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Multiple solutions of second order Hamiltonian systems

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Abstract. The existence and the multiplicity of periodic solutions for a parameter dependent second order Hamiltonian system are established via linking theorems. A monotonicity trick is adopted in order to prove the existence of an open interval of parameters for which the problem under consideration admits at least two non trivial qualified solutions.

Keywords: second order Hamiltonian systems, periodic solutions, critical points.

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1 Introduction

The study of the existence and the multiplicity of solutions for second order Hamiltonian systems of type

$$-\ddot{u}(t) = \nabla F(t, u(t)), \tag{1.1}$$

has been widely investigated in these latest years, see [1–6, 9–12, 15, 18–22, 24–26, 28–30, 32–51].

Because of its variational structure, the florid minimax methods for critical point theory, particularly with its linking theorems (see [23, 27, 31–33]) represents a fruitful tool in order to approach problem (1.1).

Recently, in [34], the following system

$$-\ddot{u}(t) = B(t)u(t) + \nabla V(t, u(t)),$$

has been studied, where

$$u(t) = (u_1(t), \dots, u_n(t))$$

is a map from $I := [0, T]$ to $\mathbb{R}^n$ such that each component $u_j(t)$ is a periodic function in H^1 with period T, and the function $V(t, x) = V(t, x_1, \dots, x_n)$ is continuous from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$\nabla V(t, x) = \nabla_x V(t, x) = (\partial V/\partial x_1, \dots, \partial V/\partial x_n) \in C(\mathbb{R}^{n+1}, \mathbb{R}^n).$$

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For each $x \in \mathbb{R}^n$, the function $V(t,x)$ is periodic in t with period T.

By assuming that the elements of the symmetric matrix $B(t)$ are to be real-valued functions $b_{jk}(t) = b_{kj}(t)$ and that

$(B1)$ *each component of $B(t)$ is an integrable function on I, i.e., for each j and k, $b_{jk}(t) \in L^1(I)$,*

it was possible to exploit the property that there is an extension of the operator

$$\mathcal{D}_0 u = -\ddot{u}(t) - B(t)u(t)$$

having a discrete, countable spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound $-L$

$$-\infty < -L \leq \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_l < \cdots$$

(cf. [30]).

Here, inspired by the arguments adopted in [34], we consider the following problem

$$\begin{cases} -\ddot{u}(t) + B(t)u = \mu \nabla V(t,u), \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{cases} \tag{1.2}$$

where B is a symmetric matrix valued function satisfying an elliptic condition (see next assumption (B_3)) and μ is a positive real parameter. In particular, first we simply require a suitable behaviour of the potential $V(t,\cdot)$ near zero in order to establish the existence of positive interval of parameters for which problem (1.2) admits at least one qualified non trivial solution (see Theorem 3.1). Then, assuming in addition that $V(t,\cdot)$ satisfies different conditions at infinity, a second non trivial solution is assured (see Theorems 3.2–3.4). The multiplicity results are obtained combining a linking theorem for functionals depending on a parameter with a monotonicity trick.

2 Variational setting and preliminary results

In the sequel we will assume the following conditions on the matrix valued function B

$(B2)$ *$B(t) = \big(b_{ij}(t)\big)$ is a symmetric matrix with $b_{ij} \in L^\infty(I)$.*

$(B3)$ *There exists a positive function $\gamma \in L^\infty(I)$ such that*

$$B(t)x \cdot x \geq \gamma(t)|x|^2$$

for every $x \in \mathbb{R}^n$ and a.e. t in I.

Thus

$$\gamma(t)|x|^2 \leq B(t)x \cdot x \leq \Lambda(t)|x|^2,$$

for every $t \in I$ and $x \in \mathbb{R}^n$, where $\Lambda(t) \in L^\infty(I)$. Following the notation of [29], let H^1_T be the Sobolev space of functions $u \in L^2(I,\mathbb{R}^n)$ having a weak derivative $\dot{u} \in L^2(I,\mathbb{R}^n)$. It is well known that H^1_T, endowed with the norm

$$\|u\|_{H^1_T} := \left(\int_0^T |u(t)|^2\, dt + \int_0^T |\dot{u}(t)|^2\, dt \right)^{1/2},$$

is a Hilbert space, compactly embedded in $C^0(I,\mathbb{R}^n)$ and $C_T^\infty \subset H_T^1$.

Because of the previous conditions, it is possible to introduce on H_T^1 the following inner product

$$(\mathcal{D}u,v) = \int_0^T B(t)u(t)\cdot v(t)\,dt + \int_0^T \dot{u}(t)\cdot\dot{v}(t)\,dt,$$

for every $u,v\in H_T^1$. The norm induced by $(\mathcal{D}u,v)$ is

$$d(u)^{1/2} := \left(\int_0^T B(t)u(t)\cdot u(t)\,dt + \int_0^T |\dot{u}(t)|^2\,dt\right)^{1/2}.$$

In fact, we have the following lemma.

Lemma 2.1. *$d(\cdot)^{1/2}$ is a norm on H_T^1. There is a constant $c_0>0$ such that*

$$\|x\|_\infty^2 \le c_0 d(x), \quad x\in H_T^1.$$

Remark 2.2. For an explicit estimate of the constant c_0 we refer to [12,21,29].

A solution of problem (1.2) is any function $u_0\in C^1(I,\mathbb{R}^n)$ such that $\dot{u}_0$ is absolutely continuous, and satisfies

$$-\ddot{u}_0 + B(t)u_0 = \mu\nabla V(t,u_0) \quad \text{a.e. in } I,$$

and

$$u_0(T)-u_0(0) = \dot{u}_0(T)-\dot{u}_0(0) = 0.$$

It follows that, if we put $\lambda = 1/\mu$, a critical point of the functional

$$G_\lambda(u) = \lambda d(u) - 2\int_I V(t,u)\,dt, \quad 0<\lambda<\infty$$

is a solution of (1.2) where the system takes the form

$$\lambda\mathcal{D}u(t) = \nabla V(t,u(t)). \tag{2.1}$$

We introduced the parameter λ to make use of the monotonicity trick. This requires us to work in an interval of the parameter λ, and it allows us to obtain solutions under very weak hypotheses. However, we obtain solutions only for almost every value of the parameter. We can then obtain solutions for all values of the parameter by introducing appropriate mild assumptions.

In proving the theorems, we shall make use of the following results of linking. Let E be a reflexive Banach space with norm $\|\cdot\|$. The set Φ of mappings $\Gamma(t)\in C(E\times[0,1],E)$ is to have following properties:

- **a)** for each $t\in[0,1), \Gamma(t)$ is a homeomorphism of E onto itself and $\Gamma(t)^{-1}$ is continuous on $E\times[0,1)$
- **b)** $\Gamma(0)=I$
- **c)** for each $\Gamma(t)\in\Phi$ there is a $u_0\in E$ such that $\Gamma(1)u=u_0$ for all $u\in E$ and $\Gamma(t)u\to u_0$ as $t\to 1$ uniformly on bounded subsets of E.

d) For each $t_0 \in [0,1)$ and each bounded set $A \subset E$ we have

$$\sup_{\substack{0 \le t \le t_0 \\ u \in A}} \{\|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\|\} < \infty.$$

A subset A of E links a subset B of E if $A \cap B = \phi$ and, for each $\Gamma(t) \in \Phi$, there is a $t \in (0,1]$ such that $\Gamma(t)A \cap B \neq \phi$.

Define

$$\mathcal{G}_\lambda(u) = \lambda \mathcal{I}(u) - \mathcal{J}(u), \quad \lambda \in \Lambda,$$

where $\mathcal{I}, \mathcal{J} \in C^1(E, \mathbb{R})$ map bounded sets to bounded sets and Λ is an open interval contained in $(0, +\infty)$. Assume one of the following alternatives holds.

(H$_1$) $\mathcal{I}(u) \ge 0$ for all $u \in E$ and $\mathcal{I}(u) + |\mathcal{J}(u)| \to \infty$ as $\|u\| \to \infty$.

(H$_2$) $\mathcal{I}(u) \le 0$ for all $u \in E$ and $|\mathcal{I}(u)| + |\mathcal{J}(u)| \to \infty$ as $\|u\| \to \infty$.

Moreover assume that

(H$_3$) there are sets A, B such that A links B and

$$a_0 := \sup_A \mathcal{G}_\lambda \le b_0 := \inf_B \mathcal{G}_\lambda$$

for each $\lambda \in \Lambda$. $a(\lambda) := \inf_{\Gamma \in \Phi} \sup_{\substack{0 \le s \le 1 \\ u \in A}} \mathcal{G}_\lambda(\Gamma(s)u)$ is finite for each $\lambda \in \Lambda$.

Theorem 2.3. *Assume that* **(H$_1$)** *(or* **(H$_2$)***) and* **(H$_3$)** *hold. Then for almost all $\lambda \in \Lambda$ there exists a bounded sequence $u_k(\lambda) \in E$ such that*

$$\|\mathcal{G}'_\lambda(u_k)\| \to 0, \quad \mathcal{G}_\lambda(u_k) \to a(\lambda) \quad \text{as } k \to \infty.$$

For a proof, cf. [33].

3 Statement of the theorems

Theorem 3.1. *Assume*

1. *There are a function $b(t) \in L^1(I)$ and positive constants m and $\theta < 2$ such that*

$$2V(t,x) \le b(t)|x|^\theta, \quad |x| \le m, \ x \in \mathbb{R}^n.$$

2. *There is a constant $M > K_0 = c_0 m^{\theta-2}\|b\|_1$ such that*

$$\liminf_{c \to 0} 2 \int_I V(t, c\varphi)/c^2 \|\varphi\|_2^2 > M\lambda_0, \tag{3.1}$$

where φ is an eigenfunction of $\mathcal{D}$ corresponding to the first eigenvalue λ_0.

Then the system (2.1) has a nontrivial solution u_λ satisfying

$$d(u_\lambda) < m^2/c_0, \quad G_\lambda(u_\lambda) < 0$$

for each $\lambda \in (K_0, M)$.

Theorem 3.2. *Assume that hypotheses (1) and (2) of Theorem 3.1 are satisfied in addition to*

$$\liminf_{c\to\infty} 2\int_I V(t,c\varphi)/c^2\|\varphi\|_2^2 > M\lambda_0. \tag{3.2}$$

Then the system (2.1) has two nontrivial solutions u_λ, v_λ satisfying

$$d(u_\lambda) < m^2/c_0, \quad G_\lambda(u_\lambda) < 0, \quad G_\lambda(v_\lambda) > 0$$

for almost all $\lambda \in (K_0, M)$.

Theorem 3.3. *Assume that hypotheses (1) and (2) of Theorem 3.1 are satisfied. Moreover,*

(3) The function V is such that

$$V(t,x)/|x|^2 \to \infty, \quad \text{as } |x| \to \infty, \tag{3.3}$$

uniformly with respect to t.

(4) There is a function $W(t) \in L^1(I)$ such that

$$2V(t,x) - 2V(t,rx) + (r^2-1)x\cdot\nabla_x V(t,x) \le W(t), \quad t\in I,\ x\in\mathbb{R}^n,\ r\in[0,1].$$

Then the system (2.1) has two nontrivial solutions u_λ, v_λ satisfying

$$d(u_\lambda) < m^2/c_0, \quad G_\lambda(u_\lambda) < 0, \quad G_\lambda(v_\lambda) > 0$$

for each $\lambda \in (K_0, M)$.

Theorem 3.4. *The conclusions of Theorem 3.3 hold if we replace Hypothesis (4) with:*
There are a constant C and a function $W(t) \in L^1(I)$ such that

$$H(t,\theta x) \le C(H(t,x) + W(t)), \quad 0\le\theta\le 1,\ t\in I,\ x\in\mathbb{R}^n,$$

where

$$H(t,x) = \nabla_x V(t,x)\cdot x - 2V(t,x).$$

4 Proofs of the theorems

Before giving the proofs, we shall prove a few lemmas.

Lemma 4.1. *If (3.4) holds, then*

$$\int_I [2V(t,u) - 2V(t,ru) + (r^2-1)u\cdot\nabla_u V(t,u)] \le C, \quad u\in H_T^1,\ r\in[0,1], \tag{4.1}$$

where the constant C does not depend on u, r.

Proof. This follows from (3.4) if we take $u = x$. □

Lemma 4.2. *If u satisfies $G'_\lambda(u) = 0$ for some $\lambda > 0$, then there is a constant C independent of u, λ, r such that*

$$G_\lambda(ru) - G_\lambda(u) \le C \tag{4.2}$$

for all $r \in [0,1]$.

Proof. From $G'_\lambda(u) = 0$ one has that

$$(G'_\lambda(u), g)/2 = \lambda(\mathcal{D}u, g) - \int_I g \cdot \nabla_u V(t,u) = 0$$

for every $g \in H^1_T$. Take

$$g = (1 - r^2)u.$$

Then we have

$$\begin{aligned} G_\lambda(ru) - G_\lambda(u) &= \lambda(r^2-1)(\mathcal{D}u,u) + \int_I [2V(t,u) - 2V(t,ru)]\,dt \\ &= \int_I [2V(t,u) - 2V(t,ru) + ((r^2-1)u \cdot \nabla_u V(t,u)]\,dt \\ &\le C \end{aligned}$$

by Lemma 4.1. □

Proof of Theorem 3.1. Fix $\lambda \in (K_0, M)$, put $r^2 = m^2/c_0$ and define

$$\mathbf{B}_r = \{u \in H^1_T : d(u) \le r^2\}, \quad \partial\mathbf{B}_r = \{u \in H^1_T : d(u) = r^2\}.$$

We claim that

$$\inf_{u \in \partial\mathbf{B}_r} G_\lambda(u) > 0. \tag{4.3}$$

Indeed, let $\delta > 0$ be such that $K_0 < K_0 + \delta < \lambda < M$, then for every $u \in \partial\mathbf{B}_r$ one has

$$G_\lambda(u) \ge \lambda d(u) - \int_I b(t)|u(t)|^\theta \ge \lambda m^2/c_0 - m^\theta \|b\|_1 \ge \delta m^2/c_0,$$

and (4.3) holds. On the other hand, from (3.1), fixed $\varepsilon \in \left(0, M\lambda_0 - 2\int_I V(t,c\varphi)/c^2\|\varphi\|_2^2\right)$, there exists $\bar\sigma > 0$ such that

$$2\int_I V(t,c\varphi)/c^2\|\varphi\|_2^2 > M\lambda_0 + \varepsilon$$

for every $|c| < \bar\sigma$. Hence, for c sufficiently small one has $c\varphi \in \mathbf{B}_r$, as well as

$$\begin{aligned} G_\lambda(c\varphi) &= c^2\|\varphi\|_2^2(\lambda\lambda_0 - 2\int_I V(t,c\varphi)/c^2\|\varphi\|_2^2) \\ &\le c^2\|\varphi\|_2^2(M\lambda_0 - 2\int_I V(t,c\varphi)/c^2\|\varphi\|_2^2) \\ &\le -c^2\|\varphi\|_2^2\varepsilon < 0. \end{aligned}$$

For each λ let $\mu(\lambda) = \inf_{\mathbf{B}_r} G_\lambda$. Then $-\infty < \mu(\lambda) < 0$. There is a minimizing sequence $(u_k) \subset \mathbf{B}_r$ such that $G_\lambda(u_k) \to \mu(\lambda)$. Consequently, there is a renamed subsequence such that $u_k \rightharpoonup u \in H^1_T$ and $u_k \to u \in L^\infty(I)$. Thus

$$\lambda d(u_k) \to \mu(\lambda) + 2\int_I V(t,u)\,dt.$$

Also $\lambda d(u) \le \liminf \lambda d(u_k) = \mu(\lambda) + 2\int_I V(t,u)\,dt$, namely $G_\lambda(u) \le \mu(\lambda) < 0$ and $u \notin \partial\mathbf{B}_r$. Hence, u is in the interior of $\mathbf{B}_r$ and we have $G'_\lambda(u) = 0$. □

Proof of Theorem 3.2. First observe that, if we define

$$\mathcal{I}(u) = d(u), \quad \mathcal{J}(u) = \int_I V(t,u)\, dt$$

for every $u \in H^1_T$ one has that $\mathcal{G}_\lambda = G_\lambda$. Hence, taking in mind that $\mathcal{I}(u) \geq 0$ for all $u \in H^1_T$, it is clear that $(\mathbf{H_1})$ holds. Moreover, as in the proof of Theorem 3.1, take $r^2 = m^2/c_0$. Then

$$\nu(\lambda) = \inf_{\partial \mathbf{B}_r} G_\lambda > 0, \quad \lambda \in (K_0, M).$$

By hypothesis, there are c_1, c_2 such that $c_1\varphi \in \mathbf{B}_r$ and $c_2\varphi \notin \mathbf{B}_r$ with $G_\lambda(c_i\varphi) < 0$, $i = 1, 2$. The set $A = (c_1\varphi, c_2\varphi)$ links $B = \partial\mathbf{B}_r$ (cf., e.g., [32]). Applying Theorem 2.3, for almost every λ we obtain a bounded sequence $(y_k) \subset H^1_T$ such that

$$G_\lambda(y_k) \to a(\lambda) := \inf_{\Gamma \in \Phi} \sup_{0 \leq s \leq 1,\, u \in A} G_\lambda(\Gamma(s)u) \geq \nu(\lambda), \quad G'_\lambda(y_k) \to 0.$$

Since the sequence is bounded, there is a renamed subsequence such that $y_k \rightharpoonup y \in H^1_T$ and $y_k \to y \in L^\infty(I)$. Since $G'_\lambda(y_k) \to 0$, we have

$$\lambda d(y_k, v) - \int_I \nabla V(t, y_k) v(t) \to 0.$$

In the limit this gives $G'_\lambda(y) = 0$. We also have $\lambda d(y_k) \to \int_I \nabla V(t,y)y = \lambda d(y)$. Consequently, we have $G_\lambda(y_k) = \lambda d(y_k) - 2\int_I V(t, y_k) \to \lambda d(y) - 2\int_I V(t,y) = G_\lambda(y)$ showing that $G_\lambda(y) = a(\lambda) \geq \nu(\lambda) > 0$. The proof is completed taking u_λ as already assured by Theorem 3.1 and $v_\lambda = y$. □

5 The remaining proofs

Proof of Theorem 3.3. Note that (3.3) implies (3.2). By Theorem 3.2, for a.e. $\lambda \in (K_0, M)$, there exists u_λ such that $G'_\lambda(u_\lambda) = 0$, $G_\lambda(u_\lambda) = a(\lambda) \geq \nu(\lambda) > 0$. Let λ satisfy $K_0 < \lambda < M$. Choose $\lambda_n \to \lambda$, $\lambda_n > \lambda$ such that there exists u_n satisfying

$$G'_{\lambda_n}(u_n) = 0, \quad G_{\lambda_n}(u_n) = a(\lambda_n) \geq \nu(\lambda_n) > 0.$$

Therefore,

$$\int_I \frac{2V(t,u_n)}{d(u_n)} dt < M.$$

Now we prove that $\{u_n\}$ is bounded in H^1_T. If $\|u_n\|_{H^1_T} \to \infty$, let $\tilde{u}_n = u_n / d^{1/2}(u_n)$. Then $d(\tilde{u}_n) = 1$ and there is a renamed subsequence such that $\tilde{u}_n \to \tilde{u}$ weakly in H^1_T, strongly in $L^\infty(I)$ and a.e. in I. Let $\Omega_0 \subset I$ be the set where $\tilde{u} \neq 0$. Then $|u_n(t)| \to \infty$ for $t \in \Omega_0$. If Ω_0 had positive measure, then, observing that (3.3) and the continuity of V assure the existence of $\beta \in \mathbb{R}$ such that

$$V(t,x) \geq \beta$$

for every $(t,x) \in I \times \mathbb{R}^n$, we would have

$$\begin{aligned} M &> \int_I \frac{2V(t,u_n)}{d(u_n)} dt = \int_{\Omega_0} \frac{2V(t,u_n)}{d(u_n)} + \int_{I \setminus \Omega_0} \frac{2V(t,u_n)}{d(u_n)} dt \\ &\geq \int_{\Omega_0} \frac{2V(t,u_n)}{|u_n|^2} |\tilde{u}_n|^2 dt + \int_{I \setminus \Omega_0} \frac{2\beta}{d(u_n)}\, dt. \end{aligned}$$

At this point, we obtain a contradiction passing to the lim inf and applying the Fatou lemma, since from (3.3) it is clear that for every $t \in \Omega_0$, $\frac{2V(t,u_n)}{|u_n|^2}|\tilde{u}_n|^2 \to +\infty$ as $n \to \infty$. This shows that $\tilde{u} = 0$ a.e. in I. Hence, $\tilde{u}_n \to 0$ in $L^\infty(I)$. For any $s > 0$ and $h_n = s\tilde{u}_n$, we have

$$\int_I V(t, h_n)\, dt \to \int_I V(t, 0)\, dt. \tag{5.1}$$

Take $r_n = s/d^{1/2}(u_n) \to 0$. By Lemma 4.2

$$G_{\lambda_n}(r_n u_n) - G_{\lambda_n}(u_n) \le C. \tag{5.2}$$

Hence,

$$G_{\lambda_n}(s\tilde{u}_n) \le C + G_{\lambda_n}(u_n) = C + a(\lambda_n) \le C + a(M). \tag{5.3}$$

But

$$\begin{aligned} G_{\lambda_n}(s\tilde{u}_n) =& \ \lambda_n s^2(\mathcal{D}\tilde{u}_n, \tilde{u}_n) - 2\int_I V(t, s\tilde{u}_n) \\ \ge& \ s^2\lambda d(\tilde{u}_n) - 2\int_I V(t, s\tilde{u}_n) \\ \to& \ \lambda s^2 \end{aligned}$$

by (5.1). This implies

$$G_{\lambda_n}(s\tilde{u}_n) \to \infty \quad \text{as } s \to \infty,$$

contrary to (5.3).

This contradiction shows that $\|u_n\|_{H^1_T} \le C$. Then there is a renamed subsequence such that $u_n \to u$ weakly in H^1_T, strongly in $L^\infty(I)$ and a.e. in I. It now follows that for the bounded renamed subsequence,

$$G'_\lambda(u_n) \to 0, \quad G_\lambda(u_n) \to \lim_{n\to\infty} a(\lambda_n) \ge \nu(\lambda).$$

We can now follow the proof of Theorem 3.2 to obtain the desired solution. □

Proof of Theorem 3.4. We follow the proof of Theorem 3.3 until we conclude that $\tilde{u}_n \to 0$ in $L^\infty(I)$ as a consequence of the fact that we assume that $\|u_n\|_{H^1_T} \to \infty$. We define $\theta_n \in [0,1]$ by

$$G_{\lambda_n}(\theta_n u_n) = \max_{\theta\in[0,1]} G_{\lambda_n}(\theta u_n).$$

For any $c > 0$ and $h_n = c\tilde{u}_n$, we have

$$\int_I V(t, h_n)\, dt \to \int_I V(t, 0)\, dt \le 0.$$

Thus, for every fixed $c > 0$, if n is large enough one has that $0 < c/d^{1/2}(u_n) < 1$ and

$$G_{\lambda_n}(\theta_n u_n) \ge G_{\lambda_n}((c/d^{1/2}(u_n))u_n) = G_{\lambda_n}(c\tilde{u}_n) = c^2\lambda_n d(\tilde{u}_n) - 2\int_I V(t, h_n)\, dt,$$

so that

$$\liminf_{n\to\infty} G_{\lambda_n}(\theta_n u_n) \ge c^2\lambda,$$

namely, $\lim_{n\to\infty} G_{\lambda_n}(\theta_n u_n) = \infty$. If there is a renamed subsequence such that $\theta_n = 1$ for every n, then

$$G_{\lambda_n}(u_n) \to \infty. \tag{5.4}$$

If $0 \le \theta_n < 1$ for all n, then we have $(G'_{\lambda_n}(\theta_n u_n), \theta_n u_n) = 0$. Indeed, defined $h(\theta) = G_{\lambda_n}(\theta u_n)$ for every $\theta \in [0,1]$, one has

$$\frac{d}{d\theta} h(\theta) = (G'_{\lambda_n}(\theta u_n), u_n)$$

.

Hence, if $\theta_n = 0$ then $(G'_{\lambda_n}(\theta_n u_n), \theta_n u_n) = 0 \cdot \frac{d}{d\theta} h(0) = 0$. Otherwise, if $0 < \theta_n < 1$, being $h(\theta_n) = \max_{\theta \in [0,1]} h(\theta)$, one has

$$(G'_{\lambda_n}(\theta_n u_n), \theta_n u_n) = \theta_n \cdot \frac{d}{d\theta} h(\theta_n) = 0.$$

Therefore,

$$\begin{aligned} \int_I H(t, \theta_n u_n)\, dt &= \int_I \Big(\nabla V(t, \theta_n u_n) \theta_n u_n - 2V(t, \theta_n u_n) \Big)\, dt \\ &= G_{\lambda_n}(\theta_n u_n) - \frac{1}{2} (G'_{\lambda_n}(\theta_n u_n), \theta_n u_n) \\ &= G_{\lambda_n}(\theta_n u_n) \to \infty. \end{aligned}$$

By hypothesis,

$$\begin{aligned} G_{\lambda_n}(u_n) &= \int_I H(t, u_n) \\ &\ge \int_I H(t, \theta_n u_n)\, dt / C - \int_I W(t)\, dt \to \infty. \end{aligned}$$

Thus, (5.4) holds in any case. But

$$G_{\lambda_n}(u_n) = a(\lambda_n) \le a(M) < \infty.$$

This contradiction shows that $\|u_n\|_{H^1_T} \le C$. It now follows that for a renamed subsequence,

$$G'_\lambda(u_n) \to 0, \quad G_\lambda(u_n) \to \lim_{n \to \infty} a(\lambda_n) \ge \nu(\lambda).$$

We can now follow the proof of Theorem 3.2 to obtain the desired solution. □

6 Some examples

Here we show that the assumptions required in the main theorems are naturally satisfied in many simple and meaningful cases.

For simplicity, in the following, we suppose that $n = 1$, $I = [0, \pi]$ and $B(t) \equiv 1$ for all $t \in I$ while $\alpha, \beta \in L^1(I)$ are two positive functions. A direct computation shows that the eigenvalues of $\mathcal{D}$, with periodic boundary conditions, are

$$\lambda_l = 4l^2 + 1.$$

Hence, $\lambda_0 = 1$ and the corresponding eigenfunctions are constants.

Example 6.1. Put

$$V(t, x) = \alpha(t) |x|^\theta$$

for every $t \in I$, $x \in \mathbb{R}$, with $1 \leq \theta < 2$. Then all the assumptions of Theorem 3.1 are satisfied. Indeed, condition (1) holds with $b(t) = 2\alpha(t)$ and for every $m > 0$. Moreover, if $\varphi(t) = k$ for every $t \in I$, with $k \in \mathbb{R} \setminus \{0\}$, one has $\|\varphi\|_2^2 = k^2\pi$ and

$$2\int_I V(t, c\varphi)/c^2\|\varphi\|_2^2 = \|b\|_1|ck|^{\theta-2}/\pi,$$

showing (2), since $\liminf_{c\to 0} 2\int_I V(t, c\varphi)/c^2\|\varphi\|_2^2 = +\infty$.

Finally, observe that in this case the interval of the parameter λ for which the conclusions of Theorem 3.1 hold is $(0, +\infty)$.

Example 6.2. Let $g : \mathbb{R} \to \mathbb{R}$ be a positive and continuously differentiable function such that

$$L = \lim_{x\to\infty} g(x) > c_0\pi$$

and

$$2g(1) + g'(1) = 2g(-1) - g'(-1) = \theta,$$

where $1 \leq \theta < 2$. Put

$$F(x) = \begin{cases} |x|^\theta & \text{if } |x| \leq 1 \\ x^2 g(x) & \text{if } |x| > 1. \end{cases}$$

Then, the function

$$V(t, x) = \alpha(t)F(x)$$

for every $t \in I$, $x \in \mathbb{R}$ satisfies all the assumptions of Theorem 3.2. Indeed, arguing as in the previous example, we see that conditions (1) and (2) hold with $m = 1$. Moreover, for $|c|$ large enough one has

$$2\int_I V(t, c\varphi)/c^2\|\varphi\|_2^2 = 2\|\alpha\|_1 g(ck)/\pi.$$

Hence,

$$\liminf_{c\to\infty} 2\int_I V(t, c\varphi)/c^2\|\varphi\|_2^2 = 2\|\alpha\|_1 L/\pi > 2\|\alpha\|_1 c_0 = K_0$$

and condition (3.2) holds.

Example 6.3. Assume that $\alpha,\ \beta \in L^\infty(I)$ and put

$$V(t, x) = \alpha(t)|x|^\theta + \beta(t)|x|^\tau$$

for every $t \in I$, $x \in \mathbb{R}$ with $1 \leq \theta < 2 < \tau$. Then all the assumptions of Theorem 3.3 are satisfied. Indeed, condition (1) holds with $b(t) = 2(\alpha(t) + \beta(t))$ and $m = 1$. Moreover, if $\varphi(t) = k$ for every $t \in I$, with $k \in \mathbb{R} \setminus \{0\}$, one has

$$2\int_I V(t, c\varphi)/c^2\|\varphi\|_2^2 = 2(\|\alpha\|_1|ck|^{\theta-2} + \|\beta\|_1|ck|^{\tau-2})/\pi.$$

Hence

$$\liminf_{c\to 0} 2\int_I V(t, c\varphi)/c^2\|\varphi\|_2^2 = +\infty$$

and (2) is verified. It is an easy matter to verify that condition (3) holds. Finally, if

$$\mathcal{V}_r(t, x) = 2V(t, x) - 2V(t, rx) + (r^2 - 1)x \cdot \nabla_x V(t, x)$$

for every $t \in I$, $x \in \mathbb{R}$ and $r \in [0,1]$, then, exploiting the choice of θ and τ and observing that $2-\tau+\tau r^2-2r^\tau \le 0$, we see that there exists $C>0$ independent from t, x and r, such that

$$\begin{aligned}\mathcal{V}_r(t,x) &= 2\alpha(t)|x|^\theta(1-r^\theta)+2\beta(t)|x|^\tau(1-r^\tau)+(r^2-1)[\alpha(t)\theta|x|^\theta+\beta(t)\tau|x|^\tau] \\ &= \alpha(t)|x|^\theta(2-\theta+\theta r^2-2r^\theta)+\beta(t)|x|^\tau(2-\tau+\tau r^2-2r^\tau)<C,\end{aligned}$$

namely (3.4) holds.

We conclude with a further example that points out how Theorem 3.3 applies to functions that do not satisfy the well known Ambrosetti–Rabinowitz condition.

Example 6.4. Let $\alpha \in L^\infty(I)$ and put

$$V(t,x)=\alpha(t)|x|^2\ln^2|x|$$

for all $t \in I$ and $x \in \mathbb{R}$ (with the meaning $V(t,0)=0$). Since

$$\lim_{x\to 0}|x|^{2-\theta}\ln^2|x|=0$$

for every $0<\theta<2$, it is clear that condition (1) holds with $b(t)=\alpha(t)$ and m small enough. Moreover, if as usual $\varphi(t)=k$ for $t \in I$ and $k \in \mathbb{R}\setminus\{0\}$, one has

$$\liminf_{c\to 0} 2\int_I V(t,c\varphi)/c^2\|\varphi\|_2^2=\liminf_{c\to 0} 2\|\alpha\|_1\ln^2|ck|/\pi=+\infty,$$

and hence (2) is verified. It is simple to check that (3) holds. Finally, if $\mathcal{V}_r$ is defined as in the previous example, for $r \in (0,1]$ one has

$$\begin{aligned}\mathcal{V}_r(t,x) &= 2\alpha(t)|x|^2\left[\ln^2|x|-r^2\ln^2|rx|+(r^2-1)(\ln^2|x|+\ln|x|)\right] \\ &= 2\alpha(t)|x|^2\left[-r^2\ln^2 r-2r^2\ln r\ln|x|+r^2\ln|x|-\ln|x|\right] \\ &\le 2\alpha(t)|x|^2\ln|x|\left[r^2-1-2r^2\ln r\right].\end{aligned}$$

Since $r^2-1-2r^2\ln r\le 0$ for every $r\in[0,1]$, there exists $C>0$ independent from t, x and r such that

$$\mathcal{V}_r(t,x)<C.$$

For $r=0$ one has

$$\mathcal{V}_0(t,x)=-2\alpha(t)|x|^2\ln|x|.$$

Thus, in any case, (3.4) holds.

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References

[1] A. Ambrosetti, V. Coti Zelati, *Periodic solutions of singular Lagrangian systems*, Progress in Nonlinear Differential Equations and their Applications, Vol. 10, Birkhäuser, Boston, 1993. MR1267225; url

[2] F. Antonacci, P. Magrone, Second order nonautonomous systems with symmetric potential changing sign, *Rend. Mat. Appl. (7)* **18**(1998), 367–379. MR1659818

[3] F. Antonacci, Periodic and homoclinic solutions to a class of Hamiltonian systems with indefinite potential in sign, *Boll. Un. Mat. Ital. B (7)* **10**(1996), 303–324. MR1397350

[4] F. Antonacci, Existence of periodic solutions of Hamiltonian systems with potential indefinite in sign, *Nonlinear Anal.* **29**(1997), 1353–1364. MR1484908; url

[5] G. Barletta, R. Livrea, Existence of three periodic solutions for a non-autonomous second order system, *Matematiche (Catania)* **57**(2002), 205–215. MR2167121

[6] M. S. Berger, M. Schechter, On the solvability of semilinear gradient operator equations, *Adv. Math.* **25**(1977), 97–132. MR0500336; url

[7] G. Bonanno, Relations between the mountain pass theorem and local minima, *Adv. Nonlinear Anal.* **1**(2012), 205–220. MR3034869; url

[8] G. Bonanno, G. D'Aguì, Two non-zero solutions for elliptic Dirichlet problems, *Z. Anal. Anwend.* **35**(2016), 449–464. MR3556756; url

[9] G. Bonanno , R. Livrea, Periodic solutions for a class of second-order Hamiltonian systems, *Electron. J. Differential Equations*, **2005**, No. 115, 1–13. MR2174547

[10] G. Bonanno, R. Livrea, Multiple periodic solutions for Hamiltonian systems with not coercive potential, *J. Math. Anal. Appl.* **363**(2010), 627–638. MR2564882; url

[11] G. Bonanno, R. Livrea, Existence and multiplicity of periodic solutions for second order Hamiltonian systems depending on a parameter, *J. Convex. Anal.* **20**(2013), 1075–1094. MR3184297

[12] G. Bonanno, R. Livrea, M. Schechter, Some notes on a superlinear second order Hamiltonian system, *Manuscripta Math.*, published online. url

[13] G. Bonanno, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.* **89**(2010), 1–10. MR2604276; url

[14] G. Bonanno, P. Jebelean, C. Şerban, Superlinear discrete problems, *Appl. Math. Lett.* **52**(2016), 162–168. MR3416401; url

[15] H. Brezis, L. Nirenberg, Remarks on finding critical points, *Comm. Pure. Appl. Math.* **44**(1991), 939–963. MR1127041; url

[16] G. D'Aguì, Existence results for a mixed boundary value problem with Sturm–Liouville equation, *Adv. Pure Appl. Math.* **2**(2011), 237–248. MR2805078; url

[17] G. D'Aguì, B. Di Bella, S. Tersian, Multiplicity results for superlinear boundary value problems with impulsive effects, *Math. Meth. Appl. Sci.* **39**(2016), 1060–1068. MR3474142; url

[18] Y. Ding, M. Girardi, Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign, *Dynam. Systems Appl.* **2**(1993), 131–145. MR1205441

[19] I. Ekeland, N. Ghoussoub, Selected new aspects of the calculus of variations in the large, *Bull. Amer. Math. Soc. (N.S.)* **39**(2002), 207–265. MR1886088; url

[20] F. Faraci, A. Iannizzotto, A multiplicity theorem for a perturbed second-order non-autonomous system, *Proc. Edinb. Math. Soc. (2)* **49**(2006), 267–275.

[21] F. Faraci, R. Livrea, Infinitely many periodic solutions for a second-order nonautonomous system, *Nonlinear Anal.* **54**(2003), 417–429. MR1978419; url

[22] G. Fei, On periodic solutions of superquadratic Hamiltonian systems, *Electron. J. Differential Equations* **2002**, No. 8, 1–12. MR1884977

[23] N. Ghoussoub, *Duality and perturbation methods in critical point theory*, Cambridge Tracts in Mathematics, Vol. 107, Cambridge Univ. Press, Cambridge, 1993. MR1251958; url

[24] M. Jiang, Periodic solutions of second order superquadratic Hamiltonian systems with potential changing sign I, *J. Differential Equations* **219**(2005), 323–341. MR2183262; url

[25] M. Jiang, Periodic solutions of second order superquadratic Hamiltonian systems with potential changing sign II, *J. Differential Equations* **219**(2005), 342–362. MR2183263; url

[26] S. Li, W. Zou, Infinitely many solutions for Hamiltonian systems, *J. Differential Equations* **186**(2002), 141–164. MR1941096; url

[27] R. Livrea, S. A. Marano, A min-max principle for non-differentiable functions with a weak compactness condition, *Commun. Pure Appl. Anal.* **8**(2009), 1019–1029. MR2476669; url

[28] J. Mawhin, Semicoercive monotone variational problems, *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **73**(1987), 118–130. MR938142

[29] J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, New York, 1989. MR982267; url

[30] J. Pipan, M. Schechter, Non-autonomous second order Hamiltonian systems, *J. Differential Equations* **257**(2014), 351–373. MR3200374; url

[31] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, Vol. 65, 1986. MR845785; url

[32] M. Schechter, *Linking methods in critical point theory*, Birkhäuser, Boston, 1999. MR1729208; url

[33] M. Schechter, Nonautonomous second order Hamiltonian systems, *Pacific J. Math.* **251**(2011), No. 2, 431–452. MR2811042; url

[34] M. Schechter, Periodic second order superlinear Hamiltonian systems, *J. Math. Anal. Appl.* **426**(2015), 546–562. MR3306387; url

[35] L. K. Shilgba, Existence result for periodic solutions of a class of Hamiltonian systems with super quadratic potential, *Nonlinear Anal.* **63**(2005), 565–574. MR2175815; url

[36] C. L. Tang, Existence and multiplicity periodic solutions of nonautonomous second order systems, *Nonlinear Anal.* **32**(1998), 299–304. MR1610641; url

[37] C. L. Tang, Periodic solutions of nonautonomous second order systems with γ-quasisubadditive potential, *J. Math. Anal. Appl.* **189**(1995), 671–675. MR1312546; url

[38] C. L. Tang, Periodic solutions for nonautonomous second order systems with sublinear nonlinearity, *Proc. Amer. Math. Soc.* **126**(1998), 3263–3270. MR1476396; url

[39] C. L. Tang, X. P. Wu, Notes on periodic solutions of subquadratic second order systems, *J. Math. Anal. Appl.* **285**(2003), 8–16. MR2000135; url

[40] X. H. Tang, J. Jiang, Existence and multiplicity of periodic solutions for a class of second-order Hamiltonian systems, *Comput. Math. Appl.* **59**(2010), 3646–3655. MR2651840; url

[41] C. L. Tang, X. P. Wu, Periodic solutions for a class of nonautonomous subquadratic second order Hamiltonian systems, *J. Math. Anal. Appl.* **275**(2002), No. 2, 870–882. MR1943785; url

[42] C. L. Tang, X. P. Wu, Periodic solutions for second order systems with not uniformly coercive potential, *J. Math. Anal. Appl.* **259**(2001), No. 2, 386–397. MR1842066; url

[43] Z. L. Tao, C. L. Tang, Periodic and subharmonic solutions of second-order Hamiltonian systems, *J. Math. Anal. Appl.* **293**(2004), 435–445. MR2053889; url

[44] M. Willem, *Oscillations forcées de systèmes hamiltoniens*, Public. Sémin. Analyse Non Linéarie, Univ. Besançon, 1981. url

[45] X. P. Wu, C. L. Tang, Periodic solutions of a class of nonautonomous second order systems, *J. Math. Anal. Appl.* **236**(1999), 227–235. MR1704579; url

[46] Z. Wang, J. Zhang, Z. Zhang, Periodic solutions of second order non-autonomous Hamiltonian systems with local superquadratic potentiall, *Nonlinear Anal.* **70**(2009), 3672–3681. MR2504454; url

[47] X. Wu, S-X Chen, F. Zhao, New existence and multiplicity theorems of periodic solutions for non-autonomous second order Hamiltonian systems, *Math. Comput. Model.* **46**(2007), 550–556. MR2329457; url

[48] L. Xiao, X. H. Tang, Existence of periodic solutions to second-order Hamiltonian systems with potential indefinite in sign, *Nonlinear Anal.* **69**(2008), 3999–4011. MR2463350; url

[49] Y. Xu, Z. Guo, Existence of periodic solutions to second-order Hamiltonian systems with potential indefinite in sign, *Nonlinear Anal.* **51**(2002), 1273–1283. MR1926629; url

[50] Y-W. Ye, C. L. Tang, Periodic solutions for some nonautonomous second order Hamiltonian systems, *J. Math. Anal. Appl.* **344**(2008), 462–471. MR2416320; url

[51] F. Zhao, J. Chen, M. Yang, A periodic solution for a second-order asymptotically linear Hamiltonian system, *Nonlinear Anal.* **70**(2009), 4021–4026. MR2515318; url