Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **38**, 1–10; doi: 10.14232/ejqtde.2016.1.38 http://www.math.u-szeged.hu/ejqtde/

Existence of entire radial solutions to a class of quasilinear elliptic equations and systems

Song Zhou ✉ [1,2]

[1]Yantai Nanshan University, No. 1, Nanshan Road, Yantai, 265713, China
[2]Yantai University, No. 30, Qingquan Road, Yantai, 264005, China

Received 28 February 2016, appeared 7 June 2016
Communicated by Patrizia Pucci

Abstract. In this paper, by a monotone iterative method and the Arzelà–Ascoli theorem, we obtain the existence of entire positive radial solutions to the following quasilinear elliptic equations

$$\operatorname{div}(\phi_1(|\nabla u|)\nabla u) + a_1(|x|)\phi_1(|\nabla u|)|\nabla u| = b_1(|x|)f(u), \qquad x \in \mathbb{R}^N,$$

and systems

$$\begin{cases} \operatorname{div}(\phi_1(|\nabla u|)\nabla u) + a_1(|x|)\phi_1(|\nabla u|)|\nabla u| = b_1(|x|)f_1(u,v), & x \in \mathbb{R}^N, \\ \operatorname{div}(\phi_2(|\nabla v|)\nabla v) + a_2(|x|)\phi_2(|\nabla v|)|\nabla v| = b_2(|x|)f_2(u,v), & x \in \mathbb{R}^N, \end{cases}$$

under simple conditions on f, f_i, a_i and b_i $(i = 1, 2)$.

Keywords: quasilinear elliptic equations, systems, entire radial solutions, existence.

2010 Mathematics Subject Classification: 35J55, 35J60, 35J65.

1 Introduction

The purpose of this paper is to investigate the existence of entire positive radial solutions to the following quasilinear elliptic equation

$$\operatorname{div}(\phi_1(|\nabla u|)\nabla u) + a_1(|x|)\phi_1(|\nabla u|)|\nabla u| = b_1(|x|)f(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

and system

$$\begin{cases} \operatorname{div}(\phi_1(|\nabla u|)\nabla u) + a_1(|x|)\phi_1(|\nabla u|)|\nabla u| = b_1(|x|)f_1(u,v), & x \in \mathbb{R}^N, \\ \operatorname{div}(\phi_2(|\nabla v|)\nabla v) + a_2(|x|)\phi_2(|\nabla v|)|\nabla v| = b_2(|x|)f_2(u,v), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where a_i, b_i, f, f_i $(i = 1, 2)$ satisfy

$(\mathbf{S_1})$ $a_i, b_i : \mathbb{R}^N \to [0, \infty)$ are continuous;

✉ Email: zhousong242727@163.com

$(\mathbf{S_2})$ $f : [0,\infty) \to [0,\infty)$ is continuous and increasing, $f_i : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and increasing (i.e., $f_i(s_2,t_2) \geq f_i(s_1,t_1)$, $\forall s_2 \geq s_1 \geq 0$ and $t_2 \geq t_1 \geq 0$),

and $\phi_i \in C^1((0,\infty),(0,\infty))$ satisfy:

$(\mathbf{S_3})$ $(t\phi_i(t))' > 0,\ \forall t > 0$;

$(\mathbf{S_4})$ there exist $p_i, q_i > 1$ such that

$$p_i \leq \frac{t\Psi_i'(t)}{\Psi_i(t)} \leq q_i, \qquad \forall t > 0,$$

where $\Psi_i(t) = \int_0^t s\phi_i(s)ds,\ t > 0$;

$(\mathbf{S_5})$ there exist $k_i, l_i > 0$ such that

$$k_i \leq \frac{t\Psi_i''(t)}{\Psi_i'(t)} \leq l_i, \qquad \forall t > 0.$$

$\Delta_{\phi_1} u = \operatorname{div}(\phi_1(|\nabla u|)\nabla u)$ is called the ϕ_1-Laplacian operator, which includes special cases appearing in mathematical models in nonlinear elasticity, plasticity, generalized Newtonian fluids, and in quantum physics, see e.g., Benci, Fortunato and Pisani [5], Cencelj, Repovš and Virk [6], Fuchs and Li [9], Fuchs and Osmolovski [10], Fukagai and Narukawa [11] and [12] and the references therein.

Some basic examples of ϕ_1-Laplacian operators are

(1) when $\phi_1(t) \equiv 2$, $\Psi_1(t) = t^2$, $t > 0$, $\Delta_{\phi_1} u = \Delta u$ is the Laplacian operator. In this case, $p_1 = q_1 = 2$ in $(\mathbf{S_4})$, and $k_1 = l_1 = 1$ in $(\mathbf{S_5})$;

(2) when $\phi_1(t) = pt^{p-2}$, $\Psi_1(t) = t^p$, $t > 0$, $p > 1$, $\Delta_{\phi_1} u = \Delta_p u$ is the p-Laplacian operator. In this case, $p_1 = q_1 = p$ in $(\mathbf{S_4})$, and $k_1 = l_1 = p - 1$ in $(\mathbf{S_5})$;

(3) when $\phi_1(t) = pt^{p-2} + qt^{q-2}$, $\Psi_1(t) = t^p + t^q$, $t > 0$, $1 < p < q$, $\Delta_{\phi_1} u = \Delta_p u + \Delta_q u$ is called as the $(p+q)$-Laplacian operator, $p_1 = p$, $q_1 = q$ in $(\mathbf{S_4})$, and $k_1 = p - 1$, $l_1 = q - 1$ in $(\mathbf{S_5})$;

(4) when $\phi_1(t) = 2p(1+t^2)^{p-1}$, $\Psi_1(t) = (1+t^2)^p - 1$, $t > 0$, $p > 1/2$, $p_1 = \min\{2, 2p\}$, $q_1 = \max\{2, 2p\}$ in $(\mathbf{S_4})$, and $k_1 = \min\{1, 2p-1\}$, $l_1 = \max\{1, 2p-1\}$ in $(\mathbf{S_5})$;

(5) when $\phi_1(t) = \frac{p(\sqrt{1+t^2}-1)^{p-1}}{\sqrt{1+t^2}}$, $\Psi_1(t) = (\sqrt{1+t^2} - 1)^p$, $t > 0$, $p > 1$, $p_1 = p$, $q_1 = 2p$ in $(\mathbf{S_4})$, and $k_1 = p - 1$, $l_1 = 2p - 1$ in $(\mathbf{S_5})$;

(6) when $\phi_1(t) = pt^{p-2}(\ln(1+t))^q + \frac{qt^{p-1}(\ln(1+t))^{q-1}}{1+t}$, $\Psi_1(t) = t^p(\ln(1+t))^q$, $t > 0$, $p > 1$, $q > 0$, $p_1 = p$, $q_1 = p + q$ in $(\mathbf{S_4})$, and $k_1 = p - 1$, $l_1 = p + q - 1$ in $(\mathbf{S_5})$.

We say that $u \in C^1(\mathbb{R}^N)$ is a solution to equation (1.1) if for each $\psi \in C_0^\infty(\mathbb{R}^N)$, it holds

$$\int_{\mathbb{R}^N} \phi_1(|\nabla u|)\nabla u \nabla \psi dx - \int_{\mathbb{R}^N} a_1(x)(\phi_1(|\nabla u|)\nabla u)\psi dx = -\int_{\mathbb{R}^N} b_1(x) f(u)\psi dx.$$

Moreover, when $\lim_{|x|\to\infty} u(x) = +\infty$, we say that u is a large solution to equation (1.1).

For convenience, for $i = 1, 2$, we denote by

$$h_i^{-1} \quad \text{the inverses of } h_i(t) = t\phi_i(t), \quad t > 0; \tag{1.3}$$

$$I_{i,\rho,g}(\infty) := \lim_{r\to\infty} I_{i,\rho,g}(r), \qquad I_{i,\rho,g}(r) := \int_0^r h_i^{-1}(\Lambda_{\rho,g}(t))dt, \qquad r \ge 0, \tag{1.4}$$

where $\rho, g \in C([0,\infty),[0,\infty))$ and

$$\Lambda_{\rho,g}(t) := \frac{1}{\Phi_g(t)} \int_0^t \Phi_g(s)\rho(s)ds, \qquad t > 0; \tag{1.5}$$

$$\Phi_g(t) := t^{N-1} \exp\left(\int_0^t g(\tau)d\tau\right), \qquad t > 0; \tag{1.6}$$

$$\theta_i(t) := \min\{t^{p_i}, t^{q_i}\}, \qquad \Theta_i(t) := \max\{t^{p_i}, t^{q_i}\}, \qquad t \ge 0; \tag{1.7}$$

$$\theta_i^{-1}(t) := \min\{t^{1/p_i}, t^{1/q_i}\}, \qquad \Theta_i^{-1}(t) := \max\{t^{1/p_i}, t^{1/q_i}\}, \qquad t \ge 0; \tag{1.8}$$

and, for an arbitrary $\alpha > 0$ and $t \ge \alpha$,

$$\Upsilon_{1,\alpha}(\infty) := \lim_{t\to\infty} \Upsilon_{1,\alpha}(t), \qquad \Upsilon_{1,\alpha}(t) := \int_\alpha^t \frac{d\tau}{\Theta_1^{-1}(f(\tau))}; \tag{1.9}$$

$$\Upsilon_{2,\alpha}(\infty) := \lim_{t\to\infty} \Upsilon_{2,\alpha}(t), \qquad \Upsilon_{2,\alpha}(t) := \int_\alpha^t \frac{d\tau}{\Theta_1^{-1}(f_1(\tau,\tau)) + \Theta_2^{-1}(f_2(\tau,\tau))}. \tag{1.10}$$

We see that for $t > \alpha$

$$\Upsilon'_{1,\alpha}(t) = \frac{1}{\Theta_1^{-1}(f(t))} > 0,$$

$$\Upsilon'_{2,\alpha}(t) = \frac{1}{\Theta_1^{-1}(f_1(t,t)) + \Theta_2^{-1}(f_2(t,t))} > 0,$$

and $\Upsilon_{1,\alpha}, \Upsilon_{2,\alpha}$ have the inverse functions $\Upsilon_{1,\alpha}^{-1}$ and $\Upsilon_{2,\alpha}^{-1}$ on $[0, \Upsilon_{1,\alpha}(\infty))$ and $[0, \Upsilon_{2,\alpha}(\infty))$, respectively.

First, let us review the following model

$$\Delta u = b_1(|x|)f(u), \qquad x \in \mathbb{R}^N. \tag{1.11}$$

For $b_1(x) \equiv 1$ on $\mathbb{R}^N$: when f satisfies (S$_2$), Keller [14] and Osserman [19] first supplied a necessary and sufficient condition

$$\int_1^\infty \frac{dt}{\sqrt{2F(t)}} = \infty, \qquad F(t) = \int_0^t f(s)ds, \tag{1.12}$$

for the existence of entire positive radial large solutions to equation (1.11).

For $N \ge 3$, $f(u) = u^\gamma$, $\gamma \in (0,1]$, and b_1 satisfies (S$_1$) with $b_1(x) = b_1(|x|)$, Lair and Wood [16] first showed that equation (1.11) has infinitely many entire positive radial large solutions if and only if

$$\int_0^\infty r b_1(r)dr = \infty. \tag{1.13}$$

The above results have been extended by many authors and in many contexts, see, for instance, [1–3, 8, 21–23] and the references therein.

Next let us review the system

$$\begin{cases} \Delta u = b_1(|x|)v^{\gamma_1}, & x \in \mathbb{R}^N, \\ \Delta v = b_2(|x|)u^{\gamma_2}, & x \in \mathbb{R}^N. \end{cases} \tag{1.14}$$

When $N \geq 3$ and $0 < \gamma_1 \leq \gamma_2$, Lair and Wood [17] have considered the existence and nonexistence of entire positive radial solutions to system (1.14).

For the further results, see, for instance, [4,7,13,15,18,24] and the references therein.

Now let us return to equation (1.1). Recently, C. A. Santos, J. Zhou, J. A. Santos [20] considered the existence of entire positive radial and nonradial large solutions to equation

$$\operatorname{div}(\phi_1(|\nabla u|)\nabla u) = b_1(x)f(u), \qquad x \in \mathbb{R}^N.$$

A basic result in [20] is the following.

Lemma 1.1 ([20, Corollary 1.2]). *Let* $(\mathbf{S_3})$–$(\mathbf{S_5})$ *hold,* f *satisfy* $(\mathbf{S_2})$*, and* b_1 *satisfy* $(\mathbf{S_1})$ *with* $b_1(x) = b_1(|x|)$, $x \in \mathbb{R}^N$. *If*

$$I_{1,b_1,0}(\infty) = \infty,$$

then equation (1.1) *admits a sequence of symmetric radial large solutions* $u_m(|x|) \in C^1(\mathbb{R}^N)$ *with* $u_m(0) \to \infty$ as $m \to \infty$ *if and only if* f *satisfies*

$$\int_1^{\infty} \frac{dt}{\Psi_1^{-1}(F(t))} = \infty,$$

where Ψ_1^{-1} *is the inverse of* Ψ_1 *which is given as in* $(\mathbf{S_4})$*, and* F *is given as in* (1.12).

Recently, when $a_i \equiv 0$ in $\mathbb{R}^N$, $f_1(u,v) = f(v)$, $f_2(u,v) = g(u)$, and g satisfies $(\mathbf{S_2})$, Zhang [25] showed existence of entire positive radial solutions to (1.1) and system (1.2).

In this paper, we extend the results of [25] and show existence of entire positive radial solutions to (1.1) and (1.2) for more general a_i and f_i.

Our main results for equation (1.1) are as follows.

Theorem 1.2. *Let the hypotheses* $(\mathbf{S_1})$–$(\mathbf{S_5})$ *hold. If*

$(\mathbf{S_6})$ $\Upsilon_{1,\alpha}(\infty) = \infty$,

then equation (1.1) *has one entire positive radial solution* $u \in C^1(\mathbb{R}^N)$. *Moreover, when* $I_{1,a_1,b_1}(\infty) < \infty$, u *is bounded, and* $\lim_{r\to\infty} u(r) = \infty$ *provided* $I_{1,a_1,b_1}(\infty) = \infty$*, where* I_{1,a_1,b_1} *is given as in* (1.4).

Theorem 1.3. *Under the hypotheses* $(\mathbf{S_1})$–$(\mathbf{S_5})$ *and*

$(\mathbf{S_7})$ $I_{1,a_1,b_1}(\infty) < \Upsilon_{1,\alpha}(\infty) < \infty$,

equation (1.1) *has one entire positive radial bounded solution* $u \in C^1(\mathbb{R}^N)$ *satisfying*

$$\alpha + \theta_1^{-1}(f(\alpha))I_{1,a_1,b_1}(r) \leq u(r) \leq \Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(r)\right), \qquad \forall r \geq 0,$$

where θ_1^{-1} *is given as in* (1.8).

Remark 1.4. When $\int_0^1 \frac{d\tau}{\Theta_1^{-1}(f(\tau))} = \infty$, one can see that there is $\alpha > 0$ sufficiently small such that $(\mathbf{S_7})$ holds provided $I_{1,a_1,b_1}(\infty) < \infty$ and $\Upsilon_{1,\alpha}(\infty) < \infty$.

Remark 1.5. For $f(s) = s^{\gamma_1}$, $s \geq 0$, $\gamma_1 > 0$, since $\Theta_1^{-1}(t) = t^{1/p_1}$, $t \geq 1$, one can see that when $\gamma_1 > p_1$, $\Upsilon_{1,\alpha}(\infty) < \infty$, and $\Upsilon_{1,\alpha}(\infty) = \infty$ provided $\gamma_1 \leq p_1$, where p_1 is given as in $(\mathbf{S_4})$.

Remark 1.6. For $f(s) = (1+s)^{\gamma_1}(\ln(1+s))^{\mu_1}$, $s \geq 0$, $\mu_1, \gamma_1 > 0$, one can see that when $\gamma_1 > p_1$ or $\gamma_1 = p_1$ and $\mu_1 > p_1$, $\Upsilon_{1,\alpha}(\infty) < \infty$, and $\Upsilon_{1,\alpha}(\infty) = \infty$ provided $\gamma_1 < p_1$ or $\gamma_1 = p_1$ and $\mu_1 \leq p_1$.

Remark 1.7. For $f(s) = \exp(c_1 s)$, $s \geq 0$, $c_1 > 0$, one can see that $\Upsilon_{1,\alpha}(\infty) < \infty$.

Our main results for system (1.2) are as follows.

Theorem 1.8. *Let the hypotheses* $(\mathbf{S_1})$–$(\mathbf{S_5})$ *hold. If*

$(\mathbf{S_8})$ $\Upsilon_{2,\alpha}(\infty) = \infty$,

then system (1.2) *has one entire positive radial solution* (u, v) *in* $C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$. *Moreover, when* $I_{1,a_1,b_1}(\infty) + I_{2,a_2,b_2}(\infty) < \infty$, *u and v are bounded; when* $I_{1,a_1,b_1}(\infty) = I_{2,a_2,b_2}(\infty) = \infty$, $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty$.

Theorem 1.9. *Under the hypotheses* $(\mathbf{S_1})$–$(\mathbf{S_5})$ *and*

$$(\mathbf{S_9}) \qquad I_{1,a_1,b_1}(\infty) + I_{2,a_2,b_2}(\infty) < \Upsilon_{2,\alpha}(\infty) < \infty,$$

system (1.2) *has one entire positive radial bounded solution* (u, v) *in* $C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$ *satisfying*

$$\alpha/2 + \theta_1^{-1}(f_1(\alpha/2, \alpha/2)) I_{1,a_1,b_1}(r) \leq u(r) \leq \Upsilon_{2,\alpha}^{-1}(I_{1,a_1,b_1}(r) + I_{2,a_2,b_2}(r)), \qquad \forall r \geq 0;$$

$$\alpha/2 + \theta_2^{-1}(f_2(\alpha/2, \alpha/2)) I_{2,a_2,b_2}(r) \leq v(r) \leq \Upsilon_{2,\alpha}^{-1}(I_{1,a_1,b_1}(r) + I_{2,a_2,b_2}(r)), \qquad \forall r \geq 0.$$

Remark 1.10. For $f_1(s,s) = s^{\gamma_1}$, $f_2(s,s) = s^{\gamma_2}$, $s \geq 0$, $\gamma_1, \gamma_2 > 0$, when $\gamma_1 > p_1$ or $\gamma_2 > p_2$, $\Upsilon_{2,\alpha}(\infty) < \infty$, and $\Upsilon_{2,\alpha}(\infty) = \infty$ provided $\gamma_1 \leq p_1$ and $\gamma_2 \leq p_2$, where p_1 and p_2 are given as in $(\mathbf{S}_4)$.

Remark 1.11. For $f_1(s,s) = (1+s)^{\gamma_1}(\ln(1+s))^{\mu_1}$, $f_2(s,s) = (1+s)^{\gamma_2}(\ln(1+s))^{\mu_2}$, $s \geq 0$, $\gamma_i, \mu_i > 0$ $(i = 1,2)$, when $\gamma_1 > p_1$ or $\gamma_2 > p_2$; or $\gamma_1 = p_1$ and $\mu_1 > p_1$; or $\gamma_2 = p_2$ and $\mu_2 > p_2$, $\Upsilon_{2,\alpha}(\infty) < \infty$, and $\Upsilon_{2,\alpha}(\infty) = \infty$ provided $\gamma_1 < p_1$ and $\gamma_2 < p_2$; or $\gamma_1 = p_1$, $\mu_1 \leq p_1$ and $\gamma_2 = p_2$, $\mu_2 \leq p_2$.

Remark 1.12. For $f_1(s,s) = \exp(c_1 s)$ or $f_2(s,s) = \exp(c_2 s)$, $s \geq 0$, $c_1, c_2 > 0$, one can see that $\Upsilon_{2,\alpha}(\infty) < \infty$.

Remark 1.13. We note that the paper [26] by X. Zhang et al. studied the nonexistence and existence of positive radial large solutions to system (1.2). But, since their basic assumption is that $\phi_i \in C^1((0,\infty),[0,\infty))$ $(i = 1,2)$ are nondecreasing and for any $c \in (0,1)$, there exist constants $\sigma_i \in (0,1)$ such that

$$\phi_i(cs) \leq c^{\sigma_i}\phi_i(s), \qquad \forall s > 0, \tag{1.15}$$

it is $c^{\sigma_i} < 1$, hence (1.15) can not be set up when $\phi_i \equiv 1$ on $(0,\infty)$ (in this case, $\Delta_{\phi_1} u = \Delta u$ is the Laplacian operator).

2 Proof of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2 and 1.3.

Lemma 2.1 ([20, Lemma 2.2]). *Let* $(\mathbf{S_3})$–$(\mathbf{S_5})$ *hold,* θ_i, Θ_i *and* $\theta_i^{-1}, \Theta_i^{-1}$ $(i = 1,2)$ *be given as in* (1.7) *and* (1.8). *We have*

(i) θ_i, Θ_i, θ_i^{-1} *and* Θ_i^{-1} *are strictly increasing on* $(0,\infty)$;

(ii) $\theta_i^{-1}(\beta)h_i^{-1}(t) \leq h_i^{-1}(\beta t) \leq \Theta_i^{-1}(\beta)h_i^{-1}(t)$, $\forall \beta, t > 0$.

Let us consider the following initial value problem

$$(\Phi_{a_1}(r)\phi_1(u'(r))u'(r))' = b_1(r)\Phi_{a_1}(r)f(u), \qquad r>0, \quad u(0)=\alpha, \quad u'(0)=0, \tag{2.1}$$

where $\Phi_{a_1}(r)$ is given as in (1.6).

By a simple calculation,

$$u'(r) = h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)f(u(s))ds\right), \qquad r>0, \quad u(0)=\alpha, \tag{2.2}$$

and thus

$$u(r) = \alpha + \int_0^r h_1^{-1}\left(\frac{1}{\Phi_{a_1}(t)}\int_0^t b_1(s)\Phi_{a_1}(s)f(u(s))ds\right)dt, \qquad r\geq 0. \tag{2.3}$$

Note that solutions in $C[0,\infty)$ to problem (2.3) are solutions in $C^1[0,\infty)$ to problem (2.1).

Let $\{u_m\}_{m\geq 1}$ be the sequence of positive continuous functions defined on $[0,\infty)$ by

$$\begin{cases} u_0(r) = \alpha, \\ u_m(r) = \alpha + \int_0^r h_1^{-1}\left(\frac{1}{\Phi_{a_1}(t)}\int_0^t b_1(s)\Phi_{a_1}(s)f(u_{m-1}(s))ds\right)dt, \quad r\geq 0. \end{cases} \tag{2.4}$$

Obviously,

$$u_m'(r) = h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)f(u_{m-1}(s))ds\right), \qquad r>0, \tag{2.5}$$

and, for all $r\geq 0$ and $m\in\mathbb{N}$, $u_m(r)\geq\alpha$, and $u_0\leq u_1$. Then $(\mathbf{S_1})$–$(\mathbf{S_3})$ and Lemma 2.1 yield $u_1(r)\leq u_2(r)$, $\forall r\geq 0$. Continuing this line of reasoning, we obtain that the sequence $\{u_m\}$ is non-decreasing on $[0,\infty)$. Moreover, we obtain by $(\mathbf{S_1})$–$(\mathbf{S_3})$ and Lemma 2.1 that for each $r>0$

$$\begin{aligned} u_m'(r) &= h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)f(u_{m-1}(s))ds\right) \\ &\leq h_1^{-1}\left(f(u_m(r))\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)ds\right) \\ &\leq \Theta_1^{-1}(f(u_m(r)))h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)ds\right), \end{aligned}$$

and

$$\int_a^{u_m(r)}\frac{d\tau}{\Theta_1^{-1}(f(\tau))}\leq I_{1,a_1,b_1}(r).$$

Consequently, for an arbitrary $R>0$,

$$\Upsilon_{1\alpha}(u_m(r))\leq I_{1,a_1,b_1}(r)\leq I_{1,a_1,b_1}(R), \qquad \forall r\in[0,R]. \tag{2.6}$$

(i) When $(\mathbf{S_6})$ holds, we see that

$$\Upsilon_{1,\alpha}^{-1}(\infty)=\infty \quad\text{and}\quad u_m(r)\leq\Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(r)\right)\leq\Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(R)\right), \qquad \forall r\in[0,R], \tag{2.7}$$

i.e., the sequence $\{u_m\}$ is bounded on $[0,R]$ for an arbitrary $R>0$.

It follows by (2.5) that $\{u_m'\}$ is bounded on $[0,R]$. By the Arzelà–Ascoli theorem, $\{u_m\}$ has a subsequence converging uniformly to u on $[0,R]$. Since $\{u_m\}$ is non-decreasing on $[0,\infty)$, we see that $\{u_m\}$ itself converges uniformly to u on $[0,R]$. By the arbitrariness of R, we see that u is an entire positive radial solution to equation (1.1). Moreover, when $I_{1,a_1,b_1}(\infty) < \infty$, we see by (2.7) that

$$u(r) \le \Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(\infty)\right), \qquad \forall r \ge 0.$$

Moreover, when $I_{1,a_1,b_1}(\infty) = \infty$, we see by ($S_2$) and Lemma 2.1 that

$$u(r) \ge \alpha + \theta_1^{-1}(f(\alpha)) I_{1,a_1,b_1}(r), \qquad \forall r \ge 0.$$

Thus $\lim_{r\to\infty} u(r) = \infty$.
(ii) When $(\mathbf{S_7})$ holds, we see by (2.6) that

$$\Upsilon_{1,\alpha}(u_m(r)) \le I_{1,a_1,b_1}(\infty) < \Upsilon_{1,\alpha}(\infty) < \infty. \tag{2.8}$$

Since $\Upsilon_{1,\alpha}^{-1}$ is strictly increasing on $[0, \Upsilon_{1,\alpha}(\infty))$, we have

$$u_m(r) \le \Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(\infty)\right) < \infty, \ \forall r \ge 0. \tag{2.9}$$

The rest part of the proof follows from (i). The proof is finished.

3 Proof of Theorems 1.8 and 1.9

In this section we prove Theorems 1.8 and 1.9.

Let us consider the following initial value problem

$$\begin{cases} \left(\Phi_{a_1}(r)\phi_1(u'(r))u'(r)\right)' = b_1(r)\Phi_{a_1}(r) f_1(u,v), & r > 0, \\ \left(\Phi_{a_2}(r)\phi_2(v'(r))v'(r)\right)' = b_2(r)\Phi_{a_2}(r) f_2(u,v), & r > 0, \\ u(0) = v(0) = \alpha/2, \quad u'(0) = v'(0) = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} u(r) = \alpha/2 + \int_0^r h_1^{-1}\left(\frac{1}{\Phi_{a_1}(t)} \int_0^t b_1(s)\Phi_{a_1}(s) f_1(u(s),v(s))ds\right)dt, & r \ge 0, \\ v(r) = \alpha/2 + \int_0^r h_2^{-1}\left(\frac{1}{\Phi_{a_2}(t)} \int_0^t b_2(s)\Phi_{a_2}(s) f_2(u(s),v(s))ds\right)dt, & r \ge 0. \end{cases}$$

Let $\{u_m\}_{m\ge1}$ and $\{v_m\}_{m\ge0}$ be the sequences of positive continuous functions defined on $[0,\infty)$ by

$$\begin{cases} u_0(r) = v_0(r) = \alpha/2, \\ u_m(r) = \alpha/2 + \int_0^r h_1^{-1}\left(\frac{1}{\Phi_{a_1}(t)} \int_0^t b_1(s)\Phi_{a_1}(s) f_1(u_{m-1}(s),v_{m-1}(s))ds\right)dt, & r \ge 0, \\ v_m(r) = \alpha/2 + \int_0^r h_2^{-1}\left(\frac{1}{\Phi_{a_2}(t)} \int_0^t b_2(s)\Phi_{a_2}(s) f_2(u_{m-1}(s),v_{m-1}(s))ds\right)dt, & r \ge 0. \end{cases}$$

Obviously, for all $r \ge 0$ and $m \in \mathbb{N}$, $u_m(r) \ge \alpha/2$, $v_m(r) \ge \alpha/2$ and $u_0 \le u_1$, $v_0 \le v_1$. $(\mathbf{S_1})$–$(\mathbf{S_3})$ and Lemma 2.1 yield $u_1(r) \le u_2(r)$ and $v_1(r) \le v_2(r)$ on $[0,\infty)$. Continuing this line of reasoning, we obtain that the sequences $\{u_m\}$ and $\{v_m\}$ are increasing on $[0,\infty)$. Moreover, we obtain by $(\mathbf{S_1})$–$(\mathbf{S_3})$ and Lemma 2.1 that for each $r > 0$

$$\begin{aligned} u_m'(r) &= h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)f_1(u_{m-1}(s),v_{m-1}(s))ds\right) \\ &\le h_1^{-1}\left(f_1(u_{m-1}(r),v_{m-1}(r))\frac{1}{\Phi_{a_1}(t)}\int_0^t b_1(s)\Phi_{a_1}(s)ds\right) \\ &\le \Theta_1^{-1}(f_1(u_m(r),v_m(r)))h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)ds\right) \\ &\le \Theta_1^{-1}(f_1(u_m(r)+v_m(r),u_m(r)+v_m(r)))\big(h_1^{-1}(\Lambda_{b_1,a_1}(r))+h_2^{-1}(\Lambda_{b_2,a_2}(r))\big), \end{aligned}$$

where $\Lambda_{b_1,a_1}(r)$ and $\Lambda_{b_2,a_2}(r)$ are given as in (1.5).

In a similar way, we can show that

$$\begin{aligned} v_m'(r) &= h_2^{-1}\left(\frac{1}{\Phi_{a_2}(t)}\int_0^t b_2(s)\Phi_{a_2}(s)f_2(u_{m-1}(s),v_{m-1}(s))ds\right)dt \\ &\le \Theta_2^{-1}(f_2(u_m(r),v_m(r)))h_2^{-1}\left(\frac{1}{\Phi_{a_2}(t)}\int_0^t b_2(s)\Phi_{a_2}(s)ds\right) \\ &\le \Theta_2^{-1}(f_2(u_m(r)+v_m(r),u_m(r)+v_m(r)))\big(h_1^{-1}(\Lambda_{b_1,a_1}(r))+h_2^{-1}(\Lambda_{b_2,a_2}(r))\big). \end{aligned}$$

Consequently,

$$\begin{aligned} u_m'(r)+v_m'(r) \le \Big(&\Theta_1^{-1}(f_1(v_m(r)+u_m(r),v_m(r)+u_m(r))) \\ &+\Theta_2^{-1}(f_2(v_m(r)+u_m(r),v_m(r)+u_m(r)))\Big) \\ &\times\big(h_1^{-1}(\Lambda_{b_1,a_1}(r))+h_2^{-1}(\Lambda_{b_2,a_2}(r))\big), \qquad r>0, \end{aligned}$$

and

$$\int_a^{u_m(r)+v_m(r)}\frac{d\tau}{\Theta_1^{-1}(f_1(\tau,\tau))+\Theta_2^{-1}(f_2(\tau,\tau))} \le I_{1,b_1,a_1}(r)+I_{2,b_2,a_2}(r), \qquad r>0,$$

$$\Upsilon_{2,\alpha}(u_m(r)+v_m(r)) \le I_{1,b_1,a_1}(r)+I_{2,b_2,a_2}(r), \qquad \forall r \ge 0.$$

The remaining proofs are similar to that for Theorems 1.2 and 1.3. Here we omit their proof.

Acknowledgements

The author is greatly indebted to the anonymous referee for the very valuable suggestions and comments which improved the quality of the presentation. This work is supported in part by NSF of P. R. China under grant 11571295.

References

[1] I. Bachar, N. Zeddini, On the existence of positive solutions for a class of semilinear elliptic equations, *Nonlinear Anal.* **52**(2003), 1239–1247. MR1941255; url

[2] N. Belhaj Rhouma, A. Drissi, Large and entire large solutions for a class of nonlinear problems, *Appl. Math. Comput.* **232**(2014), 272–284. MR3181266; url

[3] N. Belhaj Rhouma, A. Drissi, W. Sayeb, Nonradial large solutions for a class of nonlinear problems, *Complex Var. Elliptic Equations* **59**(2014), 706–722. url

[4] A. Ben Dkhil, N. Zeddini, Bounded and large radially symmetric solutions for some (p,q)-Laplacian stationary systems, *Electron. J. Differential Equations* **2012**, No. 71, 1–9. MR2928608

[5] V. Benci, D. Fortunato, L. Pisani, Solitons like solutions of a Lorentz invariant equation in dimension 3, *Rev. Math. Phys.* **10**(1998), 315–344. MR1626832; url

[6] M. Cencelj, D. Repovš, Z. Virk, Multiple perturbations of a singular eigenvalue problem, *Nonlinear Anal.* **119**(2015), 37–49. MR3334172; url

[7] F. Cîrstea, V. Rădulescu, Entire solutions blowing up at infinity for semilinear elliptic systems, *J. Math. Pures Appl. (9)* **81**(2002), 827–846. MR1940369; url

[8] L. Dupaigne, M. Ghergu, O. Goubet, G. Warnault, Entire large solutions for semilinear elliptic equations, *J. Differential Equations* **253**(2012), 2224–2251. MR2946970; url

[9] M. Fuchs, G. Li, Variational inequalities for energy functionals with nonstandard growth conditions, *Abstr. Appl. Anal.* **3**(1998), 41–64. MR1700276; url

[10] M. Fuchs, V. Osmolovski, Variational integrals on Orlicz–Sobolev spaces, *Z. Anal. Anwendungen* **17**(1998), 393–415. MR1632563; url

[11] N. Fukagai, K. Narukawa, Nonlinear eigenvalue problem for a model equation of an elastic surface, *Hiroshima Math. J.* **25**(1995), 19–41. MR1322600

[12] N. Fukagai, K. Narukawa, On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems, *Ann. Mat. Pura Appl.* **186**(2007), 539–564. MR2317653; url

[13] A. Ghanmi, H. Mâagli, V. Rădulescu, N. Zeddini, Large and bounded solutions for a class of nonlinear Schrödinger stationary systems, *Anal. Appl. (Singap.)* **7**(2009), 391–404. MR2572852; url

[14] J. B. Keller, On solutions of $\Delta u = f(u)$, *Commun. Pure Appl. Math.* **10**(1957), 503–510. MR0091407; url

[15] A. V. Lair, Entire large solutions to semilinear elliptic systems, *J. Math. Anal. Appl.* **382**(2011), 324–333. MR2805516; url

[16] A. V. Lair, A. W. Wood, Large solutions of semilinear elliptic problems, *Nonlinear Anal.* **37**(1999), 805–812. MR1692803; url

[17] A. V. Lair, A. W. Wood, Existence of entire large positive solutions of semilnear elliptic systems, *J. Diff. Equations* **164**(2000), 380–394. url

[18] H. Li, P. Zhang, Z. Zhang, A remark on the existence of entire positive solutions for a class of semilinear elliptic systems, *J. Math. Anal. Appl.* **365**(2010), 338–341. MR2585106; url

[19] R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. Math.* **7**(1957), 1641–1647. MR0098239; url

[20] C. A. Santos, J. Zhou, J. A. Santos, Necessary and sufficient conditions for existence of blow-up solutions for elliptic problems in Orlicz–Sobolev spaces. Preprint available on arXiv.

[21] S. Tao, Z. Zhang, On the existence of explosive solutions for a class of semilinear elliptic problems, *Nonlinear Anal.* **48**(2002), 1043–1050. MR1880263; url

[22] H. Yang, On the existence and asymptotic behavior of large solutions for a semilinear elliptic problem in $\mathbb{R}^N$, *Comm. Pure Appl. Anal.* **4**(2005), 197–208.

[23] D. Ye, F. Zhou, Invariant criteria for existence of bounded positive solutions, *Discrete Contin. Dyn. Syst.* **12**(2005), 413–424. MR2119248; url

[24] Z. Zhang, Existence of entire positive solutions for a class of semilinear elliptic systems, *Electron. J. Differential Equations* **2010**, No. 16, 1–5. MR2592001

[25] Z. Zhang, Existence of positive radial solutions for quasilinear elliptic equations and systems, *Electron. J. Differential Equations* **2016**, No. 50, 1–9. MR3466521

[26] X. Zhang, L. Liu, Y. Wu, L. Caccetta, Entire large solutions for a class of Schrödinger systems with a nonlinear random operator, *J. Math. Anal. Appl.* **423**(2015), 1650–1659. MR3278220; url