HALF–LINEAR DISCRETE OSCILLATION THEORY

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ABSTRACT. Oscillatory properties of the second order half-linear difference equation

$$\Delta(r_k|\Delta y_k|^{\alpha-2}\Delta y_k) + p_k|y_{k+1}|^{\alpha-2}y_{k+1} = 0,$$

where $\alpha > 1$, are investigated. A particular attention is devoted to the connection with oscillation theory of its continuous counterpart, half–linear differential equation and also with the theory of linear differential and linear difference equations. We present not only the overview of the existing results but we also establish new oscillation and nonoscillation criteria.

1. Introduction

The aim of this contribution is to present a survey of the recent results of the oscillation theory of the second order half–linear difference equation

$$\Delta(r_k\Phi(\Delta y_k)) + p_k\Phi(y_{k+1}) = 0, \tag{1}$$

where p_k and r_k are real-valued sequences with $r_k \neq 0$ and $\Phi(y) := |y|^{\alpha-1}\operatorname{sgn} y = |y|^{\alpha-2}y$, $\Phi(0) = 0$, with $\alpha > 1$. We will discuss the application of various methods in the oscillation theory of (1) which come from theory of *linear differential* equations. It is known, see [15], that the basic oscillatory properties of *half–linear difference* equation (1) ($\equiv$ HLΔE) are essentially the same as those of the *linear difference* equation ($\equiv$ LΔE)

$$\Delta(r_k\Delta y_k) + p_k y_{k+1} = 0, \tag{2}$$

which is the special case of (1) for $\alpha = 2$. Moreover, there exists a considerable similarity between the theories of difference and differential equations. This means that in this connection we are also interested in the second order *half–linear differential* equation ($\equiv$ HLDE)

$$(r(t)\Phi(y'))' + p(t)\Phi(y) = 0, \tag{3}$$

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which was very frequently investigated in the last years, see e.g. [9, 2, 3, 4, 5, 7, 8, 10, 11, 12], and its special case, namely the well–known Sturm–Liouville *linear differential* equation ($\equiv$ LDE)

$$(r(t)y')' + p(t)y = 0, \tag{4}$$

where the functions r and p are mostly considered to be continuous with $r(t) > 0$. Note that a natural requirement for the sequence r_k in (1), (2) is only to be nonzero. There exist several viewpoints which provide an explanation of this discrepancy between the continuous and the discrete cases, see e.g. [1]. It seems to be natural to require the assumption $r_k \neq 0$ also for equation (1).

Thus, our approach to the investigating of the qualitative properties of (1) can look as follows:

$$\begin{array}{ccc} \text{LDE} & \xrightarrow{\text{HL}} & \text{HLDE} \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle\Delta} \\ \text{L}\Delta\text{E} & \xrightarrow{\text{HL}} & \text{HL}\Delta\text{E}, \end{array}$$

$$\begin{aligned} \text{HL} &\equiv \text{generalization in a half–linear sense,} \\ \Delta &\equiv \text{discretization,} \end{aligned}$$

which means that we motivate ourselves by the linear continuous theory that offers many interesting topics for the discretization and the extension to the half–linear discrete case. This approach brings two types of interesting problems:

1) The first type relates to the discretization. The techniques of proofs that are needed in the discrete case are often different from the continuous case and also more complicated. This is due to the absence of the chain rule for the difference of the composite sequences and also due to some other specific properties of difference calculus.

2) The second type of problems is related to the extension to the half–linear case (both, continuous and discrete). There exist certain limitations in the use of the linear approach to the investigating of half–linear equations. These limitations are first of all the absence of transformation theory similar to that for linear equations or the impossibility of the extension of the so–called Casoratian to the half–linear discrete case. Casoratian is the discrete counterpart of Wronskian from the theory of linear differential equations.

Being motivated by the linear continuous case, we will show that one can extend some basic methods and results of oscillation theory of (4) to equation (1). In particular, this is the discrete half–linear version of the so called Roundabout theorem, see Section 2, which provides not only the Sturm type separation and comparison theorems but also two important tools for the investigating of oscillation and nonoscillation of (1). These methods are the *Riccati technique* and the *variational principle*, see Section 3. The *reciprocity principle* is also available in the oscillation theory of (1) and it is discussed in Section 3. Sections 4 and 5 contain an application of these methods, namely oscillation and nonoscillation criteria for (1).

At the end of this introductory section we recall some important concepts with some comments.

Definition 1. An interval $(m, m+1]$ is said to contain *the generalized zero* of a solution y of (1), if $y_m \neq 0$ and $r_m y_m y_{m+1} \leq 0$.

Definition 2. A nontrivial solution of (1) is called *oscillatory* if it has infinitely many generalized zeros. In view of the fact that Sturm type Separation Theorem extends to (1) (this follows from Theorem 1 – hereafter mentioned), we have the following equivalence: Any solution of (1) is oscillatory if and only if every solution of (1) is oscillatory. Hence we can speak about *oscillation* or *nonoscillation of equation* (1).

Note that authors who studied equation (2) with similar definition of generalized zero as above (but with the assumption $y_m y_{m+1} \leq 0$ instead of $r_m y_m y_{m+1} \leq 0$ – this lead to the Hartman's definition of the so called *node*, had problems with Sturmian theory since there exist simple examples such as the Fibonacci recurrence relation, where one solution seems to be oscillatory while another is nonoscillatory. This fact does not occur in the continuous case and our definition of generalized zero brings these exceptional cases into the general theory. On the other hand, it is obvious that the presence of the sequence r_k in Definition 1 is the result of the assumption $r_k \neq 0$ for equation (2) (or for (1)). Note that we cannot rewrite the Fibonacci equation into the self–adjoint form (2) with a positive r_k.

2. Roundabout Theorem

In this section we present the half–linear discrete version of the so called Roundabout Theorem (for its linear continuous and discrete version see e.g. [14] and [1], respectively, the proofs of some (nontrivial) parts of its half–linear continuous version can be found in [8, 12]) that is of the basic importance in the oscillation theory of (1). This theorem shows the connections between such concepts as disconjugacy of (1), existence of a solution of the generalized Riccati difference equation and positive definiteness of certain functional (see Definition 3 below), and therefore it provides powerful tools (see Section 3) for the investigation of oscillatory properties of equation (1). Note that by the term *generalized Riccati difference equation* we mean the following equation

$$\Delta w_k + p_k + S(w_k, r_k) = 0, \tag{5}$$

or, equivalently,

$$w_{k+1} = -p_k + \bar{S}(w_k, r_k),$$

where

$$S(w_k, r_k) = w_k \left(1 - \frac{r_k}{\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k))}\right), \tag{6}$$

$\bar{S}(w_k, r_k) = w_k - S(w_k, r_k)$ and the function Φ^{-1} being the inverse of Φ, i.e., $\Phi^{-1}(x) = |x|^{\beta-1} \operatorname{sgn} x$, where β is the conjugate number of α, i.e., $1/\alpha + 1/\beta = 1$. Equation (5) is related to (1) by the Riccati type substitution

$$w_k = \frac{r_k \Phi(\Delta y_k)}{y_k}.$$

Now we define and recall some important concepts

Definition 3. Equation (1) is said to be *disconjugate* on $[m, n]$ provided any solution of this equation has at most one generalized zero on $(m, n+1]$ and the solution $\tilde{y}$ satisfying $\tilde{y}_m = 0$ has no generalized zeros on $(m, n+1]$. Define a class U of the so called *admissible sequences* by

$$U = \{\xi \,|\, \xi : [m, n+2] \longrightarrow \mathbb{R} \text{ such that } \xi_m = \xi_{n+1} = 0\}.$$

Then define an 'α-degree" functional $\mathcal{F}$ on U by

$$\mathcal{F}(\xi; m, n) = \sum_{k=m}^{n} [r_k |\Delta \xi_k|^\alpha - p_k |\xi_{k+1}|^\alpha].$$

We say $\mathcal{F}$ is *positive definite on* U provided $\mathcal{F}(\xi) \geq 0$ for all $\xi \in U$ and $\mathcal{F}(\xi) = 0$ if and only if $\xi = 0$.

Theorem 1 (Roundabout Theorem, [15])**.** *The following statements are equivalent:*

(i) *Equation* (1) *is disconjugate on* $[m, n]$.

(ii) *Equation* (1) *has a solution* y *without generalized zeros on* $[m, n+1]$.

(iii) *The generalized Riccati difference equation* (5) *has a solution* w_k *on* $[m, n]$ *with* $r_k + w_k > 0$.

(iv) $\mathcal{F}$ *is positive definite on* U.

Proof. The proof of this theorem can be found in [15]. Note only that we use here the usual "roundabout method" that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i) and for the proof of the implication (iii) ⇒ (iv) we use the generalized Picone identity. □

3. Methods of half–linear discrete oscillation theory

In this section we describe three methods which are available in the oscillation theory of (1).

The first method is the so called *Riccati technique*. This method uses the following idea: Suppose (by contradiction) that equation (1) is nonoscillatory, i.e., there exists a solution y_k without generalized zeros (for k sufficiently large) and therefore there exists a solution of equation (5) with $r_k + w_k > 0$ according to Theorem 1 and vice versa, the existence of a solution of (5) with $r_k + w_k > 0$ in a neighborhood of infinity guarantees nonoscillation of (1). Actually, according to the following lemma, for nonoscillation of (1), it is sufficient to find a solution of the generalized Riccati difference inequality

$$\Delta w_k + p_k + S(w_k, r_k) \leq 0 \tag{7}$$

satisfying $r_k + w_k > 0$. However, this is not included in the Roundabout Theorem.

Lemma 1 ([6]). *Equation* (1) *is nonoscillatory if and only if there exists a sequence* w_k *satisfying the inequality* (7) *with* $r_k + w_k > 0$ *for* $k \geq m$ *with suitable* $m \in \mathbb{N}$.

Another method is the *variational principle* which is based on the following fact. Equation (1) is nonoscillatory if and only if the functional $\mathcal{F}$ is positive definite on the class U. But (1) is nonoscillatory if and only if it is disconjugate on a certain half–bounded discrete interval and therefore we have the equivalence (i) ⇔ (iv) from Theorem 1.

The third method which is available in the half–linear discrete oscillation theory is the *reciprocity principle*. Here we suppose that $r_k > 0,\ p_k > 0$. If we denote $u_k = r_k \Phi(\Delta y_k)$, where y is a solution of (1) then (as one can easily verify) u_k satisfies the reciprocal equation

$$\Delta(p_k^{1-\beta} \Phi_\beta(\Delta u_k)) + r_{k+1}^{1-\beta} \Phi_\beta(u_{k+1}) = 0, \tag{8}$$

where $\Phi_\beta(x) = |x|^{\beta-2}x$ and β is the conjugate number of α, i.e., $1/\alpha + 1/\beta = 1$. Conversely, if $y_k = p_{k-1}^{1-\beta} \Phi_\beta(\Delta u_{k-1})$, where u_k is a solution of (8), then y_k solves the original equation (1). Since the discrete version of the Rolle mean value theorem holds, we have the following equivalence: (1) is oscillatory [nonoscillatory] if and only if (8) is oscillatory [nonoscillatory]. Indeed, if y_k is an oscillatory solution of (1) then its difference and hence also $u_k = r_k \Phi(\Delta y_k)$ oscillates. Conversely, if u_k oscillates then $y_k = p_{k-1}^{1-\beta} \Phi_\beta(\Delta u_{k-1})$ oscillates as well.

Remark 1. Note that important concept of the oscillation theory of linear difference equations is the concept of *recessive solution* (the so called *principal solution* in the "continuous terminology"). Since the construction of principal solution and its application in oscillation theory of (3) has been already successfully made, see e.g. [2, 3, 7, 13], we would like to construct the recessive solution of half-linear difference equation and possibly apply it in oscillation theory of (1).

4. Oscillation criteria

In this section we present some already existing oscillation criteria for (1), where $r_k > 0$ for large k. We will also give one new criterion and mention some comments on the proofs and related open problems. The following notation will be used

$$P_k = \sum^k p_j, \quad \tilde{P}_k = \sum_{j=k}^{\infty} p_j, \quad R_k = \sum^k r_j^{1-\beta}, \quad \tilde{R}_k = \sum_{j=k}^{\infty} r_j^{1-\beta}, \tag{9}$$

β is the conjugate number of α, i.e., $1/\alpha + 1/\beta = 1$.

We start with the half–linear version of well–known criterion.

Theorem 2 (Leighton-Wintner type oscillation criterion, [15])**.** *Suppose that*

$$R_\infty = \infty \tag{10}$$

$$P_\infty = \infty. \tag{11}$$

Then (1) *is oscillatory.*

Proof. This statement is proved via the variational principle. Note only that according to Theorem 1, it is sufficient to find for any $K \geq m$ a sequence y satisfying $y_k = 0$ for $k \leq K$ and $k \geq N+1$, where $N > K$, such that

$$\mathcal{F}(y, K, N) = \sum_{k=K}^{N} [r_k |\Delta y_k|^\alpha - p_k |y_{k+1}|^\alpha] \leq 0.$$

□

Remark 2. In order to compare the similarity between some qualitative properties of our equation (1) and of the above mentioned equations we present here also the sufficient conditions of the same type for equations (2), (4) and (3), respectively. They are $\sum^\infty r_j^{-1} = \infty = \sum^\infty p_k$, $\int^\infty r^{-1}(t)dt = \infty = \int^\infty p(t)dt$ and $\int^\infty r^{1-\beta}(t)dt = \infty = \int^\infty p(t)dt$, respectively.

In the case when

$$\lim_{k\to\infty} \sum^k p_j \text{ is convergent,} \tag{12}$$

we can use the following criterion which is also proved via the variational principle.

Theorem 3 (Hinton-Lewis type oscillation criterion, [15])**.** *Suppose that the conditions* (10) *and* (12) *hold and*

$$\lim_{k\to\infty} R_k^{\alpha-1} \tilde{P}_k > 1. \tag{13}$$

Then (1) *is oscillatory.*

Remark 3. In the above criteria, equation (1) is essentially viewed as a perturbation of the nonoscillatory equation

$$\Delta(r_k \Phi(\Delta y_k)) = 0.$$

In the continuous case it is known (see e.g. [4, 5, 11]) that one can investigate equation (3) not as a perturbation of the nonoscillatory equation

$$(r(t)\Phi(y'))' = 0$$

but as a perturbation of a more general equation

$$(r(t)\Phi(y'))' + \tilde{p}(t)\Phi(y) = 0,$$

for example, of the generalized Euler equation

$$(r(t)\Phi(y'))' + \lambda t^{-\alpha}\Phi(y) = 0,$$

where $\lambda = ((\alpha - 1)/\alpha)^{\alpha}$ is the so–called critical constant. We conjecture that it is possible to prove some stronger criteria for (1), e.g., the assumption (13) can be replaced by the weaker one

$$\liminf_{k\to\infty} R_k^{\alpha-1}\tilde{P}_k > \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \tag{14}$$

(note we know that $\lim$ in Theorem 3 can be replaced by $\liminf$). From many other open problems we mention here e.g. the following: What are the additional conditions guaranteeing oscillation of (1) if (14) does not hold? Such types of criteria for equation (3) (with $r(t) \equiv 1$) are presented in [9]. See also Remark 6 for some other relates problems.

The next criterion is the "complementary" case of Hinton–Lewis type criterion – in the sense of the following convergence

$$\sum^{\infty} r_j^{1-\beta} < \infty. \tag{15}$$

Here we suppose not only $r_k > 0$ but also $p_k > 0$ for large k.

Theorem 4. *Suppose that* (15) *holds and*

$$\lim_{k\to\infty} \tilde{R}_{k+1}^{\alpha-1} P_k > 1. \tag{16}$$

Then (1) *is oscillatory.*

Proof. We will use the reciprocity principle. From (16) one can observe that

$$\sum^{\infty} p_j^{(1-\beta)(1-\alpha)} = \sum^{\infty} p_j = \infty.$$

Therefore according to Theorem 3, equation (8) is oscillatory if

$$\lim_{k\to\infty}\left(\sum^{k} p_j^{(1-\beta)(1-\alpha)}\right)^{\beta-1}\left(\sum_{j=k}^{\infty} r_{j+1}^{1-\beta}\right) > 1,$$

which is equivalent to (16). From here (1) is also oscillatory. □

5. NONOSCILLATION CRITERIA

This section contains nonoscillation criteria for (1), where, again, it is supposed $r_k > 0$ for large k. The first two are already proved "nonoscillatory supplements" of the meanwhile unproved Theorem 3 (but with (14) instead of (13)). Criteria that are included in Theorems 8 – 11 are new, some of them even in the linear case. Note we still use notation (9).

We start with the half–linear discrete version of quite well–known criterion for equation (4). The proof of this theorem is based on the variational principle. Note that the discrete "half–linear" version of a Wirtinger type inequality is used there.

Theorem 5 ([6]). *Suppose that* (10) *holds,* $\sum^{\infty} p_j^+ < \infty$, $p^+ := \max\{0, p\}$ *and*

$$\varphi_N := \left(\sup_{k \ge N} \frac{R_k}{R_{k-1}} \right)^{\alpha(\alpha-1)} < \infty, \quad \psi_N := \sup_{k \ge N} \left(\frac{r_k}{r_{k-1}} \right)^{1-\beta} < \infty.$$

Further suppose that

$$0 < \limsup_{N \to \infty} (1 + \psi_N)^{\alpha-1} \varphi_N =: \Psi < \infty. \tag{17}$$

If

$$\limsup_{k \to \infty} R_{k-1}^{\alpha-1} \sum_{j=k}^{\infty} p_j^+ < \frac{1}{\alpha \mu^{\alpha-1}} \left(\frac{\alpha-1}{\alpha} \right)^{\alpha-1} \frac{1}{\Psi}, \tag{18}$$

where

$$\mu := \begin{cases} \sup_{t>s>0} \frac{1}{t-s} \left[\Phi^{-1} \left(\frac{t^\alpha - s^p}{\alpha(t-s)} \right) - s \right], & \alpha \ge 2, \\ \sup_{t>s>0} \frac{1}{t-s} \left[t - \Phi^{-1} \left(\frac{t^\alpha - s^\alpha}{\alpha(t-s)} \right) \right], & \alpha \le 2, \end{cases}$$

then (1) *is nonoscillatory.*

The Riccati technique is used in the proof of the following theorem. More precisely, we use Lemma 1.

Theorem 6 ([6]). *Suppose that* (10) *and* (12) *hold and*

$$\lim_{k \to \infty} \frac{r_k^{1-\beta}}{R_{k-1}} = 0. \tag{19}$$

If

$$\limsup_{k \to \infty} R_{k-1}^{\alpha-1} \tilde{P}_k < \frac{1}{\alpha} \left(\frac{\alpha-1}{\alpha} \right)^{\alpha-1} \tag{20}$$

and

$$\liminf_{k\to\infty} R_{k-1}^{\alpha-1}\tilde{P}_k > -\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \tag{21}$$

then (1) *is nonoscillatory.*

Remark 4. 1) The above two criteria are not in fact completely the supplements of Theorem 3 (with (14)) since in contrast to the linear cases and half–linear continuous case we require additional restrictions on the sequence r_k, namely (17) and (19). However, we have an open problem in this connection: Is there really a need of these additional conditions?

2) We can see that the use of the different methods gives the different results in this case. The condition (17) is weaker than (19) but the constant at the right–hand side of (18) is less than constant in (20).

The following theorem complements the previous statement in the sense of the "complementary" case (15).

Theorem 7 ([6])**.** *Suppose that (15) holds and*

$$\lim_{k\to\infty} \frac{r_k^{1-\beta}}{\tilde{R}_k} = 0. \tag{22}$$

If

$$\limsup_{k\to\infty} \tilde{R}_k^{\alpha-1}P_{k-1} < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$$

and

$$\liminf_{k\to\infty} \tilde{R}_k^{\alpha-1}P_{k-1} > -\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \tag{23}$$

then (1) is nonoscillatory.

Next, denote by $\theta(C^{[0]})$ the greatest root of the equation

$$|x|^{\frac{1}{\beta}} + x + C^{[0]} = 0,$$

for certain $C^{[0]}$ which will be determined by the next statement and set $A_k = R_{k-1}^{\alpha-1}\tilde{P}_k$. In the case when (21) does not hold, we can use the following criterion which completes Theorem 6.

Theorem 8. *Suppose that* (10), (12) *and* (19) *hold. If*

$$\limsup_{k\to\infty} A_k < \left[\theta\left(\liminf_{k\to\infty} A_k\right)\right]^{\frac{1}{\beta}} - \theta\left(\liminf_{k\to\infty} A_k\right)$$

and

$$-\infty < \liminf_{k\to\infty} A_k \le -\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$$

then (1) *is nonoscillatory.*

Proof. The proof is similar to that of the continuous case, see [9]. □

Remark 5. It is clear that by the same way as above one can show there exists similar complementary case (i.e., when (23) fails to hold) also for Theorem 7.

The following statement is essentially a generalization of the criterion presented in [16] and at the same time the discretization of the criterion presented in [9]. First we introduce some notation. Set

$$B_k = R_{k-1}^{-1}\sum^{k-1}\left(R_{j-1}^{\alpha}p_j\right).$$

Let

$$\lambda(\alpha) = x^{[0]} + \frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1},$$

where $x^{[0]}$ is the least root of equation

$$(\alpha-1)|x|^{\beta} + \alpha x + \frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} = 0. \tag{24}$$

Theorem 9. *Suppose that* (10), (12) *and* (19) *hold. If*

$$\limsup_{k\to\infty} B_k < \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} \tag{25}$$

and

$$\liminf_{k\to\infty} B_k > \lambda(\alpha), \tag{26}$$

then (1) *is nonoscillatory.*

Proof. We show that the generalized Riccati difference inequality (7) has a solution w with $r_k + w_k > 0$ in a neighbourhood of infinity. Set

$$w_k = R_{k-1}^{1-\alpha}(C - B_k).$$

By the Lagrange mean value theorem we have

$$\Delta R_{k-1}^{1-\alpha} = (1-\alpha)r_k^{1-\beta}\eta_k^{-\alpha}$$

and

$$\Delta R_{k-1}^{\alpha} = \alpha r_k^{1-\beta}\mu_k^{\alpha-1},$$

where η_k and μ_k, respectively, are between R_{k-1} and R_k. Similarly,

$$w_k\left[\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k)) - r_k\right] = (\alpha-1)|\xi_k|^{\alpha-2}|w_k|^{\beta},$$

where ξ_k is between $\Phi^{-1}(r_k)$ and $\Phi^{-1}(r_k)+\Phi^{-1}(w_k)$, hence

$$r_k^{\beta-1}-|w_k|^{\beta-1}\le \xi_k \le r_k^{\beta-1}+|w_k|^{\beta-1}.$$

Let

$$C=\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}.$$

Then

$$\frac{|w_k|}{r_k}=\frac{\left|R_{k-1}^{1-\alpha}(C-B_k)\right|}{r_k}=\left(\frac{r_k^{1-\beta}}{R_{k-1}}\right)^{\alpha-1}|C-B_k|\to 0$$

as $k\to\infty$ according to (19) and hence $r_k+w_k=r_k(1+w_k/r_k)>0$ for large k. Further, assumptions (25) and (26) imply the existence of $\varepsilon_1>0$ such that

$$\lambda(\alpha)+\varepsilon_1<B_k<\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}-\varepsilon_1$$

for k sufficiently large, say $k\ge K_1$. From here,

$$\lambda(\alpha)-C+\varepsilon_1=x^{[0]}<B_k-C<\bar{x}^{[0]}-\bar{\varepsilon}_1, \tag{27}$$

where $\bar{x}^{[0]}=-\left((\alpha-1)/\alpha\right)^{\alpha-1}$ is the greatest root of equation (24). Therefore, from (27) it follows that there exists $\varepsilon_2>0$ such that

$$(\alpha-1)|B_k-C|^{\beta}+\alpha(B_k-C)+C+\varepsilon_2<0$$

for $k\ge K_1$, and, finally, this implies the existence of $\varepsilon>0$ such that

$$(\alpha-1)|C-B_k|^{\beta}(1+\varepsilon)-(\alpha-1)C(1-\varepsilon)+\alpha B_k(1+\varepsilon\,\mathrm{sgn}\,B_k)<0.$$

Multiplying this inequality by $r_k^{1-\beta}R_{k-1}^{-\alpha}$ we obtain

$$(\alpha-1)r_k^{1-\beta}R_{k-1}^{-\alpha}|C-B_k|^{\beta}(1+\varepsilon)-(\alpha-1)r_k^{1-\beta}CR_{k-1}^{-\alpha}(1-\varepsilon)+ \\ +\alpha r_k^{1-\beta}B_kR_{k-1}^{-\alpha}(1+\varepsilon\,\mathrm{sgn}\,B_k)<0 \tag{28}$$

for $k\ge K_1$. We can suppose that ε is at the same time such that we may add the term $p_k-p_k(1-\varepsilon\,\mathrm{sgn}\,p_k)$ to the left–hand side of (28).

Now, for a given $\varepsilon>0$ there exists $K_2\in\mathbb{N}$ such that we have $r_k>|w_k|$,

$$\left(\frac{R_{k-1}}{\eta_k}\right)^{\alpha}>1-\varepsilon,\quad (\mathrm{sgn}\,p_k)\left(\frac{R_{k-1}}{R_k}\right)^{\alpha}>(\mathrm{sgn}\,p_k)(1-\varepsilon\,\mathrm{sgn}\,p_k),$$

$$(\mathrm{sgn}\,B_k)(1+\varepsilon\,\mathrm{sgn}\,B_k)>(\mathrm{sgn}\,B_k)\left(\frac{\mu_k}{R_k}\right)^{\alpha}=\frac{(\mathrm{sgn}\,B_k)R_{k-1}^{\alpha}\mu_k^{\alpha-1}}{R_k^{\alpha}R_{k-1}^{\alpha-1}}$$

and

$$\frac{|\xi_k|^{\alpha-2}r_k^{\beta-1}}{(\Phi^{-1}(r_k)+\Phi^{-1}(w_k))^{\alpha-1}}<\frac{\left(1+\Phi^{-1}(|w_k|/r_k)\right)^{\alpha-2}}{\left(1+\Phi^{-1}(w_k/r_k)\right)^{\alpha-1}}<1+\varepsilon$$

for $k \geq K_2$. Using the above estimates, we obtain from (28)

$$\begin{aligned}
0 &> (\alpha-1)r_k^{1-\beta}R_{k-1}^{-\alpha}|C-B_k|^\beta(1+\varepsilon)- \\
&\quad -(\alpha-1)r_k^{1-\beta}CR_{k-1}^{-\alpha}(1-\varepsilon)+ \\
&\quad +\alpha r_k^{1-\beta}B_kR_{k-1}^{-\alpha}(1+\varepsilon\,\mathrm{sgn}\,B_k)+p_k-p_k(1-\varepsilon\,\mathrm{sgn}\,p_k) \\
&> -(\alpha-1)r_k^{1-\beta}C\eta_k^{-\alpha}-p_k\frac{R_{k-1}^{2\alpha}}{R_{k-1}^{\alpha}R_k^{\alpha}}+ \\
&\quad +\frac{\alpha r_k^{1-\beta}\mu_k^{\alpha-1}B_k}{R_{k-1}^{\alpha-1}R_k^{\alpha}}+p_k+ \\
&\quad +\frac{(\alpha-1)R_{k-1}^{-\alpha}|C-B_k|^\beta|\xi_k|^{\alpha-2}}{(\Phi^{-1}(r_k)+\Phi^{-1}(w_k))^{\alpha-1}} \\
&= \Delta w_k+p_k+\frac{(\alpha-1)|\xi_k|^{\alpha-2}|w_k|^\beta}{(\Phi^{-1}(r_k)+\Phi^{-1}(w_k))^{\alpha-1}} \\
&= \Delta w_k+p_k+w_k\left(1-\frac{r_k}{(\Phi^{-1}(r_k)+\Phi^{-1}(w_k))^{\alpha-1}}\right),
\end{aligned}$$

which means that inequality (7) has a solution w with $r_k+w_k>0$ in a neighbourhood of infinity and hence equation (1) is nonoscillatory by Lemma 1. □

Denote

$$\tilde{B}_k=\tilde{R}_k^{-1}\sum_k^\infty\left(\tilde{R}_j^\alpha p_j\right).$$

The following statement complements the previous one in the sense of the "complementary" case (15).

Theorem 10. *Suppose that* (15) *and* (22) *hold. If*

$$\limsup_{k\to\infty}\tilde{B}_k<\left(\frac{\alpha-1}{\alpha}\right)^\alpha \tag{29}$$

and

$$\liminf_{k\to\infty}\tilde{B}_k>\lambda(\alpha), \tag{30}$$

then (1) *is nonoscillatory.*

Proof. One can show by the similar way as in the proof of the previous theorem that under the assumptions (15), (22), (29) and (30) the sequence

$$w_k=-\left(\tilde{R}_k^{1-\alpha}\right)\left(C-\tilde{B}_k\right),$$

where $C=\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$, solves the inequality (7) with $r_k+w_k>0$. □

Remark 6. Being motivated by the half–linear continuous case, see [9, 2, 10], we conjecture that one can prove oscillatory supplements of the above criteria, namely $\liminf_{k\to\infty} B_k > \left(\frac{\alpha-1}{\alpha}\right)^\alpha$ or $\liminf_{k\to\infty} \tilde{B}_k > \left(\frac{\alpha-1}{\alpha}\right)^\alpha$ with appropriate additional conditions. Similarly as in the case of sequence $R_k^{\alpha-1}\tilde{P}_k$, see Remark 3, we can look for further oscillation criteria containing also sequence B_k. For such types of criteria for equation (3) see [9].

Denote $\Lambda(C^{[1]})$ the greatest root of equation

$$(\alpha-1)|x|^\beta + \alpha x + C^{[1]} = 0$$

and $\Gamma(C^{[2]})$ the greatest root of equation

$$(\alpha-1)|x|^\beta - (\alpha-1)x + C^{[2]} = 0$$

for certain $C^{[1]}$ and $C^{[2]}$ which are determined by the following statement that completes Theorem 9 in the case when (26) fails to hold.

Theorem 11. *Suppose that* (10), (12) *and* (19) *hold. If*

$$\limsup_{k\to\infty} B_k < \liminf_{k\to\infty} B_k + \Gamma\left(\liminf_{k\to\infty} B_k\right) + \Lambda\left[\liminf_{k\to\infty} B_k + \Gamma\left(\liminf_{k\to\infty} B_k\right)\right]$$

and

$$-\infty < \liminf_{k\to\infty} B_k \le \lambda(\alpha),$$

then (1) *is nonoscillatory.*

Proof. The proof is similar to that of the continuous case, see [9]. □

Remark 7. By the same way as above one can show that there exists similar complementary case (i.e., when (30) fails to hold) also for Theorem 10.

Remark 8. Under the assumption $r_k^{1-\beta} \to \infty$ as $k \to \infty$, the condition (19) can be replaced by more simple (but stronger) one, namely $\lim_{k\to\infty} r_{k+1}/r_k = 1$ in all above criteria where is presented. Similarly in the case when (15) holds we can suppose the same condition instead of (22).

References

[1] C. D. Ahlbrandt, A. C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations,* Kluwer Academic Publishers, Boston, 1996.

[2] O. Došlý, *Methods of oscillation theory of half-linear second order differential equations,* to appear in Czechoslovak Math. J.

[3] O. Došlý, *A remark on conjugacy of half-linear second order differential equations,* Math. Slovaca **50** (2000), 67–79.

[4] O. Došlý, *Oscillation criteria for half-linear second order differential equations,* Hiroshima J. Math. **28** (1998), 507–521.

[5] O. Došlý, A. Lomtatidze, *Oscillation and nonoscillation criteria for half-linear second order differential equations,* submitted.

[6] O. Došlý, P. Řehák, *Nonoscillation criteria for second order half-linear difference equations,* submitted.

[7] Á. Elbert, T. Kusano, *Principal solutions of nonoscillatory half-linear differential equations,* Advances in Math. Sci. and Appl. **8** (1998), 745–759.

[8] J. Jaroš, T. Kusano, *A Picone type identity for second order half-linear differential equations,* Acta Math. Univ. Comenianae **68** (1999), 137–151.

[9] N. Kandelaki, A. Lomtatidze, D. Ugulava, *Oscillation and nonoscillation criteria of a second order half–linear differential equation,* Georgian Math. J. **7** (2000), 1–19.

[10] T. Kusano, Y. Naito, *Oscillation and nonoscillation criteria for second order quasilinear differential equations,* Acta Math. Hungar. **76** (1997), 81–99.

[11] T. Kusano, Y. Naito, A. Ogata, *Strong oscillation and nonoscillation of quasilinear differential equations of second order,* Differential Equations and Dynamical Systems **2** (1994), 1–10.

[12] H. J. Li, C. C. Yeh, *Sturmian comparison theorem for half-linear second order differential equations,* Proc. Roy. Soc. Edinburgh **125A** (1996), 1193–1204.

[13] J. D. Mirzov, *Principal and nonprincipal solutions of a nonoscillatory system,* Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy **31** (1988), 100–117.

[14] W. T. Reid, *Sturmian Theory for Ordinary Differential Equations,* Springer-Verlag, New York - Berlin - Heidelberg, 1980.

[15] P. Řehák, *Oscillatory properties of second order half-linear difference equations,* to appear in Czechoslovak Math. J.

[16] P. Řehák, *Oscillation and nonoscillation criteria for second order linear difference equations,* to appear in Fasc. Mathematici.

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