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STABILITY IN NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS USING FIXED POINT THEORY

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ABSTRACT. The purpose of this paper is to use a fixed point approach to obtain asymptotic stability results of a nonlinear neutral differential equation with variable delays. An asymptotic stability theorem with a necessary and sufficient condition is proved. In our consideration we allow the coefficient functions to change sign and do not require bounded delays. The obtained results improve and generalize those due to Burton, Zhang and Raffoul. We end by giving three examples to illustrate our work.

1. INTRODUCTION

Since 1892, the Lyapunov's direct method has been successfully used in establishing stability results for a wide variety of ordinary, functional and partial differential equations. Yet, there is a large number of problems which remain unsolved. Particularly, the application of this method to problems of stability in differential equations with delay has encountered serious obstacles if the delay is unbounded or if the equation has unbounded terms $[2-4]$. Recently, investigators such as Burton, Furumochi, Zhang, Raffoul and others concentrated on new avenues and began a study in which they have noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see $[1-13,15]$). Not only the fixed point method solve problems on stability but has interesting features of averaging nature while the Lyapunov's conditions are usually pointwise (see [2]).

With this in mind, we consider, in this paper, the nonlinear neutral differential equation with variable delays

$$x'(t) = -a(t)\,x(t-\tau_1(t)) + c(t)\,x'(t-\tau_2(t)) + G(t, x(t-\tau_1(t)), x(t-\tau_2(t))), \tag{1.1}$$

with the initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(t_0), t_0],$$

where $\psi \in C([m(t_0), t_0], \mathbb{R})$ and for each $t_0 \geq 0$,

$$m_j(t_0) = \inf\{t - \tau_j(t),\ t \geq t_0\},\ m(t_0) = \min\{m_j(t_0),\ j = 1, 2\}.$$

Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$ with the supremum norm $\|.\|$. Throughout this paper we assume that $a \in C(\mathbb{R}^+, \mathbb{R})$, $c \in C^1(\mathbb{R}^+, \mathbb{R})$ and $\tau_1, \tau_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t - \tau_1(t) \to \infty$ and $t - \tau_2(t) \to \infty$ as $t \to \infty$. The function $G(t, x, y)$ is locally Lipschitz continuous in x and y. That is, there are positive constants L_1 and L_2 so that if $|x|, |y|, |z|, |w| \leq L$ for some positive constant L then

$$|G(t, x, y) - Q(t, z, w)| \leq L_1 \|x - z\| + L_2 \|y - w\| \text{ and } G(t, 0, 0) = 0. \tag{1.2}$$

Equation (1.1) and its special cases have been investigated by many authors. For example, Burton in [4], and Zhang in [15] have studied the equation

$$x'(t) = -a(t)\,x(t-\tau_1(t)), \tag{1.3}$$

and proved the following.

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Theorem A (Burton [4]). *Suppose that* $\tau_1(t) = \tau$ *and there exists a constant* $\alpha < 1$ *such that*

$$\int_{t-\tau}^{t} |a(s+\tau)|\, ds + \int_0^t |a(s+\tau)|\, e^{-\int_s^t a(u+\tau)du} \left(\int_{s-\tau}^{s} |a(u+\tau)|\, du \right) ds \le \alpha, \tag{1.4}$$

for all $t \ge 0$ *and* $\int_0^\infty a(s)\, ds = \infty$. *Then, for every continuous initial function* $\psi : [-\tau, 0] \to \mathbb{R}$, *the solution* $x(t) = x(t, 0, \psi)$ *of* (1.3) *is bounded and tends to zero as* $t \to \infty$.

Theorem B (Zhang [15]). *Suppose that* τ_1 *is differentiable, the inverse function* g *of* $t - \tau_1(t)$ *exists, and there exists a constant* $\alpha \in (0,1)$ *such that for* $t \ge 0$, $\liminf_{t\to\infty} \int_0^t a(g(s))\, ds > -\infty$ *and*

$$\int_{t-\tau_1(t)}^{t} |a(g(s))|\, ds + \int_0^t e^{-\int_s^t a(g(u))du} |a(s)|\, |\tau_1'(s)|\, ds$$
$$+ \int_0^t e^{-\int_s^t a(g(u))du} |a(g(s))| \left(\int_{s-\tau_1(s)}^{s} |a(g(u))|\, du \right) ds \le \alpha. \tag{1.5}$$

Then the zero solution of (1.3) *is asymptotically stable if and only if* $\int_0^t a(g(s))\, ds \to \infty$ *as* $t \to \infty$.

Obviously, Theorem B improves Theorem A. On the other hand, Raffoul in [13] considered the following nonlinear neutral differential equation

$$x'(t) = -a(t)x(t) + c(t)x'(t - \tau_2(t)) + G(t, x(t), x(t - \tau_2(t))), \tag{1.6}$$

and obtained the following.

Theorem C (Raffoul [13]). *Suppose* (1.2) *holds, and there exists a constant* $\alpha \in (0,1)$ *such that for* $t \ge 0$, $\int_0^t a(s)\, ds \to \infty$ *as* $t \to \infty$, *and*

$$\left| \frac{c(t)}{1 - \tau_2'(t)} \right| + \int_0^t e^{-\int_s^t a(u)du} \left[|r_2(s)| + L_1 + L_2 \right] ds \le \alpha, \tag{1.7}$$

where $r_2(t) = \dfrac{[c(t)a(t) + c'(t)](1 - \tau_2'(t)) + c(t)\tau_2''(t)}{(1 - \tau_2'(t))^2}$. *Then every solution* $x(t) = x(t, 0, \psi)$ *of* (1.6) *with a small continuous initial function* ψ *is bounded and tends to zero as* $t \to \infty$.

Our purpose here is to give, by using the contraction mapping principle, asymptotic stability results of a nonlinear neutral differential equation with variable delays (1.1). An asymptotic stability theorem with a necessary and sufficient condition is proved. In this work we do not force the delays to be bounded and allow the coefficient functions to change sign. Three examples are also given to illustrate our results. The results presented in this paper improve and generalize the main results in [4, 13, 15].

2. MAIN RESULTS

For each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, a solution of (1.1) through (t_0, ψ) is a continuous function $x : [m(t_0), t_0 + \alpha) \to \mathbb{R}$ for some positive constant $\alpha > 0$ such that x satisfies (1.1) on $[t_0, t_0 + \alpha)$ and $x = \psi$ on $[m(t_0), t_0]$. We denote such a solution by $x(t) = x(t, t_0, \psi)$. For each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, there exists a unique solution $x(t) = x(t, t_0, \psi)$ of (1.1) defined on $[t_0, \infty)$. For fixed t_0, we define $\|\psi\| = \max\{|\psi(t)| : m(t_0) \le t \le t_0\}$. Stability definitions may be found in [2], for example.

Our aim here is to generalize Theorems $A - C$ to (1.1).

Theorem 1. *Suppose* (1.2) *holds. Let* τ_1 *be differentiable and* τ_2 *be twice differentiable with* $\tau_2'(t) \ne 1$ *for all* $t \in \mathbb{R}^+$. *Suppose that there exist continuous functions* $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ *for* $j = 1, 2$ *and a constant* $\alpha \in (0,1)$ *such that for* $t \ge 0$

$$\liminf_{t\to\infty} \int_0^t H(s)\, ds > -\infty, \tag{2.1}$$

and

$$\left|\frac{c(t)}{1-\tau_2'(t)}\right| + \sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t} |h_j(s)|\,ds + \int_0^t e^{-\int_s^t H(u)du}\left\{|-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))|\right.$$
$$\left. + |h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s)| + L_1 + L_2\right\} ds$$
$$+ \sum_{j=1}^{2}\int_0^t e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,du\right) ds \le \alpha, \quad (2.2)$$

where $H(t) = \sum_{j=1}^{2} h_j(t)$ *and* $r(t) = \dfrac{[c(t)H(t) + c'(t)](1-\tau_2'(t)) + c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. *Then the zero solution of* (1.1) *is asymptotically stable if and only if*

$$\int_0^t H(s)\,ds \to \infty \ \text{as } t \to \infty. \quad (2.3)$$

Proof. First, suppose that (2.3) holds. For each $t_0 \ge 0$, we set

$$K = \sup_{t\ge 0}\left\{e^{-\int_0^t H(s)ds}\right\}. \quad (2.4)$$

Let $\psi \in C([m(t_0), t_0], \mathbb{R})$ be fixed and define

$$S_\psi = \{\varphi \in C([m(t_0), \infty), \mathbb{R}) : \varphi(t) \to 0 \text{ as } t \to \infty,\ \varphi(t) = \psi(t) \text{ for } t \in [m(t_0), t_0]\}.$$

This S_ψ is a complete metric space with metric $\rho(x, y) = \sup_{t\ge t_0}\{|x(t) - y(t)|\}$.

Multiply both sides of (1.1) by $e^{\int_{t_0}^t H(u)du}$ and then integrate from t_0 to t to obtain

$$x(t) = \psi(t_0)e^{-\int_{t_0}^t H(u)du} + \sum_{j=1}^{2}\int_{t_0}^t e^{-\int_s^t H(u)du} h_j(s)x(s)\,ds$$
$$+ \int_{t_0}^t e^{-\int_s^t H(u)du}\left\{-a(s)x(s-\tau_1(s)) + c(s)x'(s-\tau_2(s))\right.$$
$$\left. + G(s, x(s-\tau_1(s)), x(s-\tau_2(s)))\right\} ds.$$

Performing an integration by parts, we have

$$
\begin{aligned}
x(t) &= \left(\psi(t_0) - \frac{c(t_0)}{1-\tau_2'(t_0)}\psi(t_0-\tau_2(t_0))\right)e^{-\int_{t_0}^t H(u)du} \\
&+ \frac{c(t)}{1-\tau_2'(t)}x(t-\tau_2(t)) + \sum_{j=1}^{2}\int_{t_0}^{t} e^{-\int_s^t H(u)du} d\left(\int_{s-\tau_j(s)}^{s} h_j(u)\,x(u)\,du\right) \\
&+ \sum_{j=1}^{2}\int_{t_0}^{t} e^{-\int_s^t H(u)du}\left\{h_j(s-\tau_j(s))\left(1-\tau_j'(s)\right)\right\}x(s-\tau_j(s))\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)du}\{-a(s)\,x(s-\tau_1(s)) - r(s)\,x(s-\tau_2(s)) \\
&+ G(s,x(s-\tau_1(s)),x(s-\tau_2(s)))\}\,ds \\
&= \left(\psi(t_0) - \frac{c(t_0)}{1-\tau_2'(t_0)}\psi(t_0-\tau_2(t_0)) - \sum_{j=1}^{2}\int_{t_0-\tau_j(t_0)}^{t_0} h_j(s)\,\psi(s)\,ds\right)e^{-\int_{t_0}^t H(u)du} \\
&+ \frac{c(t)}{1-\tau_2'(t)}x(t-\tau_2(t)) + \sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t} h_j(s)\,x(s)\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)du}\{(-a(s)+h_1(s-\tau_1(s))(1-\tau_1'(s)))\,x(s-\tau_1(s)) \\
&+ (h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s))\,x(s-\tau_2(s)) + G(s,x(s-\tau_1(s)),x(s-\tau_2(s)))\}\,ds \\
&- \sum_{j=1}^{2}\int_{t_0}^{t} e^{-\int_s^t H(u)du}H(s)\left(\int_{s-\tau_j(s)}^{s} h_j(u)\,x(u)\,du\right)ds. \qquad (2.5)
\end{aligned}
$$

Use (2.5) to define the operator $P : S_\psi \to S_\psi$ by $(P\varphi)(t) = \psi(t)$ for $t \in [m(t_0), t_0]$ and

$$
\begin{aligned}
(P\varphi)(t) &= \left\{\psi(t_0) - \frac{c(t_0)}{1-\tau_2'(t_0)}\psi(t_0-\tau_2(t_0)) - \sum_{j=1}^{2}\int_{t_0-\tau_j(t_0)}^{t_0} h_j(s)\,\psi(s)\,ds\right\}e^{-\int_{t_0}^t H(u)du} \\
&+ \frac{c(t)}{1-\tau_2'(t)}\varphi(t-\tau_2(t)) + \sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t} h_j(s)\,\varphi(s)\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)du}\{(-a(s)+h_1(s-\tau_1(s))(1-\tau_1'(s)))\,\varphi(s-\tau_1(s)) \\
&+ (h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s))\,\varphi(s-\tau_2(s)) + G(s,\varphi(s-\tau_1(s)),\varphi(s-\tau_2(s)))\}\,ds \\
&- \sum_{j=1}^{2}\int_{t_0}^{t} e^{-\int_s^t H(u)du}H(s)\left(\int_{s-\tau_j(s)}^{s} h_j(u)\,\varphi(u)\,du\right)ds, \qquad (2.6)
\end{aligned}
$$

for $t \geq t_0$. It is clear that $(P\varphi) \in C([m(t_0), \infty), \mathbb{R})$. We now show that $(P\varphi)(t) \to 0$ as $t \to \infty$. Since $\varphi(t) \to 0$ and $t - \tau_j(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists a $T_1 > t_0$ such that $s \geq T_1$

implies that $|\varphi(s-\tau_j(s))| < \varepsilon$ for $j = 1, 2$. Thus, for $t \geq T_1$, the last term I_5 in (2.6) satisfies

$$
\begin{aligned}
|I_5| &= \left| \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)du} H(s) \left(\int_{s-\tau_j(s)}^{s} h_j(u) \varphi(u)\, du \right) ds \right| \\
&\leq \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)| |\varphi(u)|\, du \right) ds \\
&+ \sum_{j=1}^{2} \int_{T_1}^{t} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)| |\varphi(u)|\, du \right) ds \\
&\leq \sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\, du \right) ds \\
&+ \varepsilon \sum_{j=1}^{2} \int_{T_1}^{t} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\, du \right) ds.
\end{aligned}
$$

By (2.3), there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$
\begin{aligned}
&\sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\, du \right) ds \\
&\quad = \sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| e^{-\int_{T_1}^{t} H(u)du} \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^{T_1} H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\, du \right) ds < \varepsilon.
\end{aligned}
$$

Apply (2.2) to obtain $|I_5| < \varepsilon + \alpha\epsilon < 2\varepsilon$. Thus, $I_5 \to 0$ as $t \to \infty$. Similarly, we can show that the rest of the terms in (2.6) approach zero as $t \to \infty$. This yields $(P\varphi)(t) \to 0$ as $t \to \infty$, and hence $P\varphi \in S_\psi$. Also, by (2.2), P is a contraction mapping with contraction constant α. By the contraction mapping principle (Smart [14, *p.* 2]), P has a unique fixed point x in S_ψ which is a solution of (1.1) with $x(t) = \psi(t)$ on $[m(t_0), t_0]$ and $x(t) = x(t, t_0, \psi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying $2\delta K e^{\int_0^{t_0} H(u)du} + \alpha\varepsilon < \varepsilon$. If $x(t) = x(t, t_0, \psi)$ is a solution of (1.1) with $\|\psi\| < \delta$, then $x(t) = (Px)(t)$ defined in (2.6). We claim that $|x(t)| < \varepsilon$ for all $t \geq t_0$. Notice that $|x(s)| < \varepsilon$ on $[m(t_0), t_0]$. If there exists $t^* > t_0$ such that $|x(t^*)| = \varepsilon$ and

$|x(s)| < \varepsilon$ for $m(t_0) \le s < t^*$, then it follows from (2.6) that

$$|x(t^*)| \le \|\psi\| \left(1 + \left|\frac{c(t_0)}{1-\tau_2'(t_0)}\right| + \sum_{j=1}^{2} \int_{t_0-\tau_j(t_0)}^{t_0} |h_j(s)|\, ds\right) e^{-\int_{t_0}^{t^*} H(u)du}$$
$$+ \epsilon \left|\frac{c(t^*)}{1-\tau_2'(t^*)}\right| + \epsilon \sum_{j=1}^{2} \int_{t^*-\tau_j(t^*)}^{t^*} |h_j(s)|\, ds$$
$$+ \epsilon \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)du} \left\{|-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))|\right.$$
$$\left. + |h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s)| + L_1 + L_2\right\} ds$$
$$+ \epsilon \sum_{j=1}^{2} \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\, du\right) ds$$
$$\le 2\delta K e^{\int_0^{t_0} H(u)du} + \alpha\varepsilon < \epsilon,$$

which contradicts the definition of t^*. Thus, $|x(t)| < \varepsilon$ for all $t \ge t_0$, and the zero solution of (1.1) is stable. This shows that the zero solution of (1.1) is asymptotically stable if (2.3) holds.

Conversely, suppose (2.3) fails. Then by (2.1) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} H(u)\, du = l$ for some $l \in \mathbb{R}^+$. We may also choose a positive constant J satisfying

$$-J \le \int_0^{t_n} H(u)\, du \le J,$$

for all $n \ge 1$. To simplify our expressions, we define

$$\omega(s) = |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))| + |h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s)|$$
$$+ L_1 + L_2 + |H(s)| \sum_{j=1}^{2} \int_{s-\tau_j(s)}^{s} |h_j(u)|\, du,$$

for all $s \ge 0$. By (2.2), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} H(u)du} \omega(s)\, ds \le \alpha.$$

This yields

$$\int_0^{t_n} e^{\int_0^s H(u)du} \omega(s)\, ds \le \alpha e^{\int_0^{t_n} H(u)du} \le J.$$

The sequence $\left\{\int_0^{t_n} e^{\int_0^s H(u)du} \omega(s)\, ds\right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^s H(u)du} \omega(s)\, ds = \gamma,$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s H(u)du} \omega(s)\, ds < \delta_0/4K,$$

for all $n \ge m$, where $\delta_0 > 0$ satisfies $2\delta_0 K e^J + \alpha \le 1$.

By (2.1), K in (2.4) is well defined. We now consider the solution $x(t) = x(t, t_m, \psi)$ of (1.1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \le \delta_0$ for $s \le t_m$. We may choose ψ so that $|x(t)| \le 1$ for $t \ge t_m$ and

$$\psi(t_m) - \frac{c(t_m)}{1 - \tau_2'(t_m)}\psi(t_m - \tau_2(t_m)) - \sum_{j=1}^{2}\int_{t_m - \tau_j(t_m)}^{t_m} h_j(s)\psi(s)\,ds \ge \frac{1}{2}\delta_0.$$

It follows from (2.6) with $x(t) = (Px)(t)$ that for $n \ge m$

$$\begin{aligned}
&\left| x(t_n) - \frac{c(t_n)}{1 - \tau_2'(t_n)}x(t_n - \tau_2(t_n)) - \sum_{j=1}^{2}\int_{t_n - \tau_j(t_n)}^{t_n} h_j(s)x(s)\,ds \right| \\
&\ge \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} H(u)du}\omega(s)\,ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} - e^{-\int_0^{t_n} H(u)du}\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds \\
&= e^{-\int_{t_m}^{t_n} H(u)du}\left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m} H(u)du}\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds\right) \\
&\ge e^{-\int_{t_m}^{t_n} H(u)du}\left(\frac{1}{2}\delta_0 - K\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds\right) \\
&\ge \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} \ge \frac{1}{4}\delta_0 e^{-2J} > 0. \qquad (2.7)
\end{aligned}$$

On the other hand, if the zero solution of (1.1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$ and (2.2) holds, we have

$$x(t_n) - \frac{c(t_n)}{1 - \tau_2'(t_n)}x(t_n - \tau_2(t_n)) - \sum_{j=1}^{2}\int_{t_n - \tau_j(t_n)}^{t_n} h_j(s)x(s)\,ds \to 0 \text{ as } n \to \infty,$$

which contradicts (2.7). Hence condition (2.3) is necessary for the asymptotic stability of the zero solution of (1.1). The proof is complete. □

Remark 1. *It follows from the first part of the proof of Theorem* 1 *that the zero solution of* (1.1) *is stable under* (2.1) *and* (2.2). *Moreover, Theorem* 1 *still holds if* (2.2) *is satisfied for* $t \ge t_\sigma$ *for some* $t_\sigma \in \mathbb{R}^+$.

For the special case $c(t) = 0$ and $G(t, x, y) = 0$, we can get

Corollary 1. *Let* τ_1 *be differentiable, and suppose that there exist continuous function* $h_1 : [m_1(t_0), \infty) \to \mathbb{R}$ *for and a constant* $\alpha \in (0, 1)$ *such that for* $t \ge 0$

$$\liminf_{t\to\infty} \int_0^t h_1(s)\,ds > -\infty,$$

and

$$\int_{t-\tau_1(t)}^{t} |h_1(s)|\,ds + \int_0^t e^{-\int_s^t h_1(u)du}\left|-a(s) + h_1(s - \tau_1(s))(1 - \tau_1'(s))\right|ds + \int_0^t e^{-\int_s^t h_1(u)du}|h_1(s)|\left(\int_{s-\tau_1(s)}^{s} |h_1(u)|\,du\right)ds \le \alpha. \qquad (2.8)$$

Then the zero solution of (1.3) *is asymptotically stable if and only if*

$$\int_0^t h_1(s)\,ds \to \infty \text{ as } t \to \infty.$$

Remark 2. *When* $\tau_1(s) = \tau$, *a constant,* $h_1(s) = a(s+\tau)$, *Corollary* 1 *contains Theorem A. When* $h_1(s) = a(g(s))$, *where* $g(s)$ *is the inverse function of* $s - \tau_1(s)$, *Corollary* 1 *reduces to Theorem B.*

Letting $\tau_1 = 0$, we have

Corollary 2. *Suppose* (1.2) *holds. Let* τ_1 *be differentiable and* τ_2 *be twice differentiable with* $\tau_2'(t) \neq 1$ *for all* $t \in \mathbb{R}^+$. *Suppose that there exist continuous functions* $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ *for* $j = 1, 2$ *and a constant* $\alpha \in (0,1)$ *such that for* $t \geq 0$

$$\lim_{t\to\infty} \inf \int_0^t H(s)\,ds > -\infty,$$

and

$$\begin{aligned}\left|\frac{c(t)}{1-\tau_2'(t)}\right| &+ \int_{t-\tau_2(t)}^t |h_2(s)|\,ds \\ &+ \int_0^t e^{-\int_s^t H(u)du} \left(|-a(s) + h_1(s)| + |h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s)| + L_1 + L_2\right) ds \\ &+ \int_0^t e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_2(s)}^s |h_2(u)|\,du\right) ds \leq \alpha, \end{aligned} \quad (2.9)$$

where $H(t) = \sum_{j=1}^2 h_j(t)$ *and* $r(t) = \dfrac{[c(t)H(t) + c'(t)](1-\tau_2'(t)) + c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. *Then the zero solution of* (1.6) *is asymptotically stable if and only if*

$$\int_0^t H(s)\,ds \to \infty \text{ as } t \to \infty.$$

Remark 3. *When* $h_1(s) = a(s)$ *and* $h_2(s) = 0$, *Corollary* 2 *contains Theorem C.*

3. THREE EXAMPLES

In this section, we give three examples to illustrate the applications of Corollaries 1 and 2 and Theorem 1.

Example 1. *Consider the following linear delay differential equation*

$$x'(t) = -a(t)x(t-\tau_1(t)), \quad (3.1)$$

where $\tau_1(t) = 0.272t$, $a(t) = 1/(0.728t+1)$. *Then the zero solution of* (3.1) *is asymptotically stable.*

Proof. Choosing $h_1(t) = 1.31/(t+1)$ in Corollary 1, we have

$$\int_{t-\tau_1(t)}^t |h_1(s)|\,ds = \int_{0.728t}^t \frac{1.31}{s+1} ds = 1.31 \ln \frac{t+1}{0.728t+1} < 0.4159,$$

$$\int_0^t e^{-\int_s^t h_1(u)du} |h_1(s)| \left(\int_{s-\tau_1(s)}^s |h_1(u)|\,du\right) ds < \int_0^t e^{-\int_s^t (1.31/(u+1))du} \frac{1.31}{1+s} \times 0.4159 ds < 0.4159,$$

and

$$\int_0^t e^{-\int_s^t h_1(u)du} |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))|\,ds = \int_0^t e^{-\int_s^t (1.31/(u+1))du} \frac{1-1.31\times 0.728}{0.728s+1} ds$$

$$< \frac{1-1.31\times 0.728}{1.31\times 0.728} \int_0^t e^{-\int_s^t (1.31/(u+1))du} \frac{1.31}{s+1} ds < 0.0486.$$

It is easy to see that all the conditions of Corollary 1 hold for $\alpha = 0.4159 + 0.4159 + 0.0486 = 0.8804 < 1$. Thus, Corollary 1 implies that the zero solution of (3.1) is asymptotically stable.

However, Theorem B cannot be used to verify that the zero solution of (3.1) is asymptotically stable. In fact, $a(g(t)) = 1/(t+1)$. As $t \to \infty$,

$$\int_{t-\tau_1(t)}^{t} |a(g(s))|\, ds = \int_{0.728t}^{t} \frac{1}{s+1} ds = \ln \frac{t+1}{0.728t+1} \to -\ln(0.728),$$

$$\int_0^t e^{-\int_s^t a(g(u))du} |a(g(s))| \left(\int_{s-\tau_1(s)}^{s} |a(g(s))|\, du \right) ds = \int_0^t e^{-\int_s^t (1/(u+1))du} \frac{1}{1+s} \left(\int_{0.728s}^{s} \frac{1}{u+1} du \right) ds$$

$$= \frac{1}{t+1} \int_0^t [\ln(s+1) - \ln(0.728s+1)]\, ds \to -\ln(0.728),$$

$$\int_0^t e^{-\int_s^t a(g(u))du} |a(s)| |\tau_1'(s)|\, ds = \frac{0.272}{t+1} \int_0^t \frac{s+1}{0.728s+1} ds$$

$$= \frac{0.272}{0.728} \frac{t}{t+1} - \left(\frac{0.272}{0.728} \right)^2 \frac{\ln(0.728t+1)}{t+1} \to \frac{0.272}{0.728}.$$

Thus, we have

$$\limsup_{t \geq 0} \left\{ \int_{t-\tau_1(t)}^{t} |a(g(s))|\, ds + \int_0^t e^{-\int_s^t a(g(u))du} |a(s)| |\tau_1'(s)|\, ds \right.$$
$$\left. + \int_0^t e^{-\int_s^t a(g(u))du} |a(g(s))| \left(\int_{s-\tau_1(s)}^{s} |a(g(s))|\, du \right) ds \right\}$$
$$= -2\ln(0.728) + \frac{0.272}{0.728} \simeq 1.0085.$$

In addition, the left-hand side of the following inequality is increasing in $t > 0$, then there exists some $t_0 > 0$ such that for $t > t_0$,

$$\int_{t-\tau_1(t)}^{t} |a(g(s))|\, ds + \int_0^t e^{-\int_s^t a(g(u))du} |a(s)| |\tau_1'(s)|\, ds$$
$$+ \int_0^t e^{-\int_s^t a(g(u))du} |a(g(s))| \left(\int_{s-\tau_1(s)}^{s} |a(g(s))|\, du \right) ds > 1.008.$$

This implies that condition (1.5) does not hold. Thus, Theorem B cannot be applied to equation (3.1). □

Example 2. *Consider the following linear neutral delay differential equation*

$$x'(t) = -a(t)x(t) + c(t)x'(t - \tau_2(t)), \tag{3.2}$$

where $\tau_2(t) = 0.06t$, $a(t) = 1/(t+1)$ *and* $c(t) = 0.55$. *Then the zero solution of* (3.2) *is asymptotically stable.*

Proof. Choosing $h_1(t) = 1/(t+1)$ and $h_2(t) = 0.27/(t+1)$ in Corollary 2, we have $H(t) = 1.27/(t+1)$,

$$\left| \frac{c(t)}{1-\tau_2'(s)} \right| = \frac{0.55}{0.94} < 0.586,$$

$$\int_{t-\tau_2(t)}^{t} |h_2(s)|\, ds = \int_{0.94t}^{t} \frac{0.27}{s+1} ds = 0.27 \ln \frac{t+1}{0.94t+1} < 0.017,$$

$$\int_0^t e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_2(s)}^{s} |h_2(u)|\, du \right) ds < \int_0^t e^{-\int_s^t (1.27/(u+1))du} \frac{1.27}{s+1} \times 0.017 ds < 0.017,$$

and

$$\begin{aligned}
&\int_0^t e^{-\int_s^t H(u)du} \{|-a(s) + h_1(s)| + |h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s)|\} ds \\
&= \int_0^t e^{-\int_s^t (1.27/(u+1))du} \left| \frac{0.27 \times 0.94}{0.94s+1} - \frac{0.55 \times 1.27}{0.94(s+1)} \right| ds \\
&< \left(\frac{0.55}{0.94} - \frac{0.27}{1.27} \right) \int_0^t e^{-\int_s^t (1.22/(u+1))du} \frac{1.27}{s+1} ds < 0.373.
\end{aligned}$$

It is easy to see that all the conditions of Corollary 2 hold for $\alpha = 0.586 + 0.017 + 0.017 + 0.373 = 0.993 < 1.$ Thus, Corollary 2 implies that the zero solution of (3.2) is asymptotically stable.

However, Theorem C cannot be used to verify that the zero solution of (3.2) is asymptotically stable. Obviously,

$$\left| \frac{c(t)}{1-\tau_2'(s)} \right| + \int_0^t e^{-\int_s^t a(u)du} |r_2(s)| ds = \frac{0.55(2t+1)}{0.94(t+1)}. \tag{3.3}$$

Since the left-hand side of (3.3) is increasing in $t > 0$ and

$$\limsup_{t \geq 0} \left\{ \frac{0.55(2t+1)}{0.94(t+1)} \right\} \simeq 1.1702,$$

then there exists some $t_0 > 0$ such that $t \geq t_0$,

$$\left| \frac{c(t)}{1-\tau_2'(s)} \right| + \int_0^t e^{-\int_s^t a(u)du} |r_2(s)| ds > 1.17.$$

This implies that condition (1.7) does not hold. Thus, Theorem C cannot be applied to equation (3.2). □

Example 3. *Consider the following linear neutral delay differential equation*

$$x'(t) = -a(t)x(t-\tau_1(t)) + c(t)x'(t-\tau_2(t)), \tag{3.4}$$

where $\tau_1(t) = 0.05t$, $\tau_2(t) = 0.07t$, $a(t) = 0.95/(0.95t+1)$ *and* $c(t) = 0.36$. *Then the zero solution of* (3.2) *is asymptotically stable.*

Proof. Choosing $h_1(t) = 1/(t+1)$ and $h_2(t) = 0.32/(t+1)$ in Theorem 1, we have $H(t) = 1.32/(t+1)$,

$$\left| \frac{c(t)}{1-\tau_2'(s)} \right| = \frac{0.36}{0.93} < 0.388,$$

$$\begin{aligned}
\sum_{j=1}^{2} \int_{t-\tau_j(t)}^t |h_j(s)| ds &= \int_{0.95t}^t \frac{1}{s+1} ds + \int_{0.93t}^t \frac{0.32}{s+1} ds \\
&= \ln\left(\frac{t+1}{0.95t+1} \right) + 0.32 \ln\left(\frac{t+1}{0.93t+1} \right) < 0.075,
\end{aligned}$$

$$\sum_{j=1}^{2} \int_0^t e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^s |h_j(u)| du \right) ds < \int_0^t e^{-\int_s^t (1.32/(u+1))du} \frac{1.32}{s+1} \times 0.075 ds < 0.075,$$

and

$$\int_0^t e^{-\int_s^t H(u)du} \{|-a(s) + h_1(s - \tau_1(s))(1 - \tau_1'(s))| + |h_2(s - \tau_2(s))(1 - \tau_2'(s)) - r(s)|\} ds$$

$$= \int_0^t e^{-\int_s^t (1.32/(u+1))du} \left| \frac{0.32 \times 0.93}{0.93s + 1} - \frac{0.36 \times 1.32}{0.93(s+1)} \right| ds$$

$$< \left(\frac{0.36}{0.93} - \frac{0.32}{1.32} \right) \int_0^t e^{-\int_s^t (1.32/(u+1))du} \frac{1.32}{s+1} ds < 0.145.$$

It is easy to see that all the conditions of Theorem 1 hold for $\alpha = 0.388 + 0.075 + 0.075 + 0.145 = 0.683 < 1$. Thus, Theorem 1 implies that the zero solution of (3.4) is asymptotically stable. □

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