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Positive solutions for a system of higher-order singular nonlinear fractional differential equations with nonlocal boundary conditions

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Abstract. The paper deals with the existence and multiplicity of positive solutions for a system of higher-order singular nonlinear fractional differential equations with nonlocal boundary conditions. The main tool used in the proof is fixed point index theory in cone. Some limit type conditions for ensuring the existence of positive solutions are given.

Keywords: higher-order singular fractional differential equations, positive solution, cone, fixed point index.

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1 Introduction

In this paper, we discuss the following system of higher-order singular nonlinear fractional differential equations with nonlocal boundary conditions:

$$\begin{gathered} D^{\alpha}_{0+}u(x) + h_1(x)f_1(x, u(x), v(x) = 0, \\ D^{\beta}_{0+}v(x) + h_2(x)f_2(x, u(x), v(x)) = 0, \end{gathered} \tag{1.1}$$

$$\begin{gathered} u^{(i)}(0) = 0, \qquad v^{(i)}(0) = 0, \qquad 0 \le i \le n-2, \\ D^{\mu}_{0+}u(1) = \eta_1 D^{\mu}_{0+}u(\xi_1), \qquad D^{\nu}_{0+}v(1) = \eta_2 D^{\nu}_{0+}v(\xi_2), \end{gathered} \tag{1.2}$$

where $x \in (0, 1)$, D^{α}_{0+}, D^{β}_{0+} are the standard Riemann–Liouville fractional derivatives of order $\alpha, \beta \in (n-1, n]$, $1 \le \mu, \nu \le n-3$ for $n > 3$ and $n \in \mathbb{N}^+$, $\xi_1, \xi_2 \in (0, 1)$, $0 \le \eta_1 \xi_1^{\alpha-\mu-1} < 1$, $0 \le \eta_2 \xi_2^{\beta-\nu-1} < 1$, $f_j \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $h_j \in C((0, 1),\ \mathbb{R}^+)$ $(j = 1, 2), \mathbb{R}^+ = [0, +\infty)$, $h_j(x)$ is allowed to be singular at $x = 0$ and/or $x = 1$.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback

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amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Recently, the existence and multiplicity of positive solutions for the nonlinear fractional differential equations have been researched, see [3, 5, 6, 12, 18, 22, 23, 25, 31] and the references therein. Such as, C. F. Li et al. [16] studied the existence and multiplicity of positive solutions of the following boundary value problem for nonlinear fractional differential equations:

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \\ u(0) = 0, \ D^{\beta}_{0+}u(1) = aD^{\beta}_{0+}u(\xi), \end{cases}$$

where D^{α}_{0+} is the standard Riemann–Liouville fractional derivative of order $\alpha \in (1, 2]$, $\beta, a \in [0, 1]$, $\xi \in (0, 1)$, $a\xi^{\alpha-\beta-1} \le 1 - \beta$, $\alpha - \beta - 1 \ge 0$.

The existence and uniqueness of some systems for nonlinear fractional differential equations have been studied by using fixed point theory or coincidence degree theory, see [1, 10, 21, 24, 25, 34] and references therein. In [7, 17, 29, 30], authors studied the existence and multiplicity of positive solutions of two types of systems for nonlinear fractional differential equations with boundary conditions:

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda f(t, u(t), v(t)) = 0, \\ D^{\beta}_{0+}v(t) + \mu g(t, u(t), v(t)) = 0, \quad t \in (0, 1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ u(1) = \int_0^1 v(t)\, dH(t), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \ v(1) = \int_0^1 u(t)\, dK(t), \end{cases}$$

and

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda a_1(t) f(u(t), v(t)) = 0, & \\ D^{\beta}_{0+}v(t) + \mu a_2(t) g(u(t), v(t)) = 0, & t \in [0, 1], \\ u^{(i)}(0) = v^{(i)}(0) = 0, & 0 \le i \le n-2, \\ D^{\gamma}_{0+}u(1) = \phi_1(u), \ D^{\gamma}_{0+}v(1) = \phi_2(v), & 1 \le \gamma \le n-2, \end{cases}$$

where D^{α}_{0+} and D^{β}_{0+} are the standard Riemann–Liouville fractional derivatives, $\alpha, \beta \in (n-1, n]$ for $n \ge 3$, $\lambda, \mu > 0$. The sublinear or superlinear condition is used in [7, 17, 29, 30, 33]. Another example, the following extreme limits:

$$f^s_\delta =: \limsup_{u+v\to\delta} \max_{t\in[0,1]} \frac{f(t,u,v)}{u+v}, \qquad g^s_\delta =: \limsup_{u+v\to\delta} \max_{t\in[0,1]} \frac{g(t,u,v)}{u+v},$$
$$f^i_\delta =: \liminf_{u+v\to\delta} \min_{t\in[\theta,1-\theta]} \frac{f(t,u,v)}{u+v}, \qquad g^i_\delta =: \liminf_{u+v\to\delta} \min_{t\in[\theta,1-\theta]} \frac{g(t,u,v)}{u+v},$$

are used in [9, 10], where $\theta \in (0, \frac{1}{2})$, $\delta = 0^+$ or $+\infty$. For the existence of positive solutions for systems of Hammerstein integral equations, see [4, 11, 15, 28] and their references.

Motivated by the above mentioned works and continuing the paper [27], in this paper, we present some limit type conditions and discuss the existence and multiplicity of positive solutions of the singular system (1.1)–(1.2) by using of fixed point index theory in cone. Our conditions are applicable for more functions, and the results obtained here are different from those in [7, 9, 10, 17, 24, 29, 30, 33]. Some examples are also provided to illustrate our main results.

2 Preliminaries

Definition 2.1 ([19]). The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $u\colon (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

provided the right side is pointwise defined on $(0,+\infty)$. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u\colon (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided the right side is pointwise defined on $(0,+\infty)$.

Lemma 2.2 ([13]). *(i) If $x \in L^1[0,1]$, $\rho > \sigma > 0$, then*

$$I_{0+}^{\rho} I_{0+}^{\sigma} x(t) = I_{0+}^{\rho+\sigma} x(t), \qquad D_{0+}^{\sigma} I_{0+}^{\rho} x(t) = I_{0+}^{\rho-\sigma} x(t), \qquad D_{0+}^{\sigma} I_{0+}^{\sigma} x(t) = x(t).$$

(ii) If $\rho > \sigma > 0$, then $D_{0+}^{\sigma} t^{\rho-1} = \Gamma(\rho) t^{\rho-\sigma-1}/\Gamma(\rho-\sigma)$.

Lemma 2.3. *Let $\xi_1 \in (0,1)$, $\eta_1 \xi_1^{\alpha-\mu-1} \neq 1$, $n-1 < \alpha \leq n$, $1 \leq \mu \leq n-3$ $(n > 3)$. Then for any $g \in C[0,1]$, the unique solution of the following boundary value problem:*

$$D_{0+}^{\alpha} u(t) + g(t) = 0, \qquad t \in (0,1), \tag{2.1}$$

$$u^{(i)}(0) = 0 \quad (0 \leq i \leq n-2), \qquad D_{0+}^{\mu} u(1) = \eta_1 D_{0+}^{\mu} u(\xi_1) \tag{2.2}$$

is given by

$$u(t) = \int_0^1 G_1(t,s) g(s)\, ds, \tag{2.3}$$

where $d_1 = 1 - \eta_1 \xi_1^{\alpha-\mu-1}$,

$$G_1(t,s) = \begin{cases} \dfrac{t^{\alpha-1}\left[(1-s)^{\alpha-\mu-1} - \eta_1(\xi_1-s)^{\alpha-\mu-1}\right] - d_1(t-s)^{\alpha-1}}{d_1\Gamma(\alpha)}, & 0 \leq s \leq \min\{t,\xi_1\}, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\mu-1} - d_1(t-s)^{\alpha-1}}{d_1\Gamma(\alpha)}, & 0 < \xi_1 \leq s \leq t \leq 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\mu-1} - t^{\alpha-1}\eta_1(\xi_1-s)^{\alpha-\mu-1}}{d_1\Gamma(\alpha)}, & 0 \leq t \leq s \leq \xi_1 < 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\mu-1}}{d_1\Gamma(\alpha)}, & \max\{t,\xi_1\} \leq s \leq 1 \end{cases} \tag{2.4}$$

is the Green's function of the integral equation (2.3).

Proof. The equation (2.1) is equivalent to an integral equation:

$$u(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}. \tag{2.5}$$

By $u(0) = 0$, we have $c_n = 0$. Then

$$u(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_{n-1} t^{\alpha-n+1}. \tag{2.6}$$

Differentiating (2.6), we have

$$u'(t) = \frac{1-\alpha}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} g(s)\, ds + c_1(\alpha-1) t^{\alpha-2} + \cdots + c_{n-1}(\alpha-n+1) t^{\alpha-n}. \tag{2.7}$$

By (2.7) and $u'(0) = 0$, we have $c_{n-1} = 0$. Similarly, we can get that $c_2 = c_3 = \cdots = c_{n-2} = 0$. Thus

$$u(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds + c_1 t^{\alpha-1}. \tag{2.8}$$

By $D^{\mu}_{0+} u(1) = \eta_1 D^{\mu}_{0+} u(\xi_1)$ and Lemma 2.2,

$$D^{\mu}_{0+} u(t) = \frac{1}{\Gamma(\alpha-\mu)} \left[c_1 \Gamma(\alpha) t^{\alpha-\mu-1} - \int_0^t (t-s)^{\alpha-\mu-1} g(s)\, ds \right],$$

we get

$$c_1 = \frac{1}{d_1 \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\mu-1} g(s)\, ds - \frac{\eta_1}{d_1 \Gamma(\alpha)} \int_0^{\xi_1} (\xi_1 - s)^{\alpha-\mu-1} g(s)\, ds.$$

Therefore, the unique solution of the problem (2.1)–(2.2) is

$$\begin{aligned} u(t) = & \frac{t^{\alpha-1}}{d_1 \Gamma(\alpha)} \left[\int_0^1 (1-s)^{\alpha-\mu-1} g(s)\, ds - \eta_1 \int_0^{\xi_1} (\xi_1 - s)^{\alpha-\mu-1} g(s)\, ds \right] \\ & - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\, ds = \int_0^1 G_1(t,s) g(s)\, ds. \end{aligned} \tag{2.9}$$

□

Similar to the proof of Lemma 2.3 in [20], we can get the following lemma.

Lemma 2.4. *Let $0 < \eta_1 \xi_1^{\alpha-\mu-1} < 1$. The function $G_1(t,s)$ defined by (2.4) satisfies*

(i) $G_1(t,s) \geq 0$ is continuous for any $t, s \in [0,1]$.

(ii) $\max_{t\in[0,1]} G_1(t,s) = G_1(1,s)$, $G_1(t,s) \geq t^{\alpha-1} G_1(1,s)$ for $t, s \in [0,1]$, where

$$G_1(1,s) = \begin{cases} \dfrac{(1-s)^{\alpha-\mu-1} - \eta_1(\xi_1 - s)^{\alpha-\mu-1} - d_1(1-s)^{\alpha-1}}{d_1 \Gamma(\alpha)}, & 0 \leq s \leq \xi_1, \\ \dfrac{(1-s)^{\alpha-\mu-1} - d_1(1-s)^{\alpha-1}}{d_1 \Gamma(\alpha)}, & \xi_1 \leq s \leq 1. \end{cases} \tag{2.10}$$

(iii) There are $\theta \in (0, \frac{1}{2})$ and $\gamma_\alpha \in (0,1)$ such that $\min_{t\in J_\theta} G_1(t,s) \geq \gamma_\alpha G_1(1,s)$ for $s \in [0,1]$, where $J_\theta = [\theta, 1-\theta]$, $\gamma_\alpha = \theta^{\alpha-1}$.

Let $\xi_2 \in (0,1)$, $0 < \eta_2 \xi_2^{\beta-\nu-1} < 1$, $d_2 = 1 - \eta_2 \xi_2^{\beta-\nu-1}$,

$$G_2(t,s) = \begin{cases} \dfrac{t^{\beta-1}\big[(1-s)^{\beta-\nu-1} - \eta_2(\xi_2 - s)^{\beta-\nu-1}\big] - d_2(t-s)^{\beta-1}}{d_2\Gamma(\beta)}, & 0 \le s \le \min\{t, \xi_2\}, \\ \dfrac{t^{\beta-1}(1-s)^{\beta-\nu-1} - d_2(t-s)^{\beta-1}}{d_2\Gamma(\beta)}, & 0 < \xi_2 \le s \le t \le 1, \\ \dfrac{t^{\beta-1}(1-s)^{\beta-\nu-1} - t^{\beta-1}\eta_2(\xi_2 - s)^{\beta-\nu-1}}{d_2\Gamma(\beta)}, & 0 \le t \le s \le \xi_2 < 1, \\ \dfrac{t^{\beta-1}(1-s)^{\beta-\nu-1}}{d_2\Gamma(\beta)}, & \max\{t, \xi_2\} \le s \le 1. \end{cases}$$

From Lemma 2.4 we know that $G_1(t,s)$ and $G_2(t,s)$ have the same properties, and there exists $\gamma_\beta = \theta^{\beta-1}$ such that $\min_{t\in J_\theta} G_2(t,s) \ge \gamma_\beta G_2(1,s)$. Let $\gamma = \min\{\gamma_\alpha, \gamma_\beta\}$,

$$\delta_j = \int_\theta^{1-\theta} G_j(1,y)h_j(y)\,dy, \;\; \mu_j = \int_0^1 G_j(1,y)h_j(y)\,dy \qquad (j = 1,2).$$

For convenience we list the following assumptions:

(H_1) $f_j \in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R}^+)$ $\quad (j = 1,2)$.

(H_2) $h_j \in C((0,1),\mathbb{R}^+)$, $h_j(x) \not\equiv 0$ on any subinterval of $(0,1)$ and $0 < \int_0^1 G_j(1,y)h_j(y)\,dy < +\infty$ $(j = 1,2)$.

(H_3) There exist $a, b \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(1) $a(\cdot)$ is concave and strictly increasing on $\mathbb{R}^+$ with $a(0) = 0$;

(2) $f_{10} = \liminf_{v\to 0+} \frac{f_1(x,u,v)}{a(v)} > 0$, $f_{20} = \liminf_{u\to 0+} \frac{f_2(x,u,v)}{b(u)} > 0$ uniformly with respect to $(x,u) \in J_\theta \times \mathbb{R}^+$ and $(x,v) \in J_\theta \times \mathbb{R}^+$, respectively (specifically, $f_{10} = f_{20} = +\infty$);

(3) $\lim_{u\to 0+} \frac{a(Cb(u))}{u} = +\infty$ for any constant $C > 0$.

(H_4) There exists $t \in (0,+\infty)$ such that

$$f_1^\infty = \limsup_{v\to+\infty} \frac{f_1(x,u,v)}{v^t} < +\infty, \qquad f_2^\infty = \limsup_{u\to+\infty} \frac{f_2(x,u,v)}{u^{\frac{1}{t}}} = 0$$

uniformly with respect to $(x,u) \in [0,1]\times\mathbb{R}^+$ and $(x,v) \in [0,1]\times\mathbb{R}^+$, respectively (specifically, $f_1^\infty = f_2^\infty = 0$).

(H_5) There exist $p, q \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(1) p is concave and strictly increasing on $\mathbb{R}^+$;

(2) $f_{1\infty} = \liminf_{v\to+\infty} \frac{f_1(x,u,v)}{p(v)} > 0$, $f_{2\infty} = \liminf_{u\to+\infty} \frac{f_2(x,u,v)}{q(u)} > 0$ uniformly with respect to $(x,u) \in J_\theta \times \mathbb{R}^+$ and $(x,v) \in J_\theta \times \mathbb{R}^+$, respectively (specifically, $f_{1\infty} = f_{2\infty} = +\infty$);

(3) $\lim_{u\to+\infty} \frac{p(Cq(u))}{u} = +\infty$ for any constant $C > 0$.

(H_6) There exists $s \in (0, +\infty)$ such that

$$f_1^0 = \limsup_{v \to 0+} \frac{f_1(x,u,v)}{v^s} < +\infty, \qquad f_2^0 = \limsup_{u \to 0+} \frac{f_2(x,u,v)}{u^{\frac{1}{s}}} = 0$$

uniformly with respect to $(x,u) \in [0,1] \times \mathbb{R}^+$ and $(x,v) \in [0,1] \times \mathbb{R}^+$, respectively (specifically, $f_1^0 = f_2^0 = 0$).

(H_7) There exists $r > 0$ such that

$$f_1(x,u,v) \geq (\gamma\delta_1)^{-1} r, \qquad f_2(x,u,v) \geq (\gamma\delta_2)^{-1} r, \qquad \forall\, x \in J_\theta,\ \gamma r \leq u + v \leq r.$$

(H_8) $f_1(x,u,v)$ and $f_2(x,u,v)$ are increasing with respect to u and v, there exists $R > r > 0$ such that

$$4\mu_1 f_1(x,R,R) < R, \qquad 4\mu_2 f_2(x,R,R) < R, \qquad \forall\, x \in [0,1].$$

Let $E = C[0,1]$, $\|u\| = \max_{t \in [0,1]} |u(t)|$, the product space $E \times E$ be equipped with norm $\|(u,v)\| = \|u\| + \|v\|$ for $(u,v) \in E \times E$, and

$$P = \Big\{ u \in E : u(t) \geq 0,\ t \in [0,1],\ \min_{t \in J_\theta} u(t) \geq \gamma \|u\| \Big\}.$$

Then $E \times E$ is a real Banach space and $P \times P$ is a positive cone of $E \times E$. By $(H_1), (H_2)$, we can define operators

$$A_j(u,v)(x) = \int_0^1 G_j(x,y) h_j(y) f_j(y, u(y), v(y))\, dy \qquad (j = 1,2),$$

$A(u,v) = (A_1(u,v), A_2(u,v))$. Similar to the proof of Lemma 3.1 in [2], it follows from $(H_1), (H_2)$ that $A_j \colon P \times P \to P$ is a completely continuous operator and $A(P \times P) \subset P \times P$. Clearly (u,v) is a positive solution of the system (1.1) if and only if $(u,v) \in P \times P \setminus \{(0,0)\}$ is a fixed point of A (refer [9,27]).

Lemma 2.5 ([8]). *Let E be a Banach space, P be a cone in E and $\Omega \subset E$ be a bounded open set. Assume that $A \colon \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P \setminus \{0\}$ such that*

$$u \neq Au + \lambda u_0, \qquad \forall\, \lambda \geq 0,\ u \in \partial\Omega \cap P,$$

then the fixed point index $i(A, \Omega \cap P, P) = 0$.

Lemma 2.6 ([8,14]). *Let E be a Banach space, P be a cone in E and $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Assume that $A \colon \overline{\Omega} \cap P \to P$ is a completely continuous operator.*
(1) If $u \not\leq Au$ for all $u \in \partial\Omega \cap P$, then the fixed point index $i(A, \Omega \cap P, P) = 1$.
(2) If $u \not\geq Au$ for all $u \in \partial\Omega \cap P$, then the fixed point index $i(A, \Omega \cap P, P) = 0$.

In the following, we adopt the convention that $C_1, C_2, C_3, \ldots$ stand for different positive constants. Let $\Omega_\rho = \{(u,v) \in E \times E : \|(u,v)\| < \rho\}$ for $\rho > 0$.

3 Existence of a positive solution

Theorem 3.1. *Assume that the conditions* $(H_1), (H_2)$ *are satisfied and that* $(H_3), (H_4)$ *or* $(H_7), (H_8)$ *hold. Then the system* (1.1)–(1.2) *has at least one positive solution.*

Proof. Case 1. The conditions (H_3) and (H_4) hold. By (H_3), there are $\xi_1 > 0$, $\eta_1 > 0$ and a sufficiently small $\rho > 0$ such that

$$\begin{aligned} f_1(x,u,v) &\geq \xi_1 a(v), \qquad \forall\ (x,u) \in J_\theta \times \mathbb{R}^+,\ 0 \leq v \leq \rho, \\ f_2(x,u,v) &\geq \eta_1 b(u), \qquad \forall\ (x,v) \in J_\theta \times \mathbb{R}^+,\ 0 \leq u \leq \rho, \end{aligned} \tag{3.1}$$

and

$$a(K_1 b(u)) \geq \frac{2K_1}{\xi_1 \eta_1 \delta_1 \delta_2 \gamma^3} u, \qquad \forall\ u \in [0,\rho], \tag{3.2}$$

where $K_1 = \max\{\eta_1 \gamma G_2(1,y) h_2(y) : y \in J_\theta\}$. We claim that

$$(u,v) \neq A(u,v) + \lambda(\varphi,\varphi), \qquad \forall\ \lambda \geq 0,\ (u,v) \in \partial\Omega_\rho \cap (P \times P),$$

where $\varphi \in P \setminus \{0\}$. If not, there are $\lambda \geq 0$ and $(u,v) \in \partial\Omega_\rho \cap (P \times P)$ such that $(u,v) = A(u,v) + \lambda(\varphi,\varphi)$, then $u \geq A_1(u,v), v \geq A_2(u,v)$. By using the monotonicity and concavity of $a(\cdot)$, Jensen's inequality and Lemma 2.4, we have by (3.1) and (3.2),

$$\begin{aligned} u(x) &\geq \int_0^1 G_1(x,y) h_1(y) f_1(y,u(y),v(y))\,dy \\ &\geq \xi_1 \gamma_\alpha \int_0^1 G_1(1,y) h_1(y) a(v(y))\,dy \\ &\geq \xi_1 \gamma_\alpha \int_0^1 G_1(1,y) h_1(y) a\Big(\int_0^1 \eta_1 G_2(y,z) h_2(z) b(u(z))\,dz \Big)\,dy \\ &\geq \xi_1 \gamma \int_\theta^{1-\theta} G_1(1,y) h_1(y) \int_0^1 a\big(\eta_1 \gamma G_2(1,z) h_2(z) b(u(z))\big)\,dz\,dy \\ &\geq \xi_1 \gamma \int_\theta^{1-\theta} G_1(1,y) h_1(y) \int_0^1 a\big(K_1^{-1} \eta_1 \gamma G_2(1,z) h_2(z) K_1 b(u(z))\big)\,dz\,dy \\ &\geq \xi_1 \eta_1 \gamma^2 K_1^{-1} \int_\theta^{1-\theta} \int_\theta^{1-\theta} G_1(1,y) h_1(y) G_2(1,z) h_2(z) a\big(K_1 b(u(z))\big)\,dz\,dy \\ &\geq \xi_1 \eta_1 \gamma^2 \delta_1 K_1^{-1} \int_\theta^{1-\theta} G_2(1,z) h_2(z) a\big(K_1 b(u(z))\big)\,dz \\ &\geq \frac{2}{\delta_2 \gamma} \int_\theta^{1-\theta} G_2(1,z) h_2(z) u(z)\,dz \geq 2\|u\|, \qquad x \in J_\theta, \end{aligned} \tag{3.3}$$

Consequently, $\|u\| = 0$. Next, (3.1) and (3.2) yield that

$$\begin{aligned} a(v(x)) &\geq a\Big(\int_0^1 G_2(x,y) h_2(y) f_2(y,u(y),v(y))\,dy \Big) \\ &\geq \int_0^1 a\big(\eta_1 \gamma G_2(1,y) h_2(y) b(u(y))\big)\,dy \\ &\geq \eta_1 \gamma K_1^{-1} \int_\theta^{1-\theta} G_2(1,y) h_2(y) a\big(K_1 b(u(y))\big)\,dy \\ &\geq \frac{2}{\xi_1 \delta_1 \delta_2 \gamma^2} \int_\theta^{1-\theta} G_2(1,y) h_2(y) u(y)\,dy \end{aligned}$$

$$\geq \frac{2}{\delta_1\delta_2\gamma}\int_\theta^{1-\theta} G_2(1,y)h_2(y)\,dy\int_0^1 G_1(1,z)h_1(z)a(v(z))\,dz$$
$$\geq \frac{2}{\delta_1\gamma}\int_\theta^{1-\theta} G_1(1,z)h_1(z)a(v(z))\,dz \geq 2a(\|v\|), \qquad x\in J_\theta, \tag{3.4}$$

this means that $a(\|v\|)=0$. It follows from strict monotonicity of $a(v)$ and $a(0)=0$ that $\|v\|=0$. Hence $\|(u,v)\|=0$, which is a contradiction. Lemma 2.5 implies that

$$i(A,\Omega_\rho\cap(P\times P),P\times P)=0. \tag{3.5}$$

On the other hand, by (H_4), there exist $\zeta>0$ and $C_1>0, C_2>0$ such that

$$\begin{aligned} f_1(x,u,v) &\leq \zeta v^t + C_1, \qquad \forall\,(x,u,v)\in[0,1]\times\mathbb{R}^+\times\mathbb{R}^+,\\ f_2(x,u,v) &\leq \varepsilon_2 u^{\frac{1}{t}} + C_2, \qquad \forall\,(x,u,v)\in[0,1]\times\mathbb{R}^+\times\mathbb{R}^+, \end{aligned} \tag{3.6}$$

where

$$\varepsilon_2=\min\left\{\frac{1}{\mu_2(8\zeta\mu_1)^{\frac{1}{t}}},\ \frac{1}{8\mu_2(\zeta\mu_1)^{\frac{1}{t}}}\right\}.$$

Let

$$W=\{(u,v)\in P\times P:(u,v)=\lambda A(u,v),\ 0\leq\lambda\leq 1\}.$$

We prove that W is bounded. Indeed, for any $(u,v)\in W$, there exists $\lambda\in[0,1]$ such that $u=\lambda A_1(u,v), v=\lambda A_2(u,v)$. Then (3.6) implies that

$$u(x)\leq A_1(u,v)(x)\leq\zeta\int_0^1 G_1(1,y)h_1(y)v^t(y)\,dy + C_3,$$
$$v(x)\leq A_2(u,v)(x)\leq\varepsilon_2\int_0^1 G_2(1,y)h_2(y)u^{\frac{1}{t}}(y)\,dy + C_4.$$

Consequently,

$$\begin{aligned} u(x) &\leq \zeta\int_0^1 G_1(1,y)h_1(y)\,dy\left(\varepsilon_2\int_0^1 G_2(1,z)h_2(z)u^{\frac{1}{t}}(z)\,dz + C_4\right)^t + C_3\\ &\leq \zeta\mu_1\left(\varepsilon_2\int_0^1 G_2(1,z)h_2(z)\|u\|^{\frac{1}{t}}\,dz + C_4\right)^t + C_3\\ &\leq \zeta\mu_1\left[\left(\frac{\|(u,v)\|}{8\zeta\mu_1}\right)^{\frac{1}{t}} + C_4\right]^t + C_3, \end{aligned} \tag{3.7}$$

$$\begin{aligned} v(x) &\leq \varepsilon_2\int_0^1 G_2(1,y)h_2(y)\,dy\left(\zeta\int_0^1 G_1(1,z)h_1(z)v^t(z)\,dz + C_3\right)^{\frac{1}{t}} + C_4\\ &\leq \varepsilon_2\mu_2\left(\zeta\int_0^1 G_1(1,z)h_1(z)\|v\|^t\,dz + C_3\right)^{\frac{1}{t}} + C_4\\ &\leq \frac{1}{8(\zeta\mu_1)^{\frac{1}{t}}}\left(\zeta\mu_1\|(u,v)\|^t + C_3\right)^{\frac{1}{t}} + C_4. \end{aligned} \tag{3.8}$$

Since

$$\lim_{w\to+\infty}\frac{\zeta\mu_1\left[\left(\frac{w}{8\zeta\mu_1}\right)^{\frac{1}{t}} + C_4\right]^t}{w}=\frac{1}{8}, \qquad \lim_{w\to+\infty}\frac{\left(\zeta\mu_1 w^t + C_3\right)^{\frac{1}{t}}}{8(\zeta\mu_1)^{\frac{1}{t}}w}=\frac{1}{8},$$

there exists $r_1 > r$, when $\|(u,v)\| > r_1$, (3.7) and (3.8) yield that

$$u(x) \leq \frac{1}{4}\|(u,v)\| + C_3, \qquad v(x) \leq \frac{1}{4}\|(u,v)\| + C_4.$$

Hence $\|(u,v)\| \leq 2(C_3 + C_4)$ and W is bounded.

Select $G > \sup W$. We obtain from the homotopic invariant property of fixed point index that

$$i(A, \Omega_G \cap (P \times P), P \times P) = i(\theta, \Omega_G \cap (P \times P), P \times P) = 1. \tag{3.9}$$

(3.5) and (3.9) yield that

$$\begin{aligned} &i(A, (\Omega_G \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P) \\ &\quad = i(A, \Omega_G \cap (P \times P), P \times P) - i(A, \Omega_\rho \cap (P \times P), P \times P) = 1. \end{aligned}$$

So A has at least one fixed point on $(\Omega_G \setminus \overline{\Omega}_\rho) \cap (P \times P)$. This means that the system (1.1)–(1.2) has at least one positive solution.

Case 2. The conditions (H_7) and (H_8) hold. First, we prove that

$$i(A, \Omega_r \cap (P \times P), P \times P) = 0. \tag{3.10}$$

We claim that

$$(u,v) \ngeq A(u,v), \qquad \forall\ (u,v) \in \partial\Omega_r \cap (P \times P).$$

If not, there is $(u,v) \in \partial\Omega_r \cap (P \times P)$ such that $(u,v) \geq A(u,v)$. Since $\gamma r \leq u(x) + v(x) \leq r$ for $(u,v) \in \partial\Omega_r \cap (P \times P), x \in [\theta, 1-\theta]$, we know from (H_7) that

$$\begin{aligned} u(x) &\geq \int_0^1 G_1(x,y)h_1(y)f_1(y,u(y),v(y))\,dy \\ &\geq \delta_1^{-1} r \int_\theta^{1-\theta} G_1(1,y)h_1(y)\,dy = r, \qquad x \in J_\theta, \end{aligned} \tag{3.11}$$

$$\begin{aligned} v(x) &\geq \int_0^1 G_2(x,y)h_2(y)f_2(y,u(y),v(y))\,dy \\ &\geq \delta_2^{-1} r \int_\theta^{1-\theta} G_2(1,y)h_2(y)\,dy = r, \qquad x \in J_\theta. \end{aligned} \tag{3.12}$$

Hence $\|(u,v)\| \geq 2r$, which is a contradiction. As a result (3.10) is true.

It remains to prove

$$i(A, \Omega_R \cap (P \times P), P \times P) = 1. \tag{3.13}$$

(H_8) implies that

$$f_1(x,u,v) \leq f_1(x,R,R) \leq \frac{R}{4\mu_1}, \qquad f_2(x,u,v) \leq f_2(x,R,R) \leq \frac{R}{4\mu_2} \tag{3.14}$$

for any $x \in [0,1], (u,v) \in \overline{\Omega}_R$. We claim that

$$(u,v) \nleq A(u,v), \qquad \forall\ (u,v) \in \partial\Omega_R \cap (P \times P).$$

If not, there is $(u,v) \in \partial\Omega_R \cap (P \times P)$ such that $(u,v) \leq A(u,v)$, then we have by (3.14),

$$\begin{aligned} u(x) &\leq \int_0^1 G_1(1,y)h_1(y)f_1(y,u(y),v(y))\,dy \leq \frac{R}{4}, \\ v(x) &\leq \int_0^1 G_2(1,y)h_2(y)f_2(y,u(y),v(y))\,dy \leq \frac{R}{4} \end{aligned}$$

for $x \in [0,1]$. Hence $R = \|(u,v)\| = \|u\| + \|v\| \leq \frac{R}{2}$, which is a contradiction. As a result (3.13) is true. We have by (3.10) and (3.13),

$$\begin{aligned} &i(A, (\Omega_R \setminus \overline{\Omega}_r) \cap (P \times P), P \times P) \\ &\quad = i(A, \Omega_R \cap (P \times P), P \times P) - i(A, \Omega_r \cap (P \times P), P \times P) = 1. \end{aligned}$$

So A has a fixed point on $(\Omega_R \setminus \overline{\Omega}_r) \cap (P \times P)$. This means that the system (1.1)–(1.2) has at least one positive solution. □

Theorem 3.2. *Assume that the conditions $(H_1), (H_2), (H_5)$ and (H_6) are satisfied. Then the system (1.1)–(1.2) has at least one positive solution.*

Proof. By (H_5), there are $\xi_2 > 0, \eta_2 > 0, C_5 > 0, C_6 > 0$ and $C_7 > 0$ such that

$$f_1(x,u,v) \geq \xi_2 p(v) - C_5, \quad f_2(x,u,v) \geq \eta_2 q(u) - C_6, \qquad (x,u,v) \in J_\theta \times \mathbb{R}^+ \times \mathbb{R}^+,$$

and

$$p(K_2 q(u)) \geq \frac{2K_2}{\xi_2 \eta_2 \delta_1 \delta_2 \gamma^3} u - C_7, \qquad u \in \mathbb{R}^+, \tag{3.15}$$

where $K_2 = \max\{\eta_2 \gamma G_2(1,y) h_2(y) : y \in J_\theta\}$. Then we have

$$\begin{aligned} A_1(u,v)(x) &\geq \xi_2 \int_0^1 G_1(x,y) h_1(y) p(v(y))\, dy - C_8, \qquad x \in J_\theta, \\ A_2(u,v)(x) &\geq \eta_2 \int_0^1 G_2(x,y) h_2(y) q(u(y))\, dy - C_9, \qquad x \in J_\theta. \end{aligned} \tag{3.16}$$

We affirm that the set

$$W = \{(u,v) \in P \times P : (u,v) = A(u,v) + \lambda(\varphi,\varphi),\ \lambda \geq 0\}$$

is bounded, where $\varphi \in P \setminus \{0\}$. Indeed, $(u,v) \in W$ implies that $u \geq A_1(u,v), v \geq A_2(u,v)$ for some $\lambda \geq 0$. We have by (3.16),

$$u(x) \geq \xi_2 \int_0^1 G_1(x,y) h_1(y) p(v(y))\, dy - C_8, \qquad x \in J_\theta, \tag{3.17}$$

$$v(x) \geq \eta_2 \int_0^1 G_2(x,y) h_2(y) q(u(y))\, dy - C_9, \qquad x \in J_\theta. \tag{3.18}$$

By the monotonicity and concavity of $p(\cdot)$ as well as Jensen's inequality, (3.18) implies that

$$\begin{aligned} p(v(x) + C_9) &\geq p\Big(\int_0^1 \eta_2 G_2(x,y) h_2(y) q(u(y))\, dy \Big) \\ &\geq \int_0^1 p\big(\eta_2 \gamma G_2(1,y) h_2(y) q(u(y))\big)\, dy \\ &\geq \eta_2 \gamma K_2^{-1} \int_\theta^{1-\theta} G_2(1,y) h_2(y) p\big(K_2 q(u(y))\big)\, dy, \qquad x \in J_\theta. \end{aligned} \tag{3.19}$$

Since $p(v(x)) \geq p(v(x)+C_9) - p(C_9)$, we have by (3.15), (3.17) and (3.19),

$$\begin{aligned}
u(x) &\geq \xi_2\gamma \int_0^1 G_1(1,y)h_1(y)\big[p(v(y)+C_9) - p(C_9)\big]\,dy - C_8 \\
&\geq \xi_2\gamma \int_\theta^{1-\theta} G_1(1,y)h_1(y)p(v(y)+C_9)\,dy - C_{10} \\
&\geq \xi_2\eta_2\gamma^2 K_2^{-1} \int_\theta^{1-\theta} G_1(1,y)h_1(y) \int_\theta^{1-\theta} G_2(1,z)h_2(z)p\big(K_2 q(u(z))\big)\,dz\,dy - C_{10} \\
&\geq \xi_2\eta_2\gamma^2\delta_1 K_2^{-1} \int_\theta^{1-\theta} G_2(1,z)h_2(z)p\big(K_2 q(u(z))\big)\,dz - C_{10} \\
&\geq 2(\delta_2\gamma)^{-1} \int_\theta^{1-\theta} G_2(1,z)h_2(z)u(z)\,dz - C_{11} \geq 2\|u\| - C_{11}, \qquad x \in J_\theta.
\end{aligned} \tag{3.20}$$

Hence $\|u\| \leq C_{11}$.

Since $p(v(x)) \geq \gamma p(\|v\|)$ for $x \in J_\theta, v \in P$, it follows from (3.19), (3.15) and (3.17) that

$$\begin{aligned}
p(v(x)) &\geq p(v(x)+C_9) - p(C_9) \\
&\geq \eta_2\gamma K_2^{-1} \int_\theta^{1-\theta} G_2(1,y)h_2(y)p\big(K_2 q(u(y))\big)\,dy - p(C_9) \\
&\geq \frac{2}{\xi_2\delta_1\delta_2\gamma^2} \int_\theta^{1-\theta} G_2(1,y)h_2(y)u(y)\,dy - C_{12} \\
&\geq \frac{2}{\delta_1\delta_2\gamma} \int_\theta^{1-\theta} G_2(1,y)h_2(y)\,dy \int_0^1 G_1(1,z)h_1(z)p(v(z))\,dz - C_{13} \\
&\geq 2\delta_1^{-1} \int_\theta^{1-\theta} G_1(1,z)h_1(z)p(\|v\|)\,dz - C_{13} \\
&= 2p(\|v\|) - C_{13}, \qquad x \in J_\theta.
\end{aligned}$$

Hence $p(\|v\|) \leq C_{13}$. By (1) and (3) of the condition (H_5), we know that $\lim_{v\to+\infty} p(v) = +\infty$, thus there exists $C_{14} > 0$ such that $\|v\| \leq C_{14}$. This shows W is bounded. Then there exists a sufficiently large $Q > 0$ such that

$$(u,v) \neq A(u,v) + \lambda(\varphi,\varphi), \qquad \forall\, (u,v) \in \partial\Omega_Q \cap (P\times P),\ \lambda \geq 0.$$

Lemma 2.5 yields that

$$i(A, \Omega_Q \cap (P\times P), P\times P) = 0. \tag{3.21}$$

On the other hand, by (H_6), there is a $\sigma > 0$ and sufficiently small $\rho > 0$ such that

$$\begin{aligned}
f_1(x,u,v) &\leq \sigma v^s, \qquad \forall\, (x,u) \in [0,1]\times\mathbb{R}^+,\ v \in [0,\rho], \\
f_2(x,u,v) &\leq \varepsilon_1 u^{\frac{1}{s}}, \qquad \forall\, (x,v) \in [0,1]\times\mathbb{R}^+,\ u \in [0,\rho].
\end{aligned} \tag{3.22}$$

where

$$\varepsilon_1 = \min\Big\{ \big(2\sigma\mu_1\mu_2^s\big)^{-\frac{1}{s}}, \mu_2^{-1} \Big\}.$$

We claim that

$$(u,v) \not\leq A(u,v), \qquad \forall\, (u,v) \in \partial\Omega_\rho \cap (P\times P). \tag{3.23}$$

If not, there exists a $(u,v) \in \partial\Omega_\rho \cap (P \times P)$ such that $(u,v) \leq A(u,v)$, that is, $u \leq A_1(u,v), v \leq A_2(u,v)$. Then (3.22) implies that

$$\begin{aligned} u(x) &\leq \int_0^1 G_1(x,y)h_1(y)f_1(y,u(y),v(y))\,dy \\ &\leq \sigma \int_0^1 G_1(1,y)h_1(y)v^s(y)\,dy \\ &\leq \sigma \int_0^1 G_1(1,y)h_1(y)\left(\int_0^1 G_2(y,z)h_2(z)f_2(z,u(z),v(z))\,dz\right)^s dy \\ &\leq \sigma \int_0^1 G_1(1,y)h_1(y)\,dy\left(\int_0^1 G_2(1,z)h_2(z)f_2(z,u(z),v(z))\,dz\right)^s \\ &= \sigma\mu_1\left(\int_0^1 G_2(1,z)h_2(z)f_2(z,u(z),v(z))\,dz\right)^s \\ &\leq \sigma\mu_1\varepsilon_1^s\left(\int_0^1 G_2(1,z)h_2(z)u^{\frac{1}{s}}(z)\,dz\right)^s \\ &\leq \sigma\mu_1\varepsilon_1^s\mu_2^s\|u\| \leq \frac{1}{2}\|u\|, \qquad x \in [0,1], \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} v(x) &\leq \int_0^1 G_2(x,y)h_2(y)f_2(y,u(y),v(y))dy \\ &\leq \varepsilon_1 \int_0^1 G_2(1,y)h_2(y)u^{\frac{1}{s}}(y)dy \\ &\leq \varepsilon_1\mu_2\|u\|^{\frac{1}{s}} \leq \|u\|^{\frac{1}{s}}, \qquad x \in [0,1]. \end{aligned} \tag{3.25}$$

(3.24) and (3.25) imply that $\|(u,v)\| = 0$, which contradicts $\|(u,v)\| = \rho$ and the inequality (3.23) holds. Lemma 2.6 yields that

$$i(A,\Omega_\rho \cap (P \times P), P \times P) = 1. \tag{3.26}$$

We have by (3.21) and (3.26),

$$\begin{aligned} &i(A,(\Omega_Q \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P) \\ &\quad = i(A,\Omega_Q \cap (P \times P), P \times P) - i(A,\Omega_\rho \cap (P \times P), P \times P) = -1. \end{aligned}$$

Hence A has a fixed point on $(\Omega_Q \setminus \overline{\Omega}_\rho) \cap (P \times P)$. This means that the system (1.1)–(1.2) has at least one positive solution. □

4 Existence of multiple positive solutions

Theorem 4.1. *Assume that the conditions $(H_1), (H_2), (H_3), (H_5)$ and (H_8) hold. Then the system (1.1)–(1.2) has at least two positive solutions.*

Proof. We may take $Q > R > \rho$ such that both (3.5), (3.13) and (3.21) hold. Then we have

$$\begin{aligned} &i(A,(\Omega_Q \setminus \overline{\Omega}_R) \cap (P \times P), P \times P) \\ &\quad = i(A,\Omega_Q \cap (P \times P), P \times P) - i(A,\Omega_R \cap (P \times P), P \times P) = -1, \\ &i(A,(\Omega_R \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P) \\ &\quad = i(A,\Omega_R \cap (P \times P), P \times P) - i(A,\Omega_\rho \cap (P \times P), P \times P) = 1. \end{aligned}$$

Hence A has a fixed point on $(\Omega_Q \setminus \overline{\Omega}_R) \cap (P \times P)$ and $(\Omega_R \setminus \overline{\Omega}_\rho) \cap (P \times P)$, respectively. This means the system (1.1)–(1.2) has at least two positive solutions. □

Theorem 4.2. *Assume that the conditions $(H_1), (H_2), (H_4), (H_6)$ and (H_7) hold. Then the system (1.1)–(1.2) has at least two positive solutions.*

Proof. We may take $G > r > \rho$ such that both (3.9), (3.10) and (3.26) hold. Then we have

$$\begin{aligned} &i(A, (\Omega_G \setminus \overline{\Omega}_r) \cap (P \times P), P \times P) \\ &\qquad = i(A, \Omega_G \cap (P \times P), P \times P) - i(A, \Omega_r \cap (P \times P), P \times P) = 1, \\ &i(A, (\Omega_r \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P) \\ &\qquad = i(A, \Omega_r \cap (P \times P), P \times P) - i(A, \Omega_\rho \cap (P \times P), P \times P) = -1. \end{aligned}$$

Hence A has a fixed point on $(\Omega_G \setminus \overline{\Omega}_r) \cap (P \times P)$ and $(\Omega_r \setminus \overline{\Omega}_\rho) \cap (P \times P)$, respectively. This means the system (1.1)–(1.2) has at least two positive solutions. □

5 The nonexistence of positive solutions

Theorem 5.1. *Assume that the conditions (H_1) and (H_2) hold, and*

$$f_1(x, u, v) > (\gamma^2 \delta_1)^{-1} v, \quad f_2(x, u, v) > (\gamma^2 \delta_2)^{-1} u, \qquad \forall\, x \in [0, 1],\ u > 0,\ v > 0.$$

Then the system (1.1)–(1.2) has no positive solution.

Proof. Assume that (u, v) is a positive solution of the system (1.1)–(1.2), then $(u, v) \in P \times P$, $u(x) > 0, v(x) > 0$ for $x \in (0, 1)$, and for $x \in J_\theta$,

$$\begin{aligned} u(x) &= \int_0^1 G_1(x, y) h_1(y) f_1(y, u(y), v(y))\, dy \\ &\geq \gamma_\alpha \int_0^1 G_1(1, y) h_1(y) f_1(y, u(y), v(y))\, dy \\ &> \gamma (\gamma^2 \delta_1)^{-1} \int_0^1 G_1(1, y) h_1(y) v(y)\, dy \\ &\geq \gamma^2 (\gamma^2 \delta_1)^{-1} \int_\theta^{1-\theta} G_1(1, y) h_1(y)\, dy \|v\| = \|v\|. \end{aligned}$$

Hence $\|u\| > \|v\|$. Similarly, $\|v\| > \|u\|$, which is a contradiction. □

Similarly, we can obtain the following result.

Theorem 5.2. *Assume that $(H_1), (H_2)$ hold, and $f_1(x, u, v) < \mu_1^{-1} v$, $f_2(x, u, v) < \mu_2^{-1} u$ for any $x \in [0, 1]$, $u > 0$, $v > 0$, then the system (1.1)–(1.2) has no positive solution.*

Remark 5.3. If $h_j \in C([0, 1], \mathbb{R}^+)$ $(j = 1, 2)$ and $1 \leq \mu, \nu \leq n - 2$ $(n \geq 3)$ in the system (1.1)–(1.2), all our results are still true.

6 Examples

In the following examples 6.1–6.4, we select $\alpha = \beta \in (n-1, n]$, $\mu = \nu \in [1, n-3]$ for $n > 3$ and $\eta_1 = \eta_2$, $\xi_1 = \xi_2$, $d_1 = 1 - \eta_1 \xi_1^{\alpha-\mu-1}$ in the system (1.1)–(1.2).

Example 6.1. Let $h_1(x) = h_2(x) = d_1\Gamma(\alpha)/(1-x)^{\alpha-\mu-1}$, $x \in (0,1)$, $f_1(x,u,v) = e^x(1+e^{-(u+v)})$, $f_2(x,u,v) = 1 - e^{-(u+v)}$, $x \in [0,1]$, $u, v \in \mathbb{R}^+$, $a(v) = v^{\frac{1}{2}}$, $b(u) = u^{\frac{1}{2}}$, $t = 1/2$. Clearly,

$$0 < \int_0^1 G_j(1,y)h_j(y)\,dy \le 1,$$

but $\int_0^1 h_j(y)\,dy = +\infty$ $(j = 1,2)$. The results of [7,9,10,17,24,29,30] are not suitable for the problem. It is easy to verify that the conditions (H_1)–(H_4) hold, hence Theorem 3.1 implies that the system (1.1)–(1.2) has at least one positive solution. Here $f_1(x,u,v)$ and $f_2(x,u,v)$ are sublinear on u and v at 0 and $+\infty$.

Example 6.2. Let $h_j(x)$ be as in the Example 6.1, $f_1(x,u,v) = e^x(1+e^{-(u+v)})$, $f_2(x,u,v) = u^{\frac{3}{2}}$, $a(v) = v^{\frac{1}{3}}$, $b(u) = u^2$, $t = 1/2$. It is easy to verify that the conditions (H_1)–(H_4) hold, Theorem 3.1 implies that the system (1.1)–(1.2) has at least one positive solution. Here $f_1(x,u,v)$ is sublinear on u and v at 0 and $+\infty$, whereas $f_2(x,u,v)$ is superlinear on u at 0 and $+\infty$.

Example 6.3. Let $h_j(x)$ be as in the Example 6.1, $f_1(x,u,v) = (1+e^{-u})v^3$, $f_2(x,u,v) = u^3$, $p(v) = v^{\frac{1}{2}}$, $q(u) = u^3$, $s = 3$. It is easy to verify that the conditions (H_1), (H_2), (H_5) and (H_6) hold. Theorem 3.2 yields that the system (1.1)–(1.2) has at least one positive solution. Here $f_1(x,u,v)$ is superlinear on v at 0 and $+\infty$, $f_2(x,u,v)$ is superlinear on u at 0 and $+\infty$.

Example 6.4. Let $h_j(x)$ be as in the Example 6.1, $f_1(x,u,v) = (1+e^{-u})v^{\frac{2}{3}}$, $f_2(x,u,v) = (1+e^{-v})u^5$, $p(v) = v^{\frac{1}{3}}$, $q(u) = u^4$, $s = 1/3$. It is easy to see that the conditions (H_1), (H_2), (H_5) and (H_6) hold. Theorem 3.2 yields that the system (1.1)–(1.2) has at least one positive solution. Here $f_1(x,u,v)$ is sublinear on v at 0 and $+\infty$, whereas $f_2(x,u,v)$ is superlinear on u at 0 and $+\infty$.

Remark 6.5. From the Examples 6.1–6.4 we know that the conditions (H_3)–(H_6) are applicable for more general function and it is not included among the known differential system. Hence our results are different from those in [7,9,10,17,24,29,30,33].

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