Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **38**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

Green's function and positive solutions of a singular nth-order three-point boundary value problem on time scales

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Abstract

In this paper, we investigate the existence of positive solutions for a class of singular nth-order three-point boundary value problem. The associated Green's function for the boundary value problem is given at first, and some useful properties of the Green's function are obtained. The main tool is fixed-point index theory. The results obtained in this paper essentially improve and generalize some well-known results.

Mathematics Subject Classification(2000): **34B16; 34B18; 34B27**.
Keywords: Green's function; Positive solution; Three-point boundary value problem; Cone; Singular.

1. Introduction

In recent years, the existence of positive solutions for higher-order boundary value problems has been studied by many authors using various methods (see [1-4, 7-14] and the references therein). For example, in paper [3], by using the Krasnosel'skii fixed point theorem, Eloe and Ahmad established the existence of at least one positive solution for the following nth-order three-point boundary value problem

$$\begin{cases} u^{(n)}(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases} \tag{1.1}$$

where $0 < \eta < 1$, $0 < \alpha\eta^{n-1} < 1$, $a : [0,1] \to [0,\infty)$ is continuous and nonsingular, $f \in C([0,\infty),[0,\infty))$ is either superlinear or sublinear.

In this paper, we study the existence of positive solutions for a singular nth-order three-point boundary value problem as follows

$$\begin{cases} u^{(n)}(t) + h(t)f(t,u(t)) = 0, \quad t \in [a,b], \\ u(a) = \alpha u(\eta), \quad u'(a) = 0, \cdots, u^{(n-2)}(a) = 0, \quad u(b) = \beta u(\eta), \end{cases} \tag{1.2}$$

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where $a < \eta < b$, $0 \le \alpha < 1$, $0 < \beta(\eta - a)^{n-1} < (1-\alpha)(b-a)^{n-1} + \alpha(\eta - a)^{n-1}$, $f \in C([a,b] \times [0,\infty), [0,\infty))$ and $h \in C([a,b], [0,+\infty))$ may be singular at $t = a$ and $t = b$.

Up to now, no paper has appeared in the literature which discusses the existence of positive solutions for the problem (1.2). This paper attempts to fill this gap in the literature. In order to obtain our result, we give at first the associated Green's function for the problem (1.2), which is the base for further discussion. Our results extend and improve the results of Eloe and Ahmad [3] ($\alpha = a = 0$, $b = 1$ and $f(t,u) = f(u)$). Our results are obtained under certain suitable weaker conditions than that in [3]. It is also noted that our method here is different from that of Eloe and Ahmad [3].

2. Expression and properties of Green's function

Lemma 2.1. *If $y \in C[a,b]$, then the problem*

$$\begin{cases} u^{(n)}(t) + y(t) = 0, \quad a \le t \le b, \\ u(a) = 0, \quad u'(a) = 0, \cdots, u^{(n-2)}(a) = 0, \quad u(b) = 0, \end{cases} \tag{2.1}$$

has a unique solution

$$w(t) = \int_a^b H(t,s) y(s) ds, \tag{2.2}$$

where

$$H(t,s) = \begin{cases} \dfrac{(b-s)^{n-1}(t-a)^{n-1} - (b-a)^{n-1}(t-s)^{n-1}}{(b-a)^{n-1}(n-1)!}, & a \le s \le t \le b, \\ \dfrac{(b-s)^{n-1}(t-a)^{n-1}}{(b-a)^{n-1}(n-1)!}, & a \le t < s \le b. \end{cases} \tag{2.3}$$

Proof. In fact, if $w(t)$ is a solution of the problem (2.1), then we may suppose that

$$w(t) = -\int_a^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) ds + A(t-a)^{n-1} + \sum_{i=1}^{n-2} A_i (t-a)^i + B.$$

Since $w^{(i)}(a) = 0$ for $i = 0, 1, 2, \cdots, n-2$, we get $B = A_i = 0$ for $i = 1, 2, \cdots, n-2$. By $w(b) = 0$, we have

$$A = \frac{1}{(b-a)^{n-1}} \int_a^b \frac{(b-s)^{n-1}}{(n-1)!} y(s) ds.$$

Therefore, the problem (2.1) has a unique solution

$$w(t) = -\int_a^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) ds + \int_a^b \frac{(b-s)^{n-1}(t-a)^{n-1}}{(b-a)^{n-1}(n-1)!} y(s) ds = \int_a^b H(t,s) y(s) ds,$$

where $H(t,s)$ is defined by (2.3).

Lemma 2.2. *$H(t,s)$ has the following properties*

(i) $0 \le H(t,s) \le k(s)$, $\forall t, s \in [a,b]$, *where*

$$k(s) = \frac{(s-a)(b-s)^{n-1}}{(b-a)(n-2)!};$$

(ii) $H(t,s) \ge \phi(t)k(s)$, $\forall t, s \in [a,b]$, *where*

$$\phi(t) = \begin{cases} \dfrac{(t-a)^{n-1}}{(n-1)(b-a)^{n-1}}, & a \le t \le \frac{a+b}{2}, \\ \dfrac{(b-t)(t-a)^{n-2}}{(n-1)(b-a)^{n-1}}, & \frac{a+b}{2} \le t \le b. \end{cases}$$

Proof. It is obvious that $H(t,s)$ is nonnegative. Moreover,

$$H(t,s) = \begin{cases} \dfrac{(b-s)^{n-1}(t-a)^{n-1} - (b-a)^{n-1}(t-s)^{n-1}}{(b-a)^{n-1}(n-1)!}, & a \le s \le t \le b, \\ \dfrac{(b-s)^{n-1}(t-a)^{n-1}}{(b-a)^{n-1}(n-1)!}, & a \le t < s \le b, \end{cases}$$

$$= \frac{1}{(b-a)^{n-1}(n-1)!}\begin{cases} (s-a)(b-t)\{[(t-a)(b-s)]^{n-2} \\ \quad +[(t-a)(b-s)]^{n-3}(b-a)(t-s) + \cdots \\ \quad +(t-a)(b-s)[(b-a)(t-s)]^{n-3} \\ \quad +[(b-a)(t-s)]^{n-2}\}, & a \le s \le t \le b, \\ (b-s)^{n-1}(t-a)^{n-1}, & a \le t < s \le b, \end{cases}$$

$$\le \frac{1}{(b-a)^{n-1}(n-1)!}\begin{cases} (n-1)(s-a)(b-s)[(b-a)(b-s)]^{n-2}, & a \le s \le t \le b, \\ (b-s)^{n-1}(s-a)^{n-1}, & a \le t < s \le b, \end{cases}$$

$$\le \frac{(s-a)(b-s)^{n-1}}{(b-a)(n-2)!} = k(s), \quad t, s \in [a,b].$$

Thus, (i) holds.

If $s = a$ or $s = b$, we easily see that (ii) holds. If $s \in (a,b)$ and $t \in [a,b]$, we have

$$\frac{H(t,s)}{k(s)} = \begin{cases} \dfrac{(b-s)^{n-1}(t-a)^{n-1} - (b-a)^{n-1}(t-s)^{n-1}}{(n-1)(s-a)(b-s)^{n-1}(b-a)^{n-2}}, & a \le s \le t \le b, \\ \dfrac{(t-a)^{n-1}}{(n-1)(s-a)(b-a)^{n-2}}, & a \le t < s \le b, \end{cases}$$

$$= \begin{cases} \dfrac{1}{(n-1)(s-a)(b-s)^{n-1}(b-a)^{n-2}}(s-a)(b-t)\{[(t-a)(b-s)]^{n-2} \\ \quad +[(t-a)(b-s)]^{n-3}(b-a)(t-s) + \cdots \\ \quad +(t-a)(b-s)[(b-a)(t-s)]^{n-3} + [(b-a)(t-s)]^{n-2}\}, & a \le s \le t \le b, \\ \dfrac{(t-a)^{n-1}}{(n-1)(s-a)(b-a)^{n-2}}, & a \le t < s \le b, \end{cases}$$

$$\ge \begin{cases} \dfrac{(s-a)(b-t)(t-a)^{n-2}(b-s)^{n-2}}{(n-1)(s-a)(b-s)^{n-1}(b-a)^{n-2}}, & a \le s \le t \le b, \\ \dfrac{(t-a)^{n-1}}{(n-1)(s-a)(b-a)^{n-2}}, & a \le t < s \le b, \end{cases}$$

$$\geq \begin{cases} \dfrac{(b-t)(t-a)^{n-2}}{(n-1)(b-a)^{n-1}}, & a \leq s \leq t \leq b, \\ \dfrac{(t-a)^{n-1}}{(n-1)(b-a)^{n-1}}, & a \leq t < s \leq b, \end{cases}$$

since,

$$(b-t)(t-a)^{n-2} \geq (t-a)^{n-1}, \text{ for } t \in [a, \tfrac{a+b}{2}],$$

and

$$(t-a)^{n-1} \geq (b-t)(t-a)^{n-2}, \text{ for } t \in [\tfrac{a+b}{2}, b],$$

then

$$H(t,s) \geq \phi(t)k(s), \text{ for } s \in (a,b) \text{ and } t \in [a,b].$$

Thus, (ii) holds. The proof is completed.

Theorem 2.3. *Suppose that* $\Delta =: (1-\alpha)(b-a)^{n-1} + (\alpha-\beta)(\eta-a)^{n-1} \neq 0$, *then for any* $y \in C[a,b]$, *the problem*

$$\begin{cases} u^{(n)}(t) + y(t) = 0, \quad a \leq t \leq b, \\ u(a) = \alpha u(\eta), \quad u'(a) = 0, \cdots, u^{(n-2)}(a) = 0, \quad u(b) = \beta u(\eta), \end{cases} \tag{2.4}$$

has a solution

$$u(t) = \int_a^b G(t,s)y(s)ds,$$

where

$$G(t,s) = H(t,s) + \frac{1}{\Delta}\{\alpha[(b-a)^{n-1} - (t-a)^{n-1}] + \beta(t-a)^{n-1}\}H(\eta,s), \tag{2.5}$$

here $H(t,s)$ *is given by* (2.3).

Proof. The three-point boundary value problem (2.4) can be obtained from replacing $u(a) = 0$ by $u(a) = \alpha u(\eta)$ and $u(b) = 0$ by $u(b) = \beta u(\eta)$ in (2.1). Thus, we suppose the solution of the three-point boundary value problem (2.4) can be expressed by

$$u(t) = w(t) + \sum_{j=0}^{n-1} \gamma_j (t-a)^j w(\eta), \tag{2.6}$$

where $w(t)$ is given as (2.2), γ_j $(j = 0, 1, 2, \cdots, n-1)$ are constants that will be determined. By calculating, we obtain

$$u^{(i)}(t) = w^{(i)}(t) + \sum_{j=i}^{n-1} \frac{j!}{(j-i)!}\gamma_j (t-a)^{(j-i)} w(\eta). \tag{2.7}$$

Since $u^{(i)}(a) = 0$ for $i = 1, 2, \cdots, n-2$ and (2.7), we obtain

$$\gamma_j = 0, \text{ for } j = 1, 2, \cdots, n-2. \tag{2.8}$$

In view of (2.6) and (2.8), we have

$$u(t) = w(t) + [\gamma_0 + \gamma_{n-1}(t-a)^{n-1}]w(\eta). \tag{2.9}$$

Then, by $u(a) = \alpha u(\eta)$ and $u(b) = \beta u(\eta)$ (note (2.9)), we get

$$\gamma_0 w(\eta) = \alpha w(\eta) + \alpha[\gamma_0 + \gamma_{n-1}(\eta - a)^{n-1}]w(\eta),$$

and

$$[\gamma_0 + \gamma_{n-1}(b-a)^{n-1}]w(\eta) = \beta w(\eta) + \beta[\gamma_0 + \gamma_{n-1}(\eta - a)^{n-1}]w(\eta).$$

From this

$$\begin{cases} \gamma_0 = \alpha[1 + \gamma_0 + \gamma_{n-1}(\eta - a)^{n-1}], \\ \gamma_0 + \gamma_{n-1}(b-a)^{n-1} = \beta[1 + \gamma_0 + \gamma_{n-1}(\eta - a)^{n-1}]. \end{cases} \tag{2.10}$$

By calculating, we obtain by (2.10) that

$$\begin{cases} \gamma_0 = \dfrac{\alpha}{1-\alpha} + \dfrac{\alpha(\beta - \alpha)(\eta - a)^{n-1}}{(1-\alpha)\cdot\Delta}, \\ \gamma_{n-1} = \dfrac{\beta - \alpha}{\Delta}. \end{cases} \tag{2.11}$$

Hence, by (2.9) and (2.11), we obtain

$$\begin{aligned} u(t) &= w(t) + [\frac{\alpha}{1-\alpha} + \frac{\alpha(\beta-\alpha)(\eta-a)^{n-1}}{(1-\alpha)\cdot\Delta} + \frac{(\beta-\alpha)(t-a)^{n-1}}{\Delta}]w(\eta) \\ &= w(t) + \frac{1}{\Delta}\{\beta(t-a)^{n-1} + \alpha[(b-a)^{n-1} - (t-a)^{n-1}]\}w(\eta). \end{aligned}$$

Using this and (2.2), we see that

$$u(t) = \int_a^b H(t,s)y(s)ds + \frac{1}{\Delta}\{\beta(t-a)^{n-1} + \alpha[(b-a)^{n-1} - (t-a)^{n-1}]\}\int_a^b H(\eta,s)y(s)ds.$$

Thus, the Green's function $G(t,s)$ for the BVP (2.4) is described by (2.5).

By Theorem 2.3, we obtain the following corollary.

Corollary 2.4. *Suppose that* $\Delta =: (1-\alpha)(b-a)^{n-1} + (\alpha-\beta)(\eta-a)^{n-1} \neq 0$, *then for any* $y \in C[a,b]$, *the problem* (2.4) *has a unique solution*

$$u(t) = \int_a^b G(t,s)y(s)ds,$$

where $G(t,s)$ *is given as in* (2.5).

Proof. We need only prove the uniqueness. Suppose that $u_1(t)$ is also a solution of the problem

(2.4). Let

$$x(t) = u_1(t) - u(t),\ t \in [a,b].$$

Obviously,

$$x^{(n)}(t) = u_1^{(n)}(t) - u^{(n)}(t) = 0,\ t \in [a,b], \tag{2.12}$$

$$x(a) = u_1(a) - u(a) = \alpha u_1(\eta) - \alpha u(\eta) = \alpha x(\eta), \tag{2.13}$$

$$x^{(i)}(a) = u_1^{(i)}(a) - u^{(i)}(a) = 0,\ i = 1, 2, \cdots, n-2, \tag{2.14}$$

$$x(b) = u_1(b) - u(b) = \beta u_1(\eta) - \beta u(\eta) = \beta x(\eta). \tag{2.15}$$

In view of (2.12), we have

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}, \tag{2.16}$$

where c_j $(j = 0, 1, 2, \cdots, n-1)$ are undetermined constants. By calculating, one has

$$x^{(i)}(t) = \sum_{j=i}^{n-1} \frac{j!}{(j-i)!} c_j (t-a)^{(j-i)},\ i = 1, 2, \cdots, n-1. \tag{2.17}$$

From (2.14) and (2.17), we get

$$c_1 = c_2 = \cdots = c_{n-2} = 0, \tag{2.18}$$

By (2.13), (2.15), (2.16) and (2.18), one has

$$c_0 = \alpha[c_0 + c_{n-1}(\eta - a)^{n-1}], \tag{2.19}$$

and

$$c_0 + c_{n-1}(b-a)^{n-1} = \beta[c_0 + c_{n-1}(\eta - a)^{n-1}]. \tag{2.20}$$

Thus, by (2.19), (2.20) and $\Delta \neq 0$, we can easily know that $c_0 = c_{n-1} = 0$. So $x(t) = 0$, $t \in [a,b]$, which implies that the solution of the problem (2.4) is unique.

Theorem 2.5. *$G(t,s)$ has the following properties*

(i) $G(t,s) \geq 0$, $\forall t, s \in [a,b]$;

(ii) $G(t,s) \leq M_1 k(s)$, $\forall t, s \in [a,b]$, *where $k(s)$ as in Lemma 2.2, and*

$$M_1 = \max\left\{ \frac{1}{\Delta}[(b-a)^{n-1} + \alpha(\eta - a)^{n-1}], \frac{1}{\Delta}\{(b-a)^{n-1} + \beta[(b-a)^{n-1} - (\eta - a)^{n-1}]\} \right\};$$

(iii) $\min\limits_{t \in [\tau, \frac{a+b}{2}]} G(t,s) \geq M_2 k(s)$, $\forall t, s \in [a,b]$, *where $\tau \in (a, \frac{a+b}{2})$ and*

$$M_2 = \frac{(\tau - a)^{n-1}}{(n-1)(b-a)^{n-1}} + \frac{1}{\Delta}[\beta(\tau - a)^{n-1} + \frac{\alpha(2^{n-1}-1)(b-a)^{n-1}}{2^{n-1}}]\phi(\eta),$$

here $\phi(t)$ and $k(s)$ as in Lemma 2.2.

Proof. It is clear that (i) holds. Next we will divide the proof of (ii) into two cases.
Case (1) If $a \le t \le \eta$, then, by Lemma 2.2 (i), we have

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{1}{\Delta}\{\beta(t-a)^{n-1} + \alpha[(b-a)^{n-1} - (t-a)^{n-1}]\}H(\eta,s) \\ &\le k(s) + \frac{1}{\Delta}[\beta(\eta-a)^{n-1} + \alpha(b-a)^{n-1}]k(s) \\ &= \frac{1}{\Delta}[(b-a)^{n-1} + \alpha(\eta-a)^{n-1}]k(s) \le M_1 k(s). \end{aligned}$$

Case (2) If $\eta \le t \le b$, similarly, we obtain

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{1}{\Delta}\{\beta(t-a)^{n-1} + \alpha[(b-a)^{n-1} - (t-a)^{n-1}]\}H(\eta,s) \\ &\le k(s) + \frac{1}{\Delta}\{\beta(b-a)^{n-1} + \alpha[(b-a)^{n-1} - (\eta-a)^{n-1}]\}k(s) \\ &= \frac{1}{\Delta}\{(b-a)^{n-1} + \beta[(b-a)^{n-1} - (\eta-a)^{n-1}]\}k(s) \le M_1 k(s). \end{aligned}$$

By the inequality above, we know that (ii) holds.

(iii) By Lemma 2.2 (ii), we have

$$\begin{aligned} \min_{t\in[\tau,\frac{a+b}{2}]} G(t,s) &= \min_{t\in[\tau,\frac{a+b}{2}]} \{H(t,s) + \frac{1}{\Delta}\{\beta(t-a)^{n-1} + \alpha[(b-a)^{n-1} - (t-a)^{n-1}]\}H(\eta,s)\} \\ &\ge \min_{t\in[\tau,\frac{a+b}{2}]} \{\phi(t)k(s) + \frac{1}{\Delta}[\beta(\tau-a)^{n-1} + \frac{\alpha(2^{n-1}-1)(b-a)^{n-1}}{2^{n-1}}]\phi(\eta)k(s)\} \\ &= \{\frac{(\tau-a)^{n-1}}{(n-1)(b-a)^{n-1}} + \frac{1}{\Delta}[\beta(\tau-a)^{n-1} + \frac{\alpha(2^{n-1}-1)(b-a)^{n-1}}{2^{n-1}}]\phi(\eta)\}k(s) \\ &= M_2 k(s). \end{aligned}$$

Thus, (iii) holds.

3. Main results

Throughout this paper, we assume the following conditions hold.

(H_1) $h : [a,b] \to [0,+\infty)$ is continuous, and

$$0 < \int_a^b (b-s)^{n-1}h(s)ds < +\infty.$$

(H_2) $f : [a,b] \times [0,+\infty) \to [0,+\infty)$ is continuous.

Let $C[a,b]$ be Banach space with the norm $\|u\| = \max_{a\le t\le b} |u(t)|$. Define the cones K and P by

$$K = \{u \in C[a,b] : u(t) \ge 0,\ t \in [a,b]\},$$

and

$$P = \{u \in C[a,b] : u(t) \ge 0,\ t \in [a,b], \text{ and } \min_{t\in[\tau,\frac{a+b}{2}]} u(t) \ge \gamma\|u\|\},$$

where $\tau \in (a, \frac{a+b}{2})$, $\gamma := \frac{M_2}{M_1}$ (here M_1 and M_2 are defined as in Theorem 2.5). Obviously, K and P are cones of $C[a,b]$. Define the operators A_1, A_2 and T by

$$(A_1u)(t) = \int_a^b G(t,s)h(s)u(s)ds,\ \forall t \in [a,b], \tag{3.1}$$

$$(A_2u)(t) = \int_\tau^{\frac{a+b}{2}} G(t,s)h(s)u(s)ds,\ \forall t \in [a,b], \tag{3.2}$$

and

$$(Tu)(t) = \int_a^b G(t,s)h(s)f(s,u(s))ds,\ \forall t \in [a,b], \tag{3.3}$$

It is clear that the problem (1.2) has a positive solution $u = u(t)$ if and only if u is a fixed point of T.

Lemma 3.1. *Suppose* $(\mathrm{H}_0) - (\mathrm{H}_2)$ *hold. Then*

(i) *The operator $A_1 : C[a,b] \to C[a,b]$ is completely continuous and satisfies $A_1(K) \subset K$.*

(ii) *The operator $A_2 : C[a,b] \to C[a,b]$ is completely continuous and satisfies $A_2(K) \subset K$.*

(iii) *The operator $T : P \to C[a,b]$ is completely continuous and satisfies $T(P) \subset P$.*

Proof. It is obvious that (i) and (ii) hold. By Theorem 2.3, we know that $T(P) \subset P$. Next, we will prove that the operator T is completely continuous.

For $m \ge 2$, define h_m by

$$h_m(t) = \begin{cases} \inf_{a\le s<\frac{am+b}{m+1}} h(s), & a \le t < \frac{am+b}{m+1}, \\ h(t), & \frac{am+b}{m+1} \le t \le \frac{a+bm}{m+1}, \\ \inf_{\frac{a+bm}{m+1}<s\le b} h(s), & \frac{a+bm}{m+1} < t \le b, \end{cases}$$

and define the operator T_m as follows

$$(T_mu)(t) = \int_a^b G(t,s)h_m(s)f(s,u(s))ds,\ t \in [a,b].$$

It is easy to show that the operator T_m is compact on P for all $m \ge 2$ by using Arzela-Ascoli theorem. In addition, the continuity of $G(t,s)h_m(s)$ on $[a,b] \times [a,b]$ implies the continuity of $T_m : P \to P$.

Therefore, T_m is a completely continuous operator. It follows from (H_1) that

$$0 < \int_a^b k(s)h(s) < \frac{1}{(n-1)!}\int_a^b (b-s)^{n-1}h(s)ds < \infty.$$

Using this and by the absolute continuity of the integral, we have

$$\lim_{m\to\infty}\int_{e(m)} k(s)h(s)ds = 0,$$

where $k(s)$ is given as in Lemma 2.2, $e(m) = [a, \frac{am+b}{m+1}] \cup [\frac{a+bm}{m+1}, b]$.

For each $R > 0$, set $\Omega_R = \{u \in P : \|u\| \leq R\}$ and $M = \max_{t\in[a,b],u\in[0,R]} f(t,u)$. Then fix $R > 0$ and $u \in \Omega_R$, we have from Theorem 2.5 and (H_2) that

$$\begin{aligned}
|(T_m u)(t) - (Tu)(t)| &= \left|\int_a^b G(t,s)(h_m(s) - h(s))f(s,u(s))ds\right| \\
&\leq M_1 M \int_a^{\frac{am+b}{m+1}} k(s)|h(s) - h_m(s)|ds + M_1 M \int_{\frac{a+bm}{m+1}}^b k(s)|h(s) - h_m(s)|ds \\
&\leq M_1 M \int_{e(m)} k(s)h(s)ds \to 0,\ (m \to \infty).
\end{aligned}$$

Hence, the completely continuous operator T_m converges uniformly to T as $m \to \infty$ on any bounded subset of P, and $T : P \to P$ is completely continuous.

By Virtue of Krein-Rutmann theorems, It is easy to see that the following lemma holds.

Lemma 3.2 [6]. *Suppose A is a completely continuous operator and $A(K) \subset K$. If there exists $\psi \in C[a,b]\setminus(-K)$ and a constant $c > 0$ such that $cK\psi \geq \psi$, then the spectral radius $r(A) \neq 0$ and A has a positive eigenfunction φ corresponding to its first eigenvalue $\lambda = (r(A))^{-1}$, that is, $\varphi = \lambda A\varphi$.*

Lemma 3.3. *Suppose that* (H_0) *and* (H_1) *hold. Then the spectral radius $r(A_1) \neq 0$ and A_1 has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(A_1))^{-1}$.*

Proof. It follows from (2.5) that

$$G(t,t) \geq H(t,t) = \frac{(b-t)^{n-1}(t-a)^{n-1}}{(b-a)^{n-1}(n-1)!} > 0,\ \forall t \in (a,b).$$

From this and (H_1), we know that there exists $t_0 \in (a,b)$, such that $G(t_0,t_0)h(t_0) > 0$, then there is $[\alpha,\beta]$ such that $t_0 \in (\alpha,\beta)$ and $G(t,s)h(s) > 0$, $\forall t,s \in [\alpha,\beta]$. We take $\psi \in C[a,b]$ such that $\psi(t) \geq 0$, $\forall t \in [a,b]$, $\psi(t_0) > 0$ and $\psi(t) = 0$, $\forall t \notin [\alpha,\beta]$. Then for all $t \in [a,b]$, we have

$$(A_1\psi)(t) = \int_a^b G(t,s)h(s)\psi(s)ds = \int_\alpha^\beta G(t,s)h(s)\psi(s)ds > 0.$$

Then there exists a constant $c > 0$ such that $c(A_1\psi) \geq \psi$, $\forall t \in [a,b]$. By Lemma 3.2, we see that the spectral radius $r(A_1) \neq 0$ and A_1 has a positive eigenfunction corresponding to its first eigenvalue

$\lambda_1 = (r(A_1))^{-1}$.

To establish the existence of positive solutions of the problem (1.2), we will employ the following lemmas.

Lemma 3.4 [5]. *Let E be Banach space, P be a cone in E, and Ω be a bounded open set in E. Suppose that $T : P \cap \overline{\Omega} \to P$ is a completely continuous operator. If there exists $u_0 \in P\backslash\{\theta\}$ such that*

$$u - Tu \neq \mu u_0,\ \forall u \in P \cap \partial\Omega,\ \mu \geq 0.$$

Then the fixed point index $i(T, P \cap \overline{\Omega}, P) = 0$.

Lemma 3.5 [5]. *Let E be Banach space, P be a cone in E, and Ω be a bounded open set in E. Suppose that $T : P \cap \overline{\Omega} \to P$ is a completely continuous operator. If*

$$Tu \neq \mu u,\ \forall u \in P \cap \partial\Omega,\ \mu \geq 1.$$

Then the fixed point index $i(T, P \cap \Omega, P) = 1$.

For convenience, we introduce the following notations

$$f_0 = \liminf_{u\to 0^+} \min_{a\leq t\leq b}(f(t,u)/u), \quad f^0 = \limsup_{u\to 0^+} \max_{a\leq t\leq b}(f(t,u)/u),$$
$$f_\infty = \liminf_{u\to \infty} \min_{a\leq t\leq b}(f(t,u)/u), \quad f^\infty = \limsup_{u\to +\infty} \max_{a\leq t\leq b}(f(t,u)/u).$$

Theorem 3.6. *Let λ_1 be the first eigenvalue of A_1 defined as in (3.1). Suppose the previous hypotheses $(\mathrm{H}_0) - (\mathrm{H}_2)$ hold, in addition we assume $f_0 > \lambda_1$ and $f^\infty < \lambda_1$. Then problem (1.2) has at least one positive solution.*

Proof. In view of $f_0 > \lambda_1$, there exists $R_1 > 0$, such that

$$f(t,u) \geq \lambda_1 u, \text{ for all } t \in [a,b],\ u \in [0,R_1]. \tag{3.4}$$

Let $\Omega_{R_1} = \{u \in C[a,b] : \|u\| < R_1\}$, for $u \in P \cap \partial\Omega_{R_1}$, we have by (3.4) that

$$(Tu)(t) = \int_a^b G(t,s)h(s)f(s,u(s))ds \geq \lambda_1 \int_a^b G(t,s)h(s)u(s)ds = \lambda_1(A_1u)(t),\ \forall t \in [a,b]. \tag{3.5}$$

Let u^* be the positive eigenfunction of A_1 corresponding to λ_1, thus $u^* = \lambda_1 A_1 u^*$. We may suppose that T has no fixed point on $P \cap \partial\Omega_{R_1}$, otherwise, the proof is finished. In the following we will show that

$$u - Tu \neq \mu u^*,\ \forall u \in P \cap \partial\Omega_{R_1},\ \mu \geq 0. \tag{3.6}$$

If (3.6) is not true, then there is $u_0 \in P \cap \partial\Omega_{R_1}$ and $\mu_0 \geq 0$ such that $u_0 - Tu_0 = \mu_0 u^*$. It is clear that $\mu_0 > 0$ and

$$u_0 = Tu_0 + \mu_0 u^* \geq \mu_0 u^*.$$

Set

$$\mu^* = \sup\{\mu : u_0 \geq \mu u^*\}. \tag{3.7}$$

Obviously, $\mu^* \geq \mu_0 > 0$. It follows from $A_1(K) \subset K$ that

$$\lambda_1 A_1 u_0 \geq \mu^* \lambda_1 A_1 u^* = \mu^* u^*.$$

Using this and (3.5), we have

$$u_0 = Tu_0 + \mu_0 u^* \geq \lambda_1 A_1 u_0 + \mu_0 u^* \geq \mu^* u^* + \mu_0 u^*,$$

which contradicts (3.7). Thus, (3.6) holds. By Lemma 3.4, we have

$$i(T, P \cap \Omega_{R_1}, P) = 0. \tag{3.8}$$

On the other hand, it follows from $f^\infty < \lambda_1$ that there are $0 < \rho < 1$ and $R_2 > R_1$ such that

$$f(t,u) \leq \rho\lambda_1 u,\ \forall t \in [a,b],\ u \geq R_2. \tag{3.9}$$

Put

$$B = \{u \in K : u = \sigma T u,\ 0 \leq \sigma \leq 1\}\ ,\ \overline{u}(t) = \min\{u(t), R_2\} \text{ and } w(t) = \{t \in [a,b] : u(t) > R_2\}.$$

Now we will show that B is bounded. For all u in B, we have by (3.9) and Theorem 2.5 (ii) that

$$\begin{aligned}
u(t) = \sigma(Tu)(t) &\leq \int_a^b G(t,s)h(s)f(s,u(s))ds \\
&= \int_{w(t)} G(t,s)h(s)f(s,u(s))ds + \int_{[a,b]\setminus w(t)} G(t,s)h(s)f(s,u(s))ds \\
&\leq \rho\lambda_1 \int_a^b G(t,s)h(s)u(s)ds + M_1 \int_a^b k(s)h(s)f(s,\overline{u}(s))ds \\
&= \rho\lambda_1 (A_1 u)(t) + M = (\overline{A}_1 u)(t) + M.
\end{aligned}$$

where $\overline{A}_1 u = \rho\lambda_1 A_1 u$, $u \in C[a,b]$ and $M = M_1 \sup_{u \in P\cap\overline{\Omega}_{R_2}} \int_a^b k(s)h(s)f(s,\overline{u}(s))ds$.
Thus, $((I - \overline{A}_1)u)(t) \leq M$, $t \in [a,b]$. Since λ_1 is the first eigenvalue of A_1 and $0 < \rho < 1$, then the first eigenvalue of $\overline{A}_1$, $(r(\overline{A}_1))^{-1} > 1$. Thus, the inverse operator $(I - \overline{A}_1)^{-1}$ exists and

$$(I - \overline{A}_1)^{-1} = I + \overline{A}_1 + \overline{A}_1^2 + \cdots + \overline{A}_1^n + \cdots.$$

In view of $\overline{A}_1(P) \subset P$, we get that $(I - \overline{A}_1)^{-1}(P) \subset P$. Then we have $u(t) \leq (I - \overline{A}_1)^{-1}M$, $t \in [a,b]$ and B is bounded.

Take $R_3 > \max\{R_2, \sup B\}$, let $\Omega_{R_3} = \{u \in C[a,b] : \|u\| < R_3\}$. Then by Lemma 3.5, one has

$$i(T, P \cap \Omega_{R_3}, P) = i(\theta, P \cap \Omega_{R_3}, P) = 1. \tag{3.10}$$

It follows from (3.8) and (3.10) that

$$i(T, (P \cap \Omega_{R_3}) \setminus (P \cap \overline{\Omega}_{R_1}), P) = i(\theta, P \cap \Omega_{R_3}, P) - i(\theta, P \cap \Omega_{R_1}, P) = 1.$$

Hence, T has at least one fixed point on $(P \cap \Omega_{R_3}) \setminus (P \cap \overline{\Omega}_{R_1})$. This implies that the problem (1.2) has at least one positive solution.

Theorem 3.7. *Let λ_1 be the first eigenvalue of A_1 defined as in* (3.1)*, λ_2 be the first eigenvalue of A_2 defined as in* (3.2)*. Suppose the previous hypotheses* $(\mathrm{H}_0) - (\mathrm{H}_2)$ *hold, in addition we assume $f^0 < \lambda_1$ and $f_\infty > \lambda_2$. Then problem* (1.2) *has at least one positive solution.*

Proof. From $f^0 < \lambda_1$, we know that there exists $R_4 > 0$, such that

$$f(t,u) \le \lambda_1 u,\ \forall t \in [a,b],\ u \in [0, R_4]. \tag{3.11}$$

Let $\Omega_{R_4} = \{u \in C[a,b] : \|u\| < R_4\}$, for $u \in P \cap \partial\Omega_{R_4}$, we have by (3.11) that $(Tu)(t) \le \lambda_1 (A_1 u)(t)$.

Now we show that

$$i(T, P \cap \Omega_{R_4}, P) = 1. \tag{3.12}$$

We may suppose that T has no fixed point on $P \cap \partial\Omega_{R_4}$, otherwise, the proof is finished. In the following we will show that

$$Tu \ne \varrho u,\ \forall u \in P \cap \partial\Omega_{R_4},\ \varrho \ge 1. \tag{3.13}$$

If otherwise, there exist $u_0 \in P \cap \partial\Omega_{R_4}$ and $\varrho_0 \ge 1$ such that $Tu_0 = \varrho_0 u_0$. So $\varrho_0 > 1$ and $\varrho_0 u_0 = Tu_0 \le \lambda_1 A_1 u_0$. From induction, we get that $\varrho_0^j u_0 \le \lambda_1^j A_1^j u_0$, $\forall j \in N$. Therefore, we have by Gelfand's formula that

$$r(A_1) = \lim_{j\to\infty} \sqrt[j]{\|A_1^j\|} \ge \lim_{j\to\infty} \sqrt[j]{\frac{\|A_1^j u_0\|}{\|u_0\|}} \ge \lim_{j\to\infty} \sqrt[j]{\frac{\varrho_0^j \|u_0\|}{\lambda_1^j \|u_0\|}} = \frac{\varrho_0}{\lambda_1} > \frac{1}{\lambda_1},$$

which is a contradiction with $r(A_1) = \frac{1}{\lambda_1}$, and so (3.13) holds. It follows from Lemma 3.5 that (3.12) holds.

On the other hand, by $f_\infty > \lambda_2$, there exists $R_5 > R_4 > 0$ such that

$$f(t,u) \ge \lambda_2 u,\ \forall t \in [a,b],\ u \ge \gamma R_5. \tag{3.14}$$

Let $\Omega_{R_5} = \{u \in C[a,b] : \|u\| < R_5\}$, then, for all $u \in P \cap \partial\Omega_{R_5}$, we have that $\min\limits_{t \in [\tau, \frac{a+b}{2}]} u(t) \ge \gamma\|u\| = \gamma R_5$. Using this and (3.14), one has

$$(Tu)(t) \ge \lambda_2 \int_\tau^{\frac{a+b}{2}} G(t,s)h(s)u(s)ds = \lambda_2 (A_2 u)(t),\ \forall t \in [a,b].$$

Hence, by the same way as in Theorem 3.6, we obtain

$$i(T, P \cap \Omega_{R_5}, P) = 0. \tag{3.15}$$

According to (3.12) and (3.15), we have

$$i(T, (P \cap \Omega_{R_5}) \setminus (P \cap \overline{\Omega}_{R_4}), P) = i(\theta, P \cap \Omega_{R_5}, P) - i(\theta, P \cap \Omega_{R_4}, P) = -1.$$

Consequently, T has at least one fixed point on $(P \cap \Omega_{R_5}) \setminus (P \cap \overline{\Omega}_{R_4})$. Hence, the problem (1.2) has at least one positive solution.

Corollary 3.8. *Assume that* $(\mathrm{H}_0) - (\mathrm{H}_2)$ *hold. If either*

(i) $f^0 = 0$, $f_\infty = \infty$, *or*
(ii) $f_0 = \infty$, $f^\infty = 0$.

Then problem (1.2) *has at least one positive solution.*

Acknowledgements

The authors are very grateful to the referees for careful reading of the original manuscript and for valuable suggestions on improving this paper. Project supported by the National Natural Science Foundation of China (10771212) and the Natural Science Foundation of Jiangsu Education Office (06KJB110010).

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(Received October 26, 2008)