of at least one non-trivial critical point. In the case of the application of the mountain pass theorem the existence of non-trivial solution follows from the abstract result without any other assumptions than those leading to the so called mountain geometry. The application of theorem from [8] provides only the existence of some critical point and that is why one must make sure that it is non-trivial by some additional assumption. Moreover critical points obtained by both methods are located in some ball around 0.

Finally, we would like to underline that there are not many results concerning solvability of problems like (1.1) when compared to the case of a bounded interval for the reasons mentioned above. Apart from [2] we would like to mention [4, 6, 7] where also variational approaches are used but these pertain either to the critical point type result of Ricceri or else to some non-smooth setting. In none of these sources mountain pass methodology is directly applied, while some of its ideas are hidden in the approach of three critical point theorems but with different assumptions.

To the best of our knowledge, the results in Theorem 3.4 and Theorem 3.5 are new and original as we have not found any discussion in the existing literature. Also, there exists no paper concerned with the existence of at least one nontrivial solution for our problem which is posed on the half line under assumptions similar to us.

2 Preliminaries

Symbol $L^p(0,+\infty)$ for $p \geq 1$ means the space of such measurable real valued functions defined on $[0,+\infty)$ that $\int_0^\infty |u(t)|^p\,dt < +\infty$. Solutions to (1.1) will be considered in the space $H_0^1(0,+\infty)$ which is defined as follows. We say that $u \in H_0^1(0,+\infty)$ if $u \in L^2(0,+\infty)$ and if there exists a function $g \in L^2(0,+\infty)$, called a weak derivative, and such that

$$\int_0^{+\infty} u(t)\varphi'(t)dt = -\int_0^{+\infty} g(t)\varphi(t)dt$$

for all $\varphi \in C_c^\infty(0,+\infty)$, where $C_c^\infty(0,+\infty)$ is the space of compactly supported functions from $C^\infty([0,\ +\infty)),\mathbb{R})$. We denote $g := u'$. We endow the space $H_0^1(0,+\infty)$ with its natural norm

$$\|u\| = \left(\int_0^{+\infty} u^2(t)dt + \int_0^{+\infty} \left(u'(t)\right)^2 dt\right)^{\frac{1}{2}},$$

associated with the scalar product

$$(u,\ v) = \int_0^{+\infty} u(t)v(t)dt + \int_0^{+\infty} u'(t)v'(t)dt.$$

Let us also consider the space

$$C_{l,p}[0,\ +\infty) = \left\{u \in C([0,\ +\infty),\mathbb{R}) :\ \lim_{t\to+\infty} p(t)u(t) \text{ exists}\right\}$$

endowed with the norm

$$\|u\|_{\infty,p} = \sup_{t\in[0,+\infty)} p(t)|u(t)|.$$

We need some definitions and lemmas which will be used later.

Definition 2.1. Let E be a Banach space. Let $J \in C^1(E, \mathbb{R})$. For any sequence $\{u_n\} \subset E$, if $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence, then we say that J satisfies the Palais–Smale condition ((PS) condition for short).

Lemma 2.2 (Mountain pass lemma [1]). *Let $J \in C^1(E, \mathbb{R})$ satisfy the (PS) condition. Suppose that*

(1) $J(0) = 0$;

(2) there exist $\varrho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \varrho$;

(3) there exist u_1 in E with $\|u_1\| > \varrho$ such that $J(u_1) < \alpha$.

Then J has a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$\inf_{g \in \Gamma} \max_{u \in g([0,\,1])} J(u),$$

where $\Gamma = \{g \in C([0;1], E) : \ g(0) = 0, \ g(1) = u_1\}$.

We need also the following embeddings.

Lemma 2.3 ([6,7]). *Assume that **A** holds. $H_0^1(0, +\infty)$ embeds continuously in $C_{l,p}[0, +\infty)$, and we have $\|u\|_{\infty,p} \leq M\|u\|$.*

Lemma 2.4 ([6,7]). *Assume that **A** holds. The embedding*

$$H_0^1(0, +\infty) \hookrightarrow C_{l,p}[0, +\infty)$$

is compact.

We endow the space $L^\infty(0, +\infty)$ with the standard ess sup-norm. The constant of the continuous embedding $H_0^1(0, +\infty) \hookrightarrow L^\infty(0, +\infty)$ is denoted by K (see [3, Remark 10, p. 214], or else Theorem 8.8 from [3]).

Proposition 2.5. *Let $\lambda > 0$ be fixed. Assume that **A** holds. The functional J is well-defined and continuously differentiable on $H_0^1(0, +\infty)$. The derivative of J at any $u \in H_0^1(0, +\infty)$ has the following form*

$$\langle J'(u), v \rangle = \int_0^{+\infty} u'(t)v'(t)dt + \int_0^{+\infty} u(t)v(t)dt - \lambda \int_0^{+\infty} q(t)f(t, u(t))v(t)dt, \quad \forall v \in H_0^1(0, +\infty).$$

Proof. Note that the term

$$J_1(u) = \frac{1}{2} \int_0^{+\infty} \left(u'(t)\right)^2 dt + \frac{1}{2} \int_0^{+\infty} u^2(t)dt \tag{2.1}$$

is obviously well defined and C^1 since $J_1(u) = \frac{1}{2}\|u\|^2$. Thus we need to prove that

$$J_2(u) = \int_0^{+\infty} q(t)F(t, u(t))dt \tag{2.2}$$

is also C^1.

Claim 1: J_2 is well defined and Gâteaux-differentiable. Let us take any fixed $u \in H_0^1(0,+\infty)$. By Lemma 2.3 there is some $r > 0$ such that $\|u\|_{\infty,p} \le r$. By assumption A and again by Lemma 2.3 we see what follows

$$\begin{aligned}\int_0^{+\infty} q(t)F(t,u(t))dt &= \int_0^{+\infty} q(t)\int_0^{u(t)} f(t,s)dsdt \\ &\le \|u\|_{\infty,p}\int_0^{+\infty}\frac{q(t)}{p(t)}\sup_{|y|\le r} f\left(t,\frac{y}{p(t)}\right)dt \le \|u\|_{\infty,p}\int_0^{+\infty}\frac{q(t)}{p(t)}h_r(t)dt < +\infty.\end{aligned}$$

Now we turn to Gâteaux-differentiability. Indeed, let $u,v \in H_0^1(0,+\infty)$ be fixed and take any $t \in [0,+\infty)$. Then for any $\theta \in (0,1)$ and s small we have by Lemma 2.3 and Lemma 2.4,

$$\begin{aligned}p(t)|u(t)+s\theta v(t)| &\le \sup_{t\in[0,+\infty)} p(t)|u(t)| + \sup_{t\in[0,+\infty)} p(t)|v(t)| \le \|u\|_{\infty,p} + \|v\|_{\infty,p} \\ &\le M[\|u\|+\|v\|] \le 2M\max[\|u\|,\|v\|] = r_{u,v}.\end{aligned}$$

Moreover, we see by assumption **A** that

$$\begin{aligned}|q(t)f(t,u(t)+s\theta v(t))v(t)| &= q(t)\left|f\left(t,p(t)\frac{u(t)+s\theta v(t)}{p(t)}\right)v(t)\right| \\ &\le \left|\frac{q(t)}{p(t)}\sup_{y\in[-r,r]} f\left(t,\frac{y}{p(t)}\right)v(t)\right|p(t) \le \|v\|_{\infty,p}h_r(t)\frac{q(t)}{p(t)} \le M\|v\|h_r(t)\frac{q(t)}{p(t)},\end{aligned} \tag{2.3}$$

and we see that $h_r(\cdot)\frac{q(\cdot)}{p(\cdot)} \in L^1(0,+\infty)$.

Therefore we can apply the mean value theorem and then the Lebesgue dominated convergence theorem in order to pass to the limit $s \to 0$ in

$$\frac{J_2(u+sv)-J_2(u)}{s}$$

which results in

$$\langle J_2'(u),v\rangle = \int_0^{+\infty} q(t)f(t,u(t))v(t)dt, \quad \forall v \in H_0^1(0,+\infty).$$

Claim 2: J_2' is continuous. Indeed, let $(u_n) \subset H_0^1(0,+\infty)$, such that $u_n \to u$, when $n \to +\infty$. By Lemma 2.4, we have $u_n \to u$, as $n \to +\infty$ in $C_{l,p}[0,+\infty)$. Thus there is $r > 0$, such that

$$\sup_{t\in[0,+\infty)} p(t)|u_n(t)| \le r.$$

Using **A** and reasoning similar to this provided in (2.3) we have by the Lebesgue dominated convergence theorem

$$\lim_{n\to+\infty}\int_0^{\infty} q(t)f(t,u_n(t))v(t)dt = \int_0^{\infty} q(t)f(t,u(t))v(t)dt$$

uniformly for v in the unit ball. Thus we see that

$$\|J_2'(u_n)-J_2'(u)\|_{\left(H_0^1(0,+\infty)\right)^*} \to 0 \quad \text{as } n \to \infty.$$

□

Remark 2.6. We note that from the second part of the proof of the above theorem and from Lemma 2.4 it follows that J_2 is weakly continuous on $H_0^1(0,+\infty)$. Indeed, for a sequence $(u_n) \subset H_0^1(0,+\infty)$, such that $u_n \rightharpoonup u$, as $n \to +\infty$, we have by Lemma 2.4, that $u_n \to u$, as $n \to +\infty$, in $C_{l,p}[0,+\infty)$. Then we see that

$$\int_0^{+\infty} q(t)F(t,u_n(t))dt \to \int_0^{+\infty} q(t)F(t,u(t))dt$$

as $n \to +\infty$.

Proposition 2.7. *Assume that* **A** *holds. Let $\lambda > 0$ be fixed. If $u \in H_0^1(0,+\infty)$ is a solution of the Euler equation $J'(u) = 0$, then u is a classical solution of problem* (1.1).

Proof. We follow the same steps as in [6]. If u satisfies the Euler equation $J'(u) = 0$, i.e. $\langle J'(u), v\rangle = 0$ for all $v \in H_0^1(0,+\infty)$, then by (1.3) it is a weak solution of Problem (1.1). Since $C_0^\infty(0,+\infty) \subset H_0^1(0,+\infty)$ we see from the definition of the weak solution that

$$\int_0^{+\infty} u'(t)v'(t)dt = -\int_0^{+\infty} (u(t) - \lambda q(t)f(t,u(t)))v(t)dt, \quad \forall v \in C_0^\infty(0,+\infty). \tag{2.4}$$

Let us define the functions $Y : [0,+\infty) \to \mathbb{R}$ by

$$Y(t) = u(t) - \lambda q(t)f(t,u(t)), \tag{2.5}$$

and $Z : [0,+\infty) \to \mathbb{R}$ by

$$Z(t) = \int_0^t Y(s)ds.$$

Note that by **A** and by Lemma 2.4 we see that Y is $L^1_{\text{loc}}(0,+\infty)$, therefore Z is locally absolutely continuous function on $[0,+\infty)$. By using the Dirichlet formula (see [9]), we obtain

$$\begin{aligned}
\int_0^{+\infty} Z(t)v'(t)dt &= \int_0^{+\infty} \left(\int_0^t Y(s)ds\right) v'(t)dt \\
&= \int_0^{+\infty}\int_s^{+\infty} Y(s)v'(t)dtds = \int_0^{+\infty} Y(s) \left(\int_s^{+\infty} v'(t)dt\right) ds \\
&= -\int_0^{+\infty} Y(s)v(s)ds.
\end{aligned}$$

Thus using (2.4), we get

$$\int_0^{+\infty} Z(t)v'(t)dt = \int_0^{+\infty} u'(t)v'(t)dt, \quad \forall v \in C_0^\infty(0,+\infty),$$

then

$$\int_0^{+\infty} \left(u'(t) - Z(t)\right) v'(t)dt = 0, \quad \forall v \in C_0^\infty(0,+\infty).$$

Since $u \in H_0^1(0,+\infty)$, we see that $u' \in L^1_{\text{loc}}(0,+\infty)$. Thus by the fundamental theorem of the calculus of variations, we see that there exists $c \in \mathbb{R}$ such that

$$u'(t) = Z(t) + c = \int_0^t (u(s) - \lambda q(s)f(s,u(s)))ds + c,$$

for a.e. $t \in [0,+\infty)$. This means that u' is locally absolutely continuous function on $[0,+\infty)$ which implies that for a.e. $t \in [0,+\infty)$

$$\left(u'(t)\right)' = Y(t) = u(t) - \lambda q(t)f(t,u(t)),$$

and then

$$-u''(t)+u(t)=\lambda q(t)f(t,u(t)), \quad \text{for a.e. } t\in[0,+\infty). \tag{2.6}$$

On the other hand, as $u\in H_0^1(0,+\infty)$, then we obtain

$$u(0)=u(+\infty). \tag{2.7}$$

Hence, from (2.6) and (2.7), u is a classical solution of Problem (1.1) . □

We would like to note that the counterpart of the proof of Proposition 2.5 in bounded intervals is standard but when we work on infinite intervals the assertion of the proposition is not evident and for this reason we must use hypothesis **A** and utilize embeddings from Lemmas 2.3 and 2.4 to prove it.

Let E be a real reflexive Banach space, and $\Phi,\ H : E\to\mathbb{R}$ two continuously Fréchet differentiable convex functionals with derivatives $\varphi,\ h:\ E\to E^*$ respectively i.e. $\frac{d\Phi}{du}=\varphi$ and $\frac{dH}{du}=h$, we consider the problem

$$\varphi(u)=h(u), \quad u\in E. \tag{2.8}$$

We denote by $J : E\to\mathbb{R}$ the action functional connected with (2.8) , i.e. $J(u)=\Phi(u)-H(u)$, (see [8]).

Theorem 2.8 ([8]). *Let E be an infinite dimensional reflexive Banach space.*

(i) Let $X\subset E$ and let there exist $u_0, v\in X$ satisfying $\varphi(v)=h(u_0)$, and such that

$$J(u_0)\le\inf_{u\in X}J(u).$$

Then u_0 is a critical point of J, and thus it solves (2.8).

3 Applications

Now we state the following hypotheses.

($\mathbf{H_1}$) there exist positive functions $a,b:[0,+\infty)\to(0,+\infty)$ with $aq,bq\in L^1(0,+\infty)\bigcap L^2(0,+\infty)$ and $\sigma>0$ such that

$$|f(t,u)|\le a(t)|u|^\sigma+b(t), \quad \text{for a.e. } t\in[0,+\infty) \text{ and all } u\in\mathbb{R},$$

($\mathbf{H_2}$) there exist functions $c_1,c_2 : [0,+\infty)\to(0,+\infty)$ with $c_1q,\ c_2q\in L^1(0,+\infty)$, and $\theta>2$ such that

- (a) $F(t,u)\ge c_1(t)|u|^\theta-c_2(t)$, for a.e. $t\ge0$ and all $u\in\mathbb{R}$,
- (b) $\theta F(t,u)\le uf(t,u)$, for a.e. $t\ge0$ and all $u\in\mathbb{R}\backslash\{0\}$,

($\mathbf{H_3}$) $\lim_{u\to0}\frac{f(t,u)}{u}=0$ uniformly for a.e. $t\in[0,+\infty)$,

($\mathbf{H_4}$) the function $u\longmapsto F(t,u)$ is convex on $\mathbb{R}$ for a.e. $t\in[0,\ +\infty)$.

A remark is in order concerning the assumptions.

Remark 3.1. Note that from $(\mathbf{H_2})(\mathbf{b})$ we obtain a type of $(\mathbf{H_2})(\mathbf{a})$ only with fixed constants c_1 and c_2. This does not suffice for our problem so the additional assumption is crucial. Relaxed version of the A-R condition, namely condition $(\mathbf{H_2})(\mathbf{b})$, could also be assumed but these involve some technical calculations only and do not advance our main approach. We note also that it is possible to assume convexity of F at some interval centered at 0 only.

We will show now that the functional J with the above assumptions $(\mathbf{H_1})$–$(\mathbf{H_3})$ has mountain pass geometry and so at least one nontrivial solution. On the other hand assuming only $(\mathbf{H_1})$, $(\mathbf{H_4})$ and some condition at 0, we obtain the existence of at least one solution on some arbitrarily fixed closed ball for a suitable range of numerical parameter.

3.1 Results by the mountain pass lemma

Lemma 3.2. *Assume that* $\boldsymbol{A}$ *holds. Suppose also that* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *hold. Then for any* $\lambda > 0$, *the functional* J *given by* (1.2) *satisfies the PS-condition.*

Proof. Let us take a sequence $(u_k) \subset H_0^1(0,+\infty)$ such that $(J(u_k))$ is bounded and $J'(u_k) \to 0$, as $k \to \infty$. We shall show that (u_k) has a convergent subsequence.

Since $J'(u_k) \to 0$, we see that for some $\epsilon > 0$ there exists k_0 with $\|J'(u_k)\| \leq \epsilon$ for $k \geq k_0$. Note that for $k \geq k_0$

$$\left|\langle J'(u_k),\ u_k\rangle\right| \leq \epsilon\|u_k\|.$$

Observe further that by a direct calculation

$$\langle J'(u_k), u_k\rangle = \int_0^{+\infty} \left(u_k'(t)\right)^2 dt + \int_0^{+\infty} \left(u_k(t)\right)^2 dt - \lambda \int_0^{+\infty} q(t)f(t,u_k(t))u_k(t)dt.$$

Now we estimate by $(\mathbf{H_2})(\mathbf{b})$ that

$$\begin{aligned} -\lambda \int_0^{+\infty} q(t)F(t,u_k(t))dt &\geq \frac{-\lambda}{\theta}\left(\int_0^{+\infty} q(t)f(t,u_k(t))u_k(t)dt\right) \\ &= \frac{1}{\theta}\langle J'(u_k),u_k\rangle - \frac{1}{\theta}\int_0^{+\infty}\left(u_k'(t)\right)^2 dt - \frac{1}{\theta}\int_0^{+\infty}\left(u_k(t)\right)^2 dt \\ &\geq \frac{-\epsilon}{\theta}\|u_k\| - \frac{1}{\theta}\|u_k\|^2. \end{aligned} \tag{3.1}$$

Since $(J(u_k))$ is bounded, there exists a constant C such that $|J(u_k)| \leq C$, $\forall k \in \mathbb{N}$. Using (3.1), we obtain

$$C - \frac{1}{2}\|u_k\|^2 \geq \frac{-\epsilon}{\theta}\|u_k\| - \frac{1}{\theta}\|u_k\|^2$$

which results in

$$C \geq \left(\frac{\theta-2}{2\theta}\right)\|u_k\|^2 - \frac{\epsilon}{2}\|u_k\|.$$

Since $\theta > 2$, (u_k) is bounded in $H_0^1(0,+\infty)$ i.e., there is some $M_2 > 0$ such that $\|u_k\| \leq M_2$, for $k \in \mathbb{N}$.

Next, we prove that (u_k) converges strongly to some $\overline{u}$ in $H_0^1(0,+\infty)$. Since (u_k) is bounded in $H_0^1(0,+\infty)$, there exists a subsequence of (u_k), still denoted (u_k), such that (u_k) converges weakly to some $\overline{u}$ in $H_0^1(0,+\infty)$ with $\|\overline{u}\| \leq M_2$. As already mentioned, by Lemma 2.4, (u_k) converges to $\overline{u}$ on $C_{l,p}[0,+\infty)$.

Since $\lim_{k\to+\infty} J'(u_k) = 0$ and (u_k) converges weakly to some $\overline{u}$, we see that

$$\lim_{k\to+\infty} \langle J'(u_k) - J'(\overline{u}),\ u_k - \overline{u}\rangle \to\ 0. \tag{3.2}$$

Calculating in (3.2) directly we see that

$$\langle J'(u_k) - J'(\overline{u}), u_k - \overline{u}\rangle = \|u_k - \overline{u}\|^2 - \lambda \int_0^{+\infty} q(t)\left(f(t, u_k(t)) - f(t, \overline{u}(t))\right)(u_k(t) - \overline{u}(t)).$$

Since $u_k \to \overline{u}$ on $C_{l,p}(0,+\infty)$ and $p(t) > 0$, for all $t \in [0,+\infty)$ then it follows that $u_k(t) \to \overline{u}(t)$ for $t \in [0,+\infty)$ and since f is a Carathéodory function, we have $f(t, u_k(t)) \to f(t, \overline{u}(t))$ as $k \to +\infty$ for a.e. $t \in [0,+\infty)$. Using $(\mathbf{H_1})$ we have

$$\begin{aligned} q(t)\,|f(t, u_k(t))| &\le q(t)a(t)|u_k(t)|^\sigma + q(t)b(t) \\ &\le q(t)a(t)\|u_k\|_{L^\infty}^\sigma + q(t)b(t) \le K^\sigma q(t)a(t)\|u_k\|^\sigma + q(t)b(t) \\ &\le M_2^\sigma K^\sigma q(t)a(t) + q(t)b(t) \end{aligned} \tag{3.3}$$

and since $qa \in L^1(0,+\infty)$, $qb \in L^1(0,+\infty)$, we see also that

$$M_2^\sigma K^\sigma qa + qb \in L^1(0,+\infty). \tag{3.4}$$

By the Lebesgue dominated convergence theorem we now have

$$\lim_{k\to+\infty} \int_0^\infty q(t)f(t, u_k(t))dt = \int_0^\infty q(t)f(t, \overline{u}(t))dt. \tag{3.5}$$

Then (3.2) and (3.5) imply that (u_k) is strongly convergent. □

Lemma 3.3. *Assume that* $\mathbf{A}$ *holds. Suppose also that* $(\mathbf{H_2})$–$(\mathbf{H_3})$ *hold. Then for any* $\lambda > 0$ *there exist numbers* $\rho, \alpha > 0$ *such that* $J(u) \ge \alpha$ *for all* $u \in H_0^1(0,+\infty)$ *with* $\|u\| = \rho$*. Moreover, there exists an element* $z_0 \in H_0^1(0,+\infty)$ *with* $\|z_0\| > \rho$ *and such that* $J(z_0) < 0$*.*

Proof. Let us fix $\lambda > 0$ and let

$$0 < \epsilon \le \frac{1}{\lambda K^2 C_1}.$$

From $(\mathbf{H_3})$ there exists $\delta > 0$ such that $|f(t,x)| \le \epsilon|x|$ whenever $|x| \le \delta$.

Let $0 < \rho \le \dfrac{\delta}{K}$ and $\alpha = \dfrac{1}{2}(1 - \lambda\epsilon K^2 C_1)\rho^2$. Then for $\|u\| = \rho$, we have

$$\begin{aligned} \int_0^\infty q(t)|F(t,u(t))|dt &\le \frac{\epsilon}{2}\int_0^\infty q(t)|u(t)|^2 dt \\ &\le \frac{\epsilon}{2}\|u\|_{L^\infty}^2\|q\|_{L^1} \le \frac{\epsilon C_1 K^2}{2}\|u\|^2 = \frac{\epsilon C_1 K^2}{2}\rho^2 \end{aligned}$$

and

$$J(u) = \frac{1}{2}\|u\|^2 - \lambda\int_0^\infty q(t)|F(t,u(t))|dt \ge \frac{1}{2}(1 - \epsilon\lambda K^2 C_1)\rho^2 = \alpha.$$

Assumption (1) in Lemma 3.3 is then satisfied.

Now $(\mathbf{H2})(\mathbf{a})$ guarantees that for some $w_0 \in H_0^1(0,+\infty)$ with $w_0 \ne 0$ and $s \in \mathbb{R}_+$ we have the following estimation

$$\begin{aligned} J(sw_0) &= \int_0^\infty \left(\frac{1}{2}|sw_0'(t)|^2 + \frac{1}{2}|sw_0(t)|^2\right)dt - \lambda\int_0^\infty q(t)F(t, sw_0(t))dt \\ &\le \frac{1}{2}s^2\|w_0\|^2 - \lambda s^\theta \int_0^\infty c_1(t)|w_0(t)|^\theta q(t)dt + \lambda\int_0^\infty c_2(t)q(t)dt. \end{aligned}$$

Since $\theta > 2$ we see that $J(sw_0) \to -\infty$ as $s \to +\infty$. Thus there is some s_0 such that for $z_0 = s_0 w_0$ we have $J(z_0) < 0$. Therefore Assumption (2) in Lemma 3.3 is also satisfied. $\square$

Now the mountain pass lemma allows us to formulate the following existence result.

Theorem 3.4. *Assume that* **A** *holds. Suppose also that* $(\mathbf{H_1})$–$(\mathbf{H_3})$ *hold. Then for any* $\lambda > 0$, *problem* (1.1) *has at least one nontrivial solution.*

3.2 Results by the critical point theorem on a closed ball

Theorem 2.8 allows us also to obtain the existence of at least one nontrivial solution without employing mountain pass geometry. Some assumptions involved in obtaining the existence result by the mountain pass technique are to employed, namely $(\mathbf{H_1})$. However, we need no information about the behaviour of the nonlinearity around 0 apart from some assumption concerning the sign at 0 so that to ensure that the solution is nontrivial. Note that the assumption leading to the usage of the mountain pass geometry require that f is 0 at 0. This is in contrast to the previous case and so both existence results lead to the coverage of different type of nonlinear terms. Indeed, we have the following result

Theorem 3.5. *Assume that* **A** *holds. Suppose also that* $(\mathbf{H_1}), (\mathbf{H_4})$ *hold. Then there exists* $\lambda^* > 0$ *such that for all* $0 < \lambda < \lambda^*$, *problem* (1.1) *has at least one nontrivial solution provided that* $f(t, 0) \neq 0$ *on a subset of* $[0, +\infty)$ *of positive measure.*

Proof. Let us define a set $B \subset H_0^1(0, +\infty)$ as a closed ball with radius r centred at 0. Recall that by (3.3) we get what follows for any $u \in B$

$$q(t)\,|f(t, u(t))| \leq K^\sigma r^\sigma q(t)a(t) + q(t)b(t)$$

for a.e. $t \in [0, +\infty)$. Define $d := M_2^\sigma r^\sigma \|qa\|_{L^2} + \|qb\|_{L^2}$. Then we see that for any $v \in H_0^1(0, +\infty)$ by the Schwartz inequality

$$\begin{aligned} \int_0^{+\infty} q(t)\,|f(t, u(t))v(t)|\,dt &\leq M_2^\sigma r^\sigma \int_0^{+\infty} q(t)a\,(t)\,v(t)dt + \int_0^{+\infty} q(t)b(t)v(t)dt \\ &\leq \left(K^\sigma r^\sigma \|qa\|_{L^2} + \|qb\|_{L^2}\right)\|v\|_{L^2} \leq d\left(\|v\|_{L^2}^2 + \|v'\|_{L^2}^2\right)^{\frac{1}{2}} = d\|v\|. \end{aligned} \tag{3.6}$$

Put $\lambda^* = \frac{r}{d}$ and fix $\lambda \in (0,\ \lambda^*)$.

We see that $u = 0$ cannot be a solution, thus any solution which we obtain is necessarily nontrivial.

Recall that $J = J_1 - \lambda J_{2.}$, see (2.1), (2.2). Note that J_1 is weakly l.s.c. Since J_2 is weakly continuous, we see that J is weakly l.s.c. Since B is weakly compact, we obtain that J has at least one minimizer u_0 over B for any $\lambda > 0$.

We shall apply Theorem 2.8. Put $\Phi, H : H_0^1(0, +\infty) \to \mathbb{R}$ by formulas

$$\Phi(u) = \int_0^{+\infty} \frac{1}{2}\left(u'(t)\right)^2 + \int_0^{+\infty} \frac{1}{2}\left(u(t)\right)^2, \qquad H(u) = \lambda \int_0^{+\infty} q(t)F(t, u(t))dt$$

and note that these are convex C^1 functionals. Consider the auxiliary Dirichlet problem

$$\begin{cases} -u''(t) + u(t) = \lambda q(t) f(t, u_0(t)) \\ u(0) = u(+\infty) = 0. \end{cases} \tag{3.7}$$

Note that problem (3.7) is uniquely solvable by some $v \in H_0^1(0,+\infty)$. To reach this conclusion, we use the following procedure. We prove that the action functional corresponding to (3.7)

$$J_0(u) = \frac{1}{2}\int_0^{+\infty} \left((u'(t))^2 + u^2(t)\right) dt - \lambda \int_0^{+\infty} q(t)f(t,u_0(t))u(t)dt$$

is coercive, C^1, weakly l.s.c. and strictly convex. Then the direct method of the calculus of variation, see [10], provides us with exactly one solution to (3.7).

We shall prove that $v \in B$. Multiplying (3.7) with $u = v$ by v and integrating by parts we get

$$\int_0^{+\infty} (v'(t))^2\, dt + \int_0^{+\infty} (v(t))^2 dt = \lambda \int_0^{+\infty} q(t)f(t,u_0(t))v(t)dt.$$

Using (3.6) we get from the above

$$\|v\|^2 \leq \lambda d\|v\| \leq r.$$

Thus $v \in B$ and therefore Theorem 2.8 applies. □

Example

$$\begin{cases} -u''(t) + u(t) = \lambda e^{-2t}u^3\left(\dfrac{|u| + (3|u|+4)\ln(|u|+1)}{(|u|+1)^2}\right)(|\sin(t)|+1), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{3.8}$$

It can be easily checked that all conditions of Theorem 3.4 are satisfied with

$$f(t,u) = u^3\left(\frac{|u| + (3|u|+4)\ln(|u|+1)}{(|u|+1)^2}\right)(|\sin(t)|+1), \quad \sigma = 4,$$

$$a(t) = 5 + |\cos(t)|, \quad b(t) = 1 + |\sin(t)|, \quad \theta = 5/2, \quad c_1(t) = |\sin(t)| + \frac{1}{2},$$

$$c_2(t) = 4 + |\sin(t)|, \quad q(t) = e^{-2t}, \quad p(t) = e^{-t},$$

and

$$F(t,u) = u^4\frac{\ln(|u|+1)}{|u|+1}(|\sin(t)|+1).$$

Therefore problem (3.8) has at least one nontrivial solution for any $\lambda > 0$.

Concerning the usage of the theorem on a closed ball we consider the following problem for which $(\mathbf{H_3})$ is not satisfied

$$\begin{cases} -u''(t) + u(t) = \lambda e^{-2t}u^3\left(\dfrac{|u| + (3|u|+4)\ln(|u|+1)}{(|u|+1)^2}\right)(|\sin(t)|+1) \\ \qquad\qquad + \lambda e^{-2t}\left(|\sin(t)|+1\right), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{3.9}$$

It can be easily checked that all conditions of Theorem 3.5 are satisfied with

$$f(t,u) = u^3\left(\frac{|u| + (3|u|+4)\ln(|u|+1)}{(|u|+1)^2}\right)(|\sin(t)|+1) + \left(|\sin(t)|+1\right), \quad \sigma = 4,$$

$$a(t) = 5 + |\cos(t)|, \quad b(t) = 2 + |\sin(t)|, \quad \theta = 5/2$$

$$c_1(t) = |\sin(t)| + \frac{1}{2}, \quad c_2(t) = 4 + |\sin(t)|, \quad q(t) = e^{-2t}, \quad p(t) = e^{-t},$$

and

$$F(t,u) = u^4 \frac{\ln(|u|+1)}{|u|+1}(|\sin(t)|+1) + u\,(|\sin(t)|+1) - 1.$$

Then there is some $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (3.9) has at least one nontrivial solution.

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