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Positive solutions of second-order three-point boundary value problems with sign-changing coefficients

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Abstract. In this article, we investigate the boundary-value problem

$$\begin{cases} x''(t) + h(t)f(x(t)) = 0, & t \in [0,1], \\ x(0) = \beta x'(0), & x(1) = x(\eta), \end{cases}$$

where $\beta \geq 0$, $\eta \in (0,1)$, $f \in C([0,\infty),[0,\infty))$ is nondecreasing, and importantly h changes sign on $[0,1]$. By the Guo–Krasnosel'skiĭ fixed-point theorem in a cone, the existence of positive solutions is obtained via a special cone in terms of superlinear or sublinear behavior of f.

Keywords: positive solution, fixed point theorem, cone, sign-changing coefficient.

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1 Introduction

For the first time Liu [7] considered the existence of positive solutions to the following second-order three-point boundary value problems

$$\begin{cases} x''(t) + \lambda h(t)f(x(t)) = 0, & t \in [0,1], \\ x(0) = 0, & x(1) = \delta x(\eta), \end{cases} \tag{1.1}$$

where λ is a positive parameter, $\eta \in (0,1)$, $f \in C([0,\infty),[0,\infty))$ is nondecreasing, $\delta \in (0,1)$ and $h(t)$ is continuous and especially changes sign on $[0,1]$ which is different from the non-negative assumption in most of these studies.

Karaca [4] studied the problems with more general boundary conditions

$$\begin{cases} x''(t) + h(t)f(x(t)) = 0, & t \in [0,1], \\ \alpha x(0) = \beta x'(0), & x(1) = \delta x(\eta), \end{cases} \tag{1.2}$$

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where $\alpha \geq 0,\ \beta \geq 0,\ \alpha+\beta > 0$ with $0 < \delta < 1,\ f, h$ as in (1.1).

The authors of [4,7] showed the existence of at least one positive solution by applying the fixed-point theorem in a cone. Similar methods for a different problem are in [9]. Let E be a Banach space, the nonempty subset P is called a cone in E if it is a closed convex set and satisfies the properties that $\lambda x \in P$ for any $\lambda > 0,\ x \in P$ and that $\pm x \in P$ implies $x = 0$ (the zero element in E) (see [3]).

In [4] the author denoted

$$C_0^+[0,1] = \left\{ x \in C[0,1] : \min_{t\in[0,1]} x(t) \geq 0, \text{ and } \alpha x(0) = \beta x'(0),\ x(1) = \delta x(\eta) \right\}$$

and defined

$$\mathcal{P} = \left\{ x \in C_0^+[0,1] : x(t) \text{ is concave on } [0,\eta] \text{ and convex on } [\eta,1] \right\}.$$

In fact, $\mathcal{P}$ is not a cone since it is not a closed set in $C[0,1]$. For example, for $n > 3$ let

$$x_n(t) = \begin{cases} t+1, & 0 \leq t \leq \frac{1}{n}, \\ \frac{1}{n}+1, & \frac{1}{n} < t \leq \frac{1}{3}, \\ 6\left(\frac{1}{2}+\frac{1}{n}\right)\left(\frac{1}{2}-t\right)+\frac{1}{2}, & \frac{1}{3} < t \leq \frac{1}{2}, \\ \frac{3}{4}-\frac{t}{2}, & \frac{1}{2} < t \leq 1, \end{cases}$$

$$x_0(t) = \begin{cases} 1, & 0 \leq t \leq \frac{1}{3}, \\ 3\left(\frac{1}{2}-t\right)+\frac{1}{2}, & \frac{1}{3} < t \leq \frac{1}{2}, \\ \frac{3}{4}-\frac{t}{2}, & \frac{1}{2} < t \leq 1. \end{cases}$$

Obviously, $x_n \in \mathcal{P}$ for $\alpha = \beta = 1,\ \delta = 1/2$ and $x_n \to x_0$ in $C[0,1]$ since $\{x_n(t)\}$ uniformly converges to $x_0(t)$ on $[0,1]$. But $x_0 \notin \mathcal{P}$ because $x_0(0) = 1 \neq 0 = x_0'(0)$. However the conclusions in [4] are actually true only if $\alpha x(0) = \beta x'(0)$ is removed in $C_0^+[0,1]$ which is not needed in the proof of [4, Lemma 2.2] by using of the concavity.

A question is whether one can have boundary condition $x(1) = \delta x(\eta)$ with $\delta < (\beta+1)/(\beta+\eta)$ in problem (1.2) with $\alpha = 1$, which is the necessary condition when $f \geq 0$. We only consider one (less complicated) special case $\delta = 1$. If $\alpha = 0$, the corresponding linear problem for $g \in C[0,1]$ will be

$$\begin{cases} x''(t) + g(t) = 0, & t \in [0,1], \\ x'(0) = 0, & x(1) = x(\eta), \end{cases} \tag{1.3}$$

which is a resonance problem. So it is acceptable that $\alpha > 0$ and may be supposed to be $\alpha = 1$. For that reason, we investigate the existence of positive solutions to the three-point boundary-value problem

$$\begin{cases} x''(t) + h(t)f(x(t)) = 0, & t \in [0,1], \\ x(0) = \beta x'(0), & x(1) = x(\eta), \end{cases} \tag{1.4}$$

where $\beta \geq 0,\ \eta \in (0,1),\ f \in C([0,\infty),[0,\infty)),\ h(t)$ is continuous and is sign changing on $[0,1]$. The existence of positive solutions is obtained via a special cone (see (2.5)) in terms of superlinear or sublinear behavior of f by the Guo–Krasnosel'skiĭ fixed-point theorem in a cone. The ideas here are similar to the papers [4,7] and [9], but note that the signs on h are opposite to those in [4,7]. Other relevant research can be seen in [1,2,5,8,10].

2 Preliminaries

We will use the following assumptions.

(H_1) $h : [0,1] \to \mathbb{R}$ is continuous and such that $h(t) \le 0,\ t \in [0,\eta];\ h(t) \ge 0,\ t \in [\eta,1]$. Moreover, $h(t)$ does not vanish identically on any subinterval of $[0,1]$.

(H_2) $f \in C([0,\infty),[0,\infty))$ is continuous and nondecreasing.

(H_3) There exists a constant $\tau \in \left(\frac{1+\eta}{2},1\right)$ such that $A\rho h(\tau-\rho t)+h(t) \ge 0$ for $t \in [0,\eta]$ and $\rho = \frac{\tau-\eta}{\eta}$, where

$$A = \begin{cases} \frac{\beta(1-\tau)(1-\eta)}{2+\beta-\eta}, & \beta \neq 0,\\ \frac{(1-\tau)\eta^2}{1+\eta}, & \beta = 0. \end{cases} \tag{2.1}$$

Remark 2.1. The following example indicates that (H_3) is reasonable. If we take $\eta = 1/5$, $\tau = 4/5 \in (3/5,1), \rho = 3$ and

$$h(t) = \begin{cases} t - 1/5, & t \in [0,1/5],\\ (125/2)(t-1/5), & t \in (1/5,1], \end{cases}$$

then

$$A = \begin{cases} 2/125, & \beta = 1/5,\\ 1/150, & \beta = 0. \end{cases}$$

It is easy to see for $t \in [0,1/5]$ that $A\rho h(\tau-\rho t)+h(t) = 8(1/5-t) \ge 0$ when $\beta = 1/5$ and $A\rho h(\tau-\rho t)+h(t) = (11/4)(1/5-t) \ge 0$ when $\beta = 0$.

Lemma 2.2. *For $g \in C[0,1]$,*

$$\begin{cases} x''(t)+g(t) = 0, & t \in [0,1],\\ x(0) = \beta x'(0), & x(1) = x(\eta) \end{cases} \tag{2.2}$$

has the unique solution

$$x(t) = \int_0^1 G_1(t,s)g(s)ds + \frac{\beta}{1-\eta}\int_0^1 G_2(\eta,s)g(s)ds + \frac{t}{1-\eta}\int_0^1 G_1(\eta,s)g(s)ds,$$

where

$$G_1(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1,\\ (1-s)t, & 0 \le t < s \le 1, \end{cases} \qquad G_2(\eta,s) = \begin{cases} 1-\eta, & 0 \le s \le \eta,\\ 1-s, & \eta < s \le 1. \end{cases}$$

Proof. By Taylor expansion we have

$$x(t) = a_0 + a_1 t + \int_0^t (t-s)x''(s)ds = a_0 + a_1 t - \int_0^t (t-s)g(s)ds \tag{2.3}$$

and

$$x(0) = a_0,\ x(1) = a_0 + a_1 - \int_0^1 (1-s)g(s)ds,$$
$$x(\eta) = a_0 + a_1\eta - \int_0^\eta (\eta-s)g(s)ds,\ x'(0) = a_1.$$

The boundary conditions imply that $a_0 = \beta a_1$ and

$$a_0 + a_1 - \int_0^1 (1-s)g(s)ds = a_0 + a_1\eta - \int_0^\eta (\eta - s)g(s)ds,$$

thus

$$a_1 = \frac{1}{1-\eta}\int_0^1 (1-s)g(s)ds - \frac{1}{1-\eta}\int_0^\eta (\eta - s)g(s)ds,$$
$$a_0 = \frac{\beta}{1-\eta}\int_0^1 (1-s)g(s)ds - \frac{\beta}{1-\eta}\int_0^\eta (\eta - s)g(s)ds.$$

It follows from (2.3) that

$$\begin{aligned}
x(t) &= \frac{\beta+t}{1-\eta}\int_0^1 (1-s)g(s)ds - \frac{\beta+t}{1-\eta}\int_0^\eta (\eta-s)g(s)ds - \int_0^t (t-s)g(s)ds \\
&= \left(t + \frac{\beta+\eta t}{1-\eta}\right)\int_0^1 (1-s)g(s)ds + (\beta+st)\int_0^\eta g(s)ds - \frac{\beta+\eta t}{1-\eta}\int_0^\eta (1-s)g(s)ds \\
&\quad + \int_0^t (1-t)sg(s)ds - \int_0^t (1-s)tg(s)ds \\
&= \int_t^1 (1-s)tg(s)ds + \int_\eta^1 \frac{\beta+\eta t}{1-\eta}(1-s)g(s)ds \\
&\quad + \int_0^\eta (\beta+st)g(s)ds + \int_0^t (1-t)sg(s)ds \\
&= \int_0^1 G_1(t,s)g(s)ds + \frac{\beta}{1-\eta}\left(\int_0^\eta (1-\eta)g(s)ds + \int_\eta^1 (1-s)g(s)ds\right) \\
&\quad + \frac{t}{1-\eta}\left(\int_0^\eta (1-\eta)sg(s)ds + \int_\eta^1 (1-s)\eta g(s)ds\right) \\
&= \int_0^1 G_1(t,s)g(s)ds + \frac{\beta}{1-\eta}\int_0^1 G_2(\eta,s)g(s)ds + \frac{t}{1-\eta}\int_0^1 G_1(\eta,s)g(s)ds,
\end{aligned}$$

and hence the proof is complete. □

For $t, s \in [0,1]$ let

$$G(t,s) = G_1(t,s) + \frac{\beta}{1-\eta}G_2(\eta,s) + \frac{t}{1-\eta}G_1(\eta,s). \tag{2.4}$$

Lemma 2.3. *If $s_1 \in [0,\eta]$ and $s_2 \in [\eta,\tau]$, then*

$$G_1(\eta,s_2) \geq AG_1(\eta,s_1),\ G(t,s_2) \geq AG(t,s_1), \qquad \forall t \in [0,1],$$

where τ and A are as in (H_3).

Proof. In the case whether $\beta = 0$ or $\beta \neq 0$,

$$\frac{G_1(\eta,s_2)}{G_1(\eta,s_1)} = \frac{(1-s_2)\eta}{(1-\eta)s_1} \geq \frac{(1-\tau)\eta}{(1-\eta)\eta)} = \frac{1-\tau}{1-\eta} \geq A.$$

When $\beta \neq 0$,

$$\begin{aligned}
\frac{G(t,s_2)}{G(t,s_1)} &= \frac{G_1(t,s_2) + \frac{\beta}{1-\eta}G_2(\eta,s_2) + \frac{t}{1-\eta}G_1(\eta,s_2)}{G_1(t,s_1) + \frac{\beta}{1-\eta}G_2(\eta,s_1) + \frac{t}{1-\eta}G_1(\eta,s_1)} \\
&\geq \frac{\frac{\beta}{1-\eta}G_2(\eta,s_2)}{G_1(t,s_1) + \frac{\beta}{1-\eta}G_2(\eta,s_1) + \frac{t}{1-\eta}G_1(\eta,s_1)} \\
&\geq \frac{\frac{\beta}{1-\eta}(1-s_2)(1-\eta)}{(1-s_1) + \frac{\beta}{1-\eta}(1-s_1) + \frac{1}{1-\eta}(1-s_1)} \\
&= \frac{\beta(1-s_2)}{(1+\frac{\beta+1}{1-\eta})(1-s_1)} \geq \frac{\beta(1-\tau)}{1+\frac{\beta+1}{1-\eta}} = \frac{\beta(1-\tau)(1-\eta)}{2+\beta-\eta};
\end{aligned}$$

when $\beta = 0$,

$$\begin{aligned}
\frac{G(t,s_2)}{G(t,s_1)} &= \frac{G_1(t,s_2) + \frac{t}{1-\eta}G_1(\eta,s_2)}{G_1(t,s_1) + \frac{t}{1-\eta}G_1(\eta,s_1)} \geq \frac{\frac{t}{1-\eta}G_1(\eta,s_2)}{G_1(t,s_1) + \frac{t}{1-\eta}G_1(\eta,s_1)} \\
&\geq \frac{\frac{t}{1-\eta}G_1(\eta,s_2)}{(1-s_1)t + \frac{t}{1-\eta}G_1(\eta,s_1)} = \frac{\frac{1}{1-\eta}G_1(\eta,s_2)}{(1-s_1) + \frac{1}{1-\eta}G_1(\eta,s_1)} \\
&\geq \frac{\frac{1}{1-\eta}s_2\eta(1-\eta)(1-s_2)}{1+\frac{1}{1-\eta}s_1(1-\eta)} \geq \frac{(1-\tau)\eta^2}{1+\eta}.
\end{aligned}$$

Thus the proof is finished. □

In $C[0,1]$ with the norm $\|x\| = \max_{t\in[01]}|x(t)|$ for $x \in C[0,1]$, denote

$$X = \left\{ x \in C[0,1] : \min_{t\in[0,1]} x(t) \geq 0, \text{ and } x(0) \leq x(\eta),\ x(1) = x(\eta) \right\},$$

$$P = \{x \in X : x(t) \text{ is convex on } [0,\eta] \text{ and is concave on } [\eta,1]\}. \tag{2.5}$$

Obviously, P is a cone in $C[0,1]$.

Lemma 2.4. *If $x \in P$, then $x(t) \leq x(\eta) = \min_{t\in[\eta,1]} x(t)$ for $t \in [0,\eta]$.*

Lemma 2.5. *If $x \in P$, then*

$$x(t) \geq \frac{1-\tau}{2(1-\eta)}\|x\| \quad \textit{for } t \in \left[\tau, \frac{1+\tau}{2}\right],$$

where τ is as in (H_3).

Proof. By Lemma 2.4 we have $\|x\| = \max_{t\in[\eta,1]} x(t)$ and denote

$$\mu = \sup\{\xi \in [\eta,1] : x(\xi) = \|x\|\}.$$

Notice that $x(t)$ is concave on $[\eta,1]$. For $t \in [\eta,\mu)$,

$$\frac{x(\mu)-x(\eta)}{\mu-\eta} \geq \frac{x(\mu)-x(t)}{\mu-t}$$

and

$$x(t) \geq \frac{(t-\eta)x(\mu) + (\mu - t)x(\eta)}{\mu - \eta} \geq \frac{t-\eta}{\mu-\eta}\|x\| \geq \frac{t-\eta}{1-\eta}\|x\|;$$

for $t \in (\mu, 1]$,

$$\frac{x(t) - x(\mu)}{t - \mu} \geq \frac{x(1) - x(\mu)}{1 - \mu}$$

and

$$x(t) \geq \frac{(t-\mu)x(1) + (1-t)x(\mu)}{1-\mu} \geq \frac{1-t}{1-\eta}\|x\| = \left(1 - \frac{t-\eta}{1-\eta}\right)\|x\|.$$

Therefore,

$$x(t) \geq \min\left\{\frac{t-\eta}{1-\eta}, 1 - \frac{t-\eta}{1-\eta}\right\}\|x\|, \qquad \forall t \in [\eta, 1]$$

and hence

$$x(t) \geq \min\left\{\frac{\tau-\eta}{1-\eta}, \frac{1-\tau}{2(1-\eta)}\right\}\|x\| = \frac{1-\tau}{2(1-\eta)}\|x\|, \qquad \forall t \in \left[\tau, \frac{1+\tau}{2}\right]$$

since $[\tau, \frac{1+\tau}{2}] \subset [\eta, 1]$. □

Lemma 2.6. *Suppose that* (H_1)*–*(H_3) *are satisfied. If* $x \in P$*, then*

$$\int_0^\tau G(t,s)h(s)f(x(s))ds \geq 0 \qquad (\forall t \in [0,1]) \quad \textit{and} \quad \int_0^\tau G_1(\eta,s)h(s)f(x(s))ds \geq 0,$$

where τ *is as in* (H_3)*.*

Proof. For $s \in [\eta, \tau]$ let $s = \tau - \rho z$, here $\rho = (\tau - \eta)/\eta$, then $z \in [0, \eta]$. By Lemma 2.3, Lemma 2.4, (H_1) and (H_3), we have

$$\begin{aligned}
\int_\eta^\tau G(t,s)h(s)f(x(s))ds &= \rho \int_0^\eta G(t, \tau - \rho z)h(\tau - \rho z)f(x(\tau - \rho z))dz \\
&\geq A\rho \int_0^\eta G(t,z)h(\tau - \rho z)f(x(\tau - \rho z))dz \\
&\geq A\rho \int_0^\eta G(t,z)h(\tau - \rho z)f(x(z))dz \\
&\geq -\int_0^\eta G(t,z)h(z)f(x(z))dz = -\int_0^\eta G(t,s)h(s)f(x(s))ds
\end{aligned}$$

and hence

$$\int_0^\tau G(t,s)h(s)f(x(s))ds \geq 0.$$

By the same way, the other inequality holds. □

3 Main results

For $x \in P$ define the operator T as the following:

$$(Tx)(t) = \int_0^1 G(t,s)h(s)f(x(s))ds, \tag{3.1}$$

where $G(t,s)$ is in (2.4).

Lemma 3.1. *If* (H_1)–(H_3) *are satisfied, then* $T : P \to P$ *is completely continuous, where* P *is the cone defined by* (2.5) *in* $C[0,1]$.

Proof. If $x \in P$, it is clear that $(Tx)(t)$ is continuous on $[0,1]$ and for $t \in [0,1]$,

$$(Tx)(t) = \int_0^\tau G(t,s)h(s)f(x(s))ds + \int_\tau^1 G(t,s)h(s)f(x(s))ds \geq 0$$

by Lemma 2.6. Moreover, direct calculations by virtue of (2.4), (3.1) and Lemma 2.6 yield

$$(Tx)(\eta) = \frac{1}{1-\eta}\int_0^1 G_1(\eta,s)h(s)f(x(s))ds + \frac{\beta}{1-\eta}\int_0^1 G_2(\eta,s)g(s)f(x(s))ds = (Tx)(1),$$

$$\begin{aligned}(Tx)(\eta) - (Tx)(0) &= \frac{1}{1-\eta}\int_0^1 G_1(\eta,s)h(s)f(x(s))ds \\ &= \frac{1}{1-\eta}\Big(\int_0^\tau G_1(\eta,s)h(s)f(x(s))ds + \int_\tau^1 G_1(\eta,s)g(s)f(x(s))ds\Big) \geq 0.\end{aligned}$$

Meanwhile $(Tx)''(t) = -h(t)f(x(t)) \geq 0$ for $t \in [0,\eta]$ and $(Tx)''(t) \leq 0$ for $t \in [\eta,1]$, i.e., $(Tx)(t)$ is convex on $[0,\eta]$ and is concave on $[\eta,1]$ respectively. These mean that $T : P \to P$. At last, we know that T is completely continuous from the Arzelà–Ascoli theorem. □

It follows from Lemma 2.2 that there exists a positive solution to (1.4) if and only if T has a fixed point in P. In order to prove the existence of positive solution we need the following Guo-Krasnosel'skiĭ fixed point theorem in the cone [3,6].

Lemma 3.2. *Let* E *be a Banach space and* P *be a cone in* E. *Suppose that* Ω_1 *and* Ω_2 *are bounded open sets in* E *with* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *If* $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ *is a completely continuous operator and satisfies either*

(i) $\|Tx\| \leq \|x\|$ *for* $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|$ *for* $x \in P \cap \partial\Omega_2$; *or*

(ii) $\|Tx\| \geq \|x\|$ *for* $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|$ *for* $x \in P \cap \partial\Omega_2$,

then T *has a fixed point in* $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.3. *Suppose that* (H_1)–(H_3) *are satisfied. If*

$$\lim_{u \to 0^+} f(u)/u = 0, \tag{3.2}$$

$$\lim_{u \to \infty} f(u)/u = \infty, \tag{3.3}$$

then (1.4) *has at least one positive solution.*

Proof. Let P and T be respectively as (2.5) and (3.1).

By (3.2) there exists $r_1 > 0$ such that $f(u) \leq \varepsilon_1 u$ for $u \in [0,r_1]$, where $\varepsilon_1 > 0$ satisfies

$$\varepsilon_1 \max_{t \in [0,1]} \int_\eta^1 G(t,s)h(s)ds \leq 1. \tag{3.4}$$

Denote $\Omega_1 = \{x \in C[0,1] : \|x\| < r_1\}$ and hence from (H_1) and (3.4) we have that $\forall x \in P \cap \partial\Omega_1$,

$$\begin{aligned}(Tx)(t) &= \int_0^\eta G(t,s)h(s)f(x(s)) + \int_\eta^1 G(t,s)h(s)f(x(s))ds \\ &\le \int_\eta^1 G(t,s)h(s)f(x(s))ds \le \varepsilon_1 \int_\eta^1 G(t,s)h(s)x(s)ds \\ &\le \varepsilon_1 \|x\| \int_\eta^1 G(t,s)h(s)ds \le r_1,\ t \in [0,1],\end{aligned}$$

that is, $\|Tx\| \le \|x\|$.

By (3.3) there exists $\widetilde{R}_1 > 0$ such that $f(u) \ge \Lambda_1 u$ for $u \ge \widetilde{R}_1$, where $\Lambda_1 > 0$ satisfies

$$\Lambda_1 \frac{1-\tau}{2(1-\eta)} \max_{t\in[0,1]} \int_\tau^{(1+\tau)/2} G(t,s)h(s)ds \ge 1. \tag{3.5}$$

Denote $\Omega_2 = \{x \in C[0,1] : \|x\| < R_1\}$, where

$$R_1 = \max\left\{2r_1, \widetilde{R}_1 \frac{2(1-\eta)}{1-\tau}\right\}, \tag{3.6}$$

and hence by Lemma 2.5 and (3.6) we have that $\forall x \in P \cap \partial\Omega_2$,

$$x(t) \ge \frac{1-\tau}{2(1-\eta)}\|x\| = \frac{1-\tau}{2(1-\eta)} R_1 \ge \widetilde{R}_1 \quad \text{for } t \in \left[\tau, \frac{1+\tau}{2}\right]. \tag{3.7}$$

Consequently, it follows from Lemma 2.6, (3.7) and (3.5) that $\forall x \in P \cap \partial\Omega_2$,

$$\begin{aligned}\|Tx\| &= \max_{t\in[0,1]} \left(\int_0^\tau G(t,s)h(s)f(x(s)) + \int_\tau^1 G(t,s)h(s)f(x(s))ds \right) \\ &\ge \max_{t\in[0,1]} \int_\tau^1 G(t,s)h(s)f(x(s))ds \ge \max_{t\in[0,1]} \int_\tau^{(1+\tau)/2} G(t,s)h(s)f(x(s))ds \\ &\ge \max_{t\in[0,1]} \int_\tau^{(1+\tau)/2} G(t,s)h(s)\Lambda_1 x(s)ds \\ &\ge \Lambda_1 \frac{1-\tau}{2(1-\eta)} \|x\| \max_{t\in[0,1]} \int_\tau^{(1+\tau)/2} G(t,s)h(s)ds \ge \|x\|.\end{aligned}$$

By Lemma 3.1 and Lemma 3.2 T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ which is the positive solution to (1.4). □

Theorem 3.4. *Suppose that (H_1)–(H_3) are satisfied. If*

$$\lim_{u\to 0^+} f(u)/u = \infty, \tag{3.8}$$

$$\lim_{u\to\infty} f(u)/u = 0, \tag{3.9}$$

then (1.4) *has at least one positive solution.*

Proof. Let P and T be respectively as (2.5) and (3.1).

By (3.8) there exists $r_2 > 0$ such that $f(u) \ge \Lambda_2 u$ for $u \in [0, r_2]$, where $\Lambda_2 > 0$ satisfies

$$\Lambda_2 \frac{1-\tau}{2(1-\eta)} \max_{t\in[0,1]} \int_\tau^{(1+\tau)/2} G(t,s)h(s)ds \ge 1. \tag{3.10}$$

Denote $\Omega_1 = \{x \in C[0,1] : \|x\| < r_2\}$ and hence from Lemma 2.6 and Lemma 2.5 we have that $\forall x \in P \cap \partial\Omega_1$,

$$\begin{aligned}\|Tx\| &= \max_{t\in[0,1]} \left(\int_0^\tau G(t,s)h(s)f(x(s)) + \int_\tau^1 G(t,s)h(s)f(x(s))ds \right) \\ &\geq \max_{t\in[0,1]} \int_\tau^1 G(t,s)h(s)f(x(s))ds \geq \max_{t\in[0,1]} \int_\tau^{(1+\tau)/2} G(t,s)h(s)f(x(s))ds \\ &\geq \max_{t\in[0,1]} \int_\tau^{(1+\tau)/2} G(t,s)h(s)\Lambda_2 x(s)ds \\ &\geq \Lambda_2 \frac{1-\tau}{2(1-\eta)} \|x\| \max_{t\in[0,1]} \int_\tau^{(1+\tau)/2} G(t,s)h(s)ds \geq \|x\|.\end{aligned}$$

By (3.9) there exists $\widetilde{R}_2 > 0$ such that $f(u) \leq \varepsilon_2 u$ for $u \geq \widetilde{R}_2$, where $\varepsilon_2 > 0$ satisfies

$$\varepsilon_2 \max_{t\in[0,1]} \int_\eta^1 G(t,s)h(s)ds \leq 1. \tag{3.11}$$

If f is bounded, then there exists a constant $M > 0$ such that $f(u) \leq M$ for $u \geq 0$ and denote $\Omega_2 = \{x \in C[0,1] : \|x\| < R_2\}$ in this case, where

$$R_2 = \max\left\{2r_2, M \max_{t\in[0,1]} \int_\eta^1 G(t,s)h(s)ds\right\}, \tag{3.12}$$

and hence from (H_1) and (3.12) we have that $\forall x \in P \cap \partial\Omega_2$,

$$\begin{aligned}(Tx)(t) &= \int_0^\eta G(t,s)h(s)f(x(s)) + \int_\eta^1 G(t,s)h(s)f(x(s))ds \\ &\leq \int_\eta^1 G(t,s)h(s)f(x(s))ds \leq M \max_{t\in[0,1]} \int_\eta^1 G(t,s)h(s)ds \leq R_2, \qquad t \in [0,1],\end{aligned}$$

that is, $\|Tx\| \leq \|x\|$.

For the case when f is unbounded, take $R_2 = \max\{2r_2, \widetilde{R}_2\}$ and thus $f(u) \leq f(R_2)$ for $u \in [0, R_2]$ by the monotonicity of f. Therefore from (H_1) and (3.11) we have that $\forall x \in P \cap \partial\Omega_2$,

$$\begin{aligned}(Tx)(t) &= \int_0^\eta G(t,s)h(s)f(x(s)) + \int_\eta^1 G(t,s)h(s)f(x(s))ds \\ &\leq \int_\eta^1 G(t,s)h(s)f(x(s))ds \leq f(R_2) \max_{t\in[0,1]} \int_\eta^1 G(t,s)h(s)ds \\ &\leq \varepsilon_2 R_2 \max_{t\in[0,1]} \int_\eta^1 G(t,s)h(s)ds \leq R_2, \qquad t \in [0,1],\end{aligned}$$

which implies $\|Tx\| \leq \|x\|$ also.

By Lemma 3.1 and Lemma 3.2 T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ which is the positive solution to (1.4). □

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