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Singular and classical second order ϕ-Laplacian equations on the half-line with functional boundary conditions

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Abstract. This paper is concerned with the existence of bounded or unbounded solutions to regular and singular second order boundary value problem on the half-line with functional boundary conditions. These functional boundary conditions generalize the usual boundary assumptions and may be applied to a broad number of cases, such as, nonlocal, integro-differential, with delays, with maximum or minimum arguments... The arguments are based on the Schauder fixed point theorem and lower and upper solutions method.

Keywords: Half line problems, functional boundary conditions, unbounded upper and lower solutions, Schauder fixed point theory.

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1 Introduction

This paper is concerned with the study of a fully nonlinear equation on the half line

$$\big(\phi(u'(t))\big)' + q(t)f\big(t, u(t), u'(t)\big) = 0, \qquad t \in [0, +\infty), \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ and $q : (0, +\infty) \to [0, +\infty)$ are both continuous functions, verifying adequate assumptions, but q is allowed to have a singularity when $t = 0$, coupled with the functional boundary conditions

$$L\big(u, u(0), u'(0)\big) = 0, \qquad u'(+\infty) := \lim_{t\to+\infty} u'(t) = B, \tag{1.2}$$

where $L : C([0, \infty)) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with properties to be made precise later and $B \in \mathbb{R}$.

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Boundary value problems, usually, are considered on compact domains. However, problems on the half-line are becoming increasingly more popular on the literature, due to their applications to fields like engineering, chemistry and biology (see, for instance, [14–16]). Such problems require more delicate procedures to deal with the lack of compactness. In this paper, this is overcome by applying the so-called Bielecki norm and the equiconvergence at ∞, as in [5].

We point out that, in our work, we introduce a new and more general type of boundary conditions. Moreover, our method can be applied to classical or singular ϕ-Laplacian, that is, even for homeomorphism $\phi : (-a, a) \to \mathbb{R}$, with $0 < a < +\infty$ (for more details see [2,3]).

In general, the lower and upper solutions method is a very adequate and useful technique to deal with boundary value problems as it provides not only the existence of bounded or unbounded solutions but also their localization and, from that, some qualitative data about solutions, their variation and behavior (see [4,8,9,11–13]).

The technique used in this paper follows some arguments suggested in [6], combined with the upper and lower solution and a Nagumo condition to control the first derivative. The usage of such a tool helps in improving the results in the existent literature as it introduces functional boundary conditions to the problem. These boundary conditions are very general in nature. Not only they generalize most of the classical boundary conditions, but also they cover the separated and multipoint cases, nonlocal or integral conditions or other boundary conditions with maximum/minimum arguments, that is, for example, of the type

$$u(0) = \max_{t\in[0,+\infty)} u(t) \quad \text{or} \quad u'(\tau) = \min_{t\in[0,+\infty)} u'(t), \quad \text{with } \tau \in [0, +\infty),$$

provided that the assumptions on L are satisfied.

The paper is organized as it follows: in Section 2 some auxiliary result are defined such as the space, the weighted norms, lower and upper solutions to be used and the necessary Lemmas to proceed. Section 3 contains the main result: an existence and localization theorem, where it is proved the existence of a solution. Finally, two examples, which are not covered by the existent results, show the applicability of the main theorems. In the first one the Nagumo conditions are verified. On the other hand, in the second one, these assumptions are replaced by a stronger condition on lower and upper solutions together with a local monotone growth on f.

2 Definitions and preliminaries

In this section, we present some of the definitions and auxiliary results, needed for the proof of the main result. Consider the following space

$$X = \left\{ x \in C^1[0, +\infty) : \lim_{t\to+\infty} \frac{x(t)}{e^{\theta t}} \in \mathbb{R},\ \theta > 0, \text{ and } \lim_{t\to+\infty} x'(t) \in \mathbb{R} \right\}$$

equipped with a Bielecki norm type in $C^1[0, +\infty)$,

$$\|x\|_X := \max\{\|x\|_0, \|x\|_1\},$$

where

$$\|w\|_0 = \sup_{0\le t<+\infty} \frac{|w(t)|}{e^{\theta t}} \quad \text{and} \quad \|w\|_1 = \sup_{0\le t<+\infty} |w'(t)|.$$

In this way, it is clear that $(X, \|\cdot\|_X)$ is a Banach space.

In addition, the following conditions must hold:

(H1) $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$;

(H2) the function $f : (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(t, x, y)$ is uniformly bounded for $t \in (0, \infty)$ when x and y are bounded.

(H3) the function $q : (0, \infty) \to [0, \infty)$ is integrable, not identically 0 on any subinterval of $(0, \infty)$.

(H4) $L : C((0, \infty)) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, nondecreasing in the first and third variables.

The approach to the problem (1.1)–(1.2), will be from the perspective of a fixed point problem. The next lemmas establish the link between the problem (1.1)–(1.2) and its integral formulation.

Let $\gamma, \Gamma \in X$ be such that $\gamma(t) \leq \Gamma(t), \forall t \geq 0$. Consider the set, for $\theta > 0$,

$$E = \left\{ (t, x, y) \in [0, +\infty) \times \mathbb{R}^2 : \frac{\gamma(t)}{e^{\theta t}} \leq x \leq \frac{\Gamma(t)}{e^{\theta t}} \right\}.$$

The following Nagumo condition allows some *a priori* bounds on the first derivative of the solution.

Definition 2.1. A function $f : E \to \mathbb{R}$ is said to satisfy a Nagumo-type growth condition in E if, for some positive and continuous functions ψ, h , such that

$$\sup_{0 \leq t < +\infty} \psi(t) < +\infty, \qquad \int_0^{+\infty} \frac{|\phi^{-1}(s)|}{h(|\phi^{-1}(s)|)} ds = +\infty, \tag{2.1}$$

it verifies

$$|q(t) f(t, x, y)| \leq \psi(t)\, h(|y|), \qquad \forall (t, x, y) \in E. \tag{2.2}$$

Lemma 2.2. *Let $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying a Nagumo-type growth condition in E. Then there exists $N > 0$ (not depending on u) such that every solution u of* (1.1), (1.2) *with*

$$\frac{\gamma(t)}{e^{\theta t}} \leq u(t) \leq \frac{\Gamma(t)}{e^{\theta t}}, \quad \text{for } t \geq 0,\ \theta > 0,$$

we have

$$\|u\|_1 < N. \tag{2.3}$$

Proof. Let u be a solution of (1.1), (1.2) with $(t, u(t), u'(t)) \in E$. Consider $r > 0$ such that

$$r > |B|. \tag{2.4}$$

If $|u'(t)| \leq r, \forall t \geq 0$, taking $N > r$ the proof is complete as

$$\|u\|_1 = \sup_{0 \leq t < +\infty} |u'(t)| \leq r < N.$$

Suppose there exists $t_0 \geq 0$ such that $|u'(t_0)| > N$, that is $u'(t_0) > N$ or $u'(t_0) < -N$. In the first case, by (2.1), we can take $N > r$ such that

$$\int_{\phi(r)}^{\phi(N)} \frac{|\phi^{-1}(s)|}{h(|\phi^{-1}(s)|)} ds > M \left(\sup_{0 \leq t < +\infty} \frac{\Gamma(t)}{e^{\theta t}} - \inf_{0 \leq t < +\infty} \frac{\gamma(t)}{e^{\theta t}} \right) \tag{2.5}$$

with $M := \sup_{0\leq t<+\infty} \psi(t)$.

Consider $t_1, t_2 \in [t_0, +\infty)$ such that $t_1 < t_2$, $u'(t_1) = N$, $u'(t_2) = r$ and $r \leq u'(t) \leq N, \forall t \in [t_1, t_2]$. Therefore, the following contradiction with (2.5) is achieved:

$$\begin{aligned}\int_{\phi(r)}^{\phi(N)} \frac{|\phi^{-1}(s)|}{h(|\phi^{-1}(s)|)}ds &= \int_{\phi(u'(t_2))}^{\phi(u'(t_1))} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))}ds = \int_{t_2}^{t_1} \frac{u'(s)}{h(u'(s))}\left(\phi(u'(s))\right)'ds \\ &= -\int_{t_1}^{t_2} \frac{q(s)f(s,u(s),u'(s))}{h(u'(s))}\, u'(s)\, ds \\ &\leq \int_{t_1}^{t_2} \frac{|q(s)f(s,u(s),u'(s))|}{h(u'(s))}\, u'(s)\, ds \\ &\leq \int_{t_1}^{t_2} \psi(s)\, u'(s)\, ds \leq M\int_{t_1}^{t_2} u'(s)\, ds \leq M\,(u(t_2)-u(t_1)) \\ &\leq M\left(\sup_{0\leq t<+\infty}\frac{\Gamma(t)}{e^{\theta t}} - \inf_{0\leq t<+\infty}\frac{\gamma(t)}{e^{\theta t}}\right).\end{aligned}$$

So $u'(t) < N$, $\forall t \in [0, +\infty)$.

Similarly, it can be proved that $u'(t) > -N$, $\forall t \in [0, +\infty)$, and, therefore, $\|u\|_1 < N$, $\forall t \in [0, +\infty)$. □

Define a surjective homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ as

$$\varphi(y) = \begin{cases} \phi(y), & \text{if } |y| \leq R \\ \frac{\phi(R)-\phi(-R)}{2R}y + \frac{\phi(R)+\phi(-R)}{2}, & \text{if } |y| > R, \end{cases} \tag{2.6}$$

where $R > 0$ is to be defined later on.

Lemma 2.3. *Let $v \in L^1([0, +\infty))$. Then $u \in X$ such that $(\varphi(u'(t))) \in AC([0, +\infty))$ is the unique solution of*

$$\begin{aligned}\left(\varphi(u'(t))\right)' + v(t) = 0, &\qquad t \in [0, +\infty) \\ u(0) = A& \\ u'(+\infty) = B,&\end{aligned} \tag{2.7}$$

with A, $B \in \mathbb{R}$, if and only if

$$u(t) = A + \int_0^t \varphi^{-1}\left(\varphi(B) + \int_s^{+\infty} v(\tau)d\tau\right)ds \tag{2.8}$$

Proof. Let $u \in X$ be a solution of (2.7). Then

$$\left(\varphi(u'(t))\right)' = -v(t),$$

by integration we get

$$\varphi(u'(t)) = \varphi(B) + \int_t^{+\infty} v(s)ds.$$

As φ is continuous and $\varphi(\mathbb{R}) = \mathbb{R}$, then

$$u'(t) = \varphi^{-1}\left(\varphi(B) + \int_t^{+\infty} v(s)ds\right)$$

and by integration again, we obtain

$$u(t) = A + \int_0^t \varphi^{-1}\left(\varphi(B) + \int_s^{+\infty} v(\tau)d\tau\right)ds. \qquad \square$$

The lack of compactness is overcome by the following lemma, which will provide a general criteria for relative compactness.

Lemma 2.4 ([5]). *Let $M \subset X$. The set M is said to be relatively compact if the following conditions hold:*

a) M is uniformly bounded in X;

b) the functions belonging to M are equicontinuous on any compact interval of $[0, +\infty)$;

c) the functions f from M are equiconvergent at $+\infty$, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $\|f(t) - f(+\infty)\|_X < \varepsilon$ for any $t > T(\varepsilon)$ and $f \in M$.

The adaptation of the Euclidean norm of $\mathbb{R}^n$ to the weighted norms of X is an exercise and, for this reason, is omitted.

To prove the main result we will rely on the upper and lower solution method. The functions that can be considered as upper and lower solutions are defined as follows.

Definition 2.5. A function $\alpha \in X \cap C^2((0, +\infty))$ such that $\phi(\alpha') \in AC([0, +\infty))$ is said to be a lower solution of problem (1.1), (1.2) if

$$\left(\phi(\alpha')\right)'(t) + q(t) f(t, \alpha(t), \alpha'(t)) \geq 0$$

and

$$L\left(\alpha, \alpha(0), \alpha'(0)\right) \geq 0, \qquad \alpha'(+\infty) < B \tag{2.9}$$

where $B \in \mathbb{R}$.

A function β is an upper solution if it satisfies the reversed inequalities.

The following condition is applied for well ordered lower and upper solutions of problem (1.1), (1.2):

(H5) There are α and β lower and upper solutions of (1.1)–(1.2), respectively, such that

$$\alpha(t) \leq \beta(t), \qquad \forall t \in [0, +\infty). \tag{2.10}$$

Throughout the proof of the main result a modified and perturbed problem will be considered. It is given by

$$\begin{cases} (\varphi(u'(t)))' + q(t) f(t, \delta_0(t, u), \delta_1(t, u')) = 0 \\ u(0) = \delta_0(0, u(0) + L(u, u(0), u'(0))) \\ u'(+\infty) = B \end{cases} \tag{2.11}$$

with the truncation $\delta_0 : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ given by

$$\delta_0(t, y) = \begin{cases} \beta(t), & y > \beta(t) \\ y, & \alpha(t) \leq y \leq \beta(t) \\ \alpha(t), & y < \alpha(t), \end{cases} \tag{2.12}$$

and $\delta_1 : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ by

$$\delta_1(t, w) = \begin{cases} N, & w > N \\ w, & -N \leq w \leq N \\ -N, & w < -N, \end{cases} \tag{2.13}$$

where N is defined in Lemma 2.2, for functions f satisfying Nagumo's condition.

Consider $\varphi : \mathbb{R} \to \mathbb{R}$ given by (2.6) where $R := \max\{N, \|\alpha\|_1, \|\beta\|_1\}$, with N given by (2.3).

The operator $T : X \to X$, associated to (2.11) can then be defined as

$$\begin{aligned}(Tu)(t) := \delta_0\big(0, u(0) + L\big(u, u(0), u'(0)\big)\big) \\ + \int_0^t \varphi^{-1}\left(\varphi(B) + \int_s^\infty q(\tau) f\big(\tau, \delta_0(\tau, u), \delta_1(\tau, u')\big) d\tau\right) ds.\end{aligned} \tag{2.14}$$

One of the essential steps in our main result is to prove that the operator T has a fixed point. However, the function $q(t)$ may, or may not, be singular at the origin. As such two results are presented: one for the regular case, where $q(t)$ is not singular when $t = 0$, and another result for the singular case.

We will start by presenting some lemmas for the regular case.

Lemma 2.6 (Regular case). *Assume that $q : [0, \infty) \to [0, \infty)$ is continuous and that conditions (H1), (H2), (H3) and (H5) hold. Then the operator T is well defined.*

Proof. For any $u \in X$ there is $K > 0$, such that $\|u\|_X < K$.

From (2.11) and (2.12) we get

$$\begin{aligned}\lim_{t\to\infty} \frac{(Tu)(t)}{e^{\theta t}} &\leq \lim_{t\to\infty} \frac{\beta(0)}{e^{\theta t}} + \lim_{t\to\infty} \frac{\int_0^t \varphi^{-1}\big(\varphi(B) + \int_s^\infty q(\tau) f(\tau, \delta_0(\tau, u), \delta_1(\tau, u'))d\tau\big)ds}{e^{\theta t}} \\ &\leq \lim_{t\to\infty} \frac{\int_0^t \varphi^{-1}\big(\varphi(B) + \int_s^\infty q(\tau) f(\tau, \delta_0(\tau, u), \delta_1(\tau, u'))d\tau\big)ds}{e^{\theta t}}.\end{aligned}$$

As $\delta_0(\tau, u)$ and $\delta_1(\tau, u')$ are bounded, by (H2), then $f(\tau, \delta_0(\tau, u), \delta_1(\tau, u'))$ is uniformly bounded. Define

$$S_K := \sup_{t \in (0,\infty)} \{f(t, x, y), t \in (0, \infty), |x| \in (0, K_0), |y| \in (0, N)\}. \tag{2.15}$$

with

$$K_0 = \max\{\|\alpha\|_0, \|\beta\|_0\} \tag{2.16}$$

and N given by (2.3).

Remark that S_K does not depend on u.

From (H3) we can define k_1 a real number such that

$$\int_0^\infty q(\tau)|S_K| d\tau =: k_1. \tag{2.17}$$

As φ is nondecreasing, the previous inequality now becomes

$$\begin{aligned}\lim_{t\to\infty} \frac{(Tu)(t)}{e^{\theta t}} &\leq \lim_{t\to\infty} \frac{\int_0^t \varphi^{-1}\big(\varphi(B) + |S_K| \int_s^\infty q(\tau)d\tau\big)ds}{e^{\theta t}} \\ &\leq \lim_{t\to\infty} \frac{\int_0^t \varphi^{-1}(\varphi(B) + k_1)ds}{e^{\theta t}} \\ &\leq \lim_{t\to\infty} \frac{\varphi^{-1}(\varphi(B) + k_1)t}{e^{\theta t}} = 0.\end{aligned} \tag{2.18}$$

For

$$\lim_{t\to\infty}(Tu)'(t) = \varphi^{-1}\left(\varphi(B) + \int_t^\infty q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right) = B < \infty.$$

Therefore T is well defined. □

Lemma 2.7 (Regular case)**.** *Assume that $q:[0,\infty)\to[0,\infty)$ is continuous and that conditions (H1), (H2), (H3), (H4) and (H5) hold. Then the operator T is continuous.*

Proof. Consider a convergent sequence $u_n \to u \in X$.

By the arguments used in the previous lemma, the upper bounds are uniform and, therefore, do not depend on n.

By (H2) and Lebesgue's dominated convergence theorem, we have that

$$\|(Tu_n)-(Tu)\|_1 \le \sup_{0\le t<+\infty}\left|\begin{array}{c}\varphi^{-1}\left(\varphi(B)+\int_t^\infty q(\tau)f(\tau,\delta_0(\tau,u_n),\delta_1(\tau,u_n'))d\tau\right)\\ -\varphi^{-1}\left(\varphi(B)+\int_t^\infty q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)\end{array}\right| \to 0,$$

as $n\to+\infty$.

As φ is continuous, by e Lebesgue's dominated convergence theorem,

$$\|(Tu_n)-(Tu)\|_0 = \sup_{0\le t<+\infty} e^{-\theta t}\left|\begin{array}{c}\delta(0,u_n(0)+L(u_n,u_n(0),u_n'(0)))\\ +\int_0^t \varphi^{-1}\left(\varphi(B)+\int_s^\infty q(\tau)f(\tau,\delta_0(\tau,u_n),\delta_1(\tau,u_n'))d\tau\right)\\ -\delta(0,u(0)+L(u,u(0),u'(0)))\\ -\int_0^t \varphi^{-1}\left(\varphi(B)+\int_s^\infty q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)\end{array}\right| \to 0,$$

as $n\to+\infty$. Therefore T is continuous. □

Lemma 2.8. *The operator T is compact.*

Proof. The idea in this proof is to apply Lemma 2.4. For that we need to show that the operator T is equicontinuous and equiconvergent at $+\infty$.

Let us consider $t_1,t_2\in(0,T_0)$, where $T_0>0$ and $t_1<t_2$. Then, for $\theta>0$,

$$\begin{aligned}
&\left|\frac{(Tu)(t_1)}{e^{\theta t_1}} - \frac{(Tu)(t_2)}{e^{\theta t_2}}\right| \\
&\quad\le \max\{|\alpha(0)|,|\beta(0)|\}\frac{\left(e^{\theta t_2}-e^{\theta t_1}\right)}{e^{\theta(t_1+t_2)}} \\
&\qquad + \left|\frac{\left(e^{\theta t_2}-e^{\theta t_1}\right)}{e^{\theta(t_1+t_2)}}\int_0^{t_1}\varphi^{-1}\left(\varphi(B)+\int_s^\infty q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)\right| \\
&\qquad + \left|\frac{e^{\theta t_1}\int_{t_1}^{t_2}\varphi^{-1}\left(\varphi(B)+\int_s^\infty q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)}{e^{\theta(t_1+t_2)}}\right| \\
&\quad\le \max\{|\alpha(0)|,|\beta(0)|\}\frac{\left(e^{\theta t_2}-e^{\theta t_1}\right)}{e^{\theta(t_1+t_2)}} \\
&\qquad + \left|\frac{\left(e^{\theta t_2}-e^{\theta t_1}\right)\int_0^{t_1}\varphi^{-1}\left(\varphi(B)+S_K\int_s^\infty q(\tau)d\tau\right)}{e^{\theta(t_1+t_2)}}\right| \\
&\qquad + \left|\frac{e^{\theta t_1}\int_{t_1}^{t_2}\varphi^{-1}\left(\varphi(B)+S_K\int_s^\infty q(\tau)d\tau\right)}{e^{\theta(t_1+t_2)}}\right| \to 0,
\end{aligned}$$

as $t_1 \to t_2$.

Also, as φ^{-1} is continuous, by (2.15) and (2.17),

$$\left|(Tu)'(t_1)-(Tu)'(t_2)\right| = \left| \begin{array}{c} \varphi^{-1}\left(\int_{t_1}^{\infty} q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right) \\ -\varphi^{-1}\left(\int_{t_2}^{\infty} q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right) \end{array} \right| \to 0,$$

as $t_1 \to t_2$. Therefore T is equicontinuous.

As for the equiconvergency at $+\infty$ of the operator T, we have, by (2.18),

$$\begin{aligned} &\left|\frac{(Tu)(t)}{e^{\theta t}} - \lim_{t\to\infty}\frac{(Tu)(t)}{e^{\theta t}}\right| \\ &\qquad = \left|e^{-\theta t}\int_0^t \varphi^{-1}\left(\varphi(B)+\int_s^{\infty} q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)ds\right| \to 0, \end{aligned}$$

as $t \to +\infty$. For

$$\left|(Tu)'(t) - \lim_{t\to\infty}(Tu)'(t)\right| = \left| \begin{array}{c} \varphi^{-1}(\varphi(B)+\int_t^{\infty} q(\tau)f(\tau,\delta_0(\tau,u_n),\delta_1(\tau,u_n'))d\tau) \\ -\lim_{t\to\infty}\varphi^{-1}\left(\begin{array}{c}\varphi(B)+ \\ \int_t^{\infty} q(\tau)f(\tau,\delta_0(\tau,u_n),\delta_1(\tau,u_n'))d\tau\end{array}\right) \end{array} \right|$$

that tends to 0, from (H3) and the continuity of φ^{-1}.

As T is equicontinuous and equiconvergent, then from Lemma 2.4, we get that T is compact. □

We now need to consider the singular case.

Lemma 2.9 (Singular case). *Let q be singular at $t = 0$. Then the operator T given by* (2.14) *is completely continuous.*

Proof. For each $n \geq 1$ define the approximating operator T_n, such that $T_n : X \to X$ is given by

$$\begin{aligned} (T_n u)(t) := {} & \delta_0\big(0, u(0) + L\big(u, u(0), u'(0)\big)\big) \\ & + \int_{\frac{1}{n}}^{t} \varphi^{-1}\left(\varphi(B)+\int_s^{\infty} q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)ds. \end{aligned} \tag{2.19}$$

In this case it is sufficient to show that T_n tends to T on X. In fact, from (H1), (H2), (H3), (2.15) and (2.17), we get

$$\begin{aligned} \left|\frac{(Tu)(t)}{e^{\theta t}} - \frac{(T_n u)(t)}{e^{\theta t}}\right| &= \left|\frac{\int_0^{\frac{1}{n}} \varphi^{-1}(\varphi(B)+\int_s^{\infty} q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau)}{e^{\theta t}}\right| \\ &\leq \frac{\int_0^{\frac{1}{n}} \varphi^{-1}(\varphi(B)+S_K\int_s^{\infty} q(\tau)d\tau)}{e^{\theta t}} \to 0, \end{aligned}$$

as $n \to \infty$, and,

$$\left|(Tu)'(t)-(T_n u)'(t)\right| = \left| \begin{array}{l} \varphi^{-1}\left(\varphi(B)+\int_{\frac{1}{n}}^{\infty} q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right) \\ -\varphi^{-1}(\varphi(B)+\int_0^{\infty} q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau) \end{array} \right| \to 0,$$

as $n \to \infty$.

Hence the operator T is completely continuous. □

3 Main result

In this section we prove the existence and location result for (1.1)–(1.2).

Theorem 3.1. *Let $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ and $q : [0, +\infty) \to \mathbb{R}$ be both continuous functions, where q can have a singularity when $t = 0$, and f verifies the Nagumo conditions (2.1) and (2.2). If conditions (H1), (H2), (H3), (H4) and (H5) are satisfied, then problem (1.1)–(1.2) has at least a solution $u \in X$ and there exists $N > 0$ such that*

$$\alpha(t) \le u(t) \le \beta(t),$$
$$-N < u'(t) < N,$$

for every $t \in [0, +\infty)$.

Proof. Claim 1. Every solution of (2.11) verifies $\alpha(t) \le u(t) \le \beta(t)$ and there is $N > 0$ such that $-N < u'(t) < N$, $\forall t \in [0, +\infty)$.

Let $u \in X$ be a solution of the modified problem (2.11) and suppose, by contradiction, that there exists $t \in (0, +\infty)$ such that $\alpha(t) > u(t)$. Therefore

$$\inf_{t\in[0,+\infty)} u(t) - \alpha(t) < 0.$$

Suppose that this infimum is attained as $t \to +\infty$. Therefore

$$\lim_{t\to+\infty} \left(u'(t) - \alpha'(t)\right) = u'(+\infty) - \alpha'(+\infty) \le 0.$$

By Definition 2.5, we get the contradiction,

$$0 \ge u'(+\infty) - \alpha'(+\infty) = B - \alpha'(+\infty) > 0.$$

Analogously, the infimum does not happen at 0. Otherwise, the following contradiction holds:

$$0 > u(0) - \alpha(0) = \delta\big(0, u(0) + L\big(u, u(0), u'(0)\big)\big) - \alpha(0) \ge 0.$$

Therefore there are $t_* \in (0, +\infty)$ and $t_0 < t_*$ such that

$$\min_{t\in[0,+\infty)} (u(t) - \alpha(t))) := u(t_*) - \alpha(t_*) < 0,$$
$$u'(t_*) = \alpha'(t_*),$$
$$u(t) < \alpha(t),\ u'(t) < \alpha'(t), \qquad \forall t \in [t_0, t_*[,$$

and, by (H1),

$$\varphi\big(u'(t)\big) < \varphi\big(\alpha'(t)\big), \qquad \forall t \in [t_0, t_*[. \tag{3.1}$$

So, for $t \in [t_0, t_*[$, by (2.11), (2.12), (2.6) and Definition 2.5, one has

$$\begin{aligned}\big(\varphi\big(u'(t)\big)\big)' &= -q(t) f\big(t, \delta_0(t, u), \delta_1(t, u')\big) = -q(t) f\big(t, \alpha(t), \alpha'(t)\big) \\ &\le \big(\phi\big(\alpha'(t)\big)\big)' = \big(\varphi\big(\alpha'(t)\big)\big)'.\end{aligned} \tag{3.2}$$

The function $\varphi(u'(t)) - \varphi(\alpha'(t))$ is non-increasing on $[t_0, t_*[$ and

$$\varphi\big(u'(t_0)\big) - \varphi\big(\alpha'(t_0)\big) \ge \varphi\big(u'(t_*)\big) - \varphi\big(\alpha'(t_*)\big) = 0,$$

which is a contradiction with (3.1).

Therefore, $u(t) \geq \alpha(t)$, $\forall t \in [0,+\infty)$. Analogously it can be shown that $u(t) \leq \beta(t)$, $\forall t \in [0,+\infty)$.

The first derivatives inequalities are an immediate consequence of Lemma 2.2, taking

$$\gamma(t) = \frac{\alpha(t)}{e^{\theta t}} \quad \text{and} \quad \Gamma(t) = \frac{\beta(t)}{e^{\theta t}}, \quad \text{for } t \in [0,+\infty),\ \theta > 0.$$

From the lemmas in the previous section we have that the operator T is completely continuous, both for the singular and regular cases.

Claim 2. The problem (2.11) *has at least a solution $u \in X$.*

In order to apply the Schauder's fixed point theorem, we consider a closed and bounded set D defined as

$$D = \{u \in X : \|u\|_X \leq \rho\},$$

with ρ such that

$$\rho := \max\left\{K_0 + \sup_{t\in[0,+\infty)}\left(\frac{\varphi^{-1}(\varphi(B)+k_1)t}{e^{\theta t}}\right), \left|\varphi^{-1}(\varphi(B)+k_1)\right|\right\},$$

where K_0 is given by (2.16) and k_1 by (2.17).

For $u \in D$, arguing as in the proof of Lemma 2.6, as φ^{-1} is increasing, we have, for S_K given by (2.15),

$$\begin{aligned}
\|Tu\|_0 &= \sup_{t\in[0,+\infty)} \frac{|(Tu)(t)|}{e^{\theta t}} \\
&\leq \sup_{t\in[0,+\infty)}\left(K_0 + \frac{\int_0^t \varphi^{-1}(\varphi(B) + \int_s^\infty q(\tau)S_K)ds}{e^{\theta t}}\right) \\
&\leq \sup_{t\in[0,+\infty)}\left(K_0 + \frac{\int_0^t \varphi^{-1}(\varphi(B) + k_1)ds}{e^{\theta t}}\right) \\
&= \sup_{t\in[0,+\infty)}\left(K_0 + \frac{\varphi^{-1}(\varphi(B) + k_1)t}{e^{\theta t}}\right) \leq \rho,
\end{aligned}$$

and

$$\begin{aligned}
\|(Tu)\|_1 &= \sup_{t\in[0,+\infty)} \left|(Tu)'(t)\right| \\
&\leq \sup_{t\in[0,+\infty)}\left|\varphi^{-1}\left(\varphi(B) + \int_0^\infty q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)\right| \\
&\leq \sup_{t\in[0,+\infty)}\left|\varphi^{-1}(\varphi(B)+k_1)\right| \leq \rho.
\end{aligned}$$

Therefore $TD \subseteq D$. Then by Schauder's fixed point theorem, T has at least one fixed point $u \in X$, that is, the problem (2.11) has at least one solution $u \in X$.

Claim 3. Every solution u of the problem (2.11) *is a solution of* (1.1)–(1.2).

Let u be a solution of of the modified problem (2.11). By last claim, the function u verifies equation (1.1).

Then, it is enough to prove the inequalities

$$\alpha(0) \leq u(0) + L\big(u, u(0), u'(0)\big) \leq \beta(0).$$

Suppose, by contradiction, that $\alpha(0) > u(0) + L(u, u(0), u'(0))$.
By (2.11) and (2.12),

$$u(0) = \delta_0\big(0, u(0) + L\big(u, u(0), u'(0)\big)\big) = \alpha(0).$$

Therefore, by Claim 1, $u'(0) \geq \alpha'(0)$. By (H4) and Definition 2.5, the following contradiction his obtained

$$0 > u(0) + L\big(u, u(0), u'(0)\big) - \alpha(0) \geq L\big(\alpha, \alpha(0), \alpha'(0)\big) \geq 0.$$

In a similar way we can prove that $u(0) + L(u, u(0), u'(0)) \leq \beta(0)$. □

Remark 3.2. Note that Theorem 3.1 still remains true for singular ϕ-Laplacian equations. Indeed, from Nagumo condition and Lemma 2.2, for every u solution of problem (2.11), $\|u'(t)\|_1 < N$, and, therefore, considering in (2.6),$R > N$, we have

$$\phi\big(u'(t)\big) \equiv \varphi\big(u'(t)\big), \qquad \forall t \in [0, +\infty),$$

with

$$\phi :]-N, N[\to \mathbb{R}.$$

The control on the first derivative given by Nagumo condition and Lemma 2.2, can be overcome assuming stronger conditions on lower and upper solutions, as in the next theorem.

Theorem 3.3. *Let $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ and $q : [0, +\infty) \to \mathbb{R}$ be both continuous functions, where q can have a singularity when $t = 0$. Assume that there are α and β lower and upper solutions of* (1.1)–(1.2)*, respectively, such that*

$$\alpha'(t) \leq \beta'(t), \qquad \forall t \in [0, +\infty), \tag{3.3}$$

and

$$\alpha(0) \leq \beta(0). \tag{3.4}$$

If conditions (H1), (H2), (H3) and (H4) are satisfied and

$$f(t, \alpha(t), y) \leq f(t, x, y) \leq f(t, \beta(t), y), \tag{3.5}$$

for $\alpha(t) \leq x \leq \beta(t)$ and $y \in \mathbb{R}$ fixed, then problem (1.1)–(1.2) *has at least one solution $u \in X$ such that*

$$\alpha'(t) \leq u'(t) \leq \beta'(t), \qquad \forall t \in [0, +\infty).$$

Remark 3.4. Note that conditions (3.3), (3.4) imply (H5).

Proof. The proof follows analogous steps as in Claims 1 and 2 of Theorem 3.1, with φ defined by

$$R := \max\{\|\alpha\|_1, \|\beta\|_1\}. \tag{3.6}$$

It remains to prove that $\alpha'(t) \leq u'(t) \leq \beta'(t), \forall t \in [0, +\infty)$.
Assume that there is a $t \in [0, +\infty)$ such that $u'(t) < \alpha'(t)$, and define $t_0 \in [0, +\infty)$ as

$$\inf_{t \in [0, +\infty)} \big(u'(t) - \alpha'(t)\big) := u'(t_0) - \alpha'(t_0) < 0. \tag{3.7}$$

By (1.2), there is $t_1 \in (t_0, +\infty)$ such that $u'(t_1) = \alpha'(t_1)$.

By (3.5), for $t \in [t_0, t_1]$,

$$\begin{aligned}(\varphi(u'(t)))'(t) &= -q(t)f(t, \delta_0(t,u), \delta_1(t,u')) = -q(t)f(t, \delta_0(t,u), \alpha'(t)) \\ &\leq -q(t)f(t, \alpha(t), \alpha'(t)) \\ &\leq (\phi(\alpha'(t)))' = (\varphi(\alpha'(t)))'.\end{aligned}$$

Therefore, $\varphi(u'(t)) - \varphi(\alpha'(t))$ is non-increasing on $[t_0, t_1]$ and

$$\varphi(u'(t_0)) - \varphi(\alpha'(t_0)) \geq \varphi(u'(t_1)) - \varphi(\alpha'(t_1)) = 0.$$

So, $\varphi(u'(t_0)) \geq \varphi(\alpha'(t_0))$, and by (H1), $u'(t_0) \geq \alpha'(t_0)$ which contradicts (3.7). That is, $\alpha'(t) \leq u'(t)$, $\forall t \in [0, +\infty)$.

In the same way it can be shown that $u'(t) \leq \beta'(t)$, $\forall t \in [0, +\infty)$. □

Remark 3.5. Theorem 3.3 holds for singular ϕ-Laplacian equations. Considering now in (2.6), R given by (3.6), we have

$$\phi :]-R, R[\to \mathbb{R}$$

and $\phi(u'(t)) \equiv \varphi(u'(t))$, $\forall t \in [0, +\infty)$.

4 Examples

The applicability of our results is illustrated by two examples. In the first one the nonlinearity f satisfies the Nagumo conditions and, in the second one, this assumption is replaced by a monotone behavior in f.

In both cases the null function is not a solution of the referred problem.

Example 4.1. Consider for some $\theta > 0$ the nonlinear problem composed by the differential equation

$$\frac{u''(t)}{1 + (u'(t))^2} - \frac{1}{1+t^2} \frac{u(t)(u'(t))^2}{1 + u^2(t)} = 0, \quad \text{for } 0 \leq t < +\infty, \tag{4.1}$$

and the functional boundary conditions

$$\max_{t \in [0,+\infty)} \frac{|u(t)|}{e^{\theta t}} + (u'(0))^3 - u(0) = 0, \qquad u'(+\infty) = \frac{1}{2}. \tag{4.2}$$

Remark that this problem (4.1), (4.2) is a particular case of (1.1)–(1.2) with

- $\phi(v) = \arctan v$;
- $f(t, x, y) = -\frac{xy^2}{1+x^2}$;
- $q(t) = \frac{1}{1+t^2}$;
- $L(u, x, y) = \max\limits_{t \in [0,+\infty)} \frac{|u(t)|}{e^{\theta t}} + y^3 - x$;
- $B = \frac{1}{2}$.

We point out that:

- $f(t,x,y)$ and $q(t)$ verify (H2), (H3) and the Nagumo conditions (2.1) and (2.2) with $\psi(t) \equiv 1$ and $h(|y|) = y^2$;
- $L(u,x,y)$ satisfies (H4);
- the functions $\alpha(t) = \frac{1}{2}$ and $\beta(t) = t+2$ are, respectively, lower and upper solutions of (4.1), (4.2) verifying (H5);
- as ϕ is a nonsurjective homeomorphism satisfying (H1), it can be extended by a surjective homeomorphism φ, as in (2.6), that is

$$\varphi(y) = \begin{cases} \arctan(y) & \text{if } |y| \le R \\ \frac{\arctan(R)}{R} y & \text{if } |y| > R, \end{cases}$$

  with

$$R := \max\{\|\alpha'\|_1, \|\beta'\|_1\} = 1.$$

So, by Theorem 3.1, there is, at least one solution u of (4.1), (4.2) such that

$$\frac{1}{2} \le u(t) \le t+2, \qquad \forall t \ge 0.$$

Moreover, this solution is unbounded and, from the location part, strictly positive and without zeros in $[0,+\infty)$.

Example 4.2. The functional problem

$$\begin{cases} 3(u'(t))^2 u''(t) + \frac{1}{1+t^3}\left(\arctan\left((u(t))^3\right) - 2\frac{(u'(t))^5}{1+|u'(t)|^5}\right) = 0, & \text{for } 0 \le t < +\infty, \\ \int_0^1 \frac{u(t)}{e^{\theta t}} dt - 5u(0) + u'(0) = 1, \\ u'(+\infty) = B, \end{cases} \tag{4.3}$$

for some $\theta > 0$ and $B > -1$, is a particular case of (1.1)-(1.2) with

- $\phi(v) = v^3$;
- $f(t,x,y) = \arctan(x^3) - 2\frac{y^5}{1+|y|^5}$;
- $q(t) = \frac{1}{1+t^3}$;
- $L(u,x,y) = \int_0^1 \frac{u(t)}{e^{\theta t}} dt - 5x + y - 1$.

Remark that, in this case, ϕ is a surjective homeomorphism and f does not satisfy the Nagumo conditions but it verifies (3.5).

As the functions $\alpha(t) = -t-1$ and $\beta(t) \equiv 0$ are, respectively, lower and upper solutions of (4.3), satisfying assumptions (3.3) and (3.4), then, by Theorem 3.3, there is, at least, a solution u of (4.3), such that

$$-t-1 \le u(t) \le 0, \qquad \forall t \ge 0.$$

Indeed, this solution is unbounded if $B \ne 0$ and bounded if $B = 0$, and, in any case, nonpositive in $[0,+\infty)$.

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