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Nonoscillation of higher order half-linear differential equations

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Abstract. We establish nonoscillation criteria for even order half-linear differential equations. The principal tool we use is the Wirtinger type inequality combined with various perturbation techniques. Our results extend nonoscillation criteria known for linear higher order differential equations.

Keywords: even-order half-linear differential equation, Wirtinger inequality, nonoscillation, half-linear Euler equation.

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1 Introduction

In this paper we deal with the even order half-linear differential equation

$$\sum_{k=0}^{n}(-1)^k\Big(r_k(t)\Phi\big(y^{(k)}\big)\Big)^{(k)}=0 \tag{1.1}$$

where $\Phi(y)=|y|^{p-2}y$, $p>1$, is the odd power function, r_j are continuous functions, $j=0,\dots,n$, and $r_n(t)>0$ in the interval under consideration. The terminology *half-linear* equation was introduced by I. Bihari [3] and reflects the fact that the solution space of (1.1) is homogeneous, but not additive, i.e., it has just one half of the properties characterizing linearity. In the case $n=1$, equation (1.1) reduces to the classical second order half-linear differential equation

$$-\big(r_1(t)\Phi(x')\big)'+r_0(t)\Phi(x)=0 \tag{1.2}$$

whose oscillation theory is relatively deeply developed, see [1, 16] and e.g. the recent papers [11, 13, 17, 19, 24, 27, 28].

The theory of (1.1) is much less developed and as far as we known only [16, Sec. 9.4] and the paper [25] deal with this problem. The reason is that we miss the so-called Reid's roundabout theorem in the higher order case, in particular, the Riccati technique is not available for

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(1.1), in contrast to (1.2). Actually, necessary and sufficient conditions for (non)oscillation of (1.1) with $p = 2$, i.e., in the *linear case*, follow from the fact that this equation can be written as a linear Hamiltonian system (for which the Reid's roundabout theorem is well known, [26, Chap. V., Theorem 6.3]) and this enables to present oscillation and spectral theory of (1.1) with $p = 2$ as it is exhibited e.g. in the book [22], see also [20] and the references given therein.

The energy functional associated with (1.1) considered on the interval $[T, \infty)$ is

$$\mathcal{F}_n(y) = \int_T^\infty \left[\sum_{k=0}^n r_k(t)|y^{(k)}|^p\right] dt \tag{1.3}$$

(equation (1.1) is the Euler–Lagrange equation of (1.3)). If there exists a nontrivial solution $\tilde{y}$ of (1.1) with two zeros of multiplicity n in $[T, \infty)$, i.e.,

$$\tilde{y}^{(i)}(t_1) = 0 = \tilde{y}^{(i)}(t_2), \qquad i = 0, \dots, n-1, \tag{1.4}$$

for some $T \le t_1 < t_2$, then we define the function

$$y(t) = \begin{cases} \tilde{y}(t), & t \in [t_1, t_2] \\ 0 & t \in [T, \infty) \setminus [t_1, t_2], \end{cases}$$

and obviously $y \in W_0^{n,p}[T, \infty)$ (the definition of this Sobolev space will be recalled later). Multiplying (1.1) by y and integrating by parts over $[T, \infty)$ gives $\mathcal{F}_n(y) = 0$. Hence, if we show that $\mathcal{F}_n(y) > 0$ for all nontrivial functions $y \in W_0^{n,p}[T, \infty)$, we eliminate the existence of a solution of (1.1) satisfying (1.4) for some $t_1, t_2 \in [T, \infty)$.

The paper is organized as follows. In the next section we concentrate our attention on basic properties of the higher order half-linear Euler differential equation and on the so-called Wirtinger inequality which is the principal tool in our investigation. Section 3 is devoted to nonoscillation criteria for Euler type even order differential equation. Section 4 deals with nonoscillation criteria for general two-term $2n$th order half-linear differential equations and in the last section we present some remarks and comments concerning possible further investigation.

2 Preliminaries and Euler equation

The higher order Euler type half-linear differential equation is the equation

$$(-1)^n \big(t^\alpha \Phi(y^{(n)})\big)^{(n)} + (-1)^{n-1}\beta_{n-1}\big(t^{\alpha-p}\Phi(y^{(n-1)})\big)^{(n-1)} + \dots + \beta_0 t^{\alpha-np}\Phi(y) = 0, \tag{2.1}$$

where α, β_i, $i = 0, \dots, n-1$, are real constants. Moreover, it is supposed that $\alpha \notin \{p-1, 2p-1, \dots, np-1\}$ (this restriction will be explained later).

The "classical" Euler second order half-linear differential equation is the equation

$$-\big(\Phi(x')\big)' + \frac{\gamma}{t^p}\Phi(x) = 0. \tag{2.2}$$

This equation and its various perturbations were studied in detail in [18] and also in [11, 13, 14, 17, 24, 27]. It is known that the classical linear Sturmian oscillation theory extends almost verbatim to (1.2). Elbert [18] showed that (2.2) is oscillatory if and only if $\gamma < -\gamma_p$,

$\gamma_p := \left(\frac{p-1}{p}\right)^p$. In the critical case $\gamma = -\gamma_p$, equation (1.2) has a solution $x(t) = t^{\frac{p-1}{p}}$ as can be verified by a direct computation.

Concerning equation (2.1), similarly to the linear case, we look for a solution in the form $x(t) = t^\lambda$. Consider first the two-term equation

$$(-1)^n \left(t^\alpha \Phi(x^{(n)})\right)^{(n)} + \gamma t^{\alpha - np}\Phi(x) = 0, \tag{2.3}$$

with $\alpha \notin \{p-1, \dots, np-1\}$ and $\gamma \in \mathbb{R}$. Substituting into (2.3) we find that λ must be a root of the algebraic equation $G(\lambda) + \gamma = 0$ with

$$G(\lambda) = (-1)^n \Phi\big(\lambda(\lambda-1)\cdots(\lambda-n+1)\big)\big[(p-1)(\lambda-n)+\alpha\big]\cdots\big[(p-1)(\lambda-n)+\alpha-n+1\big].$$

Next we show that the function G has a stationary point $\lambda^* = \frac{np-1-\alpha}{p}$. We have the equality $\Phi'(x) = (p-1)\frac{\Phi(x)}{x}$, therefore, by a direct calculation we obtain that for $\lambda \neq j, n - \frac{\alpha - j}{p-1}$, $j = 0, \dots, n-1$,

$$G'(\lambda) = (-1)^n (p-1) G(\lambda) \left[\frac{1}{\lambda} + \frac{1}{\lambda-1} + \cdots + \frac{1}{\lambda-(n-1)} + \frac{1}{(p-1)(\lambda-n)+\alpha} \right.$$
$$\left. + \frac{1}{(p-1)(\lambda-n)+\alpha-1} + \cdots + \frac{1}{(p-1)(\lambda-n)+\alpha-(n-1)}\right].$$

Because

$$\frac{1}{\lambda^* - k} = -\frac{1}{(p-1)(\lambda^*-n)+\alpha-(n-1-k)}$$

for each $k \in \{0, \dots, n-1\}$, we have

$$G'(\lambda^*) = 0.$$

Substituting the value λ^* into G gives the value of the so-called *critical constant* in the $2n$th order Euler half-linear differential equation (2.3). We denote

$$\gamma_{n,p,\alpha} := G(\lambda^*) = \prod_{j=1}^{n} \left(\frac{|jp-1-\alpha|}{p}\right)^p.$$

The previous computation shows that the equation $G(\lambda) - \gamma_{n,p,\alpha} = 0$ has a double root $\lambda^* = \frac{np-1-\alpha}{p}$.

The terminology critical constant is used by analogy with the linear case where its value is a "borderline" between oscillation and nonoscillation of equation (2.3) with $p = 2$. In the half-linear case, we are able to prove only "one half" of conditions for an oscillation constant yet, namely that (2.3) is nonoscillatory for $\gamma > -\gamma_{n,p,\alpha}$. The proof of an "oscillation counterpart" resists our effort till now, nevertheless, it is a subject of the present investigation. More details about this problem are given in the last section.

Therefore, (2.3) with $\gamma = -\gamma_{n,p,\alpha}$ has a solution $x(t) = t^{\lambda^*}$. Note that linearly independent solutions cannot be computed explicitly even in the case $n = 1$ and $\alpha = 0$ (i.e., for second order equation (2.2) with $\gamma = -\gamma_p$, because $\gamma_p = \gamma_{1,p,0}$). Nevertheless, as shown in [18], any solution of (2.2) with $\gamma = -\gamma_p$, which is linearly independent of $x(t) = t^{\frac{p-1}{p}}$ is asymptotically equivalent to the function $\tilde{x}(t) = Ct^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$, $0 \neq C \in \mathbb{R}$. It is an open problem whether the function $\tilde{x}(t) = t^{\frac{np-1-\alpha}{p}} \log^{\frac{2}{p}} t$ is also an "approximate" solution of the equation

$$(-1)^n \left(t^\alpha \Phi(x^{(n)})\right)^{(n)} - \gamma_{n,p,\alpha} t^{\alpha-np}\Phi(x) = 0, \tag{2.4}$$

since if $p = 2$ in (2.4) then $\tilde{x}(t) = t^{\frac{2n-1-\alpha}{2}} \log t$ is a solution of this equation.

Now we recall the definition of the Sobolev space, consisting of functions with a compact support. We denote for $T \in \mathbb{R}$

$$W_0^{n,p}[T,\infty) = \Big\{ y\colon [T,\infty) \to \mathbb{R} \mid y^{(n-1)} \in \mathcal{AC}[T,\infty),\, y^{(n)} \in \mathcal{L}^p(T,\infty), \\ y^{(i)}(T) = 0 \text{ for } i = 0,1,\dots,n-1 \text{ and there exists } T_1 > T \\ \text{such that } y(t) = 0 \text{ for } t \ge T_1 \Big\},$$

where $\mathcal{AC}[T,\infty)$ is the set of absolutely continuous functions with the domain $[T,\infty)$.

We finish this section with a half-linear version of the classical Wirtinger inequality, which we use in the next sections. Its proof in the formulation presented here can be found in [7].

Lemma 2.1. *Let M be a positive continuously differentiable function for which $M'(t) \neq 0$ in $[T,\infty)$ and let $y \in W_0^{1,p}[T,\infty)$. Then*

$$\int_T^\infty |M'(t)||y|^p\,dt \le p^p \int_T^\infty \frac{M^p}{|M'(t)|^{p-1}}|y'|^p\,dt. \tag{2.5}$$

3 Euler equation

Following the linear terminology, we say that (1.1) is *nonoscillatory* if there exists $T \in \mathbb{R}$ such that no solution of this equation has two or more zeros of multiplicity n in $[T,\infty)$. In the opposite case, i.e., when for every $T \in \mathbb{R}$ there exists a nontrivial solution of (1.1) with at least two zeros of multiplicity n in $[T,\infty)$, then (1.1) is said to be *oscillatory*.

We start this section with a variational lemma which plays the fundamental role in our treatment, for its proof (whose outline we have already presented below (1.3)) see [16, Sec. 9.4].

Lemma 3.1. *Equation (1.1) is nonoscillatory if there exists $T \in \mathbb{R}$ such that*

$$\mathcal{F}_n(y) > 0$$

for every $0 \not\equiv y \in W_0^{n,p}[T,\infty)$.

The first statement of this section is a nonoscillation criterion which is essentially proved in [16, Theorem 9.4.5]. This criterion is formulated in [16] for the equation

$$(-1)^n \big(\Phi(x^{(n)})\big)^{(n)} + \frac{\gamma}{t^{np}}\Phi(x) = 0, \tag{3.1}$$

but a small modification of the proof (via Wirtinger inequality) shows that it can be extended to a more general equation (2.3).

Theorem 3.2. *Suppose that $\alpha \notin \{p-1,\dots,np-1\}$. If*

$$\gamma_{n,p,\alpha} + \gamma > 0, \qquad \gamma_{n,p,\alpha} = \prod_{j=1}^{n} \left(\frac{|jp-1-\alpha|}{p} \right)^p,$$

then (2.3) is nonoscillatory.

Proof. The proof is based on the application of the inequality

$$\int_T^\infty t^\alpha |y^{(n)}|^p \, dt \ge \gamma_{n,p,\alpha} \int_T^\infty t^{\alpha-np} |y|^p \, dt \tag{3.2}$$

for $y \in W_0^{n,p}[T,\infty)$, which is obtained by repeated application of the following Wirtinger inequality

$$\int_T^\infty t^\beta |x'|^p \, dt \ge \left(\frac{|p-1-\beta|}{p} \right)^p \int_T^\infty t^{\beta-p} |x|^p \, dt, \qquad x \in W_0^{1,p}[T,\infty) \tag{3.3}$$

for $\beta = \alpha, \alpha - p, \alpha - 2p, \dots, \alpha - (n-1)p$ and for $x' = y^{(n)}, y^{(n-1)}, \dots, y'$ respectively. Inequality (3.3) follows from inequality (2.5) in Lemma 2.1 by taking $M(t) = (|p-1-\beta|)^{p-1} t^{\beta-p+1}$ for $\beta \neq p-1$. Then for any $y \in W_0^{n,p}[T,\infty)$ such that $y \not\equiv 0$ we have

$$\begin{aligned} \mathcal{F}_n(y) &= \int_T^\infty t^\alpha |y^{(n)}|^p \, dt + \gamma \int_T^\infty t^{\alpha-np} |y|^p \, dt \\ &\ge (\gamma_{n,p,\alpha} + \gamma) \int_T^\infty t^{\alpha-np} |y|^p \, dt > 0, \end{aligned}$$

what we needed to prove, due to Lemma 3.1. □

Note that the same statement (for $\alpha = 0$) is proved via the weighted Hardy inequality in [25], we will mention this result later in our paper.

Now we turn our attention to the "full term" $2n$th order Euler differential equation.

$$(-1)^n \left(t^\alpha \Phi(y^{(n)}) \right)^{(n)} + (-1)^{n-1} \beta_{n-1} \left(t^{\alpha-p} \Phi(y^{(n-1)}) \right)^{(n-1)} + \cdots + \beta_0 t^{\alpha-np} \Phi(y) = 0, \tag{3.4}$$

with $\alpha \notin \{p-1, 2p-1, \dots, np-1\}$.

Theorem 3.3. *Suppose that $\alpha \notin \{p-1, \dots, np-1\}$ and*

$$\sum_{k=0}^{n-1} \prod_{j=1}^{n-k} \left(\frac{|(k+j)p-1-\alpha|}{p} \right)^p \beta_{n-k} + \beta_0 > 0, \qquad \beta_n := 1,$$

then equation (3.4) is nonoscillatory.

Proof. We apply the Wirtinger inequality to each term (except that one for $k = n$) in the energy functional

$$\mathcal{F}_n(y) = \int_T^\infty \left(\sum_{k=0}^n t^{\alpha-kp} |y^{(n-k)}|^p \right) dt.$$

We obtain for any $y \in W_0^{n,p}[T,\infty)$ and for $k = 0, \dots, n-1$

$$\int_T^\infty t^{\alpha-pk} |y^{(n-k)}|^p \, dt \ge \prod_{j=1}^{n-k} \left(\frac{|(k+j)p-1-\alpha|}{p} \right)^p \int_T^\infty t^{\alpha-np} |y|^p \, dt.$$

Then we have

$$\mathcal{F}_n(y) \ge \left[\sum_{k=0}^{n-1} \prod_{j=1}^{n-k} \left(\frac{|(k+j)p-1-\alpha|}{p} \right)^p \beta_{n-k} + \beta_0 \right] \int_T^\infty t^{\alpha-np} |y|^p \, dt > 0$$

for any nontrivial $y \in W_0^{n,p}[T,\infty)$. □

Remark 3.4. The reason why the case $\alpha \in \{p-1, \dots, np-1\}$ we needed to exclude from the previous considerations is the following. For $\alpha = p-1$ the Wirtinger inequality takes the form

$$\int_T^\infty t^{p-1}|y'|^p\,dt \geq \left(\frac{p-1}{p}\right)^p \int_T^\infty \frac{1}{t\log^p t}|y|^p\,dt, \tag{3.5}$$

so, a logarithmic term appears. This more difficult case is treated in the next part of this section.

We start with an auxiliary statement.

Lemma 3.5. *Let $\alpha = jp-1$ for some $j \in \{1, \dots, n\}$. Then, we have for any $y \in W_0^{n,p}[T,\infty)$*

$$\int_T^\infty t^\alpha |y^{(n)}|^p\,dt \geq \frac{[(n-j)!(j-1)!]^p}{\gamma_p^{n-j-1}} \int_T^\infty \frac{1+O(\log^{-1} t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt.$$

Proof. First we make some auxiliary computations. Integration by parts gives for $l \in \mathbb{N}$ and q the conjugate exponent of p, i.e., $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}\int t^{lq-1}\log^q t\,dt &= \frac{t^{lq}}{lq}\log^q t - \frac{1}{l}\int t^{lq-1}\log^{q-1} t\,dt\\ &= \frac{t^{lq}}{lq}\log^q t\left[1+O(\log^{-1} t)\right]\end{aligned}$$

as $t \to \infty$. This integral we use in establishing the inequality for $z \in W_0^{1,p}[T,\infty)$

$$\int_T^\infty \frac{|z'|^p}{t^{lp+1}\log^p t}\,dt \geq \frac{(l+1)^p}{\gamma_p}\int_T^\infty \frac{1+O(\log^{-1} t)}{t^{(l+1)p+1}\log^p t}|z|^p\,dt. \tag{3.6}$$

We prove (3.6) as follows. Let $r(t)>0$ be a continuous function with $\int^\infty r^{1-q}(t)\,dt = \infty$, then we have the inequality

$$\int_T^\infty r(t)|y'|^p\,dt \geq \gamma_p \int_T^\infty \frac{r^{1-q}(t)}{\left(\int_{T_0}^t r^{1-q}(s)\,ds\right)^p}|y|^p\,dt, \qquad T_0 < T, \tag{3.7}$$

which follows from (3.3) with $\beta = 0$. Indeed, let $s = \int_{T_0}^t r^{1-q}(\tau)d\tau$, i.e., $\frac{d}{dt} = r^{1-q}(t)\frac{d}{ds}$, then (3.7) is the same as

$$\int_S^\infty |\dot{y}|^p\,ds \geq \gamma_p \int_S^\infty \frac{|y|^p}{s^p}\,ds, \qquad \dot{} = \frac{d}{ds}, \qquad S = \int_{T_0}^T r^{1-q}(\tau)d\tau.$$

For $r(t) = t^{-lp-1}\log^{-p} t$ we have $r^{1-q}(t) = t^{(l+1)q-1}\log^q t$, hence

$$\int^t r^{1-q}(s)\,ds = \frac{t^{(l+1)q}}{(l+1)q}\log^q t\left(1+O(\log^{-1} t)\right)$$

as $t \to \infty$. Therefore

$$\begin{aligned}\frac{r^{1-q}(t)}{\left(\int^t r^{1-q}(s)\,ds\right)^p} &= t^{(l+1)q-1}\log^q t\left(\frac{t^{(l+1)q}}{(l+1)q}\log^q t\right)^{-p}\left(1+O(\log^{-1} t)\right)^{-p}\\ &= \frac{(l+1)^p}{\gamma_p}\frac{1}{t^{(l+1)p+1}\log^p t}\left(1+O(\log^{-1} t)\right).\end{aligned}$$

Substituting these computations into (3.7) we obtain (3.6).

Let $y \in W_0^{n,p}[T,\infty)$. Applying inequalities (3.5) and (3.6), we obtain

$$\begin{aligned}\int_T^\infty t^\alpha |y^{(n)}|^p\,dt &= \int_T^\infty t^{jp-1}|y^{(n)}|^p\,dt \geq [(j-1)!]^p \int_T^\infty t^{p-1}|y^{(n-j+1)}|^p\,dt \\ &\geq [(j-1)!]^p \gamma_p \int_T^\infty \frac{1}{t\log^p t}|y^{(n-j)}|^p\,dt \\ &\geq \frac{[(n-j)!(j-1)!]^p}{\gamma_p^{n-j-1}} \int_T^\infty \frac{1+O(\log^{-1} t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt.\end{aligned}$$

The proof is complete. □

Now we are ready to deal with the case $\alpha \in \{p-1, 2p-1, \dots, np-1\}$.

Theorem 3.6. *Let $\alpha = jp - 1$ for some $j \in \{1,\dots,n\}$ and consider the equation*

$$\begin{aligned}&(-1)^n \left(t^{jp-1}\Phi(y^{(n)})\right)^{(n)} + \sum_{i=1}^{j-1}(-1)^{n-i}\beta_{n-i}\left(t^{(j-i)p-1}\Phi(y^{(n-i)})\right)^{(n-i)} \\ &\quad + \sum_{i=0}^{n-j-1}(-1)^{n-j-i}\beta_{n-j-i}\left(\frac{\Phi(y^{(n-j-i)})}{t^{ip+1}\log^p t}\right)^{(n-j-i)} + \beta_0\frac{\Phi(y)}{t^{(n-j)p+1}\log^p t} = 0.\end{aligned} \tag{3.8}$$

If

$$\begin{aligned}L := &\frac{[(j-1)!(n-j)!]^p}{\gamma_p^{n-j-1}} + \sum_{i=1}^{j-1}\beta_{n-i}\frac{[(j-i-1)!(n-j)!]^p}{\gamma_p^{n-j-1}} \\ &+ \sum_{i=0}^{n-j-1}\beta_{n-j-i}\frac{[(i+1)\cdots(n-j)]^p}{\gamma_p^{n-j-i}} + \beta_0 > 0\end{aligned} \tag{3.9}$$

then equation (3.8) is nonoscillatory.

Proof. The energy functional corresponding to (3.8) is

$$\begin{aligned}\mathcal{F}_n(y) = \int_T^\infty \Bigg[&t^{jp-1}|y^{(n)}|^p + \sum_{i=1}^{j-1}\beta_{n-i}t^{(j-i)p-1}|y^{(n-i)}|^p \\ &+ \sum_{i=0}^{n-j-1}\beta_{n-j-i}\frac{|y^{(n-j-i)}|^p}{t^{ip+1}\log^p t} + \beta_0\frac{|y|^p}{t^{(n-j)p+1}\log^p t}\Bigg]\,dt\end{aligned}$$

The first term in the integral is estimated in Lemma 3.5. Concerning the terms under summation signs, for $i = 0,\dots,j-1$

$$\int_T^\infty t^{(j-i)p-1}|y^{(n-i)}|^p\,dt \geq \frac{[(j-i-1)!(n-j)!]^p}{\gamma_p^{n-j-1}}\int_T^\infty \frac{1+O(\log^{-1} t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt$$

and for $i = 0,\dots,n-j-1$

$$\int_T^\infty \frac{|y^{(n-j-i)}|^p}{t^{ip+1}\log^p t}\,dt \geq \frac{[(i+1)\dots(n-j)]^p}{\gamma_p^{n-j-i}}\int_T^\infty \frac{1+O(\log^{-1} t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt.$$

Substituting these computations into $\mathcal{F}_n(y)$, we have

$$\mathcal{F}_n(y) = \int_T^\infty \frac{|y|^p}{t^{(n-j)p+1}\log^p t}\,dt \times \left[L + \left(\int_T^\infty \frac{O(\log^{-1} t)}{t^{(n-j)p+1}\log^p t}|y|^p\,dt\right)\left(\int_T^\infty \frac{|y|^p}{t^{(n-j)p+1}\log^p t}\,dt\right)^{-1}\right].$$

Since the second term in the bracket tends to zero as $T \to \infty$, we have $\mathcal{F}_n(y; T, \infty) > 0$ for T sufficiently large if (3.9) holds, which means that equation (3.8) is nonocillatory by Lemma 3.1. □

4 General nonoscillation criteria

We start with two nonoscillation criteria from [25] (proved in [25] via the weighted Hardy inequality) which we later compare with our results. Both criteria are contained in the following theorem.

Theorem 4.1. *Suppose that $c(t) \le 0$ for large t and q is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. If one of the following conditions*

$$\lim_{T\to\infty}\inf_{t>T}\left(\int_T^t r^{1-q}(s)\,ds\right)^{p-1}\int_t^\infty c(s)(s-T)^{(n-1)p}\,ds > -\frac{[(n-1)!]^p}{p-1}\gamma_p \tag{4.1}$$

or

$$\lim_{T\to\infty}\inf_{t>T}\left(\int_T^t r^{1-q}(s)\,ds\right)^{-1}\int_T^t c(s)(s-T)^{(n-1)p}\left(\int_T^s r^{1-q}(u)\,du\right)^p ds > -\gamma_p[(n-1)!]^p, \tag{4.2}$$

holds, then the two-term differential equation

$$(-1)^n\big(r(t)\Phi(y^{(n)})\big)^{(n)} + c(t)\Phi(y) = 0 \tag{4.3}$$

is nonoscillatory.

In the next theorem we present a Hille–Nehari type nonoscillation criterion for (4.3) with $r(t) = t^\alpha$. This criterion extends the linear result given in [10]. We will need the following auxiliary statement, its proof can be found e.g. in [6].

Lemma 4.2. *Let $m \in \{0, \dots, n-1\}$, then we have*

$$y^{(n)} = \left\{\frac{1}{t}\left[t^{m+1}\left(\frac{y}{t^m}\right)'\right]^{(m)}\right\}^{(n-m-1)}.$$

Theorem 4.3. *Suppose that $\alpha \notin \{p-1, \dots, np-1\}$, $\int^\infty c_-(t)t^{(n-j)p}\,dt > -\infty$, where $c_-(t) = \min\{0, c(t)\}$ is the negative part of c, and*

$$\liminf_{t\to\infty} t^{jp-1-\alpha}\int_t^\infty c_-(s)s^{(n-j)p}\,ds > -\frac{\gamma_{n,p,\alpha}}{|jp-1-\alpha|} \tag{4.4}$$

for some $j \in \{1, \dots, n\}$. Then the equation

$$(-1)^n\big(t^\alpha\Phi(x^{(n)})\big)^{(n)} + c(t)\Phi(x) = 0, \tag{4.5}$$

is nonoscillatory.

Proof. Let $T \in \mathbb{R}$ be so large, that the limited expression in (4.4) is greater than

$$-\frac{\gamma_{n,p,\alpha}}{|jp-1-\alpha|}+\varepsilon =: K,$$

where $\varepsilon > 0$ is sufficiently small. Then for any $0 \not\equiv y \in W_0^{n,p}[T,\infty)$ we have with $z = y/t^{n-j}$ (using the inequality $\int_a^b fg \le \left(\int_a^b |f|^p\right)^{1/p}\left(\int_a^b |g|^q\right)^{1/q}$ between the fourth and fifth line and (3.3) (with $\beta = \alpha-(j-1)p$ and $x' = z'$) between the fifth and sixth line in the next computation)

$$\begin{aligned}
\int_T^\infty c(t)|y|^p\,dt &\ge \int_T^\infty c_-(t)t^{(n-j)p}\left|\left(\frac{y}{t^{n-j}}\right)\right|^p dt = p\int_T^\infty c_-(t)t^{(n-j)p}\left(\int_T^t \Phi(z)z'\,ds\right)dt\\
&= p\int_T^\infty \Phi(z)z'\frac{1}{t^{jp-1-\alpha}}t^{jp-1-\alpha}\left(\int_t^\infty c_-(s)s^{(n-j)p}\,ds\right)dt\\
&\ge p\int_T^\infty \frac{|\Phi(z)|\,|z'|}{t^{jp-1-\alpha}}t^{jp-1-\alpha}\left(\int_t^\infty c_-(s)s^{(n-j)p}\,ds\right)dt\\
&> pK\int_T^\infty \frac{|\Phi(z)|}{t^{\frac{jp-\alpha}{q}}}\cdot\frac{|z'|}{t^{-\frac{jp-\alpha}{q}+jp-1-\alpha}}\,dt = pK\int_T^\infty \frac{|\Phi(z)|}{t^{\frac{jp-\alpha}{q}}}\cdot\frac{|z'|}{t^{\frac{(j-1)p-\alpha}{q}}}\,dt\\
&\ge pK\left(\int_T^\infty \frac{|z|^p}{t^{jp-\alpha}}\,dt\right)^{\frac{1}{q}}\left(\int_T^\infty \frac{|z'|^p}{t^{(j-1)p-\alpha}}\,dt\right)^{\frac{1}{p}}\\
&\ge pK\left(\frac{p}{|jp-1-\alpha|}\right)^{\frac{p}{q}}\left(\int_T^\infty \frac{|z'|^p}{t^{(j-1)p-\alpha}}\,dt\right)^{\frac{1}{q}}\left(\int_T^\infty \frac{|z'|^p}{t^{(j-1)p-\alpha}}\,dt\right)^{\frac{1}{p}}\\
&= pK\left(\frac{p}{|jp-1-\alpha|}\right)^{p-1}\int_T^\infty \frac{|z'|^p}{t^{(j-1)p-\alpha}}\,dt.
\end{aligned}$$

In the previous computation, we have used the equality $|z(t)|^p = p\int_T^t \Phi(z(s))z'(s)\,ds$, which follows from the formula $\left(|z|^p\right)' = p\Phi(z)z'$ and from the definition of z ($z(T) = 0$). We have also used the relation $|\Phi(z)|^q = |z|^p$.

Now, we apply Lemma 4.2 with $m = n-j$, i.e., $n-m-1 = j-1$, and we denote

$$v = \frac{1}{t}\left[t^{n-j+1}\left(\frac{y}{t^{n-j}}\right)'\right]^{(n-j)}, \qquad u = t^{n-j+1}\left(\frac{y}{t^{n-j}}\right)'.$$

Then, using Wirtinger inequality (3.2) (in a slightly modified form), we get for $y \in W_0^{n,p}[T,\infty)$

$$\begin{aligned}
\int_T^\infty t^\alpha|y^{(n)}|^p &= \int_T^\infty t^\alpha\left|\left\{\frac{1}{t}\left[t^{n-j+1}\left(\frac{y}{t^{n-j}}\right)'\right]^{(n-j)}\right\}^{(j-1)}\right|^p dt\\
&= \int_T^\infty t^\alpha|v^{(j-1)}|^p\,dt \ge \prod_{i=1}^{j-1}\left(\frac{|ip-1-\alpha|}{p}\right)^p\int_T^\infty t^{\alpha-(j-1)p}|v|^p\,dt\\
&= \prod_{i=1}^{j-1}\left(\frac{|ip-1-\alpha|}{p}\right)^p\int_T^\infty t^{\alpha-(j-1)p}\left|\frac{1}{t}u^{(n-j)}\right|^p dt\\
&\ge \prod_{i=1}^{j-1}\left(\frac{|ip-1-\alpha|}{p}\right)^p\prod_{i=j+1}^{n}\left(\frac{|ip-1-\alpha|}{p}\right)^p\int_T^\infty t^{\alpha-np}\left|t^{n-j+1}\left(\frac{y}{t^{n-j}}\right)'\right|^p dt\\
&= \prod_{i=1}^{j-1}\left(\frac{|ip-1-\alpha|}{p}\right)^p\prod_{i=j+1}^{n}\left(\frac{|ip-1-\alpha|}{p}\right)^p\int_T^\infty t^{\alpha-(j-1)p}|z'|^p\,dt.
\end{aligned}$$

Summarizing the previous computations

$$\begin{aligned}
&\int_T^\infty t^\alpha |y^{(n)}|^p\,dt + \int_T^\infty c(t)|y|^p\,dt\\
&\quad\geq \left[\prod_{i=1,i\neq j}^{n}\left(\frac{|ip-1-\alpha|}{p}\right)^p + pK\left(\frac{p}{|jp-1-\alpha|}\right)^{p-1}\right]\int_T^\infty t^{\alpha-(j-1)p}\left|\left(\frac{y}{t^{n-j}}\right)'\right|^p dt\\
&\quad= p\left(\frac{p}{|jp-1-\alpha|}\right)^{p-1}\left[\frac{1}{|jp-1-\alpha|}\prod_{i=1}^{n}\left(\frac{|ip-1-\alpha|}{p}\right)^p + K\right]\\
&\qquad\times\int_T^\infty t^{\alpha-(j-1)p}\left|\left(\frac{y}{t^{n-j}}\right)'\right|^p dt\\
&\quad= p\left(\frac{p}{|jp-1-\alpha|}\right)^{p-1}\left[\frac{\gamma_{n,p,\alpha}}{|jp-1-\alpha|}+K\right]\int_T^\infty t^{\alpha-(j-1)p}\left|\left(\frac{y}{t^{n-j}}\right)'\right|^p dt.
\end{aligned}$$

Now, according to the definition of the constant K we see that the energy functional corresponding to (4.5) is positive for large T and hence (4.5) is nonoscillatory. □

Next we prove a statement which relates nonoscillatory behavior of a two-term $2n$th order half-linear differential equation to nonoscillation of a certain second order half-linear equation. It also presents a simpler proof of the previous theorem with $j = n$.

Theorem 4.4. *Consider equation* (4.5) *with* $\alpha \notin \{p-1,\dots,np-1\}$*. If the second order differential equation*

$$-\left(t^{\alpha-(n-1)p}\Phi(x')\right)' + \frac{c_-(t)}{\gamma_{n-1,p,\alpha}}\Phi(x) = 0 \tag{4.6}$$

is nonoscillatory, $\gamma_{n-1,p,\alpha} = \prod_{j=1}^{n-1}\left(\frac{|jp-1-\alpha|}{p}\right)^p$*,* $c_-(t) = \min\{0, c(t)\}$*, then* (4.5) *is also nonoscillatory. In particular, if* $\alpha < np-1$ *and* $\int^\infty c_-(t)\,dt > -\infty$*, equation* (4.5) *is nonoscillatory provided*

$$\liminf_{t\to\infty} t^{np-1-\alpha}\int_t^\infty c_-(s)\,ds > -\frac{\gamma_{n,p,\alpha}}{np-1-\alpha}. \tag{4.7}$$

Proof. Using the Wirtinger inequality (as in (3.2)) we can estimate the energy functional in (4.5) as follows

$$\begin{aligned}
\int_T^\infty t^\alpha|y^{(n)}|^p\,dt + \int_T^\infty c(t)|y|^p\,dt &\geq \gamma_{n-1,p,\alpha}\int_T^\infty t^{\alpha-(n-1)p}|y'|^p\,dt + \int_T^\infty c_-(t)|y|^p\,dt\\
&= \gamma_{n-1,p,\alpha}\left[\int_T^\infty t^{\alpha-(n-1)p}|y'|^p\,dt + \frac{1}{\gamma_{n-1,p,\alpha}}\int_T^\infty c_-(t)|y|^p\,dt\right].
\end{aligned}$$

The expression in brackets on the second line of the previous computation is the energy functional of (4.6) and it is positive if this equation is nonoscillatory and T is sufficiently large by [16, Theorem 2.1.1]. To prove the second statement of theorem, we apply the Hille–Nehari type nonoscillation criterion to (4.6). This criterion says (see, e.g., [16, Theorem 2.1.2]) that equation (1.2) with r_1 satisfying $\int^\infty r_1^{1-q}(t)\,dt = \infty$ and $\int_0^\infty (r_0)_-(t) > -\infty$ (where $(r_0)_-(t) = \min\{0, r_0(t)\}$) is nonoscillatory provided

$$\liminf_{t\to\infty}\left(\int^t r_1^{1-q}(s)\,ds\right)^{p-1}\int_t^\infty (r_0)_-(s)\,ds > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}. \tag{4.8}$$

Hence, for $r_1(t) = t^{\alpha-(n-1)p}$, we have

$$\left(\int_0^t r_1^{1-q}(s)\,ds\right)^{p-1} = \left(\frac{t^{\alpha(1-q)+q(n-1)+1}}{\alpha(1-q)+q(n-1)+1}\right)^{p-1} = \frac{t^{np-1-\alpha}}{\left(\frac{np-1-\alpha}{p-1}\right)^{p-1}}.$$

and $\int^\infty r_1^{1-q}(t)\,dt = \infty$, since $\alpha < np-1$. Then (4.8) reads

$$\liminf_{t\to\infty} \frac{t^{np-1-\alpha}}{\left(\frac{np-1-\alpha}{p-1}\right)^{p-1}} \int_t^\infty (r_0)_-(s)\,ds > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

which is just (4.7) with $\frac{c_-(t)}{\gamma_{n-1,p,\alpha}}$ instead of $(r_0)_-(t)$. □

Remark 4.5. Obviously, Theorem 4.4 applied to Euler type equation (2.3) gives Theorem 3.2.

Remark 4.6. Let us have a look at Theorem 4.1 with $r(t) = t^\alpha$, $\alpha \notin \{p-1, 2p-1, \dots, np-1\}$ and $c(t) \le 0$ for large t. Then $r^{1-q}(t) = t^{\alpha(1-q)}$ and for $\alpha < p-1$ (the case $\alpha > p-1$ is more complicated) we have

$$\int_0^t r^{1-q}(s)\,ds = \frac{t^{\alpha(1-q)+1}}{\alpha(1-q)+1}, \qquad \left(\int_0^t r^{1-q}(s)\,ds\right)^{p-1} = \frac{t^{p-1-\alpha}}{\left(\frac{p-1-\alpha}{p-1}\right)^{p-1}},$$

$$\left(\int_0^t r^{1-q}(s)\,ds\right)^{p} = \frac{t^{p-q\alpha}}{\left(\frac{p-1-\alpha}{p-1}\right)^{p}}, \qquad \left(\int_0^t r^{1-q}(s)\,ds\right)^{-1} = [1-(q-1)\alpha]t^{\alpha(q-1)-1}.$$

Hence, (4.1) takes the form

$$\begin{aligned}\liminf_{t\to\infty} t^{p-1-\alpha}\int_t^\infty c(s)(s-T)^{(n-1)p}\,ds &> -\left(\frac{p-1-\alpha}{p-1}\right)^{p-1}\cdot\left(\frac{p-1}{p}\right)^{p}\cdot\frac{[(n-1)!]^p}{p-1}\\ &= -\frac{1}{p-1-\alpha}\left(\frac{p-1-\alpha}{p}\right)^{p}[(n-1)!]^p.\end{aligned} \tag{4.9}$$

This condition is *more restrictive* than (4.4) with $j = 1$. Indeed, for $\alpha < p-1$ we have

$$\begin{aligned}-\frac{\gamma_{n,p,\alpha}}{p-1-\alpha} &= -\frac{1}{p-1-\alpha}\left(\frac{p-1-\alpha}{p}\right)^{p}\left(2-\frac{\alpha+1}{p}\right)^{p}\cdots\left(n-\frac{\alpha+1}{p}\right)^{p}\\ &< -\frac{1}{p-1-\alpha}\left(\frac{p-1-\alpha}{p}\right)^{p}[(n-1)!]^p\end{aligned}$$

since $\frac{\alpha+1}{p} < 1$. The difference in terms $\int_t^\infty c(s)s^{(n-1)p}\,ds$ and $\int_t^\infty c(s)(s-T)^{(n-1)p}\,ds$ in (4.4) (with $j = 1$) and (4.9), respectively, is not important since $\lim_{s\to\infty} s^{-(n-1)p}\cdot(s-T)^{(n-1)p} = 1$. Concerning (4.2), similarly as for (4.1) we obtain

$$\liminf_{t\to\infty} t^{\alpha(q-1)-1}\int_0^t c(s)s^{np-q\alpha}\,ds > -[(n-1)!]^p\left(\frac{p-1-\alpha}{p}\right)^{p}\frac{p-1}{p-1-\alpha}.$$

This condition is not covered by results presented in this paper and a subject of the present investigation is to “insert” this criterion into a general framework of even-order half-linear oscillation theory.

5 Remarks and comments

(i) In the previous parts of the paper, we have presented *nonoscillation* criteria for the investigated differential equations. The problem of *oscillation* of these equations is more complicated. In the linear case $p = 2$, we have the *equivalence* in Lemma 3.1, i.e., the differential equation

$$(-1)^n \big(r_n(t)y^{(n)}\big)^{(n)} + \cdots - \big(r_1(t)y'\big)' + r_0(t)y = 0$$

is oscillatory *if and only if* for every $T \in \mathbb{R}$ there exists $0 \not\equiv y \in W_0^{n,p}[T,\infty)$ such that

$$\int_T^\infty \Big[r_n(t)(y^{(n)})^2 + \cdots + r_1(t)y'^2 + r_0(t)y^2\Big]\,dt \le 0.$$

Such an equivalence is missing in the half-linear case and to find a general framework for the investigation of oscillation of (1.1) is a subject of the present investigation. In particular, we hope to prove that (2.3) is oscillatory if $\gamma < -\gamma_{n,p,\alpha}$, so the constant $-\gamma_{n,p,\alpha}$ really separates oscillation and nonoscillation in (2.3).

(ii) In the spectral theory of self-adjoint even order differential operators, an important role is played by the so-called *reciprocity principle* which claims that the two-term differential equation

$$(-1)^n \big(r(t)y^{(n)}\big)^{(n)} + c(t)y = 0 \tag{5.1}$$

with $r(t) > 0$ and $c(t) \neq 0$ for large t, is nonoscillatory if and only if its *reciprocal equations* (related to (5.1) by the substitution $u = ry^{(n)}$)

$$(-1)^n \left(\frac{1}{c(t)}u^{(n)}\right)^{(n)} + \frac{1}{r(t)}u = 0 \tag{5.2}$$

is also nonoscillatory, see [2]. The proof of this statement is based on the Riccati technique for Hamiltonian differential systems associated with (5.1) and (5.2) (which we miss for higher order half-linear equations as we have already mentioned in a previous part of the paper). It would be interesting to know whether a similar principle holds for the *half-linear* equation

$$(-1)^n \big(r(t)\Phi(y^{(n)})\big)^{(n)} + c(t)\Phi(y) = 0 \tag{5.3}$$

and its reciprocal equation (related to (5.3) by the substitution $u = r\Phi(y^{(n)})$)

$$(-1)^n \left(\frac{\Phi^{-1}(u^{(n)})}{\Phi^{-1}(c(t))}\right)^{(n)} + \frac{\Phi^{-1}(u)}{\Phi^{-1}(r(t))} = 0, \tag{5.4}$$

where $\Phi^{-1}(u) = |u|^{q-2}u$ is the inverse function of Φ.

A positive answer to this conjecture is partially supported by considering the pair of mutually reciprocal Euler type differential equations.

Theorem 5.1. *The reciprocal equation to Euler differential equation (2.3), which is the equation*

$$(-1)^n \Big(t^{(np-\alpha)(q-1)}\Phi^{-1}(u^{(n)})\Big)^{(n)} + \Phi^{-1}(\gamma)t^{-\alpha(q-1)}\Phi^{-1}(u) = 0, \tag{5.5}$$

is again an Euler equation. Moreover, the reciprocal equation to a critical equation is again the critical equation. In particular, if $\gamma > -\gamma_{n,p,\alpha}$, then the reciprocal equation (5.5) is also nonoscillatory.

Proof. An equation

$$(-1)^n \left(t^{\alpha_1}\Phi(y^{(n)})\right)^{(n)} + \gamma t^{\alpha_2}\Phi(y) = 0$$

is the Euler type equation if and only if $\alpha_1 - \alpha_2 = np$. Consequently, since (5.4) contains the power nonlinearity $\Phi^{-1}(u) = |u|^{q-2}u$, we compute the difference

$$(np-\alpha)(q-1) + \alpha(q-1) = (q-1)np = nq,$$

hence (5.5) is really an Euler equation. To show that the reciprocal equation to the critical equation is again a critical equation we need to show that if $\gamma = -\gamma_{n,p,\alpha}$ in (2.3), then the constant $-\Phi^{-1}(\gamma_{n,p,\alpha})$ is the critical constant for (5.5), i.e., it is

$$\Phi^{-1}(\gamma_{n,p,\alpha}) = \gamma_{n,q,\beta} = \prod_{j=1}^{n}\left(\frac{|jq-\beta-1|}{q}\right)^q$$

with $\beta = (np-\alpha)(q-1) = nq - \alpha(q-1)$. We have

$$\Phi^{-1}(\gamma_{n,p,\alpha}) = \left[\prod_{j=1}^{n}\left(\frac{|jp-1-\alpha|}{p}\right)^p\right]^{q-1}$$

On the other hand, for $j = 1,\dots,n$

$$\begin{aligned}\frac{|jq-\beta-1|}{q} &= \left|j - \frac{nq-\alpha(q-1)+1}{q}\right| = \left|j-n+\frac{\alpha}{p}-\frac{p-1}{p}\right| \\ &= \frac{|(j-n-1)p+\alpha+1|}{p} = \frac{|(n-j+1)p-\alpha-1|}{p},\end{aligned}$$

hence

$$\begin{aligned}\gamma_{n,q,\beta} = \prod_{j=1}^{n}\left(\frac{|jq-\beta-1|}{q}\right)^q &= \prod_{j=1}^{n}\left(\frac{|(n-j+1)p-\alpha-1|}{p}\right)^q \\ &= \left[\prod_{j=1}^{n}\left(\frac{|jp-1-\alpha|}{p}\right)^p\right]^{q-1},\end{aligned}$$

so really $\Phi^{-1}(\gamma_{n,p,\alpha}) = \gamma_{n,q,\beta}$. □

Note that for $n = 1$, i.e., for second order half-linear equation (1.2), the reciprocity principle holds as a simple consequence of the Rolle mean value theorem.

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