



# Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences

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**Abstract.** Well-defined solutions of the bilinear difference equation are represented in terms of generalized Fibonacci sequences and the initial value. Our results extend and give natural explanations of some recent results in the literature. Some applications concerning a two-dimensional system of bilinear difference equations are also given.

**Keywords:** bilinear difference equation, solvable difference equation, generalized Fibonacci sequence.

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## 1 Introduction

Studying difference equations and systems which are not closely related to differential ones is a topic of recent interest (see, [1–41]). Solvable difference equations attract attention of mathematicians for a long time. Some classical classes of solvable difference equations and methods for solving them can be found, for example, in [14]. Recently, there has been an increasing interest in the topic (see, for example, [1–4, 6–8, 17, 20, 21, 23–41] and the related references therein). Some of the recent papers give formulas for solutions to some very special difference equations or systems of difference equations and prove them by using only the method of induction (quite frequently the proofs of some statements are even omitted or incomplete). However, the formulas are not justified by some theoretical explanations.

In paper [20] we gave a theoretical explanation for the formula of solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad n \in \mathbb{N}_0, \quad (1.1)$$

given in [7] (in fact, a generalization of equation (1.1) was treated in [20]). Paper [20] attracted some attention among the experts in difference equations and triggered off a new interest in the area. For some results regarding solutions of various types of extensions of equation (1.1),

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see, for example, [1, 17, 24, 25, 27, 37]. Papers [3] and [4] consider also an extension of equation (1.1), but do not use formulas for their solutions. Some other explanations for the formulas of some special difference equations or systems of difference equations appearing in recent literature, can be found, for example, in papers [23], [28], [36] and [38].

Recent paper [40] is also one of those which give some formulas and prove them by the induction, but does not use any other mathematical technique in explaining the formulas.

Namely, the authors of [40] represented the general solution of the following difference equation

$$x_{n+1} = \frac{1}{1 + x_n}, \quad n \in \mathbb{N}_0, \quad (1.2)$$

in terms of the initial value  $x_0$  and the Fibonacci sequence, that is, the sequence defined as follows

$$f_{n+1} = f_n + f_{n-1}, \quad n \in \mathbb{N}, \quad (1.3)$$

$f_0 = 0, f_1 = 1$ . More precisely, it was proved by induction that every well-defined solution of equation (1.2) can be written in the following form

$$x_n = \frac{x_0 f_{n-1} + f_n}{x_0 f_n + f_{n+1}}, \quad n \in \mathbb{N}. \quad (1.4)$$

However, the authors of [40] did not explain how they come up with the formula and did not support it by any mathematical theory.

They also proved that every well-defined solution of the equation

$$x_{n+1} = \frac{1}{-1 + x_n}, \quad n \in \mathbb{N}_0, \quad (1.5)$$

can be written in the following form

$$x_n = \frac{x_0 f_{-(n-1)} + f_{-n}}{x_0 f_{-n} + f_{-(n+1)}}, \quad n \in \mathbb{N}, \quad (1.6)$$

where the terms of the Fibonacci sequence with negative indices are calculated by the formula

$$f_{-n} = f_{-n+2} - f_{-n+1}, \quad n \in \mathbb{N}, \quad (1.7)$$

and where, of course, is assumed that  $f_0 = 0$  and  $f_1 = 1$  (recurrence relation (1.7) is obtained from (1.3) when we replace  $n$  by  $-n + 1$ ).

As in the case of equation (1.2), they also did not explain how they come up with formula (1.6) nor gave any theoretical explanation for it.

The other results in [40] are folklore, that is, follow easily from well-known ones. Formulas (1.4) and (1.6) could be also known, but we are not able to find some specific references for them at the moment. Nevertheless, in our opinion, these two formulas are interesting and motivated us to explain them theoretically. Actually, our aim is to obtain, in a natural way, similar representation for a more general difference equation which includes into itself equations (1.2) and (1.5). As some applications of our main results we give explanations of some results in [8], and also obtain related results for a two-dimensional system of bilinear difference equations.

## 2 Preliminaries and some basic solvable difference equations

In this section we present some known difference equations and results related to them, and also introduce some notions which will be used in the proofs of our main results.

## 2.1 Linear first-order difference equation

Probably, the most known difference equation which can be solved is the *linear first-order difference equation*, i.e.

$$x_{n+1} = p_n x_n + q_n, \quad n \in \mathbb{N}_0, \quad (2.1)$$

where  $(p_n)_{n \in \mathbb{N}_0}$  and  $(q_n)_{n \in \mathbb{N}_0}$  are arbitrary real (or complex) sequences and  $x_0 \in \mathbb{R}$  (or  $x_0 \in \mathbb{C}$ ). Equation (2.1) can be solved in closed form in many ways, and its general solution is

$$x_n = x_0 \prod_{j=0}^{n-1} p_j + \sum_{i=0}^{n-1} q_i \prod_{j=i+1}^{n-1} p_j. \quad (2.2)$$

For example, if  $p_n \neq 0$ ,  $n \in \mathbb{N}_0$ , by dividing both sides of (2.1) by  $\prod_{j=0}^n p_j$ , we obtain

$$\frac{x_{n+1}}{\prod_{j=0}^n p_j} = \frac{x_n}{\prod_{j=0}^{n-1} p_j} + \frac{q_n}{\prod_{j=0}^n p_j}, \quad n \in \mathbb{N}_0. \quad (2.3)$$

Summing equalities in (2.3) from 0 to  $n - 1$ , we get

$$\frac{x_n}{\prod_{j=0}^{n-1} p_j} = x_0 + \sum_{i=0}^{n-1} \frac{q_i}{\prod_{j=0}^i p_j},$$

from which formula (2.2) easily follows.

It is interesting how many applications this relatively simple difference equation has. Even many recent results are essentially connected to the equation (see, for example, [6, 17, 20, 21, 23–26, 29–31, 33, 38, 39]).

## 2.2 Generalized Fibonacci sequence

Here we define an extension of the Fibonacci sequence in the following way

$$s_{n+1} = a s_n + b s_{n-1}, \quad n \in \mathbb{N}, \quad (2.4)$$

$$s_0 = 0, \quad s_1 = 1, \quad (2.5)$$

and we will call it the *generalized Fibonacci sequence* (note that for  $a = b = 1$  is obtained the Fibonacci sequence). We assume that  $b \neq 0$ , otherwise, equation (2.5) becomes a special case of the linear first-order difference equation (2.1).

Note that the characteristic polynomial associated to equation (2.4) is

$$\lambda^2 - a\lambda - b = 0,$$

so that the characteristic roots are

$$\lambda_{1,2} = \frac{a \pm \sqrt{a^2 + 4b}}{2}$$

and if  $a^2 + 4b \neq 0$ , then the solution of equation (2.4) satisfying conditions (2.5) is

$$s_n(a, b) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \quad (2.6)$$

$$= \frac{1}{\sqrt{a^2 + 4b}} \left( \left( \frac{a + \sqrt{a^2 + 4b}}{2} \right)^n - \left( \frac{a - \sqrt{a^2 + 4b}}{2} \right)^n \right). \quad (2.7)$$

The main motivation for introducing the generalized Fibonacci sequence are representations (1.4) and (1.6) of solutions of equations (1.2) and (1.5).

### 2.3 Linear second order difference equation with constant coefficients.

As is well-known, the equation

$$x_{n+2} - ax_{n+1} - bx_n = 0, \quad n \in \mathbb{N}_0, \quad (2.8)$$

(the homogeneous linear second order difference equation with constant coefficients), where  $a, x_0, x_1 \in \mathbb{R}$ , and  $b \in \mathbb{R} \setminus \{0\}$ , is usually solved by using the characteristic roots  $\lambda_1$  and  $\lambda_2$  of the characteristic polynomial  $\lambda^2 - a\lambda - b = 0$ . This standard method along with some calculations easily gives formulas for its general solution (see formulas (2.11) and (2.12)). To demonstrate the importance of equation (2.1), for the completeness and the benefit of the reader, recall, that the formulas can be also obtained by using formula (2.2). Namely, since  $a = \lambda_1 + \lambda_2$  and  $b = -\lambda_1\lambda_2$ , we have that

$$x_{n+2} - \lambda_1 x_{n+1} - \lambda_2 (x_{n+1} - \lambda_1 x_n) = 0, \quad n \in \mathbb{N}_0. \quad (2.9)$$

Using the change of variables  $y_n = x_n - \lambda_1 x_{n-1}$ ,  $n \in \mathbb{N}$ , equation (2.9) becomes

$$y_{n+2} = \lambda_2 y_{n+1}, \quad n \in \mathbb{N}_0,$$

which is equation (2.1) with  $p_n = \lambda_2$  and  $q_n = 0$ ,  $n \in \mathbb{N}$ , so its solution is  $y_n = y_1 \lambda_2^{n-1}$ ,  $n \in \mathbb{N}$ , that is

$$x_n = \lambda_1 x_{n-1} + (x_1 - \lambda_1 x_0) \lambda_2^{n-1}, \quad n \in \mathbb{N}. \quad (2.10)$$

Equation (2.10) is also equation (2.1), but with  $p_n = \lambda_1$  and  $q_n = (x_1 - \lambda_1 x_0) \lambda_2^n$ ,  $n \in \mathbb{N}_0$ . So, by formula (2.2) is obtained that the general solution of equation (2.8) is

$$x_n = x_0 \lambda_1^n + (x_1 - \lambda_1 x_0) \sum_{i=0}^{n-1} \lambda_1^{n-1-i} \lambda_2^i,$$

from which for the case  $\lambda_1 \neq \lambda_2$  is easily obtained

$$x_n = \frac{\lambda_2 x_0 - x_1}{\lambda_2 - \lambda_1} \lambda_1^n + \frac{x_1 - \lambda_1 x_0}{\lambda_2 - \lambda_1} \lambda_2^n. \quad (2.11)$$

while if  $\lambda_1 = \lambda_2$  is obtained

$$x_n = (x_1 n + \lambda_1 x_0 (1 - n)) \lambda_1^{n-1}. \quad (2.12)$$

Formulas (2.11) and (2.12) are well-known, but what is interesting to note is the fact that solution (2.11) can be written in the following form

$$x_n = x_1 s_n(a, b) + b x_0 s_{n-1}(a, b), \quad n \in \mathbb{N}, \quad (2.13)$$

and that the same formula also holds for the case  $\lambda_1 = \lambda_2$ , with

$$s_n = n \lambda_1^{n-1}.$$

**Remark 2.1.** Note that representation (2.13) holds also for  $n = 0$ , if we assume that

$$b s_{-1} = s_1 - a s_0 = 1,$$

that is, if  $s_{-1} = 1/b$ .

Now, we have all the ingredients for formulating and proving the main results in this paper.

### 3 Extensions of formulas (1.4) and (1.6) and their consequences

A natural extension of equations (1.2) and (1.5) is the bilinear difference equation

$$z_{n+1} = \frac{\alpha z_n + \beta}{\gamma z_n + \delta}, \quad n \in \mathbb{N}_0, \quad (3.1)$$

where parameters  $\alpha, \beta, \gamma, \delta$  and initial value  $z_0$  are real numbers.

We will assume that  $\gamma \neq 0$ , since for  $\gamma = 0$  equation (3.1) is reduced to a special case of equation (2.1). Beside this, we will also assume that  $\alpha\delta \neq \beta\gamma$ , since otherwise is obtained the trivial equation

$$z_{n+1} = \text{const.}, \quad n \in \mathbb{N}_0,$$

(case  $\gamma = \delta = 0$  is excluded by the first assumption). For some recent applications of equation (3.1), see, for example, [6] and [34].

Our aim is to obtain an extension of formula (1.4), for the solutions of difference equation (3.1), in terms of the initial value and a sequence of type in (2.4) satisfying the conditions in (2.5). We also want to obtain an extension of formula (1.6) for the solutions of equation (3.1).

Note that equation (3.1) can be written in the form

$$z_{n+1} = \frac{\alpha}{\gamma} + \frac{1}{\gamma} \frac{\beta\gamma - \alpha\delta}{\gamma z_n + \delta}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$\gamma z_{n+1} + \delta = \alpha + \delta + \frac{\beta\gamma - \alpha\delta}{\gamma z_n + \delta}, \quad n \in \mathbb{N}_0. \quad (3.2)$$

Since we are interested in well-defined solutions of equation (3.1) we may assume that

$$\gamma z_n + \delta \neq 0, \quad n \in \mathbb{N}_0.$$

Hence we can use the change of variables

$$b_n = \frac{1}{\gamma z_n + \delta}, \quad n \in \mathbb{N}_0, \quad (3.3)$$

in (3.2) and obtain

$$b_{n+1} = \frac{1}{\alpha + \delta + (\beta\gamma - \alpha\delta)b_n}, \quad n \in \mathbb{N}_0. \quad (3.4)$$

If we use the following change of variables

$$b_n = \frac{c_n}{c_{n+1}}, \quad n \in \mathbb{N}_0, \quad (3.5)$$

in (3.4), we get

$$c_{n+1} - (\alpha + \delta)c_n + (\alpha\delta - \beta\gamma)c_{n-1} = 0, \quad n \in \mathbb{N}. \quad (3.6)$$

Now note that equation (3.6) is nothing but equation (2.4) with

$$a = \alpha + \delta \quad \text{and} \quad b = \beta\gamma - \alpha\delta.$$

Hence, by using representation (2.13) we see that the general solution of equation (3.6) in terms of the sequence  $s_n := s_n(\alpha + \delta, \beta\gamma - \alpha\delta)$ , and initial values  $c_0$  and  $c_1$  is

$$c_n = c_1 s_n + c_0 (\beta\gamma - \alpha\delta) s_{n-1}, \quad n \in \mathbb{N}_0. \quad (3.7)$$

By using (3.7) in (3.3) we get

$$\begin{aligned}\gamma z_n + \delta &= \frac{1}{b_n} = \frac{c_1 s_{n+1} + c_0(\beta\gamma - \alpha\delta)s_n}{c_1 s_n + c_0(\beta\gamma - \alpha\delta)s_{n-1}} \\ &= \frac{(\gamma z_0 + \delta)s_{n+1} + (\beta\gamma - \alpha\delta)s_n}{(\gamma z_0 + \delta)s_n + (\beta\gamma - \alpha\delta)s_{n-1}}.\end{aligned}$$

Hence, by using (3.6) it follows that

$$\begin{aligned}z_n &= \frac{1}{\gamma} \left( \frac{(\gamma z_0 + \delta)(s_{n+1} - \delta s_n) + (\beta\gamma - \alpha\delta)(s_n - \delta s_{n-1})}{(\gamma z_0 + \delta)s_n + (\beta\gamma - \alpha\delta)s_{n-1}} \right) \\ &= \frac{1}{\gamma} \left( \frac{(\gamma z_0 + \delta)(\alpha s_n + (\beta\gamma - \alpha\delta)s_{n-1}) + (\beta\gamma - \alpha\delta)(s_n - \delta s_{n-1})}{\gamma z_0 s_n + \delta s_n + (\beta\gamma - \alpha\delta)s_{n-1}} \right) \\ &= \frac{(\alpha z_0 + \beta)s_n + z_0(\beta\gamma - \alpha\delta)s_{n-1}}{(\gamma z_0 - \alpha)s_n + s_{n+1}}.\end{aligned}$$

From all above mentioned we see that the following theorem holds.

**Theorem 3.1.** *Consider equation (3.1), with  $\gamma \neq 0$  and  $\alpha\delta \neq \beta\gamma$ . Then every well-defined solution of the equation can be written in the following form*

$$z_n = \frac{z_0(\beta\gamma - \alpha\delta)s_{n-1} + (\alpha z_0 + \beta)s_n}{(\gamma z_0 - \alpha)s_n + s_{n+1}}, \quad n \in \mathbb{N}, \quad (3.8)$$

where  $(s_n)_{n \in \mathbb{N}_0}$  is the sequence satisfying difference equation (3.6) with the initial conditions  $s_0 = 0$  and  $s_1 = 1$ .

If  $\alpha = 0$ , then from (3.8) we get

$$z_n = \beta \frac{\gamma z_0 s_{n-1} + s_n}{\gamma z_0 s_n + s_{n+1}}. \quad (3.9)$$

Hence, for  $\beta = \gamma = \delta = 1$  we have that  $s_n = f_n$ ,  $n \in \mathbb{N}_0$ , and consequently we get formula (1.4), giving a natural explanation for it.

**Corollary 3.2.** *Consider equation (3.1), with  $\beta\gamma \neq 0$  and  $\alpha = 0$ . Then for every well-defined solution of the equation the following formula holds*

$$\prod_{j=0}^n z_j = \frac{z_0 \beta^n}{\gamma z_0 s_n + s_{n+1}}, \quad (3.10)$$

where  $(s_n)_{n \in \mathbb{N}_0}$  is the sequence satisfying difference equation (3.6) with the initial conditions  $s_0 = 0$  and  $s_1 = 1$ .

*Proof.* We have

$$\prod_{j=0}^n z_j = z_0 \prod_{j=1}^n \beta \frac{\gamma z_0 s_{j-1} + s_j}{\gamma z_0 s_j + s_{j+1}} = z_0 \beta^n \frac{\gamma z_0 s_0 + s_1}{\gamma z_0 s_n + s_{n+1}},$$

from which (3.10) follows. □

Difference equation (2.4) can be naturally extended for negative indices by using the following recurrence relation

$$s_{-n} = (s_{-(n-2)} - as_{-(n-1)})/b, \quad (3.11)$$

where  $s_0 = 0$  and  $s_1 = 1$ .

It is known that its solution is

$$s_{-n} = \frac{\lambda_1^{-n} - \lambda_2^{-n}}{\lambda_1 - \lambda_2}, \quad n \geq -1,$$

from which it follows that

$$s_{-n} = -\frac{1}{(\lambda_1\lambda_2)^n} \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

Hence, for the case of difference equation (3.6), we have that

$$s_{-n} = -\frac{s_n}{(\alpha\delta - \beta\gamma)^n}, \quad n \in \mathbb{N}_0,$$

that is,

$$s_n = -s_{-n}(\alpha\delta - \beta\gamma)^n, \quad n \in \mathbb{N}_0. \quad (3.12)$$

Using (3.12) into (3.8) we get

$$z_n = \frac{-z_0s_{-(n-1)} + (\alpha z_0 + \beta)s_{-n}}{(\gamma z_0 - \alpha)s_{-n} + (\alpha\delta - \beta\gamma)s_{-(n+1)}}, \quad n \in \mathbb{N}_0, \quad (3.13)$$

which is a representation of well-defined solutions of equation (3.1) in terms of the generalized Fibonacci sequence with negative indices. Hence we have that the following theorem holds.

**Theorem 3.3.** Consider equation (3.1), with  $\gamma \neq 0$  and  $\alpha\delta \neq \beta\gamma$ . Then every well-defined solution of the equation can be written in the following form

$$z_n = \frac{-z_0s_{-(n-1)} + (\alpha z_0 + \beta)s_{-n}}{(\gamma z_0 - \alpha)s_{-n} + (\alpha\delta - \beta\gamma)s_{-(n+1)}}, \quad n \in \mathbb{N}_0, \quad (3.14)$$

where  $(s_{-n})_{n \geq -1}$  is the sequence satisfying recurrent relation (3.11) with the initial conditions  $s_0 = 0$  and  $s_1 = 1$ .

If  $\alpha = 0$ , then from (3.13) we get

$$z_n = -\frac{z_0s_{-(n-1)} - \beta s_{-n}}{\gamma(z_0s_{-n} - \beta s_{-(n+1)})}, \quad n \in \mathbb{N}_0. \quad (3.15)$$

Hence, for  $\beta = \gamma = -1$  and  $\delta = 1$  we have that  $s_{n+1} - s_n - s_{n-1} = 0$ ,  $n \in \mathbb{N}$ , so that  $s_n = f_n$ ,  $n \in \mathbb{N}_0$ . From this and since by (3.12) we have that  $(-1)^{n+1}f_n = f_{-n}$ ,  $n \in \mathbb{N}_0$ , we get  $s_{-n} = f_{-n}$ ,  $n \in \mathbb{N}_0$ , from which along with (3.15), formula (1.6) follows.

**Corollary 3.4.** Consider equation (3.1), with  $\beta\gamma \neq 0$  and  $\alpha = 0$ . Then for every well-defined solution of the equation the following formula holds

$$\prod_{j=0}^n z_j = \frac{z_0}{(-\gamma)^{n+1}(z_0s_{-n} - \beta s_{-(n+1)})}, \quad (3.16)$$

where  $(s_{-n})_{n \geq -1}$  is the sequence satisfying recurrent relation (3.11) with the initial conditions  $s_0 = 0$  and  $s_1 = 1$ .

*Proof.* We have

$$\prod_{j=0}^n z_j = \prod_{j=0}^n \left( -\frac{z_0 s_{-(j-1)} - \beta s_{-j}}{\gamma(z_0 s_{-j} - \beta s_{-(j+1)})} \right) = \frac{1}{(-\gamma)^{n+1}} \frac{z_0 s_1 - \beta s_0}{z_0 s_{-n} - \beta s_{-(n+1)}},$$

from which (3.16) follows.  $\square$

## 4 Some applications

As some applications of our main results, in this section we give theoretical explanations for the formulas presented in Theorems 4–6 in [8], and obtain some related results for a two-dimensional system of bilinear difference equations. The author of [8] formulated, among others, the following three results and proved them by induction. However, none theoretical explanations are given therein and it was also not explained how the formulas for solutions of the difference equations therein are obtained, especially since the forms of the solutions do not look simple.

**Theorem 4.1.** *Let  $(x_n)_{n \geq -1}$  be a solution of the following difference equation*

$$x_{n+1} = \frac{2x_n^2 + x_n x_{n-1}}{x_n + x_{n-1}}, \quad n \in \mathbb{N}_0. \quad (4.1)$$

*Then*

$$x_n = x_0 \prod_{j=1}^n \frac{f_{2j+1} x_0 + f_{2j} x_{-1}}{f_{2j} x_0 + f_{2j-1} x_{-1}}, \quad n \in \mathbb{N}_0. \quad (4.2)$$

**Theorem 4.2.** *Let  $(x_n)_{n \geq -1}$  be a solution of the following difference equation*

$$x_{n+1} = \frac{2x_n^2 - x_n x_{n-1}}{x_n - x_{n-1}}, \quad n \in \mathbb{N}_0. \quad (4.3)$$

*Then*

$$x_n = x_0 \prod_{j=1}^n \frac{f_{j+2} x_0 - f_j x_{-1}}{f_j x_0 - f_{j-2} x_{-1}}, \quad n \in \mathbb{N}_0. \quad (4.4)$$

**Theorem 4.3.** *Let  $(x_n)_{n \geq -1}$  be a solution of the following difference equation*

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n + x_{n-1}}, \quad n \in \mathbb{N}_0. \quad (4.5)$$

*Then*

$$x_n = \frac{x_0 x_{-1}}{f_n x_0 + f_{n+1} x_{-1}}, \quad n \in \mathbb{N}_0. \quad (4.6)$$

Now we give theoretical explanations for the formulas presented in Theorems 4.1–4.3, based on our main results. Before this, note that the author of [8] under *solutions* seems tacitly understands well-defined solutions. Hence, we will assume that the solutions we deal with are of this type. For some results in the area, see, e.g. [29].

#### 4.1 Case of equation (4.1)

First note that we may assume that  $x_n \neq 0$  for every  $n \in \mathbb{N}$ . Otherwise, if there is an  $n_0 \in \mathbb{N}$  such that  $x_{n_0} = 0$ , then if  $x_{n_0+1}$  is defined, from (4.1) we would have  $x_{n_0+1} = 0$ , which would imply that  $x_{n_0+2}$  is not defined (since in this case  $x_{n_0} + x_{n_0+1} = 0$ ). We may also assume that  $x_{-1} \neq 0$ , for if  $x_{-1} = 0$  and  $x_0 \neq 0$ , and solution  $(x_n)_{n \geq -1}$  is well-defined, then we can consider equation (4.1) for  $n \in \mathbb{N}$ , that is, to reduce the case to the previous one by scaling indices backward for one.

Hence, we can use the change of variables

$$y_n = \frac{x_{n-1}}{x_n}, \quad n \in \mathbb{N}_0, \quad (4.7)$$

and transform equation (4.1) into the following one

$$y_{n+1} = \frac{y_n + 1}{y_n + 2}, \quad n \in \mathbb{N}_0, \quad (4.8)$$

which is a special case of equation (3.1), with  $\alpha = \beta = \gamma = 1$  and  $\delta = 2$ .

Clearly, from (4.7) we have that

$$x_n = x_0 \prod_{j=1}^n \frac{1}{y_j}, \quad n \in \mathbb{N}_0. \quad (4.9)$$

By using Theorem 3.1 we have that every well-defined solution of equation (4.8) can be written in the form

$$y_n = \frac{-y_0 s_{n-1} + (y_0 + 1) s_n}{(y_0 - 1) s_n + s_{n+1}}, \quad n \in \mathbb{N}, \quad (4.10)$$

where  $(s_n)_{n \in \mathbb{N}_0}$  is the sequence satisfying the difference equation

$$s_{n+1} - 3s_n + s_{n-1} = 0, \quad n \in \mathbb{N}, \quad (4.11)$$

with the initial conditions  $s_0 = 0$  and  $s_1 = 1$ .

Employing formula (2.6) or (2.7) we have

$$s_n = \frac{\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n}{\frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2}}, \quad n \in \mathbb{N}_0. \quad (4.12)$$

Now note that

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^2 = \frac{3 \pm \sqrt{5}}{2}.$$

Using this in (4.12) we obtain

$$s_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}}{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2} = f_{2n}, \quad n \in \mathbb{N}_0. \quad (4.13)$$

Using (4.13) into (4.10), recurrent relation (1.3), and (4.7) with  $n = 0$ , we have that

$$\begin{aligned} y_n &= \frac{-y_0 f_{2n-2} + (y_0 + 1) f_{2n}}{(y_0 - 1) f_{2n} + f_{2n+2}} \\ &= \frac{-y_0 (f_{2n} - f_{2n-1}) + (y_0 + 1) f_{2n}}{(y_0 - 1) f_{2n} + f_{2n+1} + f_{2n}} \\ &= \frac{y_0 f_{2n-1} + f_{2n}}{y_0 f_{2n} + f_{2n+1}} \end{aligned} \quad (4.14)$$

$$= \frac{x_{-1} f_{2n-1} + x_0 f_{2n}}{x_{-1} f_{2n} + x_0 f_{2n+1}}, \quad n \in \mathbb{N}. \quad (4.15)$$

Employing relationship (4.15) into (4.9) and by some simple calculations formula (4.2) is obtained.

## 4.2 Case of equation (4.3)

Note that we may also assume that  $x_n \neq 0$  for every  $n \in \mathbb{N}$ . Otherwise, if there is an  $n_1 \in \mathbb{N}$  such that  $x_{n_1} = 0$ , then if  $x_{n_1+1}$  is defined, from (4.3) we would have  $x_{n_1+1} = 0$ , which would imply that  $x_{n_1+2}$  is not defined (since in this case  $x_{n_1+1} - x_{n_1} = 0$ ). We may also assume that  $x_{-1} \neq 0$ , for if  $x_{-1} = 0$  and  $x_0 \neq 0$ , and solution  $(x_n)_{n \geq -1}$  is well-defined, then we can consider equation (4.3) for  $n \in \mathbb{N}$ , that is, to reduce the case to the previous one by scaling indices backward for one.

Hence, we can use the change of variables

$$y_n = \frac{x_n}{x_{n-1}}, \quad n \in \mathbb{N}_0, \quad (4.16)$$

and transform equation (4.3) into the following one

$$y_{n+1} = \frac{2y_n - 1}{y_n - 1}, \quad n \in \mathbb{N}_0, \quad (4.17)$$

which is a special case of equation (3.1), with  $\beta = \delta = -1$ ,  $\gamma = 1$  and  $\alpha = 2$ .

Clearly, from (4.16) we have that

$$x_n = x_0 \prod_{j=1}^n y_j, \quad n \in \mathbb{N}_0. \quad (4.18)$$

By using Theorem 3.1 we have that every well-defined solution of equation (4.17) can be written in the form

$$y_n = \frac{y_0 s_{n-1} + (2y_0 - 1) s_n}{(y_0 - 2) s_n + s_{n+1}}, \quad n \in \mathbb{N}, \quad (4.19)$$

where  $(s_n)_{n \in \mathbb{N}_0}$  is the sequence satisfying the difference equation

$$s_{n+1} - s_n - s_{n-1} = 0, \quad n \in \mathbb{N},$$

with the initial conditions  $s_0 = 0$  and  $s_1 = 1$ . This means that  $(s_n)_{n \in \mathbb{N}_0}$  is the Fibonacci sequence.

Using recurrent relation (1.3) in (4.19) and the change (4.16) with  $n = 0$ , we have that

$$\begin{aligned}
 y_n &= \frac{y_0 f_{n-1} + (2y_0 - 1)f_n}{(y_0 - 2)f_n + f_{n+1}} \\
 &= \frac{y_0(f_{n+1} - f_n) + (2y_0 - 1)f_n}{(y_0 - 2)f_n + f_n + f_{n-1}} \\
 &= \frac{y_0 f_{n+1} + (y_0 - 1)f_n}{(y_0 - 1)f_n + f_{n-1}} \\
 &= \frac{y_0(f_{n+2} - f_n) + (y_0 - 1)f_n}{(y_0 - 1)f_n + f_n - f_{n-2}} \\
 &= \frac{y_0 f_{n+2} - f_n}{y_0 f_n - f_{n-2}} \\
 &= \frac{x_0 f_{n+2} - x_{-1} f_n}{x_0 f_n - x_{-1} f_{n-2}}, \quad n \in \mathbb{N}.
 \end{aligned} \tag{4.20}$$

Employing relationship (4.20) into (4.18) is obtained formula (4.4).

### 4.3 Case of equation (4.5)

As in the case of equation (4.1) it is shown that in this case we may also assume that  $x_n \neq 0$  for every  $n \geq -1$ . Hence, we can use the change of variables in (4.16), so that equation (4.5) is transformed into the following equation

$$y_{n+1} = \frac{1}{1 + y_n}, \quad n \in \mathbb{N}_0,$$

and we have that relation (4.18) holds.

By using Theorem 3.1 (or formula (1.4)) we have that

$$y_n = \frac{y_0 f_{n-1} + f_n}{y_0 f_n + f_{n+1}} = \frac{x_0 f_{n-1} + x_{-1} f_n}{x_0 f_n + x_{-1} f_{n+1}}, \quad n \in \mathbb{N}_0, \tag{4.21}$$

since equation (3.6) in this case becomes (1.3). Using (4.21) in (4.18), formula (4.6) easily follows.

### 4.4 On a bilinear system of difference equations

A natural system of difference equations related to equation (3.1) is the following

$$z_{n+1} = \frac{\alpha w_n + \beta}{\gamma w_n + \delta}, \quad w_{n+1} = \frac{a z_n + b}{c z_n + d}, \quad n \in \mathbb{N}_0, \tag{4.22}$$

where parameters  $\alpha, \beta, \gamma, \delta, a, b, c$  and  $d$ , and initial values  $z_0$  and  $w_0$  are real numbers.

If we use the second recurrent relation in (4.22) into the first one, it is obtained

$$z_{n+1} = \frac{(a\alpha + \beta c)z_{n-1} + \alpha b + \beta d}{(a\gamma + c\delta)z_{n-1} + b\gamma + d\delta}, \quad n \in \mathbb{N},$$

from which it follows that the sequences  $(z_{2n+i})_{n \in \mathbb{N}_0}$ ,  $i = 0, 1$ , satisfy the following difference equation

$$\tilde{z}_{n+1} = \frac{(a\alpha + \beta c)\tilde{z}_n + \alpha b + \beta d}{(a\gamma + c\delta)\tilde{z}_n + b\gamma + d\delta}, \quad n \in \mathbb{N}_0. \tag{4.23}$$

Analogously, if we use the first recurrent relation in (4.22) into the second one, it is obtained

$$w_{n+1} = \frac{(a\alpha + b\gamma)w_{n-1} + a\beta + b\delta}{(\alpha c + \gamma d)w_{n-1} + \beta c + d\delta'}, \quad n \in \mathbb{N},$$

from which it follows that the sequences  $(w_{2n+i})_{n \in \mathbb{N}_0}$ ,  $i = 0, 1$ , satisfy the following difference equation

$$\tilde{w}_{n+1} = \frac{(a\alpha + b\gamma)\tilde{w}_{n-1} + a\beta + b\delta}{(\alpha c + \gamma d)\tilde{w}_{n-1} + \beta c + d\delta'}, \quad n \in \mathbb{N}_0. \quad (4.24)$$

A simple calculation shows that the associated equation (3.6) to both bilinear difference equations (4.23) and (4.24) is

$$s_{n+1} - (a\alpha + b\gamma + c\beta + d\delta)s_n + (ad - bc)(\alpha\delta - \beta\gamma)s_{n-1} = 0, \quad n \in \mathbb{N}_0, \quad (4.25)$$

where  $s_0 = 0$  and  $s_1 = 1$ .

Applying Theorem 3.1 for the case of equations (4.23) and (4.24), and using the relations which are obtained from the equations in (4.22) with  $n = 0$ , after some calculation we obtain the following result.

**Theorem 4.4.** *Consider system of equations (4.22), with  $ad \neq bc$ ,  $a\gamma + c\delta \neq 0$ ,  $\alpha c + \gamma d \neq 0$  and  $\alpha\delta \neq \beta\gamma$ . Then for every well-defined solution of the system the following relations hold*

$$\begin{aligned} z_{2n} &= \frac{z_0(\beta\gamma - \alpha\delta)(ad - bc)s_{n-1} + ((a\alpha + \beta c)z_0 + \alpha b + \beta d)s_n}{((a\gamma + c\delta)z_0 - a\alpha - \beta c)s_n + s_{n+1}}, \\ z_{2n+1} &= \frac{(\alpha w_0 + \beta)(\beta\gamma - \alpha\delta)(ad - bc)s_{n-1} + ((a\alpha + \beta c)(\alpha w_0 + \beta) + (\alpha b + \beta d)(\gamma w_0 + \delta))s_n}{((a\gamma + c\delta)(\alpha w_0 + \beta) - (a\alpha + \beta c)(\gamma w_0 + \delta))s_n + (\gamma w_0 + \delta)s_{n+1}}, \\ w_{2n} &= \frac{w_0(\beta\gamma - \alpha\delta)(ad - bc)s_{n-1} + ((a\alpha + b\gamma)w_0 + a\beta + b\delta)s_n}{((\alpha c + \gamma d)w_0 - a\alpha - b\gamma)s_n + s_{n+1}}, \\ w_{2n+1} &= \frac{(az_0 + b)(\beta\gamma - \alpha\delta)(ad - bc)s_{n-1} + ((a\alpha + b\gamma)(az_0 + b) + (a\beta + b\delta)(cz_0 + d))s_n}{((\alpha c + \gamma d)(az_0 + b) - (a\alpha + b\gamma)(cz_0 + d))s_n + (cz_0 + d)s_{n+1}}, \end{aligned}$$

$n \in \mathbb{N}_0$ , where  $(s_n)_{n \in \mathbb{N}_0}$  is the sequence satisfying difference equation (4.25) with the initial conditions  $s_0 = 0$  and  $s_1 = 1$ .

The following system is a special case of system (4.22) and is a natural generalization of equation (1.2).

**Corollary 4.5.** *Consider the system of difference equations*

$$z_{n+1} = \frac{1}{1 + w_n}, \quad w_{n+1} = \frac{1}{1 + z_n}, \quad n \in \mathbb{N}_0,$$

where  $z_0$  and  $w_0$  are real numbers. Then for every well-defined solution of the system the following relations hold

$$z_{2n} = \frac{z_0 f_{2n-1} + f_{2n}}{z_0 f_{2n} + f_{2n+1}}, \quad n \in \mathbb{N}_0, \quad (4.26)$$

$$z_{2n+1} = \frac{f_{2n+1} + w_0 f_{2n}}{f_{2n+2} + w_0 f_{2n+1}}, \quad n \in \mathbb{N}_0, \quad (4.27)$$

$$w_{2n} = \frac{w_0 f_{2n-1} + f_{2n}}{w_0 f_{2n} + f_{2n+1}}, \quad n \in \mathbb{N}_0, \quad (4.28)$$

$$w_{2n+1} = \frac{f_{2n+1} + z_0 f_{2n}}{f_{2n+2} + z_0 f_{2n+1}}, \quad n \in \mathbb{N}_0. \quad (4.29)$$

*Proof.* Since the system is symmetric it is enough to prove only formulas (4.26) and (4.27) (formulas (4.28) and (4.29) follow by replacing letters  $z$  and  $w$  only). Now note that in this case the associate equation (4.25) is reduced to (4.11) and that the sequences  $(z_{2n+i})_{n \in \mathbb{N}_0}$ ,  $i = 0, 1$ , satisfy difference equation (4.8). Hence, employing formula (4.14) and equation (1.3), we obtain

$$z_{2n} = \frac{z_0 f_{2n-1} + f_{2n}}{z_0 f_{2n} + f_{2n+1}}, \quad n \in \mathbb{N},$$

and

$$z_{2n+1} = \frac{z_1 f_{2n-1} + f_{2n}}{z_1 f_{2n} + f_{2n+1}} = \frac{f_{2n-1} + (1 + w_0) f_{2n}}{f_{2n} + (1 + w_0) f_{2n+1}} = \frac{f_{2n+1} + w_0 f_{2n}}{f_{2n+2} + w_0 f_{2n+1}},$$

as desired.  $\square$

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