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Oscillation of second order neutral dynamic equations with distributed delay

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Abstract. In this paper, we establish new oscillation criteria for second order neutral dynamic equations with distributed delay by employing the generalized Riccati transformation. The obtained theorems essentially improve the oscillation results in the literature. And two examples are provided to illustrate to the versatility of our main results.

Keywords: oscillation, neutral dynamic equation, time scale, distributed delay.

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1 Introduction

In this paper, we are concerned with the oscillatory behavior of the following second order neutral dynamic equations with distributed delay

$$\left(r(t)(Z^{\Delta}(t))^{\alpha}\right)^{\Delta}+\int_c^d f(t,x(\theta(t,\xi)))\Delta\xi=0 \tag{1.1}$$

on time scales $[t_0,\infty)_{\mathbb{T}}$, where $\mathbb{T}$ is a time scale with $\sup\mathbb{T}=\infty$; $Z(t)=x(t)+p(t)x(\tau(t))$ and α is a quotient of odd positive integers.

Since we are interested in oscillation of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$ and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale we define the forward and backward jump operators by $\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}$ and $\rho(t):=\sup\{s\in\mathbb{T},s<t\}$, where $\inf\emptyset:=\sup\mathbb{T}$ and $\sup\emptyset:=\inf\mathbb{T}$; here $\emptyset$ denotes the empty set. A point $t\in\mathbb{T}$ and $t>\inf\mathbb{T}$, is said to be left-dense if $\rho(t)=t$, right-dense if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$. The graininess function μ for the time scale $\mathbb{T}$ is defined by $\mu(t):=\sigma(t)-t$, and for any function $f:\mathbb{T}\to\mathbb{R}$, the notation $f^{\sigma}(t)$

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denotes $f(\sigma(t))$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ which are differentiable and whose derivative is rd-continuous function is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$. For more details, see the monograph [5].

Throughout this paper, we always assume that

(A1) $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ with $\int_{t_0}^{\infty} r^{-1/\alpha}(t)\Delta t = \infty$;

(A2) $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ with $0 \le p(t) < 1$;

(A3) $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $\tau(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \infty$;

(A4) $c, d \in [t_0, \infty)_{\mathbb{T}}$, $\theta(t, \xi) \in C_{rd}([t_0, \infty)_{\mathbb{T}} \times [c, d]_{\mathbb{T}}, \mathbb{T})$, $[c, d]_{\mathbb{T}} = \{\xi \in \mathbb{T} : c \le \xi \le d\}$, $\theta(t, c) \le \theta(t, \xi)$ for $(t, \xi) \in [t_0, \infty)_{\mathbb{T}} \times [c, d]_{\mathbb{T}}$, and $\lim_{t\to\infty} \theta(t, c) = \infty$;

(A5) $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(t, u) > 0$ for all $u \ne 0$ and there exists a function $q(t, \xi) \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, +\infty))$ such that $|f(t, u)| \ge q(t, \xi)|u^{\alpha}|$.

By a solution of Eq. (1.1), we mean a nontrivial real-valued function $x \in C^1_{rd}([T_x, \infty), \mathbb{R})$, $T_x \ge t_0$ which has the property that $r(t)(Z^{\Delta}(t))^{\alpha} \in C^1_{rd}([T_x, \infty), \mathbb{R})$ and satisfies Eq. (1.1) on $[T_x, \infty)$. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution $x(t)$ of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

In recent years, there has been an increasing interest in studying oscillatory behavior of solutions to various classes of dynamic equations on time scales. In particular, oscillation of second order neutral dynamic equations attracted significant attention of researchers due to the fact that such equations arise in many real life problems; see, for example, [1, 2, 7, 8, 10, 11, 13–24] and the references cited therein. Chen [10], Şahiner [21], Saker et al. [18], Saker and O'Regan [20] considered the second-order nonlinear neutral dynamic equation with variable delays

$$\left(r(t)[(x(t) + p(t)x(\tau(t)))^{\Delta}]^{\gamma}\right)^{\Delta} + f(t, x(\delta(t))) = 0, \tag{1.2}$$

where $0 \le p(t) < 1$. Han et al. [13] and Saker et al. [16] examined the oscillation of Eq. (1.2) when $\gamma = 1$. In particular, Han et al. [13] investigated the case where $\gamma = 1$ and $p(t) \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, p_0])$, where p_0 is a constant.

Regarding the oscillation of dynamic equations with *distributed delay*, Candan [7] studied the oscillation of the second order neutral delay dynamic equation

$$\left(r(t)((x(t) + p(t)x(\tau(t)))^{\Delta})^{\gamma}\right)^{\Delta} + \int_c^d f(t, x(\theta(t, \xi)))\Delta\xi = 0,$$

where $f(t, u) \ge q(t)|u^{\beta}|$; $\gamma > 0$ and $\beta > 0$ are ratios of odd positive integers. He gave some oscillation results when $\theta(t, d) > t$ and $\theta(t, d) \le t$, respectively. For more related works on the oscillation of second order neutral dynamic equations with distributed delay, we refer the readers to [9, 11, 15, 22, 24].

In this paper, inspired by the works [6, 12, 24], we will study the oscillation of (1.1). Here we will employ the generalized Riccati transformation technique to establish new oscillation criteria for (1.1) when $\delta(t) \le \sigma(t)$ and $\delta(t) > \sigma(t)$, respectively, the obtained results improve the main results in [7, 15, 22]. Finally, we give two examples to illustrate the main results.

2 Main results

In what follows, we use the following notation for the convenience of the reader.

$$\delta(t) = \theta(t,c); \qquad R(t) = \int_c^d q(t,\xi)(1-p(\theta(t,\xi)))^\alpha \Delta\xi,$$

and

$$\eta_1(t,u) = \frac{r(\delta(t),u)}{r(\sigma(t),u)}, \qquad \eta_2(t,u) = \frac{r(t,u)}{r(\sigma(t),u)},$$

where

$$r(t,u) = \int_u^t \frac{1}{r^{\frac{1}{\alpha}}(s)}\Delta s, \qquad t > u \geq t_0.$$

Further, for any given functions $\eta(t), a(t) \in C^1_{rd}\big([t_0,\infty)_{\mathbb{T}}, \mathbb{R}\big)$ with $\eta(t) > 0$ and $a(t) > -1/[r(t)r^\alpha(t,T)]$, we let

$$\begin{aligned}
\phi_+(t) &:= \max\{0, \phi(t)\},\\
\varphi(t) &:= \eta^\sigma(t)\big[R(t)\eta_1^\alpha(t,T) + r(t)\eta_2^{1+\alpha}(t,T)a^{1+\frac{1}{\alpha}}(t) - (r(t)a(t))^\Delta\big],\\
\varphi_1(t) &:= \frac{\eta^\Delta(t)}{\eta(t)} + (\alpha+1)a^{\frac{1}{\alpha}}(t)\frac{\eta^\sigma(t)\eta_2^{1+\alpha}(t)}{\eta(t)},\\
\varphi_2(t) &:= \frac{\alpha\,\eta^\sigma(t)\eta_2^{1+\alpha}(t,T)}{r^{\frac{1}{\alpha}}(t)\eta^{1+\frac{1}{\alpha}}(t)}.
\end{aligned}$$

2.1 Two lemmas

In order to prove the main results, we need the following two lemmas.

Lemma 2.1. *Let $x(t)$ be an eventually positive solution of Eq. (1.1). Then there exists some $T > t_0$ large enough such that for all $t > T$,*

$$Z(t) > 0, \quad Z^\Delta(t) > 0, \quad Z(t) > r^{\frac{1}{\alpha}}(t)Z^\Delta(t)\,r(t,T), \quad Z(t) \geq \eta_2(t,T)Z(\sigma(t)). \tag{2.1}$$

Proof. Without loss of generality, we assume that there exists a $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$, $x(\tau(t))$, $x(\theta(t,\xi)) > 0$ on $[T,\infty)_{\mathbb{T}}$, then $Z(t) \geq x(t) > 0$. It follows from (1.1) and $(A5)$ that

$$\big(r(t)(Z^\Delta(t))^\alpha\big)^\Delta \leq -\int_c^d q(t,\xi)x^\alpha(\theta(t,\xi))\Delta\xi \leq 0.$$

Hence, $r(t)(Z^\Delta(t))^\alpha$ is decreasing on $[T,\infty)_{\mathbb{T}}$. We now claim that $Z^\Delta(t) > 0$ eventually on $t \in [T,\infty)_{\mathbb{T}}$. If not, then there exists a $t_1 \in [T,\infty)_{\mathbb{T}}$ such that $Z^\Delta(t_1) < 0$. Then

$$r(t)(Z^\Delta(t))^\alpha \leq r(t_1)(Z^\Delta(t_1))^\alpha := -c^* < 0, \qquad t \geq t_1,$$

i.e.,

$$Z^\Delta(t) \leq -\left(\frac{c^*}{r(t)}\right)^{\frac{1}{\alpha}}. \tag{2.2}$$

Integrating (2.2) from t_1 to t, we find from (A1) that

$$Z(t) \leq Z(t_1) - (c^*)^{\frac{1}{\alpha}}\int_{t_1}^t \frac{1}{r^{\frac{1}{\alpha}}(s)}\Delta s \to -\infty \quad \text{as } t \to \infty,$$

which implies that $Z(t)$ is eventually negative. This contradicts $Z(t) > 0$. Thus, $Z^{\Delta}(t) > 0$ on $[T, \infty)_{\mathbb{T}}$.

Since $r(t)(Z^{\Delta}(t))^{\alpha}$ is decreasing on $[T, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} Z(t) > Z(t) - Z(T) &= \int_T^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(r(s)(Z^{\Delta}(s))^{\alpha}\right)^{\frac{1}{\alpha}} \Delta s \\ &\geq \left(r(t)(Z^{\Delta}(t))^{\alpha}\right)^{\frac{1}{\alpha}} \int_T^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \Delta s \\ &= (r(t))^{1/\alpha} Z^{\Delta}(t)\, r(t, T). \end{aligned}$$

Thus, $\left(Z(t)/r(t,T)\right)^{\Delta} \leq 0$, which implies that

$$\frac{Z(t)}{r(t,T)} \geq \frac{Z^{\sigma}(t)}{r(\sigma(t),T)}.$$

This completes the proof. □

For the positive solution $x(t)$ of Eq. (1.1), it follows from the definition of $Z(t)$ and Lemma 2.1 that, for $t \geq T$,

$$x(t) = Z(t) - p(t)x(\tau(t)) \geq Z(t) - p(t)Z(\tau(t)) \geq (1 - p(t))Z(t),$$

consequently,

$$x^{\alpha}(\theta(t,\xi)) \geq (1 - p(\theta(t,\xi)))^{\alpha} Z^{\alpha}(\theta(t,\xi)). \tag{2.3}$$

Multiplying (2.3) by $q(t,\xi)$ and integrating both sides from c to d, we have

$$\int_c^d q(t,\xi) x^{\alpha}(\theta(t,\xi)) \Delta\xi \geq \int_c^d q(t,\xi)(1 - p(\theta(t,\xi)))^{\alpha} Z^{\alpha}(\theta(t,\xi)) \Delta\xi.$$

It follows from (1.1) that

$$\left(r(t)(Z^{\Delta}(t))^{\alpha}\right)^{\Delta} \leq -\int_c^d q(t,\xi)(1 - p(\theta(t,\xi)))^{\alpha} Z^{\alpha}(\theta(t,\xi)) \Delta\xi.$$

Since $\theta(t,\xi) \geq \theta(t,c)$ and $Z^{\Delta}(t) > 0$, then $Z(\theta(t,\xi)) \geq Z(\theta(t,c))$. By the definition of $R(t)$ and $\delta(t)$, we obtain

$$\left(r(t)(Z^{\Delta}(t))^{\alpha}\right)^{\Delta} \leq -R(t) Z^{\alpha}(\theta(t,c)) = -R(t) Z^{\alpha}(\delta(t)). \tag{2.4}$$

Lemma 2.2. *Let $x(t)$ be an eventually positive solution of Eq. (1.1). Then there exists some $T > t_0$ large enough such that for all $t > T$,*

$$\frac{Z(\delta(t))}{Z(\sigma(t))} \geq \begin{cases} 1, & \delta(t) > \sigma(t), \\ \eta_1(t,T), & \delta(t) \leq \sigma(t). \end{cases} \tag{2.5}$$

The proof is similar to that of [24, Lemma 2.2], we omitted the details here.

2.2 Oscillation of (1.1) for the case $\delta(t) \le \sigma(t)$

Theorem 2.3. *Assume that there exist a function $a(t)$ and a positive Δ-differentiable function $\eta(t)$, such that for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} \int_{T_1}^{t} \left(\varphi(s) - \frac{\alpha^\alpha([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\varphi_2^\alpha(s)} \right) \Delta s > \eta(T_1) \left(\frac{1}{r^\alpha(T_1, T)} + r(T_1)a(T_1) \right), \tag{2.6}$$

where $T_1 > T \ge t_0$. Then

(i) every solution $x(t)$ of (1.1) is oscillatory for $\alpha \ge 1$;

(ii) every solution $x(t)$ of (1.1) oscillates for $0 < \alpha < 1$ and $a(t) = 0$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists a $T \in [t_0, \infty)_{\mathbb{T}}$ (sufficiently large) such that $x(t)$, $x(\tau(t))$, $x(\delta(t)) > 0$ on $[T, \infty)_{\mathbb{T}}$. Then by Lemmas 2.1 and 2.2, (2.1) and (2.5) hold. Consider the generalized Riccati substitution

$$w(t) = \eta(t) \left(\frac{r(t)(Z^\Delta(t))^\alpha}{Z^\alpha(t)} + r(t)a(t) \right), \qquad t \ge T.$$

Clearly, $w(t) > 0$. In view of [5, Theorem 1.20] and (2.4), we get

$$\begin{aligned}
w^\Delta(t) &= \eta^\Delta(t) \left(\frac{r(t)(Z^\Delta(t))^\alpha}{Z^\alpha(t)} + r(t)a(t) \right) + \eta^\sigma(t) \left(\frac{r(t)(Z^\Delta(t))^\alpha}{Z^\alpha(t)} + r(t)a(t) \right)^\Delta \\
&= \frac{\eta^\Delta(t)}{\eta(t)} w(t) + \eta^\sigma(t) \left(\frac{r(t)(Z^\Delta(t))^\alpha}{Z^\alpha(t)} \right)^\Delta + \eta^\sigma(t)\big(r(t)a(t)\big)^\Delta \\
&= \frac{\eta^\Delta(t)}{\eta(t)} w(t) + \eta^\sigma(t)\big(r(t)a(t)\big)^\Delta \\
&\quad + \eta^\sigma(t) \frac{\big(r(t)(Z^\Delta(t))^\alpha\big)^\Delta Z^\alpha(t) - r(t)(Z^\Delta(t))^\alpha (Z^\alpha(t))^\Delta}{Z^\alpha(t)(Z^\sigma(t))^\alpha} \\
&\le \frac{\eta^\Delta(t)}{\eta(t)} w(t) + \eta^\sigma(t)\big(r(t)a(t)\big)^\Delta \\
&\quad - \eta^\sigma(t) \frac{R(t)Z^\alpha(\delta(t))}{(Z^\sigma(t))^\alpha} - \frac{\eta^\sigma(t) r(t)(Z^\Delta(t))^\alpha (Z^\alpha(t))^\Delta}{Z^\alpha(t)(Z^\sigma(t))^\alpha} \\
&= \frac{\eta^\Delta(t)}{\eta(t)} w(t) + \eta^\sigma(t)\big(r(t)a(t)\big)^\Delta - \eta^\sigma(t)R(t) \left(\frac{Z(\delta(t))}{Z^\sigma(t)} \right)^\alpha \\
&\quad - \eta^\sigma(t) \left(\frac{w(t)}{\eta(t)} - r(t)a(t) \right) \frac{(Z^\alpha(t))^\Delta}{(Z^\sigma(t))^\alpha}.
\end{aligned} \tag{2.7}$$

By the Pötzsche chain rule [5, Theorem 1.87], then

$$\begin{aligned}
(Z^\alpha(t))^\Delta &= \alpha \Big\{ \int_0^1 [(1-h)Z(t) + hZ(\sigma(t))]^{\alpha-1} \mathrm{d}h \Big\} Z^\Delta(t) \\
&\ge \begin{cases} \alpha(Z(t))^{\alpha-1} Z^\Delta(t), & \alpha > 1, \\ \alpha(Z^\sigma(t))^{\alpha-1} Z^\Delta(t), & 0 < \alpha \le 1. \end{cases}
\end{aligned}$$

Consequently,

$$\frac{(Z^\alpha(t))^\Delta}{Z^\alpha(t)} \ge \begin{cases} \alpha \dfrac{Z^\Delta(t)}{Z(t)}, & \alpha > 1, \\[2ex] \alpha \dfrac{(Z^\sigma(t))^{\alpha-1}}{Z^\alpha(t)} Z^\Delta(t), & 0 < \alpha \le 1. \end{cases}$$

Note that $Z(t)$ is increasing on $[T,\infty)_{\mathbb{T}}$. Then $Z(t)\le Z(\sigma(t))$ for $t\in[T,\infty)_{\mathbb{T}}$. Therefore

$$\frac{(Z^{\alpha}(t))^{\Delta}}{Z^{\alpha}(t)}\ge \alpha\frac{Z^{\Delta}(t)}{Z^{\sigma}(t)},$$

which implies

$$\begin{aligned}\frac{(Z^{\alpha}(t))^{\Delta}}{(Z^{\sigma}(t))^{\alpha}}&=\frac{(Z^{\alpha}(t))^{\Delta}}{Z^{\alpha}(t)}\frac{Z^{\alpha}(t)}{(Z^{\sigma}(t))^{\alpha}}\ge\alpha\frac{Z^{\Delta}(t)}{Z(t)}\left(\frac{Z(t)}{Z^{\sigma}(t)}\right)^{1+\alpha}\\&\ge\frac{\alpha}{r^{\frac{1}{\alpha}}(t)}\left(\frac{w(t)}{\eta(t)}-r(t)a(t)\right)^{\frac{1}{\alpha}}\eta_2^{1+\alpha}(t,T).\end{aligned}\tag{2.8}$$

Substituting (2.8) into (2.7), and by (2.5), we obtain

$$\begin{aligned}w^{\Delta}(t)&<\frac{\eta^{\Delta}(t)}{\eta(t)}w(t)+\eta^{\sigma}(t)\big(r(t)a(t)\big)^{\Delta}-\eta^{\sigma}(t)R(t)\eta_1^{\alpha}(t,T)\\&\quad-\frac{\alpha\,\eta^{\sigma}(t)\eta_2^{1+\alpha}(t,T)}{r^{\frac{1}{\alpha}}(t)}\left(\frac{w(t)}{\eta(t)}-r(t)a(t)\right)^{1+\frac{1}{\alpha}},\qquad t>T.\end{aligned}\tag{2.9}$$

For the case $\alpha\ge 1$, using the inequality (see [3, (2.18)]),

$$A^{1+\frac{1}{\alpha}}-(A-B)^{1+\frac{1}{\alpha}}\le B^{\frac{1}{\alpha}}\left(\left(1+\frac{1}{\alpha}\right)A-\frac{1}{\alpha}B\right),\qquad \alpha=\frac{\text{odd}}{\text{odd}},$$

with $A:=w(t)/\eta(t)$ and $B:=r(t)a(t)$, we get

$$\left(\frac{w(t)}{\eta(t)}-r(t)a(t)\right)^{1+\frac{1}{\alpha}}\ge\left(\frac{w(t)}{\eta(t)}\right)^{1+\frac{1}{\alpha}}+\frac{1}{\alpha}\big(r(t)a(t)\big)^{1+\frac{1}{\alpha}}-\frac{1+\alpha}{\alpha}\big(r(t)a(t)\big)^{\frac{1}{\alpha}}\frac{w(t)}{\eta(t)}.\tag{2.10}$$

Substituting (2.10) into (2.9), we obtain

$$w^{\Delta}(t)<-\varphi(t)+\varphi_1(t)w(t)-\varphi_2(t)w^{1+\frac{1}{\alpha}}(t)\tag{2.11}$$

$$\le-\varphi(t)+[\varphi_1(t)]_+w(t)-\varphi_2(t)w^{1+\frac{1}{\alpha}}(t),\qquad t>T.\tag{2.12}$$

For the case when $0<\alpha<1$ and $a(t)=0$, in view of the definitions of $w(t)$, $\varphi(t)$, $\varphi_1(t)$, $\varphi_2(t)$, we find that (2.12) also holds by (2.9). Using the inequality (see [4, (2.8)]),

$$B_1w-A_1w^{1+\frac{1}{\alpha}}\le\frac{\alpha^{\alpha}B_1^{1+\alpha}}{(1+\alpha)^{1+\alpha}A_1^{\alpha}},\tag{2.13}$$

with $B_1=[\varphi_1(t)]_+$ and $A_1=\varphi_2(t)$, we have

$$[\varphi_1(t)]_+w(t)-\varphi_2(t)w^{1+\frac{1}{\alpha}}(t)\le\frac{\alpha^{\alpha}\big([\varphi_1(t)]_+\big)^{1+\alpha}}{(1+\alpha)^{1+\alpha}(\varphi_2(t))^{\alpha}}.\tag{2.14}$$

Substituting (2.14) into (2.12) we obtain

$$w^{\Delta}(t)<-\varphi(t)+\frac{\alpha^{\alpha}\big([\varphi_1(t)]_+\big)^{1+\alpha}}{(1+\alpha)^{1+\alpha}(\varphi_2(t))^{\alpha}},\qquad t>T.\tag{2.15}$$

Integrating both sides of (2.15) from T_1 to $t(t>T_1>T)$, we have

$$\int_{T_1}^{t}\left(\varphi(t)-\frac{\alpha^{\alpha}\big([\varphi_1(t)]_+\big)^{1+\alpha}}{(1+\alpha)^{1+\alpha}(\varphi_2(t))^{\alpha}}\right)\Delta s<w(T_1)<\eta(T_1)\left(\frac{1}{r^{\alpha}(T_1,T)}+r(T_1)a(T_1)\right),$$

which contradicts (2.6). This completes the proof. □

Let $\mathbb{D}_0 \equiv \{t > s \geq t_0,\ t,s \in [t_0,\infty)_{\mathbb{T}}\}$ and $\mathbb{D} \equiv \{t \geq s \geq t_0,\ t,s \in [t_0,\infty)_{\mathbb{T}}\}$. The function $K \in C_{rd}(\mathbb{D},\mathbb{R})$ is said to belong to the class $\mathfrak{R}$ (defined by $K \in \mathfrak{R}$, for short) if

$$K(t,t) = 0, \quad t \geq t_0, \qquad K(t,s) > 0, \quad t > s \geq t_0,$$

and K has a nonpositive continuous Δ-partial derivative $K^{\Delta s}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable.

Theorem 2.4. *Assume that $K \in \mathfrak{R}$, $k \in C_{rd}(\mathbb{D}_0,\mathbb{R})$ and there exist a function $a(t)$ and a positive Δ-differentiable function $\eta(t)$, such that for sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$,*

$$K^{\Delta s}(\sigma(t),s) + K(\sigma(t),\sigma(s))\varphi_1(s) = k(t,s),$$

and

$$\limsup_{t\to\infty} \frac{1}{K(\sigma(t),T_1)} \int_{T_1}^{t} \Big(K(\sigma(t),\sigma(s))\varphi(s) - \frac{\alpha^\alpha([k(t,s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(K(\sigma(t),\sigma(s))\varphi_2(s))^\alpha} \Big)\Delta s$$
$$> \eta(T_1)\left(\frac{1}{r^\alpha(T_1,T)} + r(T_1)a(T_1)\right), \tag{2.16}$$

where $T_1 > T \geq t_0$. Then

(i) every solution $x(t)$ of (1.1) is oscillatory for $\alpha \geq 1$;

(ii) every solution $x(t)$ of (1.1) oscillates for $0 < \alpha < 1$ and $a(t) = 0$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)$ is an eventually positive. Proceeding as the proof of Theorem 2.3, we get (2.11) holds, i.e.,

$$w^\Delta(t) < -\varphi(t) + \varphi_1(t)w(t) - \varphi_2(t)w^{1+\frac{1}{\alpha}}(t), \qquad t > T_1 > T. \tag{2.11}$$

Multiplying both sides of (2.11), with t replaced by s, by $K(\sigma(t),\sigma(s))$, integrating with respect to s from T_1 to $\sigma(t)$, we get

$$\int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))\varphi(s)\Delta s$$
$$< -\int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))w^\Delta(s)\Delta s + \int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))\varphi_1(s)w(s)\Delta s$$
$$- \int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s)\Delta s. \tag{2.17}$$

Using integration by parts for the first part of the right-hand side of (2.17), we obtain

$$\int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))w^\Delta(s)\Delta s = -K(\sigma(t),T_1)w(T_1) - \int_{T_1}^{\sigma(t)} K^{\Delta_s}(\sigma(t),s)w(s)\Delta s. \tag{2.18}$$

Substitution (2.18) into (2.17) implies that

$$\int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))\varphi(s)\Delta s$$
$$< K(\sigma(t),T_1)w(T_1) + \int_{T_1}^{\sigma(t)} K^{\Delta_s}(\sigma(t),s)w(s)\Delta s$$
$$+ \int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))\varphi_1(s)w(s)\Delta s - \int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s)\Delta s$$
$$= K(\sigma(t),T_1)w(T_1) + \int_{T_1}^{\sigma(t)} k(t,s))w(s)\Delta s - \int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s).$$

Using the definition of K and

$$\int_{T_1}^{\sigma(t)} \phi(s)\Delta s = \int_{T_1}^{t} \phi(s)\Delta s + \int_{t}^{\sigma(t)} \phi(s)\Delta s = \int_{T_1}^{t} \phi(s)\Delta s + \mu(t)\phi(t),$$

we derive from $\mu(t)K^{\Delta_s}(\sigma(t),t)w(t) \leq 0$ that

$$\begin{aligned}
&\int_{T_1}^{\sigma(t)} k(t,s)w(s)\Delta s - \int_{T_1}^{\sigma(t)} K(\sigma(t),\sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s)\\
&= \int_{T_1}^{t} k(t,s)w(s)\Delta s - \int_{T_1}^{t} K(\sigma(t),\sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s)\\
&\quad + \int_{t}^{\sigma(t)} k(t,s)w(s)\Delta s - \int_{t}^{\sigma(t)} K(\sigma(t),\sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s)\\
&= \int_{T_1}^{t} k(t,s)w(s)\Delta s - \int_{T_1}^{t} K(\sigma(t),\sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s)\\
&\quad + \mu(t)\Big[K^{\Delta_s}(\sigma(t),t) + K(\sigma(t),\sigma(t))\varphi_1(t)\Big]w(t) - \mu(t)K(\sigma(t),\sigma(t))\varphi_2(t)w^{1+\frac{1}{\alpha}}(t)\\
&\leq \int_{T_1}^{t} [k(t,s)]_+ w(s)\Delta s - \int_{T_1}^{t} K(\sigma(t),\sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s).
\end{aligned}$$

Now using the inequality (2.13) with $B_1 = [k(t,s)]_+$ and $A_1 = K(\sigma(t),\sigma(s))\varphi_2(s)$, we get

$$[k(t,s)]_+ w(s) - K(\sigma(t),\sigma(s))\varphi_2(s)w^{1+\frac{1}{\alpha}}(s) \leq \frac{\alpha^\alpha\,[k(t,s)]_+^{1+\alpha}}{(1+\alpha)^{1+\alpha}\big(K(\sigma(t),\sigma(s))\varphi_2(s)\big)^\alpha}.$$

Hence, noting that $K(\sigma(t),\sigma(t)) = 0$, we get

$$\int_{T_1}^{t} \left(K(\sigma(t),\sigma(s))\varphi(s) - \frac{\alpha^\alpha([k(t,s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(K(\sigma(t),\sigma(s))\varphi_2(s))^\alpha}\right)\Delta s \leq K(\sigma(t),T_1)w(T_1).$$

Thus,

$$\begin{aligned}
&\frac{1}{K(\sigma(t),T_1)}\int_{T_1}^{t} \left(K(\sigma(t),\sigma(s))\varphi(s) - \frac{\alpha^\alpha([k(t,s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(K(\sigma(t),\sigma(s))\varphi_2(s))^\alpha}\right)\Delta s\\
&\quad \leq w(T_1) < \eta(T_1)\left(\frac{1}{r^\alpha(T_1,T)} + r(T_1)a(T_1)\right),
\end{aligned}$$

which contradicts (2.16), and then the proof is complete. □

Remark 2.5. If (2.6) and (2.16) are replaced respectively by

$$\limsup_{t\to\infty} \int_{T_1}^{t} \left(\varphi(s) - \frac{\alpha^\alpha([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\varphi_2^\alpha(s)}\right)\Delta s = \infty,$$

$$\limsup_{t\to\infty} \frac{1}{K(\sigma(t),T_1)}\int_{T_1}^{t} \left(K(\sigma(t),\sigma(s))\varphi(s) - \frac{\alpha^\alpha([k(t,s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(K(\sigma(t),\sigma(s))\varphi_2(s))^\alpha}\right)\Delta s = \infty,$$

then the conclusions of Theorems 2.3, 2.4 are also true which are special cases of Theorems 2.3, 2.4. Thus, Theorems 2.3, 2.4 essentially improve the related results established by [7,15,22].

Remark 2.6. The assumption (H_6) in D. Chen [10] required $b^\Delta(t) > 0$ and $b(\sigma(t)) = \sigma(b(t))$, the function $\psi(t)$ in Theorem 3.1 and 3.2 required $\psi(t) \geq 0$ which are stronger than that of $(H4)$ and $a(t)$ in our work, respectively. Therefore, Theorem 2.4 improves Theorem 3.2 in D. Chen [10]

2.3 Oscillation of (1.1) for the case $\delta(t) > \sigma(t)$

In this case when $\delta(t) > \sigma(t)$, by Lemma 2.2, we get $Z(\delta(t))/Z(\sigma(t)) > 1$. Now we replace $Z(\delta(t)/Z(\sigma(t)))$ by 1 in (2.7), and similarly to the proof of Theorems 2.3–2.4, we can obtain following results.

Theorem 2.7. *Assume that there exist a function $a(t)$ and a positive Δ-differentiable function $\eta(t)$, such that for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} \int_{T_1}^{t} \left(\overline{\varphi}(s) - \frac{\alpha^\alpha([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\varphi_2^\alpha(s)} \right) \Delta s > \eta(T_1) \left(\frac{1}{r^\alpha(T_1,T)} + r(T_1)a(T_1) \right).$$

where $T_1 > T \geq t_0$, and

$$\overline{\varphi}(t) = \eta^\sigma(t)\left[R(t) + r(t)\eta_2^{1+\alpha}(t,T)a^{1+\frac{1}{\alpha}}(t) - (r(t)a(t))^\Delta\right].$$

Then

(i) every solution $x(t)$ of (1.1) is oscillatory for $\alpha \geq 1$;

(ii) every solution $x(t)$ of (1.1) oscillates for $0 < \alpha < 1$ and $a(t) = 0$.

Theorem 2.8. *Assume that $K \in \mathfrak{R}$, $k \in C_{rd}(\mathbb{D}_0, \mathbb{R})$ and there exist a function $a(t)$ and a positive Δ-differentiable function $\eta(t)$, such that for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,*

$$K^{\Delta s}(\sigma(t), s) + K(\sigma(t), \sigma(s))\varphi_1(s) = k(t,s),$$

and

$$\limsup_{t\to\infty} \frac{1}{K(\sigma(t),T_1)} \int_{T_1}^{t} \left(K(\sigma(t),\sigma(s))\overline{\varphi}(s) - \frac{\alpha^\alpha([k(t,s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(K(\sigma(t),\sigma(s))\varphi_2(s))^\alpha} \right) \Delta s$$
$$> \eta(T_1) \left(\frac{1}{r^\alpha(T_1,T)} + r(T_1)a(T_1) \right),$$

where $T_1 > T \geq t_0$, and $\overline{\varphi}(s)$ is defined as in Theorem 2.7. Then

(i) every solution $x(t)$ of (1.1) is oscillatory for $\alpha \geq 1$;

(ii) every solution $x(t)$ of (1.1) oscillates for $0 < \alpha < 1$ and $a(t) = 0$.

3 Two examples

In this section, we give two examples to illustrate our main results.

Example 3.1. Consider the neutral dynamic equation

$$\left(\frac{1}{(t+\sigma(t))^\alpha} (Z^\Delta(t))^\alpha \right)^\Delta + \int_c^d \frac{1+\sin^2(t\xi)}{t\sigma(t)} x^\alpha(t-\xi)\Delta\xi = 0, \tag{3.1}$$

where $1/2 \leq \alpha < 1$ with $\alpha = \text{odd}/\text{odd}$, $\tau(t)$ satisfies (A3), and $Z(t) = x(t) + \frac{1}{2}x(\tau(t))$.

For (1.1), we let

$$r(t) = \frac{1}{(t+\sigma(t))^\alpha}, \qquad p(t) = \frac{1}{2}, \qquad q(t,\xi) = \frac{1}{t\sigma(t)},$$

and

$$\theta(t,\xi) = t - \xi, \qquad \delta(t) = \theta(t,c) = t - c.$$

Obviously, (A1) holds and we have

$$r(t,T) = t^2 - T^2, \qquad \eta_1(t,T) = \frac{\delta^2(t) - T^2}{\sigma^2(t) - T^2} = \frac{(t-c)^2 - T^2}{\sigma^2(t) - T^2}.$$

Then the function $\eta_1(t,T)$ is strictly increasing and we have $1 \geq \eta_1(t,T) \geq 1/3$ for sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$. Then

$$\begin{aligned} R(t) &= \int_c^d q(t,\xi)(1 - p(\theta(t,\xi)))^\alpha \Delta\xi \\ &= \frac{1}{2^\alpha}\int_c^d \frac{1}{t\sigma(t)}\Delta\xi \\ &= \frac{d-c}{2^\alpha\, t\sigma(t)}. \end{aligned}$$

Let $a(t) = 0$ and $\eta(t) = 1 + 1/t$, choosing $T_1 = 2T$, then

$$\frac{d-c}{2^{\alpha-1}\, t\sigma(t)} > \varphi(t) = \eta^\sigma(t)R(t)\eta_1^\alpha(t,T) > \frac{d-c}{6^\alpha\, t\sigma(t)},$$
$$[\varphi_1(s)]_+ = 0, \qquad \frac{\eta(T_1)}{r^\alpha(T_1,T)} = \left(1 + \frac{1}{T_1}\right)\frac{1}{(T_1^2 - T^2)^\alpha} < \frac{1}{3^\alpha\, T}.$$

Hence,

$$\begin{aligned} \infty &> \limsup_{t\to\infty}\int_{T_1}^t \frac{d-c}{2^{\alpha-1}t\sigma(t)}\Delta s > \limsup_{t\to\infty}\int_{T_1}^t \left(\varphi(s) - \frac{\alpha^\alpha([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\varphi_2^\alpha(s)}\right)\Delta s \\ &> \limsup_{t\to\infty}\int_{T_1}^t \frac{d-c}{6^\alpha\, t\sigma(t)}\Delta s = \limsup_{t\to\infty}\frac{d-c}{6^\alpha}\Big(\frac{1}{2T} - \frac{1}{t}\Big). \end{aligned}$$

We can choose c, d such that

$$\frac{d-c}{6^\alpha} > \frac{2}{3^\alpha} + 2.$$

Consequently,

$$\limsup_{t\to\infty}\int_{T_1}^t \left(\varphi(s) - \frac{\alpha^\alpha([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\varphi_2^\alpha(s)}\right)\Delta s > \frac{\eta(T_1)}{r^\alpha(T_1,T)}.$$

Thus, by Theorem 2.3, Eq. (3.1) is oscillatory.

Example 3.2. Let $\mathbb{T} = 2^{\mathbb{N}}$, $\alpha > \gamma + 1 > 2$ and $\alpha = \text{odd}/\text{odd}$. Consider the neutral dynamic equation

$$\left((Z^\Delta(t))^\alpha\right)^\Delta + \int_c^d \frac{(|\sin t| + 1)^\alpha}{t^\gamma(2 - \sin^2(t\xi))}x^\alpha(t)\Delta\xi = 0, \tag{3.2}$$

where $\tau(t)$ satisfies (A3), and $Z(t) = x(t) + \big[|\sin t|/(|\sin t| + 1)\big]x(\tau(t))$. Here

$$r(t) = 1, \qquad p(t) = \frac{|\sin t|}{|\sin t| + 1}, \qquad q(t,\xi) = \frac{(|\sin t| + 1)^\alpha}{2\, t^\gamma},$$

and

$$\theta(t,\xi) = t, \qquad \delta(t) = \theta(t,c) = t.$$

Clearly (A1) holds. Noting that the function $r(t,T)/r(\sigma(t),T) = (t-T)/(\sigma(t)-T)$ is strictly increasing. Hence, $\eta_1(t,T) = \eta_2(t,T) \geq 1/3$ for sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$ and $t \geq \sigma(T) = 2T$. Then

$$\begin{aligned} R(t) &= \int_c^d q(t,\xi)(1-p(\theta(t,\xi)))^\alpha \Delta\xi \\ &= \int_c^d \frac{(|\sin t|+1)^\alpha}{2\,t^\gamma}\frac{1}{(|\sin t|+1)^\alpha}\Delta\xi \\ &= \frac{d-c}{2\,t^\gamma}, \end{aligned}$$

Let $a(t) = 0$ and $\eta(t) = 1$ in Theorem 2.3, we have $\varphi_1(t) = 0$ for $t > T$. Choosing $T_1 = 2T$, then

$$\frac{d-c}{2\,t^\gamma} > \varphi(t) = \eta^\sigma(t)R(t)\eta_1^\alpha(t,T) \geq \frac{d-c}{2(3^\alpha\,t^\gamma)},$$

and

$$[\varphi_1(t)]_+ = 0, \qquad \frac{\eta(T_1)}{r^\alpha(T_1,T)} = \frac{1}{T^\alpha}.$$

Hence

$$\begin{aligned} \infty > \limsup_{t\to\infty}\int_{T_1}^t \frac{d-c}{2\,s^\gamma}\Delta s &> \limsup_{t\to\infty}\int_{T_1}^t \left(\varphi(s) - \frac{\alpha^\alpha([\varphi_1(s)]_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\varphi_2^\alpha(s)}\right)\Delta s \\ \geq \limsup_{t\to\infty}\int_{T_1}^t \frac{d-c}{2(3^\alpha\,s^\gamma)}\Delta s &> \frac{\eta(T_1)}{r^\alpha(T_1,T)}, \end{aligned}$$

Thus, by Theorem 2.3, Eq. (3.2) is oscillatory.

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