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Multiple Positive Solutions for Boundary Value Problems of Second-Order Differential Equations System on the Half-Line*

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Abstract: In this paper, we study the existence of positive solutions for boundary value problems of second-order differential equations system with integral boundary condition on the half-line. By using a three functionals fixed point theorem in a cone and a fixed point theorem in a cone due to Avery-Peterson, we show the existence of at least two and three monotone increasing positive solutions with suitable growth conditions imposed on the nonlinear terms.

MSC: 34B10; 34B18

Keywords: Boundary value problems; Monotone increasing positive solutions; Fixed-point theorem in a cone; Half-line

1 Introduction

In this paper, we consider the existence of monotone increasing positive solutions for second-order boundary value problems of differential equations system with integral boundary condition on the half-line:

$$\begin{aligned} U''(t) + F(t,U) &= \mathbf{0}, \\ U(0) &= \mathbf{0}, \\ U'(\infty) &= \int_0^\infty g(s)U(s)ds \end{aligned} \tag{1.1}$$

where

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \; F(t,U) = \begin{pmatrix} f_1(t,u_1,u_2,\cdots,u_n) \\ f_2(t,u_1,u_2,\cdots,u_n) \\ \vdots \\ f_n(t,u_1,u_2,\cdots,u_n) \end{pmatrix}, \; U'(\infty) = \lim_{t\to\infty} U'(t),$$

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$$g(s) = \begin{pmatrix} g_1(s) & 0 & \cdots & 0 \\ 0 & g_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_n(s) \end{pmatrix}, \; f_i \in C(\mathbb{R}_+^{n+1}, \mathbb{R}_+) \text{ and } g_i \in L^1(\mathbb{R}_+)$$

$(i = 1, 2, \cdots, n)$ are nonnegative and $\mathbb{R}_+ = [0, +\infty)$.

This work is a continuation of our previous paper [16] where we considered the existence of one positive solution for a system of two equations. Boundary value problems with Riemann-Stieltjes integral boundary conditions are now being studied since they include boundary value problems with two-point, multi-point and integral boundary conditions as special cases, see for example [1, 2, 13, 14, 15, 26, 27].

In [13], Ma considered the existence of positive solutions for second-order ordinary differential equations while the nonlinear term is either superlinear or sublinear. This was improved by Webb and Infante in [14, 15] who used fixed point index theory and gave a general method for solving problems with integral boundary conditions of Riemann-Stieltjes type. In [16], by using the fixed point theorem in a cone, we studied the existence of positive solutions of boundary value problem for systems of second-order differential equations with integral boundary condition on the half-line. In fact, the result in [16] holds for n terms in the system.

Boundary value problems on the half-line have been applied in unsteady flow of gas through a semi-infinite porous medium, the theory of drain flows, etc. They have received much attention in recent years, and there are many results in these areas (See [3]-[12] and the references therein). In [3]-[8], authors studied two-point boundary value problems on the half-line by using different method. By using fixed point theorem, Tian [9] studied three-point boundary-value problem, and then she studied multi-point boundary value problem on the half-line, see [11].

Zhang [12] investigated the existence of positive solutions of singular multi-point boundary value problems for systems of nonlinear second-order differential equations on infinite intervals in Banach space by using the Monch fixed point theorem and a monotone iterative technique

$$\begin{aligned} x''(t) + f(t, x(t), x'(t), y(t), y'(t)) &= 0, \\ y''(t) + g(t, x(t), x'(t), y(t), y'(t)) &= 0, \\ x(0) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i), x'(\infty) = x_\infty, \\ y(0) &= \sum_{i=1}^{m-2} \beta_i y(\xi_i), y'(\infty) = y_\infty \end{aligned}$$

In [20], by using a new twin fixed point theorem due to Avery and Henderson (see [17], [18]), He and Ge studied twin positive solutions of nonlinear differential equations of the form

$$(\phi(u'))' + e(t)f(u) = 0$$

with boundary conditions including the following

$$\begin{aligned} u(0) - B_0(u'(0)) = 0, u(1) + B_1(u'(1)) = 0 \\ u(0) - B_0(u'(0)) = 0, u'(1) = 0 \\ u'(0) = 0, u(1) + B_1(u'(1)) = 0. \end{aligned}$$

Liang [21] used the same method and studied four-point boundary value problem with a p-Laplacian operator.

By using a fixed point theorem in a cone due to Avery-Peterson, see [19], Pang [24] and Zhao [23] investigated the existence of multiple positive solutions to four-point boundary value problems with one-dimensional p-Laplacian. Afterwards, Feng [25] studied the existence of at least three positive solutions to the m-point boundary value problems with one-dimensional p-Laplacian by using the same method. In 2007, Lian [22] studied two-point boundary value problems on the half-line by using the Avery-Peterson fixed point theorem.

Motivated by these works, we use the three functionals fixed point theorem in a cone due to Avery and Henderson (see [17], [18]) and the fixed point theorem due to Avery-Peterson (see [19]) to investigate the boundary value problem (1.1).

We define $U, V \in \mathbb{R}_+^n, U \geq V$ if and only if $u_i \geq v_i, i = 1, 2, \cdots, n$. $U > V$ if and only if $u_i > v_i, i = 1, 2, \cdots, n$.

Throughout the paper, we assume that the following conditions hold.

(H1) $1 - \int_0^1 sg_i(s)ds > 0, i = 1, \cdots, n$;

(H2) F is an L^1-Carathéodory function, that is,

(1) $F(\cdot, U)$ is measurable for any $U \in \mathbb{R}_+^n$;

(2) $F(t, \cdot)$ is continuous for almost every $t \in \mathbb{R}_+$;

(3) For each $r_1, r_2, \cdots, r_n > 0$, there exists $\phi_{r_1, r_2, \cdots, r_n} \in L^1(\mathbb{R}_+, \mathbb{R}_+^n)$ such that

$$\mathbf{0} \leq F\big(t, (1+t)U\big) \leq \phi_{r_1, r_2, \cdots, r_n}(t),$$

for all $u_i \in [0, r_i], i = 1, 2, \cdots, n$ and almost every $t \in \mathbb{R}_+$.

2 Preliminaries

In this section, we first give the two fixed point theorems which will be used in the following proof.

Definition 2.1. Let E be a real Banach space and $P \subset E$ be a cone. We denote the partial order induced by P on E by $\leq$. That is, $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. Given a cone P in a real Banach space E, a functional $\psi : P \to \mathbb{R}$ is said to be increasing on P, provided $\psi(x) \leq \psi(y)$, for all $x, y \in P$ with $x \leq y$.

Definition 2.3. Given a cone P in a real Banach space E, a continuous map ψ is called a concave (convex) functional on P if and only if for all $x, y \in P$ and $0 \leq \lambda \leq 1$, it holds

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y),$$

$$(\psi(\lambda x + (1-\lambda)y) \leq \lambda\psi(x) + (1-\lambda)\psi(y).)$$

Let γ be a nonnegative continuous functional on a cone P. For each $c > 0$, we define the set

$$P(\gamma, c) = \{x \in P | \gamma(x) < c\}.$$

Let $\sigma, \alpha, \varphi, \psi$ be nonnegative continuous maps on P with σ concave, α, φ convex. Then for positive numbers a, b, c, d, we define the following subset of P

$$\begin{aligned}
&P(\alpha^d) = \{x \in P | \alpha(x) \le d\}, \\
&P(\sigma_b, \alpha^d) = \{x \in P | b \le \sigma(x), \alpha(x) \le d\}, \\
&P(\sigma_b, \varphi^c, \alpha^d) = \{x \in P | b \le \sigma(x), \varphi(x) \le c, \alpha(x) \le d\}, \\
&R(\psi_a, \alpha^d) = \{x \in P | a \le \psi(x), \alpha(x) \le d\}.
\end{aligned}$$

Theorem 2.1 ([17], [18]) *Let P be a cone in a real Banach space E. Let α and γ be increasing, nonnegative, continuous functionals on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$, such that, for some $c > 0$ and $M > 0$, $\gamma(x) \le \theta(x) \le \alpha(x)$ and $\|x\| \le M\gamma(x)$, for all $x \in \overline{P(\gamma, c)}$. Suppose there exists a completely continuous operator $T : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that*

$$\theta(\lambda x) \le \lambda\theta(x), \quad \textit{for} \quad 0 \le \lambda \le 1 \quad \textit{and} \quad x \in \partial P(\theta, b),$$

and

(i) $\gamma(T(x)) < c$, for all $x \in \partial P(\gamma, c)$;

(ii) $\theta(T(x)) > b$, for all $x \in \partial P(\theta, b)$;

(iii) $P(\alpha, a) \ne \emptyset$, and $\alpha(T(x)) < a$, for all $x \in \partial P(\alpha, a)$.

Then T has at least two fixed points x_1, x_2 belonging to $\overline{P(\gamma, c)}$ such that

$$a < \alpha(x_1), \textit{ with } \theta(x_1) < b,$$

and

$$b < \theta(x_2), \textit{ with } \gamma(x_2) < c.$$

Theorem 2.2 ([19]) *Let P be a cone in a real Banach space E. Let α and φ be nonnegative continuous convex functionals on P, σ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \le \lambda\psi(x)$ for $0 \le \lambda \le 1$, such that for some positive numbers M and d,*

$$\sigma(x) \le \psi(x) \textit{ and } \|x\| \le M\alpha(x),$$

for all $x \in P(\alpha^d)$. Suppose $T : P(\alpha^d) \to P(\alpha^d)$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(1) $\{x \in P(\sigma_b, \varphi^c, \alpha^d) | \sigma(x) > b\} \ne \emptyset$ and $\sigma(Tx) > b$ for $x \in P(\sigma_b, \varphi^c, \alpha^d)$;

(2) $\sigma(Tx) > b$ for $x \in P(\sigma_b, \alpha^d)$ with $\varphi(Tx) > c$;

(3) $0 \notin R(\psi_a, \alpha^d)$ and $\psi(Tx) < a$ for $x \in R(\psi_a, \alpha^d)$ with $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in P(\alpha^d)$, such that

$$\alpha(x_i) \le d \textit{ for } i = 1, 2, 3;\ \psi(x_1) < a;\ \psi(x_2) > a \textit{ with } \sigma(x_2) < b;\ \sigma(x_3) > b.$$

Let $C(\mathbb{R}_+) = \{x : \mathbb{R}_+ \to \mathbb{R} | x \text{ is continuous on } \mathbb{R}_+ \text{ and } \sup_{t\in\mathbb{R}_+} \frac{|x(t)|}{1+t} < +\infty\}$. Define $\|x\|_1 = \sup_{t\in\mathbb{R}_+} \frac{|x(t)|}{1+t}$. Then $(C(\mathbb{R}_+), \|\cdot\|_1)$ is a Banach space (refer to [16]).

Let $X = \{U = (u_1, u_2, \cdots, u_n) : u_i \in C(\mathbb{R}_+), i = 1, 2, \cdots, n\}$ with the norm $\|U\| = \sum_{i=1}^{n} \|u_i\|_1$, where $\|u_i\|_1 = \sup_{t\in\mathbb{R}_+} \frac{|u_i(t)|}{1+t}$, and it is easy to prove that $(X, \|\cdot\|)$ is a Banach space.

Let $\delta \in (0, 1)$ be a constant and

$$P = \{U \in X : U(t) \geq \mathbf{0}, t \in \mathbb{R}_+, \sum_{i=1}^{n} \min_{\delta\leq t\leq \frac{1}{\delta}} u_i(t) \geq \delta\|U\|\}.$$

Then P is a cone in X.

The following lemmas 2.3 and 2.4 are proved in [16].

Lemma 2.3 *Assume that* $(H1)$ *holds. Then for any* $y \in L^1(\mathbb{R}_+^n) \cap C(\mathbb{R}_+^n)$*, the boundary value problem*

$$U''(t) + y(t) = \mathbf{0} \tag{2.1}$$

$$U(0) = \mathbf{0}, U'(\infty) = \int_0^\infty g(s)U(s)ds \tag{2.2}$$

has a unique solution $U \in X$*, and*

$$U(t) = \int_0^\infty H(t, s)y(s)ds,$$

where

$$H(t, s) = \begin{pmatrix} H_1(t, s) & 0 & \cdots & 0 \\ 0 & H_2(t, s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_n(t, s) \end{pmatrix},$$

$$H_i(t, s) = G(t, s) + \frac{t\int_0^\infty g_i(r)G(s, r)dr}{1 - \int_0^\infty sg_i(s)ds}, i = 1, 2, \cdots, n.$$

$$G(t, s) = \min\{t, s\}.$$

Lemma 2.4 *Assume that* $(H1)$ *holds. If* $y \in L^1(\mathbb{R}_+^n) \cap C(\mathbb{R}_+^n), \delta \in (0, 1), y \geq \mathbf{0}$*, then the unique solution* U *of the boundary value problem* $(2.1) - (2.2)$ *satisfies* $U(t) \geq \mathbf{0}$ *for* $t \in \mathbb{R}_+$ *and* $\sum_{i=1}^{n} \min_{\delta\leq t\leq \frac{1}{\delta}} u_i(t) \geq \delta\|U\|$.

From [16], we know $F(t, U) \in L^1(\mathbb{R}_+ \times \mathbb{R}_+^n) \cap C(\mathbb{R}_+ \times \mathbb{R}_+^n)$. Hence, the solution of the boundary value problem(1.1) is equivalent to

$$U(t) = \int_0^\infty H(t, s)F(s, U)ds.$$

Define $T_i : P \to C(\mathbb{R}_+)$ by

$$(T_iU)(t) = \int_0^\infty H_i(t, s)f_i(s, u_1(s), \cdots, u_n(s))ds, i = 1, 2, \cdots, n.$$

Let

$$(TU)(t) = ((T_1U)(t), \cdots, T_n(U)(t))^T.$$

Then $T : P \to C(\mathbb{R}_+^n)$, and

$$(TU)(t) = \int_0^\infty H(t, s)F(s, U)ds.$$

It is easy to get the following lemma.

Lemma 2.5 *Assume that $(H1)$ and $(H2)$ hold. Then $T : P \to X$ is completely continuous.*

Lemma 2.6 *Suppose $(H1)$ and $(H2)$ hold. If $U = (u_1(t), \cdots, u_n(t)) \geq \mathbf{0}$ is a solution of boundary value problem (1.1), then $u_i (i = 1, 2, \cdots, n)$ are increasing.*

Proof. For $i = 1, 2, \cdots, n$ and $f_i \geq 0$, we can get $U''(t) \leq \mathbf{0}$. So $U'(t)$ is decreasing.

Noticing the boundary condition $U'(\infty) = \int_0^\infty g(s)U(s)ds$, we can obtain $U'(\infty) \geq \mathbf{0}$. So $U'(t) \geq 0, t \in \mathbb{R}_+$.

This proves the lemma.

3 Existence of two positive solutions of (1.1)

We define the nonnegative, increasing continuous functionals γ, α and $\theta : P \to \mathbb{R}_+$ by

$$\gamma(U) = \sum_{i=1}^{n} \min_{\delta \leq t \leq \frac{1}{\delta}} \frac{u_i(t)}{1+t},$$

$$\alpha(U) = \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+} \frac{u_i(t)}{1+t},$$

$$\theta(U) = \frac{1}{2}\Big(\sum_{i=1}^{n} \frac{u_i(\delta)}{1+\delta} + \sum_{i=1}^{n} \frac{u_i(\frac{1}{\delta})}{1+\frac{1}{\delta}} \Big)$$

For every $U \in P$,

$$\gamma(U) \leq \theta(U) \leq \alpha(U).$$

It follows from Lemma 2.4, for each $U \in P$, we have $\gamma(U) \geq \frac{\delta^2}{1+\delta}\|U\|$, so

$$\|U\| \leq \frac{1+\delta}{\delta^2}\gamma(U), \ U \in P.$$

We also notice that $\theta(\lambda U) = \lambda\theta(U)$, for $\lambda \geq 0$ and all $U \in P$.

For convenience, we denote

$$L_1 = \sum_{i=1}^{n} \min_{\delta \leq t \leq \frac{1}{\delta}} \int_0^\infty \frac{H_i(t,s)}{1+t} a_i(s)ds,$$

$$L_2 = \min\Big\{ \sum_{i=1}^{n} \int_\delta^{\frac{1}{\delta}} \frac{H_i(\delta,s)}{1+\delta}ds, \sum_{i=1}^{n} \int_\delta^{\frac{1}{\delta}} \frac{H_i(\frac{1}{\delta},s)}{1+\frac{1}{\delta}}ds \Big\},$$

$$L_3 = \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+} \int_0^\infty \frac{H_i(t,s)}{1+t} a_i(s)ds.$$

We will also use the following hypothesis:

(H3) There exist nonnegative functions $a_i \in L^1(\mathbb{R}_+)$, $a_i(t) \not\equiv 0$ on $\mathbb{R}_+$, and continuous functions $h_i \in C[\mathbb{R}_+^n, \mathbb{R}_+]$, $i = 1, 2, \cdots, n$, such that

$$f_i(t, u_1, u_2, \cdots, u_n) \leq a_i(t)h_i(u_1, u_2, \cdots, u_n), \quad t, u_1, u_2, \cdots, u_n \in \mathbb{R}_+,$$

$$\text{with} \int_0^\infty s a_i(s)ds < +\infty.$$

Obviously, if (H1) and (H3) hold, then $\int_0^\infty \frac{H_i(t,s)}{1+t}(1+s)a_i(s)ds < +\infty$ for $t \in \mathbb{R}_+$ and $i = 1, 2, \cdots, n$. We denote

$$h(u_1, u_2, \cdots, u_n) = \begin{pmatrix} h_1(u_1, u_2, \cdots, u_n) \\ h_2(u_1, u_2, \cdots, u_n) \\ \cdots \\ h_n(u_1, u_2, \cdots, u_n) \end{pmatrix}.$$

Theorem 3.1 *Assume that (H1), (H2) and (H3) hold, suppose that there exist positive constants a, b, c such that $0 < a < \frac{\delta^2 b}{1+\delta} < \frac{\delta^2 L_2 c}{(1+\delta)L_1}$, and*

(H4) $h((1+t)u_1, \cdots, (1+t)u_n) < (\frac{a}{L_3})_{n\times 1}$, for $0 \le \sum_{i=1}^n u_i \le a, t \in \mathbb{R}_+$;

(H5) $F(t, (1+t)u_1, \cdots, (1+t)u_n) > (\frac{b}{L_2})_{n\times 1}$, for $\frac{\delta^2 b}{1+\delta} \le \sum_{i=1}^n u_i \le \frac{(1+\delta)b}{\delta^2}, t \in [\delta, \frac{1}{\delta}]$;

(H6) $h((1+t)u_1, \cdots, (1+t)u_n) < (\frac{c}{L_1})_{n\times 1}$, for $0 \le \sum_{i=1}^n u_i \le \frac{(1+\delta)c}{\delta^2}, t \in \mathbb{R}_+$.

Then, the boundary value problem (1.1) has at least two monotone increasing positive solutions U^* and U^{**} such that

$$a < \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \frac{u_i^*(t)}{1+t}, \ \ \textit{with} \ \ \frac{1}{2}\Big(\sum_{i=1}^n \frac{u_i^*(\delta)}{1+\delta} + \sum_{i=1}^n \frac{u_i^*(\frac{1}{\delta})}{1+\frac{1}{\delta}}\Big) < b,$$

and

$$b < \frac{1}{2}\Big(\sum_{i=1}^n \frac{u_i^{**}(\delta)}{1+\delta} + \sum_{i=1}^n \frac{u_i^{**}(\frac{1}{\delta})}{1+\frac{1}{\delta}}\Big), \ \ \textit{with} \ \ \sum_{i=1}^n \min_{\delta\le t\le\frac{1}{\delta}} \frac{u_i^{**}(t)}{1+t} < c.$$

Proof. For $U \in P$, it follows from Lemma 2.4

$$TU(t) \ge \mathbf{0},$$

and

$$\sum_{i=1}^n \min_{\delta\le t\le\frac{1}{\delta}} (T_iU)(t) \ge \delta\|TU\|.$$

By Lemma 2.5, we know $T : P \to P$ is completely continuous.

We now show that the conditions of Theorem 2.1 are satisfied.

(i) For every $U \in \partial P(\gamma, c)$, $\gamma(U) = \sum_{i=1}^n \min_{\delta\le t\le\frac{1}{\delta}} \frac{u_i(t)}{1+t} = c$, so

$$\|U\| \le \frac{1+\delta}{\delta^2}\gamma(U) = \frac{1+\delta}{\delta^2}c,$$

therefore

$$0 \le \sum_{i=1}^n \frac{u_i(t)}{1+t} \le \|U\| \le \frac{1+\delta}{\delta^2}c.$$

By (H6) of Theorem 3.1,

$$h\big(u_1(t), \cdots, u_n(t)\big) = h\Big((1+t)\frac{u_1(t)}{1+t}, \cdots, (1+t)\frac{u_n(t)}{1+t}\Big) < \Big(\frac{c}{L_1}\Big)_{n\times 1}.$$

We have

$$\begin{aligned}\gamma(TU) &= \sum_{i=1}^{n} \min_{\delta\le t\le \frac{1}{\delta}} \frac{(T_iU)(t)}{1+t} \\ &= \sum_{i=1}^{n} \min_{\delta\le t\le \frac{1}{\delta}} \int_0^\infty \frac{H_i(t,s)}{1+t} f_i(s,u_1(s),\cdots,u_n(s))ds \\ &\le \sum_{i=1}^{n} \min_{\delta\le t\le \frac{1}{\delta}} \int_0^\infty \frac{H_i(t,s)}{1+t} a_i(s)h_i(u_1(s),\cdots,u_n(s))ds \\ &< \frac{c}{L_1}\sum_{i=1}^{n} \min_{\delta\le t\le \frac{1}{\delta}} \int_0^\infty \frac{H_i(t,s)}{1+t} a_i(s)ds \\ &= c.\end{aligned}$$

Therefore, condition (i) of the Theorem 2.1 is satisfied.

(ii) For $U \in \partial P(\theta, b)$, $\theta(U) = \frac{1}{2}\Big(\sum_{i=1}^{n}\frac{u_i(\delta)}{1+\delta} + \sum_{i=1}^{n}\frac{u_i(\frac{1}{\delta})}{1+\frac{1}{\delta}}\Big) = b$. We can obtain that $\|U\| \ge \theta(U) = b$.

Noting that $\|U\| \le \frac{1+\delta}{\delta^2}\gamma(U) \le \frac{1+\delta}{\delta^2}\theta(U) = \frac{1+\delta}{\delta^2}b$, for $t \in [\delta, \frac{1}{\delta}]$, we have

$$\frac{\delta^2 b}{1+\delta} \le \frac{\delta^2}{1+\delta}\|U\| \le \sum_{i=1}^{n}\frac{u_i(t)}{1+t} \le \|U\| \le \frac{1+\delta}{\delta^2}b$$

and now from (H5) we get

$$F(t,u_1(t),\cdots,u_n(t)) = F(t,(1+t)\frac{u_1(t)}{1+t},\cdots,(1+t)\frac{u_n(t)}{1+t}) > \Big(\frac{b}{L_2}\Big)_{n\times 1}.$$

So

$$\begin{aligned}\theta(TU) &= \frac{1}{2}\Big(\sum_{i=1}^{n}\frac{(T_iU)(\delta)}{1+\delta} + \sum_{i=1}^{n}\frac{(T_iU)(\frac{1}{\delta})}{1+\frac{1}{\delta}}\Big) \\ &= \frac{1}{2}\Big(\sum_{i=1}^{n}\int_0^\infty \frac{H_i(\delta,s)}{1+\delta} f_i(s,u_1(s),\cdots,u_n(s))ds \\ &\quad + \sum_{i=1}^{n}\int_0^\infty \frac{H_i(\frac{1}{\delta},s)}{1+\frac{1}{\delta}} f_i(s,u_1(s),\cdots,u_n(s))ds\Big) \\ &\ge \frac{1}{2}\Big(\sum_{i=1}^{n}\int_\delta^{\frac{1}{\delta}} \frac{H_i(\delta,s)}{1+\delta} f_i(s,u_1(s),\cdots,u_n(s))ds \\ &\quad + \sum_{i=1}^{n}\int_\delta^{\frac{1}{\delta}} \frac{H_i(\frac{1}{\delta},s)}{1+\frac{1}{\delta}} f_i(s,u_1(s),\cdots,u_n(s))ds\Big) \\ &> \frac{1}{2}\Big(\sum_{i=1}^{n}\int_\delta^{\frac{1}{\delta}} \frac{H_i(\delta,s)}{1+\delta}ds + \sum_{i=1}^{n}\int_\delta^{\frac{1}{\delta}} \frac{H_i(\frac{1}{\delta},s)}{1+\frac{1}{\delta}}ds\Big)\frac{b}{L_2} \\ &\ge b.\end{aligned}$$

Hence, condition (ii) of the Theorem 2.1 holds.

(iii) Finally, we verify the condition (iii) of the Theorem 2.1 satisfied.

We choose $U_0(t) = (1+t)\begin{pmatrix} \frac{a}{2n} \\ \vdots \\ \frac{a}{2n} \end{pmatrix}$, $t \in \mathbb{R}_+$, then $\alpha(U_0) = \frac{a}{2}$, and $U_0 \in P$, so $P(\alpha, a) \neq \emptyset$.

For $U \in \partial P(\alpha, a)$, $\alpha(U) = \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \frac{u_i(t)}{1+t} = a$. It implies

$$0 \leq \sum_{i=1}^n \frac{u_i(t)}{1+t} \leq a, t \in \mathbb{R}_+.$$

From (H4), we have

$$h(u_1(t), \cdots, u_n(t)) < \left(\frac{a}{L_3}\right)_{n\times 1}.$$

Therefore

$$\begin{aligned}
\alpha(TU) &= \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \frac{(T_iU)(t)}{1+t} \\
&= \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \int_0^\infty \frac{H_i(t,s)}{1+t} f_i(s, u_1(s), \cdots, u_n(s))ds \\
&\leq \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \int_0^\infty \frac{H_i(t,s)}{1+t} a_i(s)h_i(u_1(s), \cdots, u_n(s))ds \\
&< \frac{a}{L_3} \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \int_0^\infty \frac{H_i(t,s)}{1+t} a_i(s)ds \leq a.
\end{aligned}$$

Thus, all of the conditions of the Theorem 2.1 are satisfied, from Lemma 2.6, we complete the proof of the Theorem 3.1.

4 Existence of three positive solutions of (1.1)

We define the nonnegative continuous functionals φ, σ, ψ on P by

$$\begin{aligned}
\varphi(U) &= \sum_{i=1}^n \max_{\delta\leq t\leq \frac{1}{\delta}} \frac{u_i(t)}{1+t}, \\
\sigma(U) &= \frac{1}{1+\delta} \sum_{i=1}^n \min_{\delta\leq t\leq \frac{1}{\delta}} u_i(t), \\
\psi(U) &= (1+\delta) \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \frac{u_i(t)}{(1+t)^2}.
\end{aligned}$$

The definitions of the nonnegative continuous functional α and L_3 are the same as that in Section 3.

For every $U \in P$ and $0 \leq \lambda \leq 1$, $\psi(\lambda U) = \lambda\psi(U)$ and

$$\frac{\min_{\delta\leq t\leq \frac{1}{\delta}} u_i(t)}{1+\delta} \leq (1+\delta)\frac{u_i(\delta)}{(1+\delta)^2} \leq (1+\delta) \sup_{t\in\mathbb{R}_+} \frac{u_i(t)}{(1+t)^2}.$$

So

$$\sigma(U) \leq \psi(U), \|U\| = \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \frac{u_i(t)}{1+t} = \alpha(U). \tag{4.1}$$

Denote

$$L_4=\sum_{i=1}^{n}\max_{\delta\le t\le\frac{1}{\delta}}\int_{\delta}^{\frac{1}{\delta}}\frac{H_i(t,s)}{1+t}ds,$$
$$L_5=\sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\int_{0}^{\infty}\frac{H_i(t,s)}{(1+t)^2}a_i(s)ds.$$

Lemma 4.1 *For all $U\in P$, we have $\sigma(U)\ge\frac{\delta}{1+\delta}\varphi(U)$.*

Proof.

$$\sigma(U)=\frac{1}{1+\delta}\sum_{i=1}^{n}\min_{\delta\le t\le\frac{1}{\delta}}u_i(t)\ge\frac{\delta}{1+\delta}\|U\|\ge\frac{\delta}{1+\delta}\varphi(U).$$

Theorem 4.2 *Assume that $(H1)$, $(H2)$ and $(H3)$hold, suppose that there exist positive constants a,b,d such that $0<a<b<\frac{\delta d}{2+\delta}$, and*

(H7) $h((1+t)u_1,\cdots,(1+t)u_n)<(\frac{d}{L_3})_{n\times1}$, *for* $0\le\sum_{i=1}^{n}u_i\le d, t\in\mathbb{R}_+$*;*

(H8) $F(t,(1+t)u_1,\cdots,(1+t)u_n)>(\frac{b(1+\delta)}{\delta L_4})_{n\times1}$, *for* $\delta b\le\sum_{i=1}^{n}u_i\le\frac{(2+\delta)b}{\delta}, t\in[\delta,\frac{1}{\delta}]$*;*

(H9) $h((1+t)u_1,\cdots,(1+t)u_n)<(\frac{a}{L_5(1+\delta)})_{n\times1}$, *for* $0\le\sum_{i=1}^{n}u_i\le\frac{1+\delta}{\delta}a, t\in\mathbb{R}_+$.

Then, the boundary value problem (1.1) has at least three monotone increasing positive solutions $U^{\langle1\rangle}$, $U^{\langle2\rangle}$ and $U^{\langle3\rangle}$ such that

$$\sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\frac{u_i^{\langle j\rangle}(t)}{1+t}\le d, j=1,2,3;$$

and

$$(1+\delta)\sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\frac{u_i^{\langle1\rangle}(t)}{(1+t)^2}<a;$$

$$(1+\delta)\sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\frac{u_i^{\langle2\rangle}(t)}{(1+t)^2}>a\quad\text{with}\quad\frac{1}{1+\delta}\sum_{i=1}^{n}\min_{\delta\le t\le\frac{1}{\delta}}u_i^{\langle2\rangle}(t)<b;$$

$$\frac{1}{1+\delta}\sum_{i=1}^{n}\min_{\delta\le t\le\frac{1}{\delta}}u_i^{\langle3\rangle}(t)>b.$$

Proof. For $U\in P(\alpha^d)$, we have

$$\alpha(U)=\sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\frac{u_i(t)}{1+t}\le d,$$

so, we can get

$$0\le\sum_{i=1}^{n}\frac{u_i(t)}{1+t}\le d.$$

Notice (H7), and we can obtain $\alpha(TU)\le d$ whose proof is similar to that in Theorem 3.1.

From Lemma 2.5, $T:P(\alpha^d)\to P(\alpha^d)$ is completely continuous.

Next, we show that conditions (1)-(3) of the Theorem 2.2 hold.

(1) Take $c = \frac{2+\delta}{\delta}b$. We choose $U(t) = (\frac{1+\delta}{n}(1+t)b)_{n\times 1}, t \in \mathbb{R}_+$, then $\sigma(U) = (1+\delta)b > b, \varphi(U) = (1+\delta)b < c, \alpha(U) = (1+\delta)b < d$. So $U \in P(\sigma_b, \varphi^c, \alpha^d)$ with $\sigma(U) > b$. Hence, $\{U \in P(\sigma_b, \varphi^c, \alpha^d) | \sigma(U) > b\} \neq \emptyset$.

For $U \in P(\sigma_b, \varphi^c, \alpha^d)$, we have

$$\delta b \leq \frac{1}{1+\frac{1}{\delta}} \sum_{i=1}^{n} \min_{\delta \leq t \leq \frac{1}{\delta}} u_i(t) \leq \sum_{i=1}^{n} \frac{u_i(t)}{1+\frac{1}{\delta}} \leq \sum_{i=1}^{n} \frac{u_i(t)}{1+t} \leq \frac{2+\delta}{\delta} b, t \in [\delta, \frac{1}{\delta}].$$

From (H8) and Lemma 4.1, we can get

$$\begin{aligned}
\sigma(TU) &\geq \frac{\delta}{1+\delta}\varphi(TU) = \frac{\delta}{1+\delta} \sum_{i=1}^{n} \max_{\delta \leq t \leq \frac{1}{\delta}} \frac{T_i U(t)}{1+t} \\
&= \frac{\delta}{1+\delta} \sum_{i=1}^{n} \max_{\delta \leq t \leq \frac{1}{\delta}} \int_0^{\infty} \frac{H_i(t,s)}{1+t} f_i(s, u_1(s), \cdots, u_n(s)) ds \\
&\geq \frac{\delta}{1+\delta} \sum_{i=1}^{n} \max_{\delta \leq t \leq \frac{1}{\delta}} \int_{\delta}^{\frac{1}{\delta}} \frac{H_i(t,s)}{1+t} f_i(s, u_1(s), \cdots, u_n(s)) ds \\
&> \frac{\delta}{1+\delta} \frac{(1+\delta)b}{\delta L_4} \sum_{i=1}^{n} \max_{\delta \leq t \leq \frac{1}{\delta}} \int_{\delta}^{\frac{1}{\delta}} \frac{H_i(t,s)}{1+t} ds \\
&\geq b.
\end{aligned}$$

Therefore, $\sigma(TU) > b$, for $U \in P(\sigma_b, \varphi^c, \alpha^d)$.

(2) For $U \in P(\sigma_b, \alpha^d)$ with $\varphi(TU) > c$,

$$\sigma(TU) \geq \frac{\delta}{1+\delta}\varphi(TU) > \frac{\delta}{1+\delta} c > b.$$

Thus, $\sigma(TU) > b$, for $U \in P(\sigma_b, \alpha^d)$ with $\varphi(TU) > c$.

(3) It is clear that $0 \notin R(\psi_a, \alpha^d)$. For $U \in R(\psi_a, \alpha^d)$ with $\psi(U) = a$, from (4.1) we have $\frac{\delta}{1+\delta}\|U\| \leq \sigma(U) \leq \psi(U) = a$. Hence,

$$0 \leq \sum_{i=1}^{n} \frac{u_i(t)}{1+t} \leq \frac{1+\delta}{\delta} a, t \in \mathbb{R}_+.$$

From (H9),

$$\begin{aligned}
\psi(TU) &= (1+\delta) \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+} \frac{T_i U(t)}{(1+t)^2} \\
&= (1+\delta) \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+} \int_0^{\infty} \frac{H_i(t,s)}{(1+t)^2} f_i(s, u_1(s), \cdots, u_n(s)) ds \\
&\leq (1+\delta) \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+} \int_0^{\infty} \frac{H_i(t,s)}{(1+t)^2} a_i(s) h_i(u_1(s), \cdots, u_n(s)) ds \\
&< (1+\delta) \frac{a}{L_5(1+\delta)} \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+} \int_0^{\infty} \frac{H_i(t,s)}{(1+t)^2} a_i(s) ds \\
&\leq a.
\end{aligned}$$

Hence, from Theorem 2.2 and Lemma 2.6, the boundary value problem (1.1) has at least three monotone increasing positive solutions $U^{\langle 1\rangle}$, $U^{\langle 2\rangle}$ and $U^{\langle 3\rangle}$ such that

$$\sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\frac{u_i^{\langle j\rangle}(t)}{1+t}\le d, j=1,2,3;$$

and

$$(1+\delta)\sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\frac{u_i^{\langle 1\rangle}(t)}{(1+t)^2}<a;$$

$$(1+\delta)\sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\frac{u_i^{\langle 2\rangle}(t)}{(1+t)^2}>a \text{ with } \frac{1}{1+\delta}\sum_{i=1}^{n}\min_{\delta\le t\le\frac{1}{\delta}}u_i^{\langle 2\rangle}(t)<b;$$

$$\frac{1}{1+\delta}\sum_{i=1}^{n}\min_{\delta\le t\le\frac{1}{\delta}}u_i^{\langle 3\rangle}(t)>b.$$

Remark 4.1 It follows F$\not\equiv 0$ from the conditions (H8) of the theorem 4.2. As a result, the solutions of the boundary value problems cannot be zero.

5 Illustration

We give an example to illustrate our results.

Example We consider the boundary value problem

$$\begin{aligned} U''(t)+F(t,U)&=\mathbf{0},\\ U(0)&=\mathbf{0},\\ U'(\infty)&=\int_0^\infty g(s)U(s)\mathrm{d}s, \end{aligned}$$

where

$$U=\begin{pmatrix}u_1\\u_2\end{pmatrix},\quad F(t,U)=\begin{pmatrix}f_1(t,u_1,u_2)\\f_2(t,u_1,u_2)\end{pmatrix},\quad g(s)=\begin{pmatrix}e^{-2s}&0\\0&e^{-3s}\end{pmatrix},$$

$$f_1(t,u_1,u_2)=e^{-t}\begin{cases}\frac{4}{5},\ 0\le u_1+u_2<\frac{1}{2},\\ \frac{1}{(u_1+u_2)^2+1},\ \frac{1}{2}\le u_1+u_2<\frac{2\sqrt{3}}{3},\\ \frac{3}{7}+500\big(\frac{3}{4}-\frac{1}{(u_1+u_2)^2}\big)^2,\ \frac{2\sqrt{3}}{3}\le u_1+u_2<4,\\ 236\frac{339}{448}-\frac{200}{u_1+u_2},\ u_1+u_2\ge 4,\end{cases}$$

$$f_2(t,u_1,u_2)=e^{-2t}\begin{cases}\frac{4}{5}\big(1-\frac{\frac{1}{4}-(u_1+u_2)}{(u_1+u_2)^2+1}\big),\ 0\le u_1+u_2<\frac{1}{4},\\ \frac{1}{u_1+u_2+1},\ \frac{1}{4}\le u_1+u_2<\frac{4}{3},\\ \frac{3}{7}+680\big(\frac{9}{16}-\frac{1}{(u_1+u_2)^2}\big)^{\frac{1}{2}},\ \frac{4}{3}\le u_1+u_2<2,\\ \frac{3}{7}+170\sqrt{5},\ u_1+u_2\ge 2.\end{cases}$$

Let

$$a_1(t)=e^{-t}, a_2(t)=e^{-2t}.$$

For the constants r_1, $r_2>0$, we take $\phi_{r_1,r_2}(t)=\begin{pmatrix}600e^{-t}\\600e^{-2t}\end{pmatrix}$.

Let $h(u_1,u_2) = \begin{pmatrix} h_1(u_1,u_2) \\ h_2(u_1,u_2) \end{pmatrix}$, where

$$h_1(u_1,u_2) = h_2(u_1,u_2) = \begin{cases} \frac{4}{5}, & 0 \le u_1 + u_2 < \frac{2\sqrt{3}}{3}, \\ \frac{4}{5} + 600\sqrt{3} - \frac{1200}{u_1+u_2}, & \frac{2\sqrt{3}}{3} \le u_1 + u_2 < 2, \\ 600\sqrt{3} - 599\frac{1}{5}, & u_1 + u_2 \ge 2. \end{cases}$$

We take $\delta = \frac{1}{2}$. By calculating, we can get $L_1 \approx 0.443194$, $L_2 \approx 1.20668$, $L_3 \approx 0.571763$, $L_4 \approx 1.77563$, $L_5 \approx 0.30729$.

Let $a = 1$, $b = 10$, $c = 220$. Then $\frac{a}{L_3} \approx 1.74898$, $\frac{b}{L_2} \approx 8.28719$, $\frac{c}{L_1} \approx 496.397$. It is easy to verify that the conditions in Theorem 3.1 are all satisfied. By Theorem 3.1, the boundary value problem mentioned in the example above has at least two monotone increasing positive solutions.

On the other hand, let $a = \frac{2\sqrt{3}}{9}$, $b = 4$, $d = 300$. Then $\frac{d}{L_3} \approx 524.693$, $\frac{b(1+\delta)}{\delta L_4} \approx 6.75816$, $\frac{a}{(1+\delta)L_5} \approx 0.835043$. Then the conditions in Theorem 4.2 are all satisfied. By Theorem 4.2, the boundary value problem mentioned in the example above has at least three monotone increasing positive solutions.

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