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WEAK SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS ON REFLEXIVE BANACH SPACES

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Abstract. The aim of this paper is to investigate a class of boundary value problem for fractional differential equations involving nonlinear integral conditions. The main tool used in our considerations is the technique associated with measures of weak noncompactness.

1. Introduction

The theory of fractional differential equations has been emerging as an important area of investigation in recent years. In this paper, we investigate the existence of solutions for the boundary value problem with fractional order differential equations and nonlinear integral conditions of the form

$$^cD^\alpha x(t) = f(t, x(t)), \quad \text{for each } t \in I = [0, T], \tag{1}$$

$$x(0) - x'(0) = \int_0^T g(s, x(s))ds, \tag{2}$$

$$x(T) + x'(T) = \int_0^T h(s, x(s))ds, \tag{3}$$

where $^cD^\alpha, 1 < \alpha \leq 2$, is the Caputo fractional derivative, f, g and $h : I \times E \to E$ are given functions satisfying some assumptions that will be specified later, and E is a reflexive Banach space with norm $\| \cdot \|$.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point, and nonlocal boundary value problems as special cases. Integral boundary conditions are often encountered in various applications, it is worthwhile mentioning the applications of those conditions in the study of population dynamics [10], and cellular systems [1].

Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, for instance, Arara and Benchohra [2], Benchohra *et al.* [8], Infante [14], Peciulyte *et al.* [21], and the references therein.

In our investigation we apply the method associated with the technique of measures of weak noncompactness and a fixed point theorem of Mönch type. This technique was mainly initiated in the monograph of Banaŝ and Goebel [4] and subsequently developed and used in many papers; see, for example, Banaŝ *et al.* [5], Guo *et al.* [13], Krzyska

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and Kubiaczyk [16], Lakshmikantham and Leela [17], Mönch [18], O'Regan [19, 20], Szufla [25], Szufla and Szukala [26], and the references therein. In [6, 9], Benchohra *et al.* considered some classes of boundary value problems for fractional order differential equations in a Banach space by means of the strong measure of noncompactness.

Our goal is to prove the existence of solutions to the problem (1)–(3) under a weakly sequentially continuity assumption imposed on f, g and h. Recall that a weakly continuous operator is weakly sequentially continuous but the converse is not true in general [3]. We note that no compactness condition will be assumed on the nonlinearity of f. This is due to the fact that a subset of a reflexive Banach space is weakly compact if and only if it is weakly closed and norm bounded. As far as we know, there are very few papers (see [23]) related to the application of the measure of weak noncompactness to fractional differential equations on Banach spaces. This paper complements the corresponding literature.

2. Preliminaries

This section is devoted to recalling some notations and results that will be used throughout this paper.

We set $I = [0, T]$ and let $L^1(I)$ denote the Banach space of real-valued Lebesgue integrable functions on the interval I and $L^\infty(I, E)$ denote the Banach space of real-valued essentially bounded and measurable functions defined over I with the norm $\|\cdot\|_{L^\infty}$.

Let E be a real reflexive Banach space with norm $\|\cdot\|$ and dual E^*, and let $(E, w) = (E, \sigma(E, E^*))$ denote the space E with its weak topology. Here, $C(I, E)$ is the Banach space of continuous functions $x : I \to E$ with the usual supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| \ : \ t \in I\}.$$

Definition 2.1. *A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any $(x_n)_n$ in E with $x_n(t) \to x(t)$ in (E, w) for each $t \in I$ then $h(x_n(t)) \to h(x(t))$ in (E, w) for each $t \in I$).*

Definition 2.2. ([22]) *The function $x : I \to E$ is said to be Pettis integrable on I if and only if there is an element $x_J \in E$ corresponding to each $J \subset I$ such that $\varphi(x) = \int_J \varphi(x(s))ds$ for all $\varphi \in E^*$ where the integral on the right is assumed to exist in the sense of Lebesgue. By definition, $x_J = \int_J x(s)ds$.*

Let $P(I, E)$ be the space of all E-valued Pettis integrable functions in the interval I.

Proposition 2.3. [12, 22]) *If $x(\cdot)$ is Pettis integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then $x(\cdot)h(\cdot)$ is Pettis integrable.*

Definition 2.4. ([11]) *Let E be a Banach space, Ω_E be the set of all bounded subsets of E, and B_1 be the unit ball in E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty]$ defined by*

$$\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

Properties: The the Blasi measure of noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$;
(b) $\beta(A) = 0 \iff A$ is relatively compact;
(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;
(d) $\beta(\overline{A}^{\omega}) = \beta(A)$, where $\overline{A}^{\omega}$ denotes the weak closure of A;
(e) $\beta(A + B) \leq \beta(A) + \beta(B)$;
(f) $\beta(\lambda A) = |\lambda|\beta(A)$;
(g) $\beta(conv(A)) = \beta(A)$;
(h) $\beta(\cup_{|\lambda| \leq h} \lambda A = h\beta(A)$.

The following result follows directly from the Hahn-Banach theorem.

Proposition 2.5. *Let E be a normed space with $x_0 \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For completeness, we recall the definitions of the Pettis-integral and the Caputo derivative of fractional order.

Definition 2.6. ([24]) *Let $h : I \to E$ a function. The fractional Pettis integral of the function h of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^{\alpha} h(t) = \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} h(s) ds$$

where the sign "$\int$" denotes the Pettis integral and Γ is the Gamma function.

Definition 2.7. ([15]) *For a function $h : I \to E$, the Caputo fractional-order derivative of h is defined by*

$$^{c}D^{\alpha} h(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{h^{(n)}(s) ds}{(t - s)^{1 - n + \alpha}}$$

here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Theorem 2.8. ([19]) *Let Q be a closed convex and equicontinuous subset of a metrizable locally convex vector space $C(I, E)$ such that $0 \in Q$. Assume that $T : Q \to Q$ is weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \text{ is relatively weakly compact,} \tag{4}$$

holds for every subset $V \subset Q$, then T has a fixed point.

3. Existence of solutions

Let us start by defining what we mean by a solution of the problem (1)–(3).

Definition 3.1. *A function $x \in AC^1(I, E_w)$ is said to be solution of (1)–(3) if x satisfies (1)–(3).*

Let σ, σ_1, $\sigma_2 : I \to E$ be continuous functions and consider the linear boundary value problem

$$^{c}D^{\alpha}x(t) = \sigma(t), \quad t \in I, \tag{5}$$

$$x(0) - x'(0) = \int_0^T \sigma_1(s)ds, \tag{6}$$

$$x(T) + x'(T) = \int_0^T \sigma_2(s)ds. \tag{7}$$

Lemma 3.2. ([7]) *Let $1 < \alpha \le 2$ and let σ, σ_1, $\sigma_2 : I \to E$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = P(t) + \int_0^T G(t,s)\sigma(s)ds \tag{8}$$

with

$$P(t) = \frac{(T+1-t)}{T+2}\int_0^T \sigma_1(s)ds + \frac{(t+1)}{T+2}\int_0^T \sigma_2(s)ds \tag{9}$$

and

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(T-s)^{\alpha-1}}{(T+2)\Gamma(\alpha)} - \frac{(1+t)(T-s)^{\alpha-2}}{(T+2)\Gamma(\alpha-1)}, & 0 \le s \le t, \\ -\frac{(1+t)(T-s)^{\alpha-1}}{(T+2)\Gamma(\alpha)} - \frac{(1+t)(T-s)^{\alpha-2}}{(T+2)\Gamma(\alpha-1)}, & t \le s < T, \end{cases} \tag{10}$$

if and only if x is a solution of the fractional boundary value problem (5)-(7).

Let

$$\tilde{G} = \sup\left\{\int_0^T |G(t,s)|ds, \ t \in I\right\}.$$

To establish our main result concerning existence of solutions of (1)–(3), we impose suitable conditions on the functions involved in that problem, namely, we assume that the following conditions hold.

(H1) For each $t \in I$, $f(t,\cdot)$, $g(t,\cdot)$ and $h(t,\cdot)$ are weakly sequentially continuous.

(H2) For each $x \in C(I,E)$, $f(\cdot,x(\cdot))$, $g(\cdot,x(\cdot))$, and $h(\cdot,x(\cdot))$ are Pettis integrable on I.

(H3) There exist p_g, $p_h \in L^1(I,\mathbb{R}^+)$ and $p_f \in L^\infty(I,\mathbb{R}^+)$ such that:

$$\|f(t,x)\| \le p_f(t)\|x\|, \text{ for a.e. } t \in I \text{ and each } x \in E,$$

$$\|g(t,x)\| \le p_g(t)\|x\|, \text{ for a.e. } t \in I \text{ and each } x \in E,$$

$$\|h(t,x)\| \le p_h(t)\|x\|, \text{ for a.e. } t \in I \text{ and each } x \in E.$$

Theorem 3.3. *Let E be a reflexive Banach space and assume that (H1)–(H3) hold. If*

$$\frac{T+1}{T+2}\int_0^T [p_g(s)+p_h(s)]ds + \tilde{G}\|p_f\|_{L^\infty} < 1, \tag{11}$$

then the boundary value problem (1)–(3) has at least one solution.

Proof. We shall reduce the existence of solutions of the boundary value problem (1)–(3) to a fixed point problem. To this end, we consider the operator $T : C(I,E) \to C(I,E)$ defined by

$$(Tx)(t) = P_x(t) + \int_0^T G(t,s)f(s,x(s))ds \tag{12}$$

with

$$P_x(t) = \frac{(T+1-t)}{T+2}\int_0^T g(s,x(s))ds + \frac{(t+1)}{T+2}\int_0^T h(s,x(s))ds$$

and where $G(\cdot,\cdot)$ is the Green's function defined by (10). Clearly, the fixed points of the operator T are solutions of the problem (1)–(3).

First notice that, for $x \in C(I,E)$, we have $f(\cdot,x(\cdot)) \in P(I,E)$ by (H2). Since, $s \mapsto G(t,s) \in L^\infty(I)$, $G(t,\cdot)f(\cdot,x(\cdot))$, is Pettis integrable for all $t \in I$ by Proposition 2.3, and so the operator T is well defined.

Let $R \in \mathbb{R}_+^*$, and consider the set

$$\begin{aligned} Q \;=\; & \{x \in C(I,E) : \|x\|_\infty \le R \\ & \text{and } \|x(t_1)-x(t_2)\| \le \frac{|t_1-t_2|R}{T+2}\int_0^T (p_h(s)+p_g(s))ds \\ & + R\|p_f\|_{L^\infty}\int_0^T \|G(t_2,s)-G(t_1,s)\|ds \text{ for } t_1,t_2 \in I\}. \end{aligned}$$

Clearly, the subset Q is closed, convex and equicontinuous. We shall show that T satisfies the assumptions of Theorem 2.8.

Step 1: T maps Q into itself.

Take $x \in Q$ and assume that $Tx(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|Tx(t)\| = \varphi(Tx(t))$. Thus,

$$\begin{aligned}
\|Tx(t)\| &= \varphi(Tx(t)) \\
&= \varphi(P_x(t) + \int_0^T G(t,s)f(s,x(s))ds) \\
&\leq \varphi(P_x(t)) + \varphi(\int_0^T G(t,s)f(s,x(s))ds) \\
&\leq \|P_x(t)\| + \int_0^T \|G(t,s)\|\varphi(f(s,x(s)))ds \\
&\leq \frac{T+1}{T+2}R\int_0^T [p_g(s) + p_h(s)]ds + \tilde{G}R\|p_f\|_{L^\infty} \\
&\leq R.
\end{aligned}$$

Let t_1, $t_2 \in I$, $t_1 < t_2$, and $x \in Q$ so that $Tx(t_2) - Tx(t_1) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|Tx(t_2) - Tx(t_1)\| = \varphi(Tx(t_2) - Tx(t_1))$. Hence,

$$\begin{aligned}
&\|Tx(t_2) - Tx(t_1)\| \\
=\ & \varphi(P_x(t_2) - P_x(t_1) + \int_0^T (G(t_2,s) - G(t_1,s))f(s,x(s))ds) \\
=\ & \varphi(P_x(t_2) - P_x(t_1)) + \varphi(\int_0^T (G(t_2,s) - G(t_1,s))f(s,x(s))ds) \\
\leq\ & \|P_x(t_2) - P_x(t_1)\| + \int_0^T \|G(t_2,s) - G(t_1,s)\|\|f(s,x(s)\|ds \\
\leq\ & \frac{(t_2 - t_1)R}{T+2}\int_0^T (p_h(s) + p_g(s))ds \\
+\ & R\|p_f\|_{L^\infty}\int_0^T \|G(t_2,s) - G(t_1,s)\|ds.
\end{aligned}$$

Thus, $T(Q) \subset Q$.

Step 2: T is weakly sequentially continuous.

Let (x_n) be a sequence in Q and let $(x_n(t)) \to x(t)$ in (E, w) for each $t \in I$. Fix $t \in I$. Since f, g, and h satisfy assumption (H1), we have $f(t, x_n(t))$, $g(t, x_n(t))$, and $h(t, x_n(t))$ converge weakly uniformly to $f(t, x(t))$, $g(t, x(t))$, and $h(t, x(t))$, respectively. Hence, the Lebesgue Dominated Convergence Theorem for Pettis integrals implies $Tx_n(t)$ converges weakly uniformly to $Tx(t)$ in E_w. Repeating this for each $t \in I$ shows $Tx_n \to Tx$. Thus, $T : Q \to Q$ is weakly sequentially continuous.

Now let V be a subset of Q such that $V = \overline{conv}(T(V) \cup \{0\})$. Clearly, $V(t) \subset \overline{conv}(T(V) \cup \{0\})$ for all $t \in I$. Hence, $TV(t) \subset TQ(t)$, $t \in I$, is bounded in E. Thus,

$TV(t)$ is weakly relatively compact since a subset of a reflexive Banach space is weakly relatively compact if and only if it is bounded in the norm topology. Therefore,

$$\begin{aligned}\beta(V(t)) &\leq \beta(\overline{conv}(T(V)) \cup \{0\}) \\ &\leq \beta(TV(t)) \\ &= 0.\end{aligned}$$

Thus, V is relatively weakly compact.

Applying now Theorem 2.8, we conclude that T has a fixed point which is an solution of the problem (1)–(3). □

4. An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem,

$$^cD^r y(t) = \frac{2}{19+e^t}|y(t)|, \quad t \in J := [0,1], \quad 1 < r \leq 2, \tag{13}$$

$$y(0) - y'(0) = \int_0^1 \frac{1}{5+e^{5s}}|y(s)|ds,, \tag{14}$$

$$y(1) + y'(1) = \int_0^1 \frac{1}{3+e^{3s}}|y(s)|ds. \tag{15}$$

Set

$$f(t,x) = \frac{2}{19+e^t}x, \quad (t,x) \in J \times [0,\infty),$$

$$g(t,x) = \frac{1}{5+e^{5t}}x, \ (t,x) \in [0,1] \times [0,\infty),$$

$$h(t,x) = \frac{1}{3+e^{3t}}x, \ (t,x) \in [0,1] \times [0,\infty).$$

Clearly, conditions (H1)–(H2) hold with

$$p_f(t) = \frac{2}{19+e^t}, \ p_g(t) = \frac{1}{5+e^{5t}}, \quad \text{and} \quad p_h(t) = \frac{1}{3+e^{3t}}.$$

¿From (10) the function G is given by

$$G(t,s) = \begin{cases} \frac{(t-s)^{r-1}}{\Gamma(r)} - \frac{(1+t)(1-s)^{r-1}}{3\Gamma(r)} - \frac{(1+t)(1-s)^{r-2}}{3\Gamma(r-1)}, & 0 \leq s \leq t \\ -\frac{(1+t)(1-s)^{r-1}}{3\Gamma(r)} - \frac{(1+t)(1-s)^{r-2}}{3\Gamma(r-1)}, & t \leq s < 1. \end{cases} \tag{16}$$

From (16), we have

$$\begin{aligned}
\int_0^1 G(t,s)ds &= \int_0^t G(t,s)ds + \int_t^1 G(t,s)ds \\
&= \frac{t^r}{\Gamma(r+1)} + \frac{(1+t)(1-t)^r}{3\Gamma(r+1)} - \frac{(1+t)}{3\Gamma(r+1)} \\
&\quad + \frac{(1+t)(1-t)^{r-1}}{3\Gamma(r)} - \frac{(1+t)}{3\Gamma(r)} \\
&\quad - \frac{(1+t)(1-t)^r}{3\Gamma(r+1)} - \frac{(1+t)(1-t)^{r-1}}{3\Gamma(r)}.
\end{aligned}$$

A simple computation gives

$$\tilde{G} < \frac{3}{\Gamma(r+1)} + \frac{2}{\Gamma(r)}.$$

Now

$$\begin{aligned}
\tfrac{T+1}{T+2}[\|p_g\|_{L^\infty} + \|p_h\|_{L^\infty}] + \tilde{G}\|p_f\|_{L^\infty} &< \tfrac{2}{3}[\tfrac{1}{6} + \tfrac{1}{4}] + \tfrac{3}{10\Gamma(r+1)} + \tfrac{2}{10\Gamma(r)} \\
&= \tfrac{5}{18} + \tfrac{3}{10\Gamma(r+1)} + \tfrac{1}{5\Gamma(r)} < 1
\end{aligned}$$

for each $r \in (1,2]$, so condition (11) is satisfied with $T = 1$. By Theorem 3.3, the problem (13)–(15) has a solution on $[0,1]$.

5. Concluding remarks

In this paper, we presented an existence result for weak solutions of the boundary value problem (1)–(3) in the case where the Banach space E is reflexive. However, in the nonreflexive case, conditions (H1)–(H3) are not sufficient for the application of Theorem 2.8; the difficulty is with condition (4). Let us introduce the following conditions.

- (C1) For each bounded set $Q \subset E$ and each $t \in I$, the sets $f(t,Q)$, $g(t,Q)$, and $h(t,Q)$ are weakly relatively compact in E.
- (C2) For each bounded set $Q \subset E$ and each $t \in I$,

$$\beta(f(t,Q)) \le p_f(t)\beta(Q),$$

$$\beta(g(t,Q)) \le p_g(t)\beta(Q),$$

$$\beta(h(t,Q)) \le p_h(t)\beta(Q).$$

We then have the following results.

Theorem 5.1. *Let E be a Banach space, and assume that (H1)–(H3) and (C1) hold. If (11) holds, then the boundary value problem (1)–(3) has at least one solution.*

Theorem 5.2. *Let E be a Banach space, and assume that (H1)–(H3) and (C2) hold. If (11) holds, then the boundary value problem (1)–(3) has at least one solution.*

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