Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **49**, 1–10; doi: 10.14232/ejqtde.2016.1.49 http://www.math.u-szeged.hu/ejqtde/

Analytic reducibility of nondegenerate centers: Cherkas systems

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Received 13 October 2015, appeared 13 July 2016
Communicated by Gabriele Villari

Abstract. In this paper we study the center problem for polynomial differential systems and we prove that any center of an analytic differential system is analytically reducible. We also study the centers for the Cherkas polynomial differential systems

$$\dot{x} = y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2,$$

where $P_i(x)$ are polynomials of degree n, $P_0(0) = 0$ and $P_0'(0) < 0$. Computing the focal values we find the center conditions for such systems for degree 3, and using modular arithmetics for degree 4. Finally we do a conjecture about the center conditions for Cherkas polynomial differential systems of degree n.

Keywords: center problem, analytic integrability, polynomial Cherkas differential systems.

2010 Mathematics Subject Classification: 34C05, 37C10.

1 Introduction

The well-known polynomial Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.1}$$

where $f(x)$ and $g(x)$ are polynomials, which we can be rewritten as the differential system in the plane

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x), \tag{1.2}$$

can be generalized into what we call the polynomial *Cherkas polynomial differential equation*

$$\ddot{x} + h(x)\dot{x}^2 + f(x)\dot{x} + g(x) = 0, \tag{1.3}$$

where now $f(x)$, $g(x)$ and $h(x)$ are polynomials. This Cherkas equation can be also transformed into the differential system in the plane

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x) - y^2h(x). \tag{1.4}$$

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System (1.4) has associated the differential equation

$$y' = \frac{dy}{dx} = \frac{-g(x) - yf(x) - y^2h(x)}{y}.$$

which can be written as

$$yy' = -g(x) - yf(x) - y^2h(x). \tag{1.5}$$

Systems (1.2) and (1.4) arise frequently in the study of various mathematical models of physical, chemical, biological, physiological, economical and other processes, see for instance [16, 18] and references therein.

For the Liénard system (1.2) Cherkas [1] gave necessary and sufficient conditions for the existence of a center at the origin. Christopher [4] extended this result and obtained global conditions on the form of f and g to have a center. In fact the Liénard systems (1.2) with a center are time-reversible (see below the definition) through an analytic invertible transformation followed by a rescalling of time. This type of symmetry after transformation is called *generalized symmetry*.

Cherkas in [1] was the first in study differential equation (1.5) that he wrote into the form $yy' = P(x) + Q(x)y^2 + R(x)y^2$ having a singular point at the origin with purely imaginary eigenvalues and where P, Q and R are real rational functions. Cherkas gave necessary and sufficient conditions for the existence of a center at the origin of equation (1.5). Later on Cherkas in [2, 3] considered the more general case

$$\dot{x} = P_3(x)y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2, \tag{1.6}$$

where here $P_i(x)$ are polynomials, with $P_3(0) \neq 0$, $P_0(0) = 0$ and $P_3(0)P_0'(0) < 0$. Such systems include the so-called reduced Kukles systems (Kukles systems with $a_7 = 0$), see [19, 24] and also [14] with references therein.

In [6] it is shown that the centers of system (1.6) arise either from a Darboux first integral of the form

$$H = \exp(D/E) \prod C_i^{\alpha_i}, \tag{1.7}$$

where D, E and the C_i are polynomials in $\mathbb{C}[x, y]$ and $\alpha_i \in \mathbb{C}$, or from a simple form of algebraic reversibility (see definition below).

In [5], at the end of chapter 5 of the part I, it is noted that the same results can be obtained for more general systems of the form

$$\dot{x} = P_4(x)y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2 + P_3(x)y^3, \tag{1.8}$$

where P_i are polynomial. It is claimed without proof that the results are essentially the same i.e. either there is a Darboux first integral, or there is an algebraic reversibility with an algebraic symmetry. We recall that system (1.8) include the Kukles systems, see [14, 19, 24]. Unfortunately this claim is not strictly correct because there are centers in the Kukles family with a Liouvillian first integral which is not a Darboux first integral, see [19, 24]. However the general idea that the results must be essentially the same seems be true with extending the first integrals to the Liouvillian ones. Nevertheless up to know no proof is known.

These results permit to check if a particular system of the form (1.2), (1.4), (1.6), and (1.8) has a center at the origin. However, in practice from the conditions obtained it is not easy to get the explicit form of the families with center even for systems of small degree.

2 Preliminary results

A *center* for a real analytic differential system in the plane is an isolated singularity p for which there exist a neighborhood U such that $U \setminus \{p\}$ is filled with periodic orbits. A singularity is *nondegenerate* if the eigenvalues of its linear part are purely imaginary. To detect nodegenerate center is a very classical problem in qualitative theory of differential equations, see for instance [11,20,21,23]. For a nondegenerate singularity (either a focus or a center) a theorem of Poincaré–Liapunov [23] says that this singularity is a center if, and only if, the system has a nonconstant analytic first integral in a neighborhood of it.

It seems natural to think that this analytic first integral must be of algebraic nature if the differential system is polynomial attending to the algebraic nature of the necessary conditions to have a center. This is true for lower degree (quadratic and cubic symmetric systems) where the first integral of each family of centers are Darboux first integrals of the form (1.7). However for general cubic system this is not true and more general mechanisms for producing centers must be introduced, see [6].

The first mechanism is the generalization of the Darboux first integrals to Liouville first integrals which in fact is a straightforward generalization because all these systems have a Darboux integrating factor of the form (1.7), see [12] and references therein. Nevertheless there are centers of polynomial differential systems without a Liouville first integral. For instance the polynomial Liénard system

$$\dot{x} = y + x^4, \qquad \dot{y} = -x, \tag{2.1}$$

has neither any invariant algebraic curve, nor an integrating factor of the form (1.7) and consequently is not Liouvillian integrable, see [9].

System (2.1) is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$ hence the phase portrait is symmetric respect to the y axis what is called a *time-reversible* system. This example leads immediately to the second mechanism to produce centers. This mechanism, which is of algebraic nature also, produces centers by pulling back a nonsingular differential system via an algebraic map which allows to obtain a symmetric differential system. For system (2.1) the map is polynomial $(\bar{x}, \bar{y}) \mapsto (x^2, y)$ and we obtain the nonsingular differential system

$$\dot{\bar{x}} = 2(\bar{y} + \bar{x}^2), \qquad \dot{\bar{y}} = -1,$$

that is, a differential system without singular points.

This second mechanism is called *algebraic reducibility*, see [6]. Also in [6] it is mentioned the *algebraic reversible* mechanism. This method consists in to find an algebraic map that transforms our original system into a time-reversible system. In [26] it is introduced the *rationally reversibility* mechanism, which is a particular case of the algebraic reversible mechanism because in this case the map is assumed rational. However any algebraic reversible system or rational reversible system is also algebraic reducible, see [6].

In [6] it is proved that these two general mechanisms 1) by finding a first integral using Darboux methods, and 2) by producing a system with the help of a pull-back of a nonsingular differential system along a map of algebraic nature are the two general methods that give all the centers of the Liénard and Cherkas polynomial differential systems, i.e. systems (1.2) and (1.4) respectively. The main result proved in [6] is the following.

Theorem 2.1. *System (1.6) with $P_0(0) = 0$, $P_3(0) > 0$ and $P_0'(0) < 0$ has a center at the origin if and only if it satisfies one of the following (possibly overlapping) conditions.*

(i) The system is algebraic reducible via the map $(x, y) \mapsto (x, y^2)$ and thus it has a symmetry with respect to the x-axis.

(ii) The system is algebraic reducible via the map $(x, y) \mapsto (M(x)^{1/r}, yR(x)^{1/q})$, where M and R are rational functions in x over $\mathbb{R}$ with $r, q \in \mathbb{Z}$ and $M(x)^{1/r}$ of order 2 at $x = 0$ (i.e. $M(x)^{1/r} = O(x^2)$) and $R(0) \neq 0$.

(iii) There is a local first integral of Darboux type.

However these two mechanisms, Liouville integrability and algebraic reducibility are both of algebraic nature because in the first case the map is algebraic and in the second case the system has an integrating factor of the form (1.7) and both methods can be unified in a unique way as the following result shows.

3 Statement of the main results

The next result summarizes the results obtained up to now for polynomial differential (1.2) and (1.4). The following result was proved in [1,6].

Theorem 3.1. *Any center of a polynomial differential system (1.2) or (1.4) is Liouville integrable or algebraic reducible.*

Here reducible means that the polynomial differential system with a center is the pull-back of a nonsingular differential system via a map of algebraic nature. We note that Theorem 3.1 is a particular case of the next result where both mechanisms are unified. This is due, first to the fact that all the centers of systems (1.2) or (1.4) which are algebraic reducible are analytical reducible, and second the centers of systems (1.2) or (1.4) which are Liouville integrable can be written in the Poincaré normal form and following the steps of the proof of Theorem 3.2 it follows that they analytical reducible.

Theorem 3.2. *Any nondegenerate center of an analytic differential system is analytically reducible.*

The most important thing in Theorem 3.1 with respect to Theorem 3.2 is its applicability, because once we know that the centers of Theorem 3.1 can be Liouville integrable we can look for their invariant algebraic curves and their exponential factors, and using the Darboux theory of integrability (see for more details Chapter 8 of [7]) find an integrating factor for those systems, and after the Liouvillian first integral associated to this integrating factor. On the other hand Theorem 3.2 is difficult to apply because in general we only can compute some of the terms of the Taylor series of the analytic change for writing the system into its Poincaré normal form and actually we do not know the analytic change, see for more for instance [11,23].

Given a differential system we can propose a transformation to the Poincaré normal form and find the necessary conditions but it never gives you the sufficient conditions. However the algebraic nature of the algebraic reducibility, or of the Liouville integrability allows to find, in general, the transformation to the nonsingular differential system, or to the Liouville first integral providing the sufficient conditions. Hence Theorem 3.1 can be used to classify the centers of a family of differential systems when we have found the necessary conditions, see for instance the next Theorems 3.3 and 3.4.

An open question is: *Are there nondegenerate centers of polynomial differential systems which are neither Liouville integrable, nor algebraic reducible?*

In this paper we give the center conditions for systems of the form (1.4) with f, g and h of degree ≤ 4, that is, a system of the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - b_2x^2 - b_3x^3 - b_4x^4 - (a_1x + a_2x^2 + a_3x^3 + a_4x^4)y, \\ &\quad -(c_1x + c_2x^2 + c_3x^3 + c_4x^4)y^2, \end{aligned} \tag{3.1}$$

where a_i, b_i and $c_i \in \mathbb{R}$.

The next two theorems are the main results of this paper.

Theorem 3.3. *Cherkas polynomial differential systems* (3.1) *with* $a_4 = b_4 = c_4 = 0$ *have a center if and only if one of the following conditions holds.*

(a) $a_2 = b_2 = c_2 = 0$,

(b) $a_1 = a_2 = a_3 = 0$,

(c) $a_2 = a_1b_2$, $a_3 = a_1b_3$, $c_2 = c_1b_2$, $c_3 = c_1b_3$.

In [13] were found necessary and sufficient conditions for the origin of system (3.1) with $c_1 = c_2 = c_3 = c_4 = 0$ to be a center, i.e. the Liénard systems where f and g are of degree ≤ 4. In fact the Liénard systems where f and g are of degree ≤ 5 are classified.

Theorem 3.4. *Cherkas polynomial differential systems* (3.1) *have a center if one of the following conditions holds.*

(a) $a_2 = b_2 = c_2 = a_4 = b_4 = c_4 = 0$,

(b) $a_1 = a_2 = a_3 = a_4 = 0$,

(c) $a_2 = a_1b_2$, $a_3 = a_1b_3$, $a_4 = a_1b_4$, $c_2 = c_1b_2$, $c_3 = c_1b_3$, $c_4 = c_1b_4$.

Using modular arithmetics we have proved that with very high probability the unique center cases of system (1.4) when f, g and h are of degree 4 are the given in Theorem 3.4. From the previous results and other partial computations for bigger degree using modular arithmetics we can establish the following conjecture.

Conjecture 3.5. *Cherkas polynomial differential systems* (1.4) *with* f, g *and* $h \not\equiv 0$ *of degree* n *have a center if, and only if, one of the following conditions holds.*

(a) $a_i = b_i = c_i = 0$ *for* i *even,*

(b) $a_i = 0$ *for all* i,

(c) $a_i = a_1b_i$ *and* $c_i = c_1b_i$ *for* $i \geq 2$.

We have excluded from Conjecture 3.5 the Liénard systems that have any other centers, see [4, 13]. For instance the conditions $a_2 = a_1b_2$, $a_5 = a_4b_5/b_4$, $a_4 = a_3b_4/b_3$, $b_5 = 2b_2b_4/5$, $b_4 = 5b_2b_3/3$, $a_i = b_i = 0$ for $i \geq 6$ and $c_i = 0$ for all i give a center for system (1.4) that indeed is a center of a Liénard system because $h \equiv 0$.

The proofs of Theorems 3.3 and 3.4 are given in Sections 5 and 6 respectively. In fact the Cherkas polynomial differential systems with a center of fixed degree are more simple than the families with centers described in Theorem 2.1 as you can see in Section 6.

4 Proof of Theorem 3.2

Poincaré [21] showed that for a nondegenerated center of an analytic system there is always a local analytic change of coordinates of the form $u = x + o(|(x,y)|)$, $v = y + o(|(x,y)|)$ and an analytic function ψ which transforms the analytic system into

$$\dot{u} = -v(1 + \psi(u^2 + v^2)), \qquad \dot{v} = u(1 + \psi(u^2 + v^2)). \tag{4.1}$$

Hence the Poincaré normal form of any center of an analytic differential system is analytically reversible because it is invariant under the symmetry $(u, v, t) \to (-u, v, -t)$. Now doing a scaling of time we obtain the linear system $\dot{u} = -v$ and $\dot{v} = u$. This linear system is reducible by the polynomial map $(\bar{u}, \bar{v}) \mapsto (u^2, v)$. Deriving we have $\dot{\bar{u}} = 2u\dot{u} = -2uv = -2u\bar{v}$, and $\dot{\bar{v}} = u$. Finally doing a scaling of time we obtain the nonsingular differential equation

$$\dot{\bar{u}} = -2\bar{v}, \qquad \dot{\bar{v}} = 1.$$

Consequently any analytic differential system with a non degenerate center is analytically reducible in a neighborhood of this center because by an analytic change we arrive to the normal form (4.1) and later via an algebraic map and a scaling of time we reduce the system to a nonsingular differential equation.

5 Proof of Theorem 3.3

In order to compute the necessary conditions we use the classical method of construction of a formal first integral. In system (3.1) we introduce the change of variables $x = r\cos\theta$ and $y = r\sin\theta$ and we propose the Poincaré power series

$$H(r, \theta) = \sum_{m=2}^{\infty} H_m(\theta) r^m,$$

where $H_2(\theta) = 1/2$ and $H_m(\theta)$ are homogeneous trigonometric polynomials respect to θ of degree m. Imposing that this power series is a formal first integral of the transformed system we obtain

$$\dot{H}(r, \theta) = \sum_{k=2}^{\infty} V_{2k} r^{2k},$$

where V_{2k} are the *focal values* which are polynomials in the parameters of system (3.1), see [15, 23]. The first nonzero focal value is $V_4 = -a_2 + a_1 b_2$. The second nonzero focal value is

$$V_6 = 4a_1^2 a_2 - 6a_4 - 4a_1^3 b_2 + 10a_3 b_2 + 5a_2 b_2^2 - 5a_1 b_2^3 + 3a_2 b_3 - 13a_1 b_2 b_3 + 6a_1 b_4 + 3a_2 c_1 - 5a_1 b_2 c_1 + 2a_1 c_2.$$

The size of the next focal values sharply increases so we do not present the other polynomials here but the reader can easily compute them. Due to the Hilbert basis theorem, the ideal $J = \langle V_4, V_6, \dots \rangle$ generated by the focal values is finitely generated, i.e. there exist $v_1, v_2, \dots, v_k$ in J such that $J = \langle v_1, v_2, \dots, v_k \rangle$. Such set of generators is a basis of J and the conditions $v_j = 0$ for $j = 1, \dots, k$ provide a finite set of necessary and sufficient conditions to have a center for system (3.1). We compute a certain number of focal values thinking that inside these number there is the set of generators. We decompose this algebraic set into its irreducible components using a computer algebra system. The computational tool used is the routine `minAssGTZ` [8] of the computer algebra system SINGULAR [17] which is based on the Gianni–Trager–Zacharias algorithm [10]. The computations have been completed in the field of rational numbers so we know that the decomposition of the center variety is complete.

To verify if the number of focal values computed a priori is enough to generate the full ideal $B := \langle V_{2k} : k \in \mathbb{N} \rangle$ we proceed as follows.

Let B_i be the ideal generated only by the first i focal values, i.e. $B_i = \langle V_4, \ldots, V_{2i} \rangle$. We want to determine s so that $V(B) = V(B_s)$, being V the variety of the ideals B and B_s, respectively. Using the Radical Membership Test [23] we can find when the computation stabilizes in the sense that $\sqrt{B_{s-1}} \subset \sqrt{B_s}$ but $\sqrt{B_s} = \sqrt{B_{s+1}}$. It is clear that $V(B) \subset V(B_s)$. However to verify the opposite inclusion we need to obtain the irreducible decomposition of the variety of $V(B_s)$ (given by the cases presented in the statement of the theorem) and check that any point of each component corresponds to a system having a center at the origin.

As all the centers are particular cases of Theorem 3.4 the proof of the sufficient conditions can be followed in the proof of Theorem 3.4 given in the next section.

6 Proof of Theorem 3.4

For statement (a) we have that system (3.1) with $a_2 = b_2 = a_4 = b_4 = a_6 = b_6 = 0$ is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$ and therefore the phase portrait is symmetric respect to a line passing through the origin and consequently it has a center at the origin.

Under the assumptions of statement (b) system (3.1) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - b_2x^2 - b_3x^3 - b_4x^4 - (c_1x + c_2x^2 + c_3x^3 + c_4x^4)y^2, \end{aligned} \tag{6.1}$$

which is invariant by the symmetry $(x, y, t) \to (x, -y, -t)$ and therefore it has a center at the origin.

Under the assumptions of statement (c) system (3.1) takes the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + b_2x + b_3x^2 + b_4x^3)(1 + a_1y + c_1y^2), \end{aligned} \tag{6.2}$$

which is a system that defines an equation of separable variables and has a first integral of the form

$$H(x, y) = e^{c_1(30x^2+20b_2x^3+15b_3x^4+12b_4x^5) - \frac{60a_1 \arctan \frac{a_1+2c_1y}{\sqrt{4c_1-a_1^2}}}{\sqrt{4c_1-a_1^2}}} (1 + a_1y + c_1y^2)^{30},$$

if $4c_1 - a_1^2 \neq 0$ and if $4c_1 - a_1^2 = 0$ it has a first integral of the form

$$H(x, y) = e^{a_1^2x^2(30+20b_2x+15b_3x^2+12b_4x^3) + \frac{480}{2+a_1y}} (2 + a_1y)^{240}.$$

Both first integrals are well-defined in a neighborhood of the origin, and then system (3.1) has a center at the origin.

7 Necessary conditions for system (3.1)

Due to its complexity, we are not able to compute the decomposition of the ideal generated only by the first i focal values B_i over the rational field for system (3.1). Hence we use modular arithmetics. In fact the decomposition is obtained over characteristic 32003. We go back to the rational numbers using the rational reconstruction algorithm of Wang et al. [25].

Because we have used modular arithmetics we must check if the decomposition is complete and no component is lost. In order to do that let P_i denotes the polynomial defining each component. Using the instruction `intersect` of Singular we compute the intersection $P = \cap_i P_i = \langle p_1, \ldots, p_m \rangle$. By the Strong Hilbert Nullstellensatz (see for instance [23]) to check whether $V(B_j) = V(P)$ it is sufficient to check if the radicals of the ideals are the same, that is, if $\sqrt{B_j} = \sqrt{P}$. Computing over characteristic 0 reducing Gröbner bases of ideals $\langle 1 - wV_{2k}, P : V_{2k} \in B_j \rangle$ we find that each of them is $\{1\}$. By the Radical Membership Test this implies that $\sqrt{B_j} \subseteq \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subseteq \sqrt{B_j}$ it is sufficient to check that

$$\langle 1 - wp_k, B_j : k = 1, \ldots, m \rangle = \langle 1 \rangle. \tag{7.1}$$

Using the Radical Membership Test to check if (7.1) is true, we were not able to complete computations working in the field of characteristic zero. However we have checked that (7.1) holds in several polynomial rings over fields of finite characteristic. It means that (7.1) and consequently $V(B_j) = V(P)$ holds with high probability, see [22].

We have performed the same computations for higher-degree Cherkas polynomial differential systems with restrictions in the parameters getting the same results which has led us to establish Conjecture 3.5.

Acknowledgements

The first author is partially supported by a MINECO/FEDER grant number MTM2014-53703-P and an AGAUR (Generalitat de Catalunya) grant number 2014SGR 1204. The second author is partially supported by a MINECO grant MTM2013-40998-P, an AGAUR grant 2014SGR 568, and two grants FP7-PEOPLE-2012-IRSES numbers 316338 and 318999.

References

[1] L. A. Cherkas, On the conditions for a center for certain equations of the form $yy' = P(x) + Q(x)y + R(x)y^2$, *Differ. Uravn.* **8**(1972), 1435–1439; *Differ. Equ.* **8**(1972), 1104–1107. MR0308510

[2] L. A. Cherkas, Conditions for a center for the equation $P_3(x)yy' = \sum_{i=0}^{2} P_i(x)y^i$, *Differ. Uravn.* **10**(1974), 367–368; *Differ. Equ.* **10**(1974), 276–277. MR0340708

[3] L. A. Cherkas, Conditions for a center for a certain Liénard equation, *Differ. Uravn.* **12**(1976), 292–298; *Differ. Equ.* **12**(1976), 201–206. MR0404755

[4] C. J. Christopher, An algebraic approach to the classification of centres in polynomial Liénard systems, *J. Math. Anal. Appl.* **229**(1999), 319–329. MR1664344

[5] C. J. Christopher, C. Li, *Limit cycles of differential equations*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser-Verlag, Basel, 2007. MR2325099

[6] C. J. Christopher, D. Schlomiuk, On general algebraic mechanisms for producing centers in polynomial differential systems, *J. Fixed Point Theory Appl.* **3**(2008), No. 2, 331–351. MR2434452

[7] F. Dumortier, J. Llibre, J. C. Artés, *Qualitative theory of planar differential systems*, UniversiText, Springer-Verlag, New York, 2006. MR2256001

[8] W. Decker, S. Laplagne, G. Pfister, H. A. Schonemann, SINGULAR 3-1 library for computing the prime decomposition and radical of ideals, primdec.lib, 2010.

[9] I. A. García, J. Giné, Generalized cofactors and nonlinear superposition principles, *Appl. Math. Lett.* **16**(2003), No. 7, 1137–1141. MR2013085

[10] P. Gianni, B. Trager, G. Zacharias, Gröbner bases and primary decompositions of polynomials, *J. Symbolic Comput.* **6**(1988) 146–167. MR0988410

[11] J. Giné, On some open problems in planar differential systems and Hilbert's 16th problem, *Chaos Solitons Fractals* **31**(2007), No. 5, 1118–1134. MR2261479

[12] J. Giné, Reduction of integrable planar polynomial differential systems, *Appl. Math. Lett.* **25**(2012), No. 11, 1862–1865. MR2957768

[13] J. Giné, Center conditions for polynomial Liénard systems, *Qual. Theory Dyn. Syst.*, to appear. url

[14] J. Giné, J. Llibre, C. Valls, Centers for the Kukles homogeneous systems with odd degree, *Bull. Lond. Math. Soc.* **47**(2015), No. 2, 315–324. MR3335125

[15] J. Giné, X. Santallusia, Implementation of a new algorithm of computation of the Poincaré–Liapunov constants, *J. Comput. Appl. Math.* **166**(2004), No. 2, 465–476. MR2041193

[16] N. Glade, L. Forest, J. Demongeot, Liénard systems and potential-Hamiltonian decomposition. III. Applications, *C. R. Math. Acad. Sci. Paris* **344**(2007), No. 4, 253–258. MR2292997

[17] G. M. Greuel, G. Pfister, H. A. Schönemann, SINGULAR 3.0 *A computer algebra system for polynomial computations*, Centre for Computer Algebra, University of Kaiserlautern, http://www.singular.uni-kl.de, 2005.

[18] J. Llibre, A survey on the limit cycles of the generalization polynomial Liénard differential equations, in: *Mathematical models in engineering, biology and medicine*, AIP Conf. Proc., Vol. 1124, Amer. Inst. Phys., Melville, NY, 2009, pp. 224–233. MR2657149

[19] N. G. Lloyd, J. M. Pearson, Computing centre conditions for certain cubic systems, *J. Comp. Appl. Math.* **40**(1992), 323–336. MR1170911

[20] G. R. Nicklason, Constant invariant solutions of the Poincaré center-focus problem, *Electron. J. Differential Equations* **2010**, No. 130, 1–11. MR2685040

[21] H. Poincaré, Sur l'integration algébrique des équations différentielles du premier ordre et du premier degré (I–II) (in French) [On the algebraic integration of differential equations of the first order and first degree], *Rend. Circ. Mat. Palermo* **5**(1891), 161–191, url; **11**(1897), 193–239, url.

[22] V. G. Romanovski, M. Prešern, An approach to solving systems of polynomials via modular arithmetics with applications, *J. Comput. Appl. Math.* **236**(2011), No. 2, 196–208. MR2827401

[23] V. G. Romanovski, D. S. Shafer, *The center and cyclicity problems: a computational algebra approach*, Birkhäuser, Boston, 2009. MR2500203

[24] A. P. Sadovskii, Solution of the center and focus problem for a cubic system of nonlinear oscillations (in Russian), *Differ. Uravn.* **33**(1997), No. 2, 236–244, 286; *Differential Equations* **33**(1997), No. 2, 236–244. MR1609880

[25] P. S. Wang, M. J. T. Guy, J. H. Davenport, P-adic reconstruction of rational numbers, *SIGSAM Bull.* **16**(1982), No. 2, 2–3. url

[26] H. Żołądek, The solution of the center problem, preprint, 1992.