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On the construction of the approximate solution of a special type integral boundary value problem

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Abstract. We consider the integral boundary value problem (BVP) for a certain class of non-linear system of ordinary differential equations of the form

$$\frac{dx(t)}{dt} = f(t, x(t)),$$
$$Ax(0) + \int_0^T P(s)k(s, x(s))ds + Cx(T) = d,$$

where $t \in [0, T]$, $x \in \mathbb{R}^n$, $f : [0, T] \times D \to \mathbb{R}^n$ and $k : [0, T] \times D \to \mathbb{R}^n$ are continuous vector functions, $D \subset \mathbb{R}^n$ is a closed and bounded domain, A, C and d are arbitrary matrices and vector with real components, $\det C \neq 0$.
We give a new approach for studying this problem, namely by using an appropriate parametrization technique the original BVP is reduced to the equivalent parametrized two-point one with linear restrictions without integral term.
To study the transformed problem we use a method based upon a special type of successive approximations constructed analytically.

Keywords: integral boundary value problems, parametrization, numerical–analytic technique, successive approximations.

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1 Notations

- Operations $=, <, >, \leq, \geq$, max, min between matrices and vectors mean component-wise;
- $L(\mathbb{R}^n)$ is an algebra of n-dimensional matrices with real components;
- I_n and O_n are unit and zero n-dimensional matrices, respectively;
- for any vector $u \in \mathbb{R}^n$ and non-negative vector $r \in \mathbb{R}^n$ we write

$$B(u, r) := \{\xi \in \mathbb{R}^n : |\xi - u| \leq r\}$$

as an r-neighborhood of $u \in \mathbb{R}^n$;
- $r(K)$ – spectral radius of matrix K.

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2 Problem setting and parametrization of the integral boundary conditions

Let us investigate the solutions of the system of nonlinear differential equations subjected to the special type integral boundary conditions of the form:

$$\frac{dx(t)}{dt} = f(t, x(t)), \tag{2.1}$$

$$Ax(0) + \int_0^T P(s)k(s, x(s))ds + Cx(T) = d, \tag{2.2}$$

where $t \in [0, T]$, $f : [0, T] \times D \to \mathbb{R}^n$, $A, C \in L(\mathbb{R}^n)$, $\det C \neq 0$, $k : [0, T] \times D \to \mathbb{R}^n$, $d \in \mathbb{R}^n$ are some given matrices and vector and P is a continuous n-dimensional matrix function.

Suppose that the vector function f in the right-hand side of the system of differential equations is continuous, where $D \subset \mathbb{R}^n$ is a closed and bounded domain, and let us put

$$D_0 := \left\{ \int_0^T P(s)k(s, x(s))ds \,\middle|\, P(\cdot)k(\cdot, x(\cdot)) \in C(\mathbb{R}^n) \right\}.$$

The problem is to find the continuously differentiable solution $x : [0, T] \to D$ of the system of differential equations (2.1) satisfying integral boundary restrictions (2.2).

To study this problem we use a technique suggested in [1–10].

Using the main ideas from [4], let us introduce the following parameters by putting

$$z := x(0) = \operatorname{col}(x_1(0), x_2(0), \dots, x_n(0)) = \operatorname{col}(z_1, z_2, \dots, z_n), \tag{2.3}$$

$$\lambda := \int_0^T P(s)k(s, x(s))ds = \operatorname{col}(\lambda_1, \lambda_2, \dots, \lambda_n). \tag{2.4}$$

Taking into account (2.3), the integral boundary restrictions (2.2) can be written as the linear ones:

$$Ax(0) + Cx(T) = d(\lambda), \tag{2.5}$$

where $d(\lambda) := d - \lambda$ and λ is the vector parameter given by (2.3).

So, instead of the original BVP with integral boundary conditions (2.1), (2.2) we study an equivalent parametrized one, containing already linear restrictions (2.1), (2.5).

Remark 2.1. The set of the solutions of the non-linear BVP with integral boundary conditions (2.1), (2.2) coincides with the set of the solutions of the parametrized problem (2.1) with linear boundary restrictions (2.5), satisfying additional conditions (2.3).

3 Construction of the successive approximations and their uniform convergence

Let us introduce the vector

$$\delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t, x) - \min_{(t,x)\in[0,T]\times D} f(t, x)\right], \tag{3.1}$$

and assume that the BVP (2.1), (2.5) satisfies following conditions:

A) there exists a set $D_\beta \subset D$ such that

$$D_\beta := \left\{ z \in D : B\left(z + \frac{t}{T} C^{-1} \left[d(\lambda) - (A + C) z\right], \ \frac{T}{2}\delta_D(f)\right) \subset D, \ \forall \lambda \in D_0, \ t \in [0, T] \right\}$$

is non-empty, i.e.,

$$D_\beta \neq \emptyset, \tag{3.2}$$

for all $\lambda \in D_0$, $t \in [0, T]$;

B) function f in the right-hand side of (2.1) satisfies Lipschitz condition of the form

$$|f(t, u) - f(t, v)| \leq K |u - v|, \tag{3.3}$$

for all $t \in [0, T]$, $\{u, v\} \subset D$ with some non-negative constant matrix $K = (k_{ij})_{i,j=1}^{n}$;

C) the spectral radius $r(Q)$ satisfies the inequality

$$r(Q) < 1, \tag{3.4}$$

where

$$Q := \frac{3TK}{10}. \tag{3.5}$$

Let us connect with the parametrized BVP (2.1), (2.5) the sequence of functions:

$$\begin{aligned} x_m(t, z, \lambda) := z &+ \int_0^t f(s, x_{m-1}(s, z, \lambda)) ds \\ &- \frac{t}{T} \int_0^T f(s, x_{m-1}(s, z, \lambda)) ds \\ &+ \frac{t}{T} C^{-1} \left[d(\lambda) - (A + C) z\right], \end{aligned} \tag{3.6}$$

where $m \in \mathbb{N}$,

$$x_0(t, z, \lambda) := z + \frac{t}{T} C^{-1} \left[d(\lambda) - (A + C) z\right],$$

$x_m(t, z, \lambda) = \text{col}\,(x_{m,1}(t, z, \lambda), x_{m,2}(t, z, \lambda), \ldots, x_{m,n}(t, z, \lambda))$ and z, λ are considered as parameters.

Note that the functions x_m of the sequence (3.6) were built from the linear parametrized boundary conditions (2.5), so they satisfy them for all $m \in \mathbb{N}$, $z \in D_\beta$, $\lambda \in D_0$.

Similarly to [4], let us establish the uniform convergence of the sequence (3.6).

Theorem 3.1. *Assume that for the system of differential equations (2.1) and the parametrized boundary restrictions (2.5) conditions A)–C) are satisfied.*

Then for all fixed $z \in D_\beta$, $\lambda \in D_0$ the following hold.

1. *The functions of the sequence (3.6) are continuously differentiable and satisfy the parametrized boundary conditions (2.5):*

$$A x_m(0, z, \lambda) + C x_m(T, z, \lambda) = d(\lambda), \qquad m \in \mathbb{N}.$$

2. *The sequence of functions (3.6) for $t \in [0, T]$ converges uniformly as $m \to \infty$ to the limit function*

$$x_\infty(t, z, \lambda) = \lim_{m \to \infty} x_m(t, z, \lambda). \tag{3.7}$$

3. *The limit function x_∞ satisfies the parametrized linear two-point boundary conditions:*

$$Ax_\infty(0,z,\lambda) + Cx_\infty(T,z,\lambda) = d(\lambda).$$

4. *The limit function (3.7) is a unique continuously differentiable solution of the integral equation*

$$\begin{aligned} x(t) = z &+ \int_0^t f(s,x(s))ds - \frac{t}{T}\int_0^T f(s,x(s))ds \\ &+ \frac{t}{T}C^{-1}\left[d(\lambda) - (A+C)\,z\right], \end{aligned} \tag{3.8}$$

i.e., it is the unique solution on $[0,T]$ of the Cauchy problem for the modified system of differential equations:

$$\frac{dx}{dt} = f(t,x) + \Delta(z,\lambda), \tag{3.9}$$

$$x(0) = z, \tag{3.10}$$

where

$$\Delta(z,\lambda) := \frac{1}{T}\left[C^{-1}\left[d(\lambda) - (A+C)\,z\right] - \int_0^T f(s,x_\infty(s,z,\lambda))ds\right]. \tag{3.11}$$

5. *The following error estimation holds:*

$$|x_\infty(t,z,\lambda) - x_m(t,z,\lambda)| \le \frac{20}{9}t\left(1-\frac{t}{T}\right)Q^m(I_n - Q)^{-1}\delta_D(f), \tag{3.12}$$

where $\delta_D(f)$ and Q are given by (3.1), (3.5).

Proof. Let us prove that the sequence of functions (3.6) is a Cauchy sequence in the Banach space $C([0,T],\mathbb{R}^n)$. First we show that $x_m(t,z,\lambda) \in D$, for all $(t,z,\lambda) \in [0,T] \times D_\beta \times D_0$, $m \in \mathbb{N}$. Let us note, that the function $x_0(t,z,\lambda) \in D$ for all $(t,z,\lambda) \in [0,T] \times D_\beta \times D_0$.

Then, using the estimation from [5]:

$$\left|\int_0^t \left[h(\tau) - \frac{1}{T}\int_0^T h(s)ds\right]d\tau\right| \le \frac{1}{2}\alpha_1(t)\left[\max_{t\in[0.T]} h(t) - \min_{t\in[0,T]} h(t)\right], \tag{3.13}$$

where

$$\alpha_1(t) := 2t\left(1-\frac{t}{T}\right), \qquad |\alpha_1(t)| \le \frac{T}{2}, \qquad t \in [0,T], \tag{3.14}$$

relation (3.6) for $m = 0$ implies that:

$$\begin{aligned} &|x_1(t,z,\lambda) - x_0(t,z,\lambda)| \\ &\quad \le \left|\int_0^t \left[f(s,x_0(s,z,\lambda)) - \frac{1}{T}\int_0^T f(\xi,x_0(\xi,z,\lambda))d\xi\right]ds\right| \\ &\quad \le \alpha_1(t)\delta_D(f) \le \frac{T}{2}\delta_D(f). \end{aligned} \tag{3.15}$$

Therefore, by virtue of (3.15), we conclude that $x_1(t,z,\lambda) \in D$ whenever $(t,z,\lambda) \in [0,T] \times D_\beta \times D_0$.

By induction we can easily establish that all functions (3.6) are also contained in the domain D, $\forall m \in \mathbb{N}$, $t \in [0,T]$, $z \in D_\beta$, $\lambda \in D_0$.

Now, consider the difference of functions:

$$\begin{aligned}
&x_{m+1}(t,z,\lambda) - x_m(t,z,\lambda)\\
&\quad = \int_0^t \left[f(s,x_m(s,z,\lambda)) - f(s,x_{m-1}(s,z,\lambda))\right]ds\\
&\quad\quad -\frac{t}{T}\int_0^T \left[f\left(s,x_m(s,z,\lambda)\right) - f\left(s,x_{m-1}(s,z,\lambda)\right)\right]ds, \qquad m\in\mathbb{N},
\end{aligned}$$

and introduce the notation:

$$r_m(t,z,\lambda) := \left|x_m(t,z,\lambda) - x_{m-1}(t,z,\lambda)\right|, \qquad m\in\mathbb{N}.$$

By virtue of the estimation (3.13) and of the Lipschitz condition (3.3), we have:

$$r_{m+1}(t,z,\lambda) \le K\left[\left(1-\frac{t}{T}\right)\int_0^t r_m(s,z,\lambda)ds + \frac{t}{T}\int_t^T r_m(s,z,\lambda)ds\right], \qquad \forall\, m\in\mathbb{N}. \tag{3.16}$$

According to (3.15),

$$r_1(t,z,\lambda) = |x_1(t,z,\lambda) - x_0(t,z,\lambda)| \le \alpha_1(t)\delta_D(f).$$

Using the inequality from [5]

$$\alpha_{m+1}(t) \le \frac{10}{9}\left(\frac{3}{10}T\right)^m \alpha_1(t), \qquad m\in\mathbb{N}, \tag{3.17}$$

obtained for the sequence of functions

$$\alpha_{m+1}(t) = \left(1-\frac{t}{T}\right)\int_0^t \alpha_m(s)ds + \frac{t}{T}\int_t^T \alpha_m(s)ds, \qquad m\in\mathbb{N}, \tag{3.18}$$

from (3.16) for $m=1$ follows:

$$r_2(t,z,\lambda) \le K\delta_D(f)\left[\left(1-\frac{t}{T}\right)\int_0^t \alpha_1(s)ds + \frac{t}{T}\int_t^T \alpha_1(s)ds\right] \le K\alpha_2\left(t\right)\delta_D(f).$$

By induction using (3.18), we can easily obtain that

$$r_{m+1}(t,z,\lambda) \le K^m\alpha_{m+1}(t)\delta_D(f), \qquad m=0,1,2,\ldots, \tag{3.19}$$

where α_{m+1} is calculated according to (3.18), and $\delta_D(f)$ is given by (3.1).

By virtue of the estimate (3.17) from (3.19) we get:

$$r_{m+1}(t,z,\lambda) \le \frac{10}{9}\alpha_1(t)Q^m\delta_D(f), \tag{3.20}$$

$\forall m\in\mathbb{N}$, where matrix Q is given by (3.5).

Therefore, in view of (3.20):

$$\begin{aligned}
&\left|x_{m+j}(t,z,\lambda) - x_m(t,z,\lambda)\right|\\
&\quad\le \left|x_{m+j}(t,z,\lambda) - x_{m+j-1}(t,z,\lambda)\right|\\
&\quad\quad + \left|x_{m+j-1}(t,z,\lambda) - x_{m+j-2}(t,z,\lambda)\right| + \cdots\\
&\quad\quad + \left|x_{m+1}(t,z,\lambda) - x_m(t,z,\lambda)\right|\\
&\quad = \sum_{i=1}^{j} r_{m+i}(t,z,\lambda) \le \frac{10}{9}\alpha_1(t)\sum_{i=1}^{j} Q^{m+i-1}\delta_D(f)\\
&\quad = \frac{10}{9}\alpha_1(t)Q^m\sum_{i=0}^{j-1} Q^i\delta_D(f).
\end{aligned} \tag{3.21}$$

Since, due to the condition (3.4), the maximum eigenvalue of the matrix Q of the form (3.5) does not exceed the unity, we have

$$\sum_{i=0}^{j-1} Q^i \leq (I_n - Q)^{-1}, \qquad \lim_{m\to\infty} Q^m = O_n.$$

Therefore, we conclude from (3.21) that, according to the Cauchy criterion, the sequence $\{x_m\}$ of the form (3.6) uniformly converges in the domain $(t, z, \lambda) \in [0, T] \times D_\beta \times D_0$ to the limit function x_∞. Since all functions x_m of the sequence (3.6) satisfy the boundary conditions (2.5) for all values of the artificially introduced parameters, the limit function x_∞ also satisfies these conditions. Passing to the limit as $m \to \infty$ in equality (3.6) we show that for all $z \in D_\beta$ and $\lambda \in D_0$ the limit function $x_\infty(\cdot, z, \lambda)$ is a solution of both integral equation (3.8) and the Cauchy problem (3.9), (3.10) with Δ given by (3.11). The uniqueness of $x_\infty(\cdot, z, \lambda)$ follows from the Lipschitz condition imposed on the function f. □

4 Connection of the limit function x_∞ with the solution of the BVP (2.1), (2.2)

Consider the Cauchy problem

$$\frac{dx}{dt} = f(t, x) + \mu, t \in [0, T], \tag{4.1}$$

$$x(0) = z, \tag{4.2}$$

where $\mu \in \mathbb{R}^n$ is a control parameter and $z \in D_\beta$.

By analogy to [4] let us prove the control parameter theorem.

Theorem 4.1. *Let $z \in D_\beta$, $\lambda \in D_0$ and $\mu \in \mathbb{R}^n$ be some given vectors. Suppose that for the system of differential equations (2.1) all conditions of Theorem 3.1 hold.*

Then for the solution $x = x(\cdot, z, \mu)$ of the initial value problem (4.1), (4.2) to be defined on $[0, T]$ and to satisfy boundary conditions (2.5), it is necessary and sufficient that μ satisfies

$$\mu := \mu_{z,\lambda}, \tag{4.3}$$

where

$$\mu_{z,\lambda} := \frac{1}{T}\left[C^{-1}[d(\lambda) - (A + C)z] - \int_0^T f(s, x_\infty(s, z, \lambda))\right] ds \tag{4.4}$$

and $x_\infty(\cdot, z, \lambda)$ is a function from the assertion 2. of Theorem 3.1.

In that case

$$x(t, z, \mu) = x_\infty(t, z, \lambda) \quad \text{for } t \in [0, T]. \tag{4.5}$$

Proof. Sufficiency. Let us suppose that $\mu = \mu_{z,\lambda}$ on the right-hand side of the system of differential equations (4.1) is given by (4.4). By virtue of Theorem 3.1, the limit function (3.7) of the sequence (3.6) is the unique solution of the initial value problem (4.1), (4.2). Furthermore, the limit function x_∞ satisfies (2.5).

Thus we have found the value of the parameter μ given by (4.4), for which (4.5) holds.

Necessity. Now we show that the parameter value (4.4) is unique, i.e., that for any $\mu = \bar{\mu} \neq \mu_{z,\lambda}$ the solution $x(t, z, \bar{\mu})$ of the initial value problem (4.6), (4.2), where

$$\frac{dx}{dt} = f(t, x) + \bar{\mu}, \tag{4.6}$$

does not satisfy boundary condition (2.5).

Indeed, assume the contrary. Then there exists $\bar{\mu} \in \mathbb{R}^n$ such that $\mu_{z,\lambda} \neq \bar{\mu}$ and the solution $\bar{x}(\cdot) := x(\cdot, z, \bar{\mu})$ of the Cauchy problem (4.1)–(4.2) is defined on $[0, T]$ and satisfies boundary condition (2.5).

Moreover, put $x_{z,\lambda}(\cdot) := x(\cdot, z, \mu_{z,\lambda})$.

It is obviously that the functions $x_{z,\lambda}$ and $\bar{x}$ satisfy the following integral equations

$$x_{z,\lambda}(t) = z + \int_0^t f(s, x_{z,\lambda}(s))ds + \mu_{z,\lambda} t \tag{4.7}$$

and

$$\bar{x}(t) = z + \int_0^t f(s, \bar{x}(s))ds + \bar{\mu} t. \tag{4.8}$$

By assumption, the functions $x_{z,\lambda}$, $\bar{x}$ satisfy parametrized boundary conditions (2.5) and the initial conditions (4.2). That is why we have

$$Ax_{z,\lambda}(0) + Cx_{z,\lambda}(T) = d(\lambda), \tag{4.9}$$

$$x_{z,\lambda}(0) = z, \tag{4.10}$$

$$A\bar{x}(0) + C\bar{x}(T) = d(\lambda), \tag{4.11}$$

$$\bar{x}(0) = z. \tag{4.12}$$

Taking into account (4.9)–(4.12) we get

$$x_{z,\lambda}(T) = C^{-1}[d(\lambda) - Az], \tag{4.13}$$

$$\bar{x}(T) = C^{-1}[d(\lambda) - Az]. \tag{4.14}$$

By virtue of (4.7), (4.8) for $t = T$, $\mu_{z,\lambda}$ and μ can be written as

$$\mu_{z,\lambda} = \frac{1}{T}\left[C^{-1}[d(\lambda) - (A + C)z] - \int_0^T f(s, x_{z,\lambda}(s))ds\right], \tag{4.15}$$

$$\bar{\mu} = \frac{1}{T}\left[C^{-1}[d(\lambda) - (A + C)z] - \int_0^T f(s, \bar{x}(s))ds\right]. \tag{4.16}$$

Substituting (4.15), (4.16) into the integral equations (4.7), (4.8), it follows that for all $t \in [0, T]$

$$x_{z,\lambda}(t) = z + \int_0^t f(s, x_{z,\lambda}(s))ds + \frac{t}{T}\left[C^{-1}[d(\lambda) - (A + C)z] - \int_0^T f(s, x_{z,\lambda}(s))ds\right], \tag{4.17}$$

$$\bar{x}(t) = z + \int_0^t f(s, \bar{x}(s))ds + \frac{t}{T}\left[C^{-1}[d(\lambda, \eta) - (A + C)z] - \int_0^T f(s, \bar{x}(s))ds\right]. \tag{4.18}$$

Using (4.17), (4.18) it is obvious that

$$x_{z,\lambda}(t) - \bar{x}(t) = \int_0^t [f(s, x_{z,\lambda}(s)) - f(s, \bar{x}(s))]ds - \frac{t}{T}\int_0^T [f(s, x_{z,\lambda}(s)) - f(s, \bar{x}(s))]ds. \tag{4.19}$$

On the basis of the Lipschitz condition (3.3), from the relation (4.19) we get that the function

$$\omega(t) = |x_{z,\lambda}(t) - \bar{x}(t)|, \qquad t \in [0, T], \tag{4.20}$$

satisfies integral inequalities:

$$\omega(t) \le K\left[\int_0^t \omega(s)ds + \frac{t}{T}\int_0^T \omega(s)ds\right] \le K\alpha_1(t)\max_{s\in[0,T]}\omega(s), \qquad t\in[0,T], \tag{4.21}$$

where α_1 is given by (3.14).

Using (4.19) recursively, we come to an inequality:

$$\omega(t) \le K^m\alpha_m(t)\max_{s\in[0,T]}\omega(s), \qquad t\in[0,T], \tag{4.22}$$

where $m \in \mathbb{N}$ and functions α_m are given by the formula (3.18).

Taking into account (3.17), from (4.22) for each $m \in \mathbb{N}$ we get an estimation:

$$\omega(t) \le K\alpha_1(t)\frac{10}{9}\left(\frac{3T}{10}K\right)^{m-1} \cdot \max_{s\in[0,T]}\omega(s), \qquad t\in[0,T].$$

By passing to the limit as $m \to \infty$ in the last inequality and by virtue of (3.4), we come to the conclusion that

$$\max_{s\in[0,T]}\omega(s) \le Q^m \max_{s\in[0,T]}\omega(s) \to 0.$$

It means, according to (4.20), that the function $x_{z,\lambda}$ coincides with $\bar{x}$. Starting with (4.15) and (4.16), we get that $\mu_{z,\lambda} = \bar{\mu}$. The contradiction we received proves the necessity part of the theorem. □

Let us find out the relation of the limit function $x = x_\infty(\cdot, z, \lambda)$ of the sequence (3.6) to the solution of the parametrized two-point BVP (2.1) with linear boundary conditions (2.5) or the equivalent non-linear problem (2.1) with integral conditions (2.2) [4].

Theorem 4.2. *Let the conditions A)–C) hold for the original BVP (2.1), (2.2).*

Then $x_\infty(\cdot, z^*, \lambda^*)$ is a solution of the integral BVP (2.1), (2.2) if and only if the pair (z^*, λ^*) is a solution of the determining system of algebraic or transcendental equations:

$$\Delta(z,\lambda) = 0, \tag{4.23}$$

$$V(z,\lambda) = 0, \tag{4.24}$$

where

$$\Delta(z,\lambda) := \frac{1}{T}\left[C^{-1}\left[d(\lambda) - (A+C)z\right] - \int_0^T k(s, x_\infty(s,z,\lambda))ds\right], \tag{4.25}$$

$$V(z,\lambda) := \int_0^T P(s)k(s, x_\infty(s,z,\lambda))ds - \lambda. \tag{4.26}$$

Proof. It suffices to apply Theorem 3.1 and notice that $x_\infty(\cdot, z^*, \lambda^*)$ is a solution of (2.1) if and only if the pair (z^*, λ^*) satisfies the equation

$$\Delta(z^*, \lambda^*) = 0.$$

Moreover, taking into account (2.3), it is clear that $x_\infty(\cdot, z^*, \lambda^*)$ satisfies (2.2), if and only if

$$\int_0^T P(s)k(s, x_\infty(\cdot, z^*, \lambda^*))ds = \lambda^*,$$

It means that $x_\infty(\cdot, z^*, \lambda^*)$ is a solution of the integral BVP (2.1), (2.2) if and only if (z^*, λ^*) is a solution of system (4.23), (4.24).

□

The next statement proves that the system of determining equations (4.23), (4.24) defines all possible solutions of the original non-linear BVP (2.1) with integral boundary restrictions (2.2).

Theorem 4.3. *Let all the assumptions of Theorem 3.1 be satisfied. Then the following assertions hold.*

1. *If vectors $z \in D_\beta$, $\lambda \in D_0$ satisfy the system of determining equations (4.23), (4.24), then the non-linear BVP (2.1) with integral boundary conditions (2.2) has the solution $x(\cdot)$ such that*

$$x(0) = z,$$
$$\int_0^T P(s)k(s, x(s))ds = \lambda.$$

Moreover, this solution is given by formula

$$x(t) = x_\infty(t, z, \lambda), \qquad t = [0, T], \tag{4.27}$$

where x_∞ is the limit function of the sequence (3.6).

2. *If BVP (2.1), (2.2) has a solution $x(\cdot)$, then this solution is given by (4.27), and the system of determining equations (4.23), (4.24) is satisfied with*

$$z = x(0),$$
$$\lambda = \int_0^T P(s)k(s, x(s))ds.$$

Proof. We will apply Theorems 4.1 and 4.2. If there exist such $z \in D_\beta$, $\lambda \in D_0$ that satisfy determining system (4.23), (4.24), then according to Theorem 4.2, function (4.27) is a solution of the given BVP (2.1), (2.2) and, in view of Theorem 3.1, we get $x(0) = z$ and $\int_0^T P(s)k(s, x(s))ds = \lambda$. On the other hand, if $x(\cdot)$ is the solution of the original BVP (2.1), (2.2), then this function is the solution of the Cauchy problem (4.1), (4.2) for

$$\mu = 0,$$
$$z = x(0).$$

As $x(\cdot)$ satisfies integral boundary restrictions (2.2) and equivalent condition (2.5) with

$$\lambda := \int_0^T P(s)k(s, x(s))ds, \tag{4.28}$$

by virtue of Theorem 4.1, equality (4.27) is holds. Besides,

$$\mu = \mu_{z,\lambda},$$
$$z = x(0),$$

where $\mu_{z,\lambda}$ is given by (4.4). Therefore, the first equation (4.23) of the determining system is satisfied.

Taking into account the above-proved equality (4.27), it follows from (4.28) that the second equation of the determining system also holds. □

5 Remarks on the constructive applications of the method

Although Theorem 4.2 gives sufficient and necessary conditions for the solvability and construction of the solution of the given BVP, its application faces with difficulties due the fact that the explicit forms of the functions

$$\Delta : D_\beta \times D_0 \to \mathbb{R}^n,$$
$$V : D_\beta \times D_0 \to \mathbb{R}^n,$$
$$x_\infty(\cdot, z, \lambda) = \lim_{m\to\infty} x_m(\cdot, z, \lambda),$$

in (4.23), (4.24) are usually unknown.

This complication can be overcome by using the properties of the function $x_m(\cdot, z, \lambda)$ of the form (3.6) for a fixed m, which will lead one instead of the exact determining system (4.23), (4.24) to the m-th approximate system of determining equations of the form:

$$\Delta_m(z, \lambda) = 0, \tag{5.1}$$
$$V_m(z, \lambda) = 0, \tag{5.2}$$

where $\Delta_m, V_m : D_\beta \times D_0 \to \mathbb{R}^n$ are defined by the determining function given by formulae

$$\Delta_m(z, \lambda) := \frac{1}{T}\left[C^{-1}\left[d(\lambda) - (A + C)z\right] - \int_0^T f(s, x_m(s, z, \lambda))ds\right],$$
$$V_m(z, \lambda) := \int_0^T P(s)k(s, x_m(s, z, \lambda))ds - \lambda,$$

and $x_m(\cdot, z, \lambda)$ is a vector function, that is defined by the recursive relation (3.6).

It is important to note that, unlike to system (4.23), (4.24), the m-th approximate determining system (5.1), (5.2) contains only terms involving the function x_m and, thus known explicitly.

6 An illustrative example

Let us apply the numerical–analytic scheme described above to the system of differential equations

$$\begin{cases} x_1'(t) = x_2, \\ x_2'(t) = x_1 + \frac{5}{2}x_2^2, \end{cases} \tag{6.1}$$

considered for $t \in [0, 1]$ with the two-point integral boundary conditions

$$Ax(0) + \int_0^1 P(s)f(s, x(s))ds + Cx(1) = d, \tag{6.2}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad d = \begin{pmatrix} 3/5 \\ 13/30 \end{pmatrix},$$

and

$$P(t) = \begin{pmatrix} 0 & t \\ t/2 & 1 \end{pmatrix}, \qquad t \in [0, 1].$$

It is easy to check that the pair of functions

$$x_1^*(t) = 0.1t^2 + 0.2, \qquad x_2^*(t) = 0.2t$$

is an exact solution of the problem (6.1), (6.2).

Suppose that the BVP (6.1), (6.2) is considered in the domain

$$D = \{(x_1, x_2) : |x_1| \leq 0.32,\ |x_2| \leq 0.25\}.$$

Following (2.3), introduce the parameters:

$$\text{col}\,(x_1(0), x_2(0)) =: \text{col}\,(z_1, z_2), \tag{6.3}$$

$$\int_0^T P(s) f(s, x(s)) ds =: \text{col}\,(\lambda_1, \lambda_2). \tag{6.4}$$

The formal substitution (6.3) transforms the boundary restrictions (6.2) to the linear conditions

$$Ax(0) + Cx\,(1) = d(\lambda), \tag{6.5}$$

where $d(\lambda) := d - \lambda$.

Put

$$f_1(t, x_1, x_2) := x_2,$$
$$f_2(t, x_1, x_2) := x_1 - \frac{5}{2} x_2^2.$$

Then (6.1) takes the form (2.1) with $T = 1$, $n = 2$, and it is then easy to check that the matrix K from the Lipschitz condition (3.3) can be taken as

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 1.25 \end{pmatrix}.$$

Calculations show that matrix $Q = \left(\begin{smallmatrix} 0 & 0.3 \\ 0.3 & 0.375 \end{smallmatrix}\right)$ and

$$r\,(Q) < 0.55 < 1.$$

The vector $\delta_D\,(f)$ can be estimated as

$$\delta_D\,(f) \leq \begin{pmatrix} 0.26 \\ 0.4 \end{pmatrix}.$$

The role of D_β is played by the domain defined by inequalities:

$$-0.5666666667 + \lambda_1 + 2z_1 \leq 0.125,$$
$$-0.3625 + \lambda_2 + z_2 \leq 0.1990625.$$

The domain D_0 is such that

$$D_0 := \{(\lambda_1, \lambda_2) : |\lambda_1| \leq 0.2,\ |\lambda_2| \leq 0.27\}.$$

One can verify that, for the parametrized BVP (6.1), (6.5), all the needed conditions are fulfilled, and we can proceed with application of the numerical–analytic scheme described above. As a result, we construct the sequence of approximate solutions.

The components of the iteration sequence (3.6) for the boundary value problem (6.1) under the linear parametrized two-point boundary conditions (6.5) have the form

$$
\begin{aligned}
x_{m,1}(t,z,\lambda) := z_1 &+ \int_0^t f_1\left(s, x_{m-1,1}(s,z,\lambda), x_{m-1,2}(s,z,\lambda)\right) ds \\
&- t\int_0^1 f_1\left(s, x_{m-1,1}(s,z,\lambda), x_{m-1,2}(s,z,\lambda)\right) ds \\
&+ t(0.5666666667 - \lambda_1 - 2z_1), \qquad (6.6)\\
x_{m,2}(t,z,\lambda) := z_2 &+ \int_0^t f_2\left(s, x_{m-1,1}(s,z,\lambda), x_{m-1,2}(s,z,\lambda)\right) ds \\
&- t\int_0^1 f_2\left(s, x_{m-1,1}(s,z,\lambda), x_{m-1,2}(s,z,\lambda)\right) ds \\
&+ t(0.3625 - \lambda_2 - z_2), \qquad (6.7)
\end{aligned}
$$

for $m = 1, 2, 3, \ldots,$ where

$$
\begin{aligned}
x_{0,1}(t,z,\lambda) &:= z_1 + t(0.5666666667 - \lambda_1 - 2z_1), \qquad (6.8)\\
x_{0,2}(t,z,\eta,\lambda) &:= z_2 + t(0.3625 - \lambda_2 - z_2). \qquad (6.9)
\end{aligned}
$$

The system of approximate determining equations of the form (5.1), (5.2) for the given example at the m-th step is

$$
\begin{aligned}
\Delta_{m,1}(z,\lambda) &= 0, \qquad (6.10)\\
\Delta_{m,2}(z,\lambda) &= 0, \qquad (6.11)\\
\int_0^1 P(s) f(s, x_m(s,z,\lambda)) ds &= \lambda, \qquad (6.12)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_{m,1}(z,\lambda,\eta) &:= -\int_0^1 f_1\left(s, x_{m-1,1}(s,z,\lambda), x_{m-1,2}(s,z,\lambda)\right) ds + 0.5666666667 - \lambda_1 - 2z_1,\\
\Delta_{m,2}(z,\lambda) &= -2\int_0^1 f_2\left(s, x_{m-1,1}(s,z,\lambda), x_{m-1,2}(s,z,\lambda)\right) ds + 0.3625 - \lambda_2 - z_2.
\end{aligned}
$$

Using (6.6)–(6.9) at the first iteration $(m = 1)$ and applying Maple 13, we get

$$
x_{11} = z_1 + 0.5z_2t + 0.18125t^2 - 0.5t^2\lambda_2 - 0.5t^2z_2 + 0.3854166667t + 0.5t\lambda_2 - t\lambda_1 - 2tz_1,
$$

and

$$
\begin{aligned}
x_{12} = {} & 0.1886718749t - 0.6979166666z_2t - 0.90625t^2z_2 - 1.604166667t\lambda_2\\
&+ 0.5t\lambda_1 + tz_1 - 1.666666667z_2^2t + 0.6041666666t^3\lambda_2\\
&+ 0.6041666666t^3z_2 - 0.8333333333t^3\lambda_2^2 - 0.8333333333t^3z_2^2\\
&- 0.5t^2\lambda_1 - t^2z_1 + 2.5t^2z_2^2 + 0.8333333333t\lambda_2^2\\
&+ 0.2833333334t^2 - 1.666666667t^3\lambda_2z_2 + 2.5t^2\lambda_2z_2 - 0.8333333334t\lambda_2z_2\\
&- 0.1095052083t^3 + z_2
\end{aligned}
$$

for all $t \in [0,1]$.

Here and below, we omit the obvious arguments reflecting the dependence on $z_1, z_2, \lambda_1, \lambda_2$.

The computation shows that the approximate solutions of the determining system (6.10)–(6.12) for $m = 1$ are

$$z_1 \approx z_{11} = 0.1997985545,$$
$$z_2 \approx z_{12} = 0.0003290208687,$$
$$\lambda_1 \approx \lambda_{11} = 0.06696248863,$$
$$\lambda_2 \approx \lambda_{12} = 0.1625831391.$$

Hence, the components of the first approximation to the first and second components of solution are

$$x_{11} = 0.1997985545 + 0.0003131491t + 0.09979392005t^2,$$

and

$$x_{12} = 0.0003290208687 + 0.1828945650t + 0.04988936318t^2 - 0.03319608821t^3.$$

The graphs of the first approximation and the exact solution of the original BVP are shown on Figure 6.1.

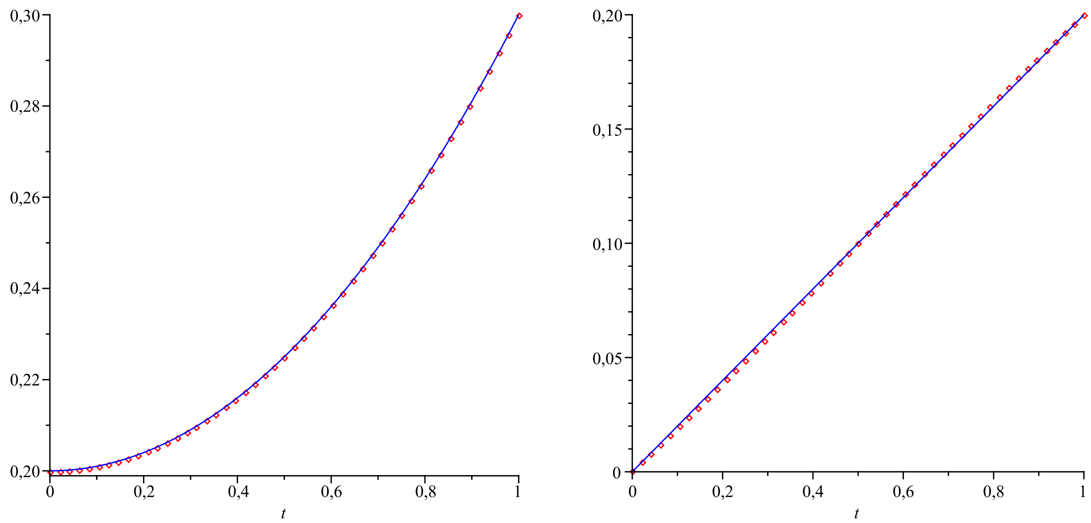

Figure 6.1: The first components of the exact solution (solid line) and its first approximation (drawn with dots).

The error of the first approximation is

$$\max_{t\in[0,1]} |x_1^*(t) - x_{11}(t)| \leq 2.1 \cdot 10^{-4},$$
$$\max_{t\in[0,1]} |x_2^*(t) - x_{12}(t)| \leq 1.4 \cdot 10^{-3}.$$

Similarly, the error of the third approximation is

$$\max_{t\in[0,1]} |x_1^*(t) - x_{31}(t)| \leq 2.1 \cdot 10^{-5},$$
$$\max_{t\in[0,1]} |x_2^*(t) - x_{32}(t)| \leq 5.1 \cdot 10^{-5}.$$

Continuing calculations, one can get approximate solutions of the original BVP with higher precision.

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