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SOLUTION TO A TRANSMISSION PROBLEM FOR QUASILINEAR PSEUDOPARABOLIC EQUATIONS BY THE ROTHE METHOD

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Abstract. In this paper, we deal with a transmission problem for a class of quasilinear pseudoparabolic equations. Existence, uniqueness and continuous dependence of the solution upon the data are obtained via the Rothe method. Moreover, the convergence of the method and an error estimate of the approximations are established.

1. Introduction

Let Ω be a bounded open domain in the space $\mathbb{R}^N$ of points $x = (x_1, ..., x_N)$ with the Lipschitz boundary $\partial\Omega$, such that $\partial\Omega = \overline{\Gamma}^0 \cup \overline{\Gamma}^1$, where Γ^0 and Γ^1 are open complementary parts, each consisting of an integer number of parts. Assume that Ω consists of M subdomains Ω_k, $1 \leq k \leq M$, (see *fig.1*), with respective boundaries $\partial\Omega_k$.
We first introduce some notations. Let

$$\mathcal{N}\left(\Gamma^\mu\right) = \left\{k, 1 \leq k \leq M / \text{meas}_{N-1}\left(\Gamma^\mu \cap \partial\Omega_k\right) > 0\right\},$$

$$\Gamma_k^\mu = \begin{cases} \Gamma^\mu \cap \partial\Omega_k, & k \in \mathcal{N}\left(\Gamma^\mu\right), \\ \phi, & k \notin \mathcal{N}\left(\Gamma^\mu\right), \end{cases}$$

for $\mu = 0, 1$. Moreover, let for $k = 1, ..., M$,

$$\mathcal{N}_k = \left\{\ell, 1 \leq \ell \leq M / \text{meas}_{N-1}\left(\partial\Omega_k \cap \partial\Omega_\ell\right) > 0\right\}$$

$$\Gamma_{k,\ell} = \begin{cases} \partial\Omega_k \cap \partial\Omega_\ell, & \ell \in \mathcal{N}_k, \\ \phi, & \ell \notin \mathcal{N}_k. \end{cases}$$

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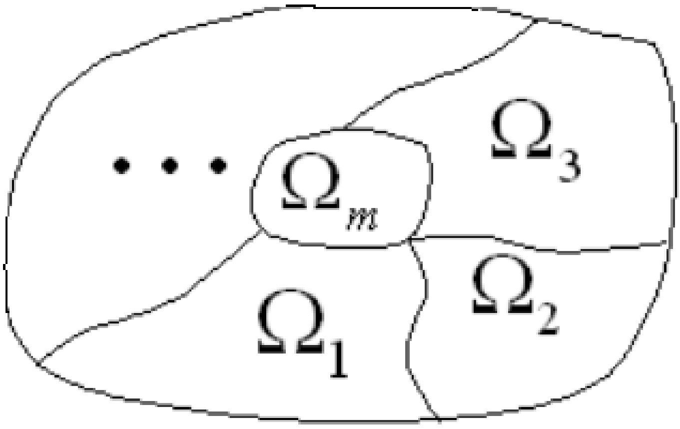

fig. 1

For instance, in the situation illustrated by *fig.* 1, we have

$$\mathcal{N}_2 = \{1, 2, 3\}$$

and then:

$$\Gamma_{2,1} = \partial\Omega_2 \cap \partial\Omega_1,$$

$$\Gamma_{2,2} = \partial\Omega_2,$$

$$\Gamma_{2,3} = \partial\Omega_2 \cap \partial\Omega_3,$$

$$\Gamma_{2,\ell} = \phi, \quad \forall \ell = 4, ..., M.$$

One should note that $\ell \in \mathcal{N}_k$ iff $k \in \mathcal{N}_\ell$ and hence $\Gamma_{k,\ell} = \Gamma_{\ell,k},\ 1 \le k, \ell \le M.$

Then we consider the following problem:

Determine $u^k = u^k(x,t)\ (k = 1, ..., M)$, $x \in \Omega_k,\ t \in I = (0, T]$, which obey the respective quasilinear third order pseudoparabolic equations

$$\begin{aligned} &\frac{\partial u^k}{\partial t} + Au^k + \alpha A \frac{\partial u^k}{\partial t} + a_0^k(x) u^k + \alpha a_0^k(x) \frac{\partial u^k}{\partial t} \\ = \ & f^k(x, t, u^k, \nabla u^k), \qquad \text{in } \Omega_k \times I, \qquad (k = 1, ..., M), \end{aligned}$$

where

$$Au^k = -\sum_{p,q=1}^{N} \frac{\partial}{\partial x_p} \left(a_{pq}^k(x) \frac{\partial u^k}{\partial x_q} \right),$$

along with the initial conditions

$$u^k(x,0) = u_0^k(x), \qquad x \in \Omega_k, \quad (k = 1, ..., M), \tag{1.1}$$

together with the Dirichlet and Neumann boundary conditions

$$\begin{aligned} u^k &= 0, \qquad \text{on } \Gamma_k^0 \times I \ \ (k = 1, ..., M)\,, \\ \frac{\partial u^k}{\partial \vartheta_A} &= 0, \qquad \text{on } \Gamma_k^1 \times I \ \ (k = 1, ..., M)\,, \end{aligned} \tag{1.2}$$

as well as with transmission conditions

$$\begin{aligned} & u^k = u^\ell, \qquad \text{on } \Gamma_{k,\ell} \times I, \qquad \forall \ell \in \mathcal{N}_k \ \ (k = 1, ..., M)\,, \\ & \left(\frac{\partial u^k}{\partial \vartheta_A} + \alpha \frac{\partial^2 u^k}{\partial t \partial \vartheta_A} \right) + \left(\frac{\partial u^\ell}{\partial \vartheta_A} + \alpha \frac{\partial^2 u^\ell}{\partial t \partial \vartheta_A} \right) = 0, \\ & \qquad \text{on } \Gamma_{k,\ell} \times I, \ \forall \ell \in \mathcal{N}_k \ \ (k = 1, ..., M)\,, \end{aligned} \tag{1.3}$$

where $\frac{\partial u^k}{\partial \vartheta_A}$ is the conormal derivative defined by:

$$\frac{\partial u^k(x)}{\partial \vartheta_A} = \sum_{p,q=1}^{N} a_{pq}^k(x) \frac{\partial u^k}{\partial x_q} \cos\left(\vartheta^k, x_p\right),$$

with $\cos\left(\vartheta^k, x_p\right)$ denotes the $p - th$ component of the outward unit normal vector ϑ^k to $\partial\Omega_k$, k stands for the superscript in u^k, f^k and ϑ^k, not an exponent.

Equation (1.1) can be classified as a pseudoparabolic equation because of its close link with the corresponding parabolic equation. In fact, in several cases, the solution of parabolic problem can be obtained as a limit of solutions to the corresponding problem for (1.1) when $\alpha \to 0$ [28]. It can be also classified as a hyperbolic equation with a dominate derivative [4].

Particular cases of problem (1.1)-(1.4) arise in various physical phenomena, for instance, in the theory of seepage of homogeneous liquids in fissured rocks [2, 8, 9], in the nonsteady flows of second order fluids [28], in the diffusion of imprisoned resonant radiation through a gas [20, 21, 27] which has applications in the analysis of certain laser systems [24] and in the modelling of the heat conduction involving a thermodynamic temperature $\theta = u - \alpha\Delta u$ along with a conductive temperature u, see [7]. An important particular case of problem (1.1)-(1.4) is which related to the Benjamin-Bona-Mahony equation

$$\frac{\partial u}{\partial t} - \eta \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} = 0 \tag{1.4}$$

proposed in [3]. Taking into account dissipative phenomena, equation (1.5) is modified to the so-called Benjamin-Bona-Mahony-Burgers equation [1]

$$\frac{\partial u}{\partial t} - \eta \frac{\partial^3 u}{\partial x^2 \partial t} - \alpha \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} = 0. \tag{1.5}$$

Let us cite some interesting papers dealing with transmission problems. The first of them is that of Gelfend [12], who attracted the attention on these problems, by showing their motivation. In [23], von Petersdorff used a boundary integral method to study a transmission problem for the Helmholtz equation in a number of adjacent Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, on the boundaries of which inhomogeneous Dirichlet, Neumann or transition conditions are imposed. Gaiduk [10], considered a linear problem about transverse vibrations of a uniform rectangular viscoelastic plate with supported boundaries caused by an impact. By means of the contour integral method, due to Rasulov [25], in combination with the method of separation of variables, it is shown the solvability and properties of the solution. In [15], Kačur-van Keer established a numerical solution for a transmission of linear parabolic problem, which is encountered in the context of transient temperature distribution in composite media consisting of several regions in contact, by applying a Rothe-Galerkin finite element method. Along a different line, transmission problems for parabolic-hyperbolic equations were considered by Ostrovsky [22], Ladyžhenskaya [18], Korzyuk [13], Lions [17] and Bouziani [5].

In this paper, we present the Rothe time-discretisation method, as a suitable method for both theoretical and numerical analysis of problem (1.1)-(1.4). Actually, in addition to providing the first step towards a fully discrete approximation scheme, it gives a constructive proof of the existence, uniqueness and continuous dependence of the solution upon the data.

Let us mention that the present work can be considered as a continuation of the previous works of the authors [6, 19], where linear single pseudoparabolic equations were studied. It can also be viewed as a companion of paper [15] by Kačur and van Keer.

An outline of the paper is as follows. In Section 2, we give the basic assumptions, notations and the appropriate function spaces. We also recall some auxiliary results used in the rest of the paper. The variational formulation of problem (1.1)-(1.4) as well as the concept of the solution we are considering and the solvability of the time discretized problem corresponding to (1.1)-(1.4) are given in Section 3. In Section 4, we establish some a priori estimates for the discretized problem, while convergence results and error estimate are given in Section 5. Section 6 is devoted to the existence, uniqueness and continuous dependence of the solution upon the data of problem (1.1)-(1.4). Finally, we establish in Section 7 the error estimate.

2. Preliminaries

Let $H^1(\Omega_k)$ be the first order Sobolev space on Ω_k with scalar product $(\cdot,\cdot)_{1,\Omega_k}$ and corresponding norm $\|\cdot\|_{1,\Omega_k}$, and let $(\cdot,\cdot)_{0,\Omega_k}$ and $\|\cdot\|_{0,\Omega_k}$ be the scalar product and corresponding norm respectively in $L^2(\Omega_k)$. Let the space of functions defined by:

$$V \ : \ = \Big\{ v = \left(v^1, ..., v^M\right) / \, v^k \in V^k, \, v^k\big|_{\Gamma_{k,\ell}} = v^\ell\big|_{\Gamma_{k,\ell}} , \quad \forall \ell \in \mathcal{N}_k \ \ (k = 1, ..., M) \Big\} , \tag{2.1}$$

where

$$V^k := \left\{ v^k \in H^1\left(\Omega_k\right) / \, v^k = 0 \text{ on } \Gamma_k^0 \right\} . \tag{2.2}$$

The space V is equipped with the norm $\|\cdot\|_{1,\Omega}$, namely

$$\|v\|_{1,\Omega}^2 = \sum_{k=1}^{M} \left\|v^k\right\|_{1,\Omega_k}^2 .$$

We identify $v \in V$ with a function $v : \Omega \to \mathbb{R}$, for which $v|_{\Omega_k} = v^k, \ (k = 1, ..., M)$. Similarly, we introduce the product space $\mathbb{L}^2(\Omega) = L^2(\Omega_1) \times \cdots \times L^2(\Omega_M)$ equipped with the scalar product and the associated norm

$$(u, v)_{0,\Omega} = \sum_{k=1}^{M} \left(u^k, v^k\right)_{0,\Omega_k} \text{ and } \|u\|_{0,\Omega}^2 = \sum_{k=1}^{M} \left\|u^k\right\|_{0,\Omega_k}^2 ,$$

respectively.

Now, we state the following hypotheses which are assumed to hold for $k = 1, ..., M$:

A1. $a_{pq}^k \in L^\infty(\Omega_k) ; \exists \varkappa > 0, \forall \xi \in \mathbb{R}^N : \sum_{p,q=1}^N a_{pq}^k(x)\xi_p\xi_q \geq \varkappa \sum_{p=1}^N \xi_p^2$ *a.e.* in Ω_k,

A2. $a_{pq}^k(x) = a_{qp}^k(x)$, *a.e.* in Ω_k,

A3. $a_0^k \in L^\infty(\Omega_k) ; \exists \beta > 0 : a_0^k(x) \geq \beta$, *a.e.* in Ω_k,

A4. $f^k\left(t,u^k,v^k\right):I\times\left(L^2\left(\Omega_k\right)\right)^2\rightarrow L^2\left(\Omega_k\right)$ is bounded in $L^2\left(\Omega_k\right)$ and fulfills the Lipschitz condition:

$$\begin{aligned}&\left\|f^k\left(t,u^k,v^k\right)-f^k\left(t',u'^k,v'^k\right)\right\|_{0,\Omega_k}\\ \leq\;& L\left(\left|t-t'\right|+\left\|u^k-u'^k\right\|_{0,\Omega_k}+\left\|v^k-v'^k\right\|_{0,\Omega_k}\right)\end{aligned}$$

for all $t,t'\in I$, and $u^k,u'^k,v^k,v'^k\in L^2\left(\Omega_k\right).$

A5. $u_0^k\in V^k$, $k=1,...,M.$

Let us define the symmetrical integro-differential form

$$a\left(u,v\right)=\sum_{k=1}^{M}a_k\left(u^k,v^k\right),$$

where

$$a_k\left(u^k,v^k\right)=\sum_{p,q=1}^{N}\int_{\Omega_k}\left(a_{pq}^k\frac{\partial u^k}{\partial x_q}\frac{\partial v^k}{\partial x_p}+a_0^k u^k v^k\right)dx$$

with a_{pq}^k satisfy assumptions **A1-A2**, then the form $a\left(u,v\right)$ fulfills the following properties:

P1. $\forall u,v\in V,\ \left|a\left(u,v\right)\right|\leq\kappa_0\left\|u\right\|_{1,\Omega}\left\|v\right\|_{1,\Omega},\ \kappa_0=cste,$

P2. There exists a sufficiently large constant $\beta_0\left(\geq 1\right)$ such that

$$a\left(v,v\right)\geq\beta_0\left\|v\right\|_{1,\Omega}^2,\ \forall u\in V.$$

Throughout, we will identify any function $(x,t)\in\Omega\times I\mapsto g(x,t)\in\mathbb{R}$ with the associated abstract function $t\mapsto g(t)$ defined from I into certain function space on Ω by setting $g(t):x\in\Omega\mapsto g(x,t)$. Moreover, we will use the standard functional spaces $L^2(I,H)$, $C(\overline{I},H)$, $L^\infty(I,H)$ and $\mathrm{Lip}(\overline{I},H)$, where H is a Banach space. For their properties, we refer the reader, for instance, to [16].

In order to solve the stated problem by the Rothe method, we divide the interval I into n subintervals by points $t_j=jh_n$, $j=0,...,n$, where $h_n:=T/n$ is a time-step. Set

$$\begin{aligned}u_j&=\left(u_j^1,u_j^2,...,u_j^M\right), & u_j^k\left(x\right)&:\simeq u^k\left(x,t_j\right),\\ \delta u_j&=\left(\delta u_j^1,\delta u_j^2,...,\delta u_j^M\right), & \delta u_j^k\left(x\right)&:=\frac{u_j^k\left(x\right)-u_{j-1}^k\left(x\right)}{h_n},\\ f_j&=\left(f_j^1,f_j^2,...,f_j^M\right), & f_j^k\left(x\right)&:=f^k\left(x,t_j,u_{j-1}^k,\nabla u_{j-1}^k\right),\end{aligned}$$

for $k=1,...,M$ and $j=1,...,n$. Introduce now functions obtained from the approximates u_j by piecewise linear interpolation and piecewise constant with respect to the time, respectively:

$$u^{(n)}(t) = \left(u^{1,n}(t), u^{2,n}(t), \cdots, u^{M,n}(t)\right),$$

$$\overline{u}^{(n)}(t) = \left(\overline{u}^{1,n}(t), \overline{u}^{2,n}(t), \cdots, \overline{u}^{M,n}(t)\right),$$

where

$$u^{k(n)}(t) := u_{j-1}^k + \delta u_j^k \left(t - t_{j-1}\right), \quad t \in \left[t_{j-1}, t_j\right], \tag{2.3}$$

and

$$\overline{u}^{k(n)}(t) := \begin{cases} u_j^k, & \text{for } t \in \left(t_{j-1}, t_j\right], \\ u_0^k, & \text{for } t \in \left[-h_n, 0\right], \end{cases} \tag{2.4}$$

for $j = 1, ..., n$, and $k = 1, ..., M$. Moreover, we use the notation:

$$\tau_{h_n} u^{k(n)}(x,t) = u^{k(n)}(x, t - h_n), \qquad k = 1, ..., M,$$

then for $w = \left(w^1, w^2, \cdots, w^M\right) \in \mathbb{H}^1(\Omega)$, we write

$$\overline{f}^{(n)}(t, w, \nabla w) = \left(\overline{f}^{1(n)}\left(t, w^1, \nabla w^1\right), \cdots, \overline{f}^{M(n)}\left(t, w^M, \nabla w^M\right)\right)$$

with

$$\overline{f}^{k(n)}(t, w^k, \nabla w^k) := f^k(t_j, w^k, \nabla w^k), \tag{2.5}$$

$t \in (t_{j-1}, t_j],\ k = 1, ..., M$, thus

$$\overline{f}^{k(n)}(t, \tau_{h_n}\overline{u}^{k(n)}, \nabla \tau_{h_n}\overline{u}^{k(n)}) := f^k(t_j, u_{j-1}^k, \nabla u_{j-1}^k) = f_j^k, \tag{2.6}$$

for $t \in (t_{j-1}, t_j],\ j = 1, ..., n$.

Finally, the following lemmas are used in this paper. We list them for convenience:

Lemma 1 (**An analogue of Gronwall's Lemma in continuous form** [11])**.** *Let $f_i(t)$ $(i = 1, 2)$ be real continuous functions on the interval $(0, T)$, $f_3(t) \geq 0$ nondecreasing function on t, and $C > 0$. Then the inequality*

$$\int_0^t f_1(s)ds + f_2(t) \leq f_3(t)e^{Ct}, \quad \forall t \in (0, T),$$

is a consequence of the inequality

$$\int_0^t f_1(s)ds + f_2(t) \leq f_3(t) + C\int_0^t f_2(s)ds.$$

Lemma 2 (**Gronwall's Lemma in discret form** [14]). *Let* $\{a_i\}$ *be a sequence of real, nonnegative numbers, and* A, C *and* h *be positive constants. If the inequality*

$$a_j \leq A + Ch \sum_{i=1}^{j-1} a_i,$$

takes place for all $j = 1, 2, \ldots, n$*, then the estimate*

$$a_i \leq Ae^{C(j-1)h},$$

holds for all $j = 2, \ldots n$.

3. Variational formulation

First, we take the scalar product in $L^2(\Omega_k)$ of equation (1.1) with $v^k \in V^k$, we have

$$\begin{aligned} & \left(\frac{\partial u^k(t)}{\partial t}, v^k\right)_{0,\Omega_k} + \left(Au^k(t) + \alpha A \frac{\partial u^k(t)}{\partial t}, v^k\right)_{0,\Omega_k} \\ & + \left(a_0^k \left(u^k(t) + \alpha \frac{\partial u^k(t)}{\partial t}\right), v^k\right)_{0,\Omega_k} \\ = \; & \left(f^k(t, u^k(t), \nabla u^k(t)), v^k\right)_{0,\Omega_k}. \end{aligned} \tag{3.1}$$

Applying the Green formula to the second term of the above identity, by taking into account condition (1.3b) and (2.6), we get

$$\begin{aligned} & \left(Au^k(t) + \alpha A \frac{\partial u^k(t)}{\partial t}, v^k\right)_{0,\Omega_k} \\ = \; & -\sum_{\ell \in \mathcal{N}_k} \int_{\Gamma_{k,\ell}} \left(\frac{\partial u^k(t)}{\partial \vartheta_A} + \alpha \frac{\partial^2 u^k(t)}{\partial t \partial \vartheta_A}\right) v^k dx \\ & + a_k\left(u^k(t), v^k\right) + \alpha a_k \left(\frac{du^k(t)}{dt}, v^k\right). \end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), it yields

$$\begin{aligned} & \left(\frac{du^k(t)}{dt}, v^k\right)_{0,\Omega_k} + a_k\left(u^k(t), v^k\right) + \alpha a_k \left(\frac{du^k(t)}{dt}, v^k\right) \\ = \; & \left(f^k(t, u^k(t), \nabla u^k(t)), v^k\right)_{0,\Omega_k}. \end{aligned} \tag{3.3}$$

Thus, summing up (3.3) for $k = 1, ..., M$ and invoking (2.5) and (1.4), we obtain the desired variational formulation of problem (1.1)-(1.4)

$$\begin{aligned} &\left(\frac{du(t)}{dt}, v\right)_{0,\Omega} + a\left(u(t), v\right) + \alpha a\left(\frac{du(t)}{dt}, v\right) \qquad (3.4)\\ = &\ \left(f(t, u\left(t\right), \nabla u\left(t\right)), v\right)_{0,\Omega}, \ \forall v \in V, \ a.e. t \in I, \ u(t) \in V, t \in I. \end{aligned}$$

Now, we are able to make precise the concept of the solution of problem (1.1)-(1.4) we are considering:

Definition 1. *A function $u : I \to \mathbb{L}^2(\Omega)$ is called a weak solution of problem (1.1)-(1.4), if*
(i) $u \in \mathrm{Lip}\left(\overline{I}, V\right)$;
(ii) *u has a strong derivative (a.e. in I)* $\frac{du}{dt} \in L^\infty(I, V)$;
(iii) $u(0) = \left(u_0^1, u_0^2, ..., u_0^M\right) = u_0$ *in* V;
(iv) *the identity (3.4) holds for all $v \in V$ and a.e. $t \in I$.*

Consider now the following linearized problem, obtained by discretization with respect to the time of (3.4)

$$\left\{\begin{array}{c} u_j \in V, \\ (\delta u_j, v)_{0,\Omega} + a(u_j, v) + \alpha a(\delta u_j, v) \\ = (f(t_j, u_{j-1}, \nabla u_{j-1}), v)_{0,\Omega}, \ \forall v \in V, \ (j = 1, \cdots, n), \end{array}\right. \qquad (3.5)$$

and consider the auxiliary functions

$$y_j = u_j + \alpha \delta u_j \quad (j = 1, ..., n), \qquad (3.6)$$

then, we can easily get

$$u_j = \frac{h_n}{\alpha + h_n} y_j + \frac{\alpha}{\alpha + h_n} u_{j-1} \quad (j = 1, ..., n),$$

from which, it follows

$$\delta u_j = \frac{1}{\alpha + h_n}\left(y_j - u_{j-1}\right) \quad (j = 1, ..., n). \qquad (3.7)$$

Therefore to prove the solvability of problem (3.5) it suffices to establish the proof for the following problem:

Find, successively for $j = 1, ..., n$, the functions $y_j \in V$ verifying:

$$\begin{aligned} &a\left(y_j, v\right) + \frac{1}{\alpha + h_n}\left(y_j, v\right)_{0,\Omega} \qquad (3.8)\\ = &\ \left(f_j + \frac{1}{\alpha + h_n} u_{j-1}, v\right)_{0,\Omega}, \qquad \forall v \in V, \end{aligned}$$

with

$$u_j = \frac{h_n}{\alpha + h_n} y_j + \frac{\alpha}{\alpha + h_n} u_{j-1}. \tag{3.9}$$

In light of properties **P1-P2**, a successive application of the Lax-Milgram Theorem to the coupled problem (3.8)-(3.9) leads to:

Lemma 3. *Under properties **P1-P2**, problem (3.8)-(3.9) admits for all $j = 1, ..., n$, a unique solution $y_j \in H^2(\Omega) \cap V$.*

As a consequence, we have

Corollary 4. *Problem (3.5) admits for all $j = 1, ..., n$, a unique solution $u_j \in H^2(\Omega) \cap V$.*

4. A priori estimates for the discretized problem

Let us now derive some a priori estimates:

Lemma 5. *Let assumption **A4** and properties **P1-P2** be fulfilled. Then, for $n \in \mathbb{N}^*$, the solutions u_j of the semi-discretized problem (3.5), satisfy:*

$$\|u_j\|_{1,\Omega} \leq C_1, \tag{4.1}$$

for all $j = 1, \ldots, n$, where C_1 is a positive constant independent of h_n and j.

Proof. Take $v = y_j$ in the integral identity (3.9), it yields

$$a(y_j, y_j) + \frac{1}{\alpha + h_n} \|y_j\|_{0,\Omega}^2 = \left(f_j + \frac{1}{\alpha + h_n} u_{j-1}, y_j \right)_{0,\Omega}.$$

Thanks to the Schwarz inequality, we obtain

$$\begin{aligned} & a(y_j, y_j) + \frac{1}{\alpha + h_n} \|y_j\|_{0,\Omega}^2 \\ \leq & \left(\|f_j\|_{0,\Omega} + \frac{1}{\alpha + h_n} \|u_{j-1}\|_{0,\Omega} \right) \|y_j\|_{0,\Omega}. \end{aligned} \tag{4.2}$$

Invoking (**P2**) and omitting the second term on the left-hand side of (4.2), it comes

$$\alpha\beta_0 \|y_j\|_{1,\Omega} \leq (\alpha + h_n) \beta_0 \|y_j\|_{1,\Omega} \leq (\alpha + h_n) \|f_j\|_{0,\Omega} + \|u_{j-1}\|_{0,\Omega},$$

therefore

$$\|y_j\|_{1,\Omega} \leq (\alpha + h_n) \|f_j\|_{0,\Omega} + \|u_{j-1}\|_{1,\Omega}. \tag{4.3}$$

According to (3.9), we have

$$\|u_j\|_{1,\Omega} \leq \frac{h_n}{\alpha + h_n} \|y_j\|_{1,\Omega} + \frac{\alpha}{\alpha + h_n} \|u_{j-1}\|_{1,\Omega} \quad (j = 1, ..., n). \quad (4.4)$$

Substituting (4.3) into (4.4), yields

$$\|u_j\|_{1,\Omega} \leq h_n \|f_j\|_{0,\Omega} + \|u_{j-1}\|_{1,\Omega}. \quad (4.5)$$

Iterating, we get

$$\|u_j\|_{1,\Omega} \leq h_n \sum_{i=1}^{j} \|f_i\|_{0,\Omega} + \|u_0\|_{1,\Omega}, \quad \forall j = 1, ..., n. \quad (4.6)$$

According to assumption A4, the following inequality holds

$$\begin{aligned} \|f_i\|_{0,\Omega} &\leq \|f(t_i, u_{i-1}, \nabla u_{i-1}) - f(t_i, 0, 0)\|_{0,\Omega} + \|f(t_i, 0, 0)\|_{0,\Omega} \\ &\leq \sqrt{2}L \|u_{i-1}\|_{1,\Omega} + M', \end{aligned} \quad (4.7)$$

where $M' := \max_{t \in I} \|f(t, 0, 0)\|_{0,\Omega} < \infty$.

Inserting (4.7) into (4.6), it comes

$$\begin{aligned} \|u_j\|_{1,\Omega} &\leq h_n \sum_{i=1}^{j} \left(\sqrt{2}L \|u_{i-1}\|_{1,\Omega} + M' \right) + \|u_0\|_{1,\Omega} \\ &= M' j h_n + \|u_0\|_{1,\Omega} + \sqrt{2} h_n L \sum_{i=0}^{j-1} \|u_i\|_{1,\Omega} \\ &\leq M'T + \left(1 + \sqrt{2}LT\right) \|u_0\|_{1,\Omega} + \sqrt{2} h_n L \sum_{i=1}^{j-1} \|u_i\|_{1,\Omega}. \end{aligned}$$

Therefore, owing to Lemma 2.2, we obtain

$$\|u_j\|_{1,\Omega} \leq \left(M'T + \left(1 + \sqrt{2}LT\right) \|u_0\|_{1,\Omega} \right) e^{\sqrt{2}L(j-1)h_n} \leq C_1,$$

where $C_1 := \left(M'T + \left(1 + \sqrt{2}LT\right) \|u_0\|_{1,\Omega} \right) e^{\sqrt{2}LT}$, which concludes the proof. ∎

Lemma 6. *Under assumptions of Lemma 4.1, the following estimates*

$$\|\delta u_j\|_{1,\Omega} \leq C_2, \quad (4.8)$$

$$\|\delta u_j\|_{0,\Omega} \leq C_3, \quad (4.9)$$

hold for $j = 1, \ldots, n$, where C_2 and C_3 are positive constants independent of h_n and j.

Proof. According to (4.1) and (4.7), inequality (4.3) becomes

$$\|y_j\|_{1,\Omega} \leq (\alpha + h_n)\left(\sqrt{2}L\,\|u_{j-1}\|_{1,\Omega} + M'\right) + \|u_{j-1}\|_{1,\Omega} \leq C_4,$$

where

$$C_4 := \left((\alpha + T)\,\sqrt{2}L + 1\right)C_1 + (\alpha + T)\,M'.$$

Therefore by virtue of (3.6), we have

$$\|\delta u_j\|_{1,\Omega} \leq \frac{1}{\alpha}\left(\|y_j\|_{1,\Omega} + \|u_{j-1}\|_{1,\Omega}\right),$$

from where, due to (4.1) and (4.10), we deduce

$$\|\delta u_j\|_{1,\Omega} \leq C_2, \quad \forall j = 1, ..., n, \tag{4.10}$$

where

$$C_2 := \frac{1}{\alpha}\left(C_1 + C_4\right).$$

On the other hand, invoking (4.2) we get, in consequence of the positivity of the first term

$$\|y_j\|_{1,\Omega} \leq (\alpha + h_n)\,\|f_j\|_{0,\Omega} + \|u_{j-1}\|_{0,\Omega} \tag{4.11}$$

and due to (3.9), it comes

$$\|u_j\|_{0,\Omega} \leq \frac{h_n}{\alpha + h_n}\,\|u_j\|_{0,\Omega} + \frac{h_n}{\alpha + h_n}\,\|u_{j-1}\|_{0,\Omega} \qquad (j = 1, ..., n), \tag{4.12}$$

then, combining (4.12) and (4.13), and summing up the resulting inquality for $i = 1, ..., j$, to obtain

$$\|u_j\|_{0,\Omega} \leq h_n \sum_{i=1}^{j} \|f_\iota\|_{0,\Omega} + \|u_0\|_{0,\Omega}$$

consequently, by (4.1) and (4.7), we conclude

$$\|u_j\|_{0,\Omega} \leq C_{5,} \qquad \forall j = 1, .., n, \tag{4.13}$$

with

$$C_5 := \sqrt{2}LC_1T + M'T + \|u_0\|_{0,\Omega}.$$

However, from (4.1), (4.7) and (4.12), it follows that

$$\|y_j\|_{0,\Omega} \leq (\alpha + h_n)\left(\sqrt{2}L\,\|u_{j-1}\|_{1,\Omega} + M'\right) + \|u_{j-1}\|_{0,\Omega},$$

from which, we have

$$\|y_j\|_{0,\Omega} \leq C_{6,} \qquad \forall j = 1, .., n, \tag{4.14}$$

with

$$C_6 := (\alpha + T)\left(\sqrt{2}LC_1 + M'\right) + C_5.$$

Finally, it follows from (3.6) that

$$\|\delta u_j\|_{0,\Omega} \le \frac{1}{\alpha}\left(\|y_j\|_{0,\Omega} + \|u_j\|_{0,\Omega}\right)$$

from which, due to (4.14) and (4.15), we find

$$\|\delta u_j\|_{0,\Omega} \le C_{3,} \qquad \forall j = 1, .., n,$$

where

$$C_3 := \frac{1}{\alpha}\left(C_5 + C_6\right).$$

This achieves the proof. ∎

5. Convergence results

The variational equation (3.5) may be written in terms of $u^{(n)}$ and $\overline{u}^{(n)}$:

$$\begin{aligned} &\left(\frac{du^{(n)}(t)}{dt}, v\right)_{0,\Omega} + a\left(\overline{u}^{(n)}(t), v\right) + \alpha a\left(\frac{du^{(n)}(t)}{dt}, v\right) \qquad (5.1) \\ = \; &\left(\overline{f}^{(n)}\left(t, \tau_{h_n}\overline{u}^{(n)}(t), \nabla\tau_{h_n}\overline{u}^{(n)}(t)\right), v\right)_{0,\Omega}, \qquad \forall v \in V, \quad \forall t \in I. \end{aligned}$$

For the functions $u^{(n)}$ and $\overline{u}^{(n)}$, we derive from results of Section 4 the following obvious properties:

Corollary 7. *For all $n \in \mathbb{N}^*$, the functions $u^{(n)}$ and $\overline{u}^{(n)}$ satisfy the following estimates:*

$$\left\|u^{(n)}(t)\right\|_{1,\Omega} \le C_1, \qquad \left\|\overline{u}^{(n)}(t)\right\|_{1,\Omega} \le C_1, \qquad \forall t \in I, \qquad (5.2)$$

$$\left\|\frac{du^{(n)}(t)}{dt}\right\|_{1,\Omega} \le C_2, \qquad \left\|\frac{du^{(n)}(t)}{dt}\right\|_{0,\Omega} \le C_3, \qquad a.e.\ in\ I, \qquad (5.3)$$

$$\left\|\overline{u}^{(n)}(t) - u^{(n)}(t)\right\|_{1,\Omega} \le C_2 h_n, \qquad \forall t \in I, \qquad (5.4)$$

$$\left\|\overline{u}^{(n)}(t) - \tau_{h_n}\overline{u}^{(n)}(t)\right\|_{1,\Omega} \le C_2 h_n, \qquad \forall t \in I, \qquad (5.5)$$

where C_1, C_2 and C_3 are the same constants given in Lemmas 4.1 and 4.2.

Proof. Estimates (5.2) follow directly from (4.1). To establish estimates (5.3), we differentiate identity (2.1) with respect to t and take into account (4.8) and (4.9), it yields

$$\left\|\frac{du^{(n)}(t)}{dt}\right\|_{1,\Omega} \le C_2, \qquad \left\|\frac{du^{(n)}(t)}{dt}\right\|_{0,\Omega} \le C_3, \quad \text{for } a.e.\, t \in I,$$

from where we get

$$\int_I \left\|\frac{du^{(n)}(t)}{dt}\right\|^2_{1,\Omega} dt \le C_8, \qquad \int_I \left\|\frac{du^{(n)}(t)}{dt}\right\|^2_{0,\Omega} dt \le C_7, \tag{5.6}$$

where $C_7 = C_3^2 T$ and $C_8 = C_2^2 T$. As for estimate (5.4), it suffices to observe that

$$\overline{u}^{(n)}(t) - u^{(n)}(t) = (t_j - t)\delta u_j, \quad \forall t \in (t_{j-1}, t_j] \quad (j = 1, ..., n)\,,$$

so that

$$\left\|\overline{u}^{(n)}(t) - u^{(n)}(t)\right\| \le h_n \max_{1\le j\le n} \|\delta u_j\|_{1,\Omega} \quad \forall\, t \in I \quad (j = 1, ..., n)\,,$$

therefore, due to (4.8), we obtain (5.4). Finally, it follows from

$$\overline{u}^{(n)}(t) - \tau_{h_n}\overline{u}^{(n)}\,(t) = u_j - u_{j-1}, \quad \forall t \in (t_{j-1}, t_j] \quad (j = 1, ..., n)\,,$$

that

$$\left\|\overline{u}^{(n)}(t) - \tau_{h_n}\overline{u}^{(n)}\,(t)\right\|_{1,\Omega} \le h_n \max_{1\le j\le n} \|\delta u_j\|_{1,\Omega}\,, \quad \forall\, t \in I.$$

Consequently, owing to (4.8), we get estimate (5.4). ∎

To continue, we have need to establish the following lemma:

Lemma 8. *Let assumption **A4** and property **P1** be fulfilled. Then the following estimate*

$$\left|a\left(\overline{u}^{(n)}(t), v\right)\right| \le C_9 \|v\|_{1,\Omega}\,, \tag{5.7}$$

holds for all $v \in V$ and a.e. $t \in I$.

Proof. Identity (5.1) can be written

$$a\left(\overline{u}^{(n)}(t), v\right) = \left(\overline{f}^{(n)}\left(t, \tau_{h_n}\overline{u}^{(n)}(t), \nabla\tau_{h_n}\overline{u}^{(n)}(t)\right) - \frac{du^{(n)}(t)}{dt}, v\right)_{0,\Omega}$$
$$-\alpha a\left(\frac{du^{(n)}(t)}{dt}, v\right), \qquad \forall t \in I,\ \forall v \in V. \tag{5.8}$$

In light of **P1** and the Schwarz inequality, the right-hand side of (5.8) is then dominated as follows

$$\left|a\left(\overline{u}^{(n)}(t), v\right)\right| \leq \left(\left\|\overline{f}^{(n)}\left(t, \tau_{h_n}\overline{u}^{(n)}(t), \nabla\tau_{h_n}\overline{u}^{(n)}(t)\right)\right\|_{0,\Omega} \right. \tag{5.9}$$
$$\left. + \left\|\frac{du^{(n)}(t)}{dt}\right\|_{0,\Omega} + \alpha\kappa_0 \left\|\frac{du^{(n)}(t)}{dt}\right\|_{1,\Omega}\right) \|v\|_{1,\Omega}.$$

However due to (2.4) and (4.7), it yields for all $j = 1, ..., n$:

$$\left\|\overline{f}^{(n)}\left(t, \tau_{h_n}\overline{u}^{(n)}(t), \nabla\tau_{h_n}\overline{u}^{(n)}(t)\right)\right\|_{0,\Omega} = \|f_j\|_{0,\Omega}$$
$$\leq \sqrt{2}L\,\|u_{j-1}\|_{1,\Omega} + M', \quad \forall t \in (t_{j-1}, t_j]\,;$$

therefore, owing to (4.1)

$$\left\|\overline{f}^{(n)}\left(t, \tau_{h_n}\overline{u}^{(n)}(t), \nabla\tau_{h_n}\overline{u}^{(n)}(t)\right)\right\|_{0,\Omega} \leq \sqrt{2}LC_1 + M', \quad \forall t \in I. \tag{5.10}$$

Inserting (5.3) and (5.10) into (5.9), we obtain (5.7), with

$$C_9 := \left(\sqrt{2}LC_1 + M' + C_3 + \alpha\kappa_0 C_2\right).$$

Let us subtract from identity (5.1) the similar identity for m, and set $v = u^{(n)}(t) - u^{(m)}(t)\ (\in V)$, we have for all $t \in I$

$$\begin{aligned}
&\left(\frac{d}{dt}\left(u^{(n)}(t) - u^{(m)}(t)\right), u^{(n)}(t) - u^{(m)}(t)\right)_{0,\Omega} \\
&+a\left(\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t), u^{(n)}(t) - u^{(m)}(t)\right) \\
&+\alpha a\left(\frac{d}{dt}\left(u^{(n)}(t) - u^{(m)}(t)\right), u^{(n)}(t) - u^{(m)}(t)\right) \\
= &\left(\overline{f}^{(n)}\left(t, \tau_{h_n}\overline{u}^{(n)}(t), \nabla\tau_{h_n}\overline{u}^{(n)}(t)\right)\right. \\
&\left. - \overline{f}^{(m)}\left(t, \tau_{h_m}\overline{u}^{(m)}(t), \nabla\tau_{h_m}\overline{u}^{(m)}(t)\right), u^{(n)}(t) - u^{(m)}(t)\right)_{0,\Omega}.
\end{aligned}$$

Observing that

$$u^{(n)} - u^{(m)} = \left(\overline{u}^{(n)} - \overline{u}^{(m)}\right) - \left(\overline{u}^{(n)} - u^{(n)}\right) - \left(u^{(m)} - \overline{u}^{(m)}\right),$$

the last equality can be written

$$\begin{aligned} & \frac{d}{dt}\left\|u^{(n)}(t)-u^{(m)}(t)\right\|_{0,\Omega}^{2} \qquad (5.11)\\ & +2a\left(\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t),\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t)\right)\\ & +\alpha\frac{d}{dt}a\left(u^{(n)}(t)-u^{(m)}(t),u^{(n)}(t)-u^{(m)}(t)\right)\\ = & \ 2\Big(\overline{f}^{(n)}\left(t,\tau_{h_n}\overline{u}^{(n)}(t),\nabla\tau_{h_n}\overline{u}^{(n)}(t)\right)\\ & \qquad -\overline{f}^{(m)}\left(t,\tau_{h_m}\overline{u}^{(m)}(t),\nabla\tau_{h_m}\overline{u}^{(m)}(t)\right),u^{(n)}(t)-u^{(m)}(t)\Big)_{0,\Omega}\\ & +2a\left(\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t),\left(\overline{u}^{(n)}(t)-u^{(n)}(t)\right)+\left(u^{(m)}(t)-\overline{u}^{(m)}(t)\right)\right),\end{aligned}$$

for $a.e.\ t\in I$.

We now estimate the terms on the right-hand side of (5.11). In light of Lemma 5.2 with $v=\left(\overline{u}^{(n)}(t)-u^{(n)}(t)\right)+\left(u^{(m)}(t)-\overline{u}^{(m)}(t)\right)$, it comes by taking into account (5.5):

$$\begin{aligned} & \left|a\left(\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t),\left(\overline{u}^{(n)}(t)-u^{(n)}(t)\right)+\left(u^{(m)}(t)-\overline{u}^{(m)}(t)\right)\right)\right|\\ \leq & \ 2C_9\left(\left\|\overline{u}^{(n)}(t)-u^{(n)}(t)\right\|_{1,\Omega}+\left\|u^{(m)}(t)-\overline{u}^{(m)}(t)\right\|_{1,\Omega}\right) \qquad (5.12)\\ \leq & \ C_{10}\left(h_n+h_m\right),\end{aligned}$$

where $C_{10}:=2C_2C_9$.
The first term on the right-hand side of (5.11) can be estimated as follows:

$$\begin{aligned} & 2\Big(\overline{f}^{(n)}\left(t,\tau_{h_n}\overline{u}^{(n)}(t),\nabla\tau_{h_n}\overline{u}^{(n)}(t)\right) \qquad (5.13)\\ & \qquad -\overline{f}^{(m)}\left(t,\tau_{h_m}\overline{u}^{(m)}(t),\nabla\tau_{h_m}\overline{u}^{(m)}(t)\right),u^{(n)}(t)-u^{(m)}(t)\Big)_{0,\Omega}\\ \leq & \ \Big\|\overline{f}^{(n)}\left(t,\tau_{h_n}\overline{u}^{(n)}(t),\nabla\tau_{h_n}\overline{u}^{(n)}(t)\right)\\ & \qquad -\overline{f}^{(m)}\left(t,\tau_{h_m}\overline{u}^{(m)}(t),\nabla\tau_{h_m}\overline{u}^{(m)}(t)\right)\Big\|_{0,\Omega}^{2}\\ & +\left\|u^{(n)}(t)-u^{(m)}(t)\right\|_{1,\Omega}^{2}.\end{aligned}$$

But for all $t \in I$, there exist two integers j and i such that $t \in (t_{j-1}, t_j] \cap (t_{i+1}, t_i]$

$$\begin{aligned}
& \left\| \overline{f}^{(n)} \left(t, \tau_{h_n} \overline{u}^{(n)}(t), \nabla \tau_{h_n} \overline{u}^{(n)}(t) \right) - \overline{f}^{(m)} \left(t, \tau_{h_m} \overline{u}^{(m)}(t), \nabla \tau_{h_m} \overline{u}^{(m)}(t) \right) \right\|_{0,\Omega} \\
= & \left\| f \left(t_j, \tau_{h_n} \overline{u}^{(n)}(t), \nabla \tau_{h_n} \overline{u}^{(n)}(t) \right) - \overline{f} \left(t_k, \tau_{h_m} \overline{u}^{(m)}(t), \nabla \tau_{h_m} \overline{u}^{(m)}(t) \right) \right\|_{0,\Omega} \\
\leq & \sqrt{3} L \left(\sqrt{M'} (h_n + h_m) + \left\| \tau_{h_n} \overline{u}^{(n)}(t) - \tau_{h_m} \overline{u}^{(m)}(t) \right\|_{1,\Omega} \right) \\
\leq & \sqrt{3} L \left(\sqrt{M'} (h_n + h_m) + \left\| \tau_{h_n} \overline{u}^{(n)}(t) - \overline{u}^{(n)}(t) \right\|_{1,\Omega} \right. \\
& + \left\| \overline{u}^{(n)}(t) - u^{(n)}(t) \right\|_{1,\Omega} + \left\| u^{(n)}(t) - u^{(m)}(t) \right\|_{1,\Omega} \\
& \left. + \left\| u^{(m)}(t) - \overline{u}^{(m)}(t) \right\|_{1,\Omega} + \left\| \overline{u}^{(m)}(t) - \tau_{h_m} \overline{u}^{(m)}(t) \right\|_{1,\Omega} \right) \\
\leq & \sqrt{3} L \left(\left(\sqrt{M'} + 2C_2 \right) (h_n + h_m) + \left\| u^{(n)}(t) - u^{(m)}(t) \right\|_{1,\Omega} \right).
\end{aligned}$$

Consequently, inequality (5.13) becomes

$$\begin{aligned}
& 2 \left(\overline{f}^{(n)} \left(t, \tau_{h_n} \overline{u}^{(n)}(t), \nabla \tau_{h_n} \overline{u}^{(n)}(t) \right) \right. \qquad (5.14) \\
& \quad \left. - \overline{f}^{(m)} \left(t, \tau_{h_m} \overline{u}^{(m)}(t), \nabla \tau_{h_m} \overline{u}^{(m)}(t) \right), u^{(n)}(t) - u^{(m)}(t) \right)_{0,\Omega} \\
\leq & \ 6L^2 \left(\sqrt{M'} + 2C_2 \right)^2 (h_n + h_m)^2 \\
& + \left(6L^2 + 1 \right) \left\| u^{(n)}(t) - u^{(m)}(t) \right\|_{1,\Omega}^2.
\end{aligned}$$

Substituting (5.12) and (5.14) into (5.11), integrating the result over $(0, t)$ by taking into account the fact that $u^{(n)}(0) = u(0) = u_0$, and applying property **P2**, we obtain, by omitting the first term on the left hand-side and

$$\begin{aligned}
& \int_0^t \left\| \overline{u}^{(n)}(s) - \overline{u}^{(m)}(s) \right\|_{1,\Omega}^2 ds + \left\| u^{(n)}(t) - u^{(m)}(t) \right\|_{1,\Omega}^2 \\
\leq & \frac{2T}{\beta_0 \min(2, \alpha)} \left(C_{10} (h_n + h_m) + 3L^2 \left(\sqrt{M'} + 2C_2 \right)^2 (h_n + h_m)^2 \right) \\
& + \frac{(6L^2 + 1)}{\beta_0 \min(2, \alpha)} \int_0^t \left\| u^{(n)}(s) - u(s) \right\|_{1,\Omega}^2 ds,
\end{aligned}$$

for all $t \in I$. Hence, by virtue of Lemma 2.1, we get

$$\begin{aligned} & \int_0^t \left\| \overline{u}^{(n)}(s) - \overline{u}^{(m)}(s) \right\|_{1,\Omega}^2 ds + \left\| u^{(n)}(t) - u^{(m)}(t) \right\|_{1,\Omega}^2 \\ \leq \; & \frac{2T}{\beta_0 \min(2,\alpha)} \left(C_{10} \left(h_n + h_m \right) \right. \\ & \left. + 3L^2 \left(\sqrt{M'} + 2C_2 \right)^2 \left(h_n + h_m \right)^2 \right) \exp \left(\frac{\left(6L^2 + 1 \right) t}{\beta_0 \min(2,\alpha)} \right), \end{aligned}$$

for all $t \in I$. Consequently,

$$\begin{aligned} & \int_0^t \left\| \overline{u}^{(n)}(s) - \overline{u}^{(m)}(s) \right\|_{1,\Omega}^2 ds + \left\| u^{(n)}(t) - u^{(m)}(t) \right\|_{1,\Omega}^2 \\ \leq \; & \frac{2T}{\beta_0 \min(2,\alpha)} \left(C_{10} \left(h_n + h_m \right) \right. \\ & \left. + 3L^2 \left(\sqrt{M'} + 2C_2 \right)^2 \left(h_n + h_m \right)^2 \right) \exp \left(\frac{\left(2L^2 + 1 \right)}{\beta_0 \min(2,\alpha)} T \right). \end{aligned}$$

Since the right-hand side of the above inequality is independent of t; hence, replacing the left-hand side by its upper bound with respect to t from 0 to T, we obtain

$$\begin{aligned} & \left\| \overline{u}^{(n)} - \overline{u}^{(m)} \right\|_{L^2(I,V)}^2 + \left\| u^{(n)} - u^{(m)} \right\|_{C(\overline{I},V)}^2 \\ \leq \; & \frac{2T}{\beta_0 \min(2,\alpha)} \left(C_{10} \left(h_n + h_m \right) \right. \\ & \left. + 3L^2 \left(\sqrt{M'} + 2C_2 \right)^2 \left(h_n + h_m \right)^2 \right) \exp \left(\frac{\left(2L^2 + 1 \right) T}{\alpha \beta_0} \right), \end{aligned}$$

which implies that $\left\{ \overline{u}^{(n)} \right\}$ and $\left\{ u^{(n)} \right\}$ are Cauchy sequences in the Banach spaces $L^2(I,V)$ and $C\left(\overline{I},V\right)$, respectively. Consequently, having in mind (5.5), there exists some function $u \in C\left(\overline{I},V\right)$ such that:

$$u^{(n)} \to u \qquad \text{in } C\left(\overline{I},V\right) \tag{5.15}$$

$$\overline{u}^{(n)} \to u \qquad \text{in } L^2(I,V) \tag{5.16}$$

as n tends to infinity. ∎

According to (5.2b), (5.3) and (5.15), we get, by taking into account [14, Lemma 1.3.15], the following results formulated in:

Theorem 9. *Let the assumption **A4** et properties **P1-P2** be hold. Then, the function u possesses the following properties:*

$$u \in Lip\left(\overline{I}, V\right), \tag{5.17}$$

$$u \text{ is strongly differentiable a.e. in } I \text{ and } \frac{du}{dt} \in L^{\infty}(I, V), \tag{5.18}$$

$$\overline{u}^{(n)}(t) \to u(t), \qquad \text{in } V, \quad \forall t \in I, \tag{5.19}$$

$$\frac{du^{(n)}}{dt} \rightharpoonup \frac{du}{dt}, \qquad \text{in } L^2(I, V). \tag{5.20}$$

6. Existence, uniqueness and continuous dependence

Theorem 10. *Under assumptions of Theorem 5.3, the limit function u is the weak solution of problem (1.1)-(1.4) in the sense of Definition 3.1.*

Proof. In light of (5.17) and (5.18) the points (i)-(ii) of Definition 3.1 are verified. Furthermore, since by definition $u^{(n)}(0) = u_0$, it then follows from (5.15) that the point (iii) of Definition 3.1 is fulfilled. It remains to prove that the limit function $u = u(x,t)$ satisfies the integral identity (3.4). To this end, integrate identity (5.1) over $(0,t) \subset I$

$$\begin{aligned} &\left(u^{(n)}(t), v\right)_{0,\Omega} + \int_0^t a\left(\overline{u}^{(n)}(s), v\right) ds + \alpha a\left(u^{(n)}(t), v\right) \\ = &\int_0^t \left(\overline{f}^{(n)}\left(s, \tau_{h_n}\overline{u}^{(n)}(s), \nabla\tau_{h_n}\overline{u}^{(n)}(s)\right), v\right)_{0,\Omega} ds \\ &+ (u_0, v)_{0,\Omega} + \alpha a(u_0, v), \end{aligned} \tag{6.1}$$

which can be written

$$\begin{aligned} &\left(u^{(n)}(t) - u(t), v\right)_{0,\Omega} + (u(t), v)_{0,\Omega} \\ &+ \int_0^t a\left(\overline{u}^{(n)}(s) - u(s), v\right) ds + \int_0^t a(u(s), v)\, ds \\ &+ \alpha a\left(u^{(n)}(t) - u(t), v\right) + \alpha a(u(t), v) \\ = &\int_0^t \Big(\overline{f}^{(n)}\left(s, \tau_{h_n}\overline{u}^{(n)}(s), \nabla\tau_{h_n}\overline{u}^{(n)}(s)\right) \\ &\qquad - f(s, u(s), \nabla u(s)), v)_{0,\Omega}\, ds \\ &+ \int_0^t (f(s, u(s), \nabla u(s)), v)_{0,\Omega}\, ds + (u_0, v)_{0,\Omega} + \alpha a(u_0, v). \end{aligned} \tag{6.2}$$

We must show that

$$\lim_{n\to\infty}\Big(\big(u^{(n)}(t)-u(t),v\big)_{0,\Omega}+\int_0^t a\left(u^{(n)}(s)-u(s),v\right)ds \quad (6.3)$$
$$+\alpha a\left(u^{(n)}(t)-u(t),v\right)=0$$

and

$$\lim_{n\to\infty}\int_0^t\Big(\overline{f}^{(n)}\left(s,\tau_{h_n}\overline{u}^{(n)}(s),\nabla\tau_{h_n}\overline{u}^{(n)}(s)\right) \quad (6.4)$$
$$-f\left(t,u(s),\nabla u(s)\right),v)_{0,\Omega}\,ds=0.$$

It is easy to check

$$\begin{aligned}&\left(u^{(n)}(t)-u(t),v\right)_{0,\Omega}+\alpha a\left(u^{(n)}(t)-u(t),v\right) \\ \leq\ &(1+\alpha\kappa_0)\left\|u^{(n)}(t)-u(t)\right\|_{1,\Omega}\|v\|_{1,\Omega},\ \forall t\in I.\end{aligned} \quad (6.5)$$

Therefore invoking (5.15) and passing to the limit in (6.5), when n tends to infinity, we get

$$\lim_{n\to\infty}\Big(\left(u^{(n)}(t)-u(t),v\right)_{0,\Omega}+\alpha a\left(u^{(n)}(t)-u(t),v\right)\Big)=0. \quad (6.6)$$

However, owing to (5.16), property **P1** and Lemma 5.2 we have

$$\left|\int_0^t a\left(\overline{u}^{(n)}(s)-u(s),v\right)ds\right|\leq\kappa_0\sqrt{T}\,\|v\|_{1,\Omega}\left\|\overline{u}^{(n)}-u\right\|_{L^2(I,V)},$$

hence

$$\lim_{n\to\infty}\int_0^t a\left(\overline{u}^{(n)}(s)-u(s),v\right)ds=0,\ \forall v\in V,\ \forall t\in I. \quad (6.7)$$

Consequently, by combining (6.6) and (6.7), we obtain (6.3).

On the other hand, owing to assumption A4, it comes

$$\begin{aligned}&\left\|\overline{f}^{(n)}\left(t,\tau_{h_n}\overline{u}^{(n)}(t),\nabla\tau_{h_n}\overline{u}^{(n)}(t)\right)-f\left(t,u(t),\nabla u(t)\right)\right\|_{0,\Omega} \\ =\ &\left\|f\left(t_j,\tau_{h_n}\overline{u}^{(n)}(t),\nabla\tau_{h_n}\overline{u}^{(n)}(t)\right)-f\left(t,u(t),\nabla u(t)\right)\right\|_{0,\Omega} \\ \leq\ &\sqrt{3}L\left(\sqrt{M'}\,|t_j-t|+\left\|\tau_{h_n}\overline{u}^{(n)}(t)-u(t)\right\|_{1,\Omega}\right),\end{aligned}$$

for all $t \in (t_{j-1}, t_j]$ $(j = 1, ..., n)$. Consequently, according to (5.5)

$$\begin{aligned} & \left\| \overline{f}^{(n)} \left(t, \tau_{h_n} \overline{u}^{(n)} (t), \nabla \tau_{h_n} \overline{u}^{(n)} (t) \right) - f \left(t, u (t), \nabla u (t) \right) \right\|_{0,\Omega} \\ \leq & \sqrt{3} L \left(\sqrt{M'} h_n + \left\| \tau_{h_n} \overline{u}^{(n)} (t) - \overline{u}^{(n)}(t) \right\|_{1,\Omega} + \left\| \overline{u}^{(n)} (t) - u(t) \right\|_{1,\Omega} \right) \\ \leq & \sqrt{3} L h_n \left(\sqrt{M'} + C_2 \right) + \sqrt{3} L \left\| \overline{u}^{(n)} (t) - u(t) \right\|_{1,\Omega}, \end{aligned}$$

from which, we conclude, in view of (5.20), that

$$\overline{f}^{(n)} \left(t, \tau_{h_n} \overline{u}^{(n)} (t), \nabla \tau_{h_n} \overline{u}^{(n)} (t) \right) \underset{n \to \infty}{\longrightarrow} f \left(t, u (t), \nabla u (t) \right), \tag{6.8}$$

in V, $\forall t \in I$. Moreover, by virtue of the Schwarz inequality and (5.10), we have

$$\begin{aligned} & \left(\overline{f}^{(n)} \left(t, \tau_{h_n} \overline{u}^{(n)} (t), \nabla \tau_{h_n} \overline{u}^{(n)} (t) \right), v \right)_{0,\Omega} \\ \leq & \left(\sqrt{2} L C_1 + M' \right) \| v \|_{0,\Omega}, \qquad \forall t \in I, \ \forall n. \end{aligned}$$

Therefore the application of the Lebesgue Theorem of dominate convergence leads to (6.4). Hence, by passing to the limit in (6.2) when n tends to infinity, we get

$$\begin{aligned} & (u(t), v)_{0,\Omega} + \int_0^t a \left(u(s), v \right) ds + \alpha a \left(u(t), v \right) \\ = & \int_0^t \left(f \left(s, u (s), \nabla u (s) \right), v \right)_{0,\Omega} ds + (u_0, v)_{0,\Omega} + \alpha a \left(u_0, v \right). \end{aligned}$$

Differentiating the above identity with respect to t we obtain the integral identity (3.4), thanks to the identities

$$\frac{d}{dt} (u(t), v)_{0,\Omega} = \left(\frac{du(t)}{dt}, v \right)_{0,\Omega}, \ \forall v \in V, \ a.e. \ t \in I,$$

and

$$\frac{d}{dt} a \left(u(t), v \right) = a \left(\frac{du(t)}{dt}, v \right), \ \forall v \in V, \ a.e. \ t \in I.$$

This completes the proof of Theorem 6.1. ∎

Theorem 11. *Under assumptions of Theorem 5.3, the weak solution of problem (1.1)-(1.4) is unique.*

Proof. Assume that (1.1)-(1.4) possesses two weak solutions u^1 and u^2. Taking the difference of identities (3.4) corresponding to u^1 and u^2, and $v = u$, where $u = u^1 - u^2$, it yields

$$\begin{aligned} & \frac{d}{dt}\|u(t)\|_{0,\Omega}^2 + 2a\left(u(t), u(t)\right) + \alpha\frac{d}{dt}a\left(u(t), u(t)\right) \\ = \; & 2\left(f\left(t, u^1(t), \nabla u^1(t)\right) - f\left(t, u^2(t), \nabla u^2(t)\right), u(t)\right)_{0,\Omega}, \end{aligned}$$

a.e. $t \in I$. Integrating over $(0,t) \subset I$, we get by taking into account that $u(0) = 0$

$$\begin{aligned} & \|u(t)\|_{0,\Omega}^2 + 2\int_0^t a\left(u(s), u(s)\right) ds + \alpha a\left(u(t), u(t)\right) \\ = \; & 2\int_0^t \left(f\left(s, u^1(s), \nabla u^1(s)\right) - f\left(s, u^2(s), \nabla u^2(s)\right), u(s)\right)_{0,\Omega} ds, \end{aligned}$$

so that, owing to the Schwarz inequality and assumption **A4**

$$\begin{aligned} & \|u(t)\|_{0,\Omega}^2 + 2\int_0^t a\left(u(s), u(s)\right) ds + \alpha a\left(u(t), u(t)\right) \\ \leq \; & 2\int_0^t \left\|f\left(s, u^1(s), \nabla u^1(s)\right) - f\left(s, u^2(s), \nabla u^2(s)\right)\right\|_{0,\Omega} \|u(s)\|_{0,\Omega} ds \\ \leq \; & 2\sqrt{2}L\int_0^t \|u(s)\|_{1,\Omega}^2 ds, \quad \forall t \in I. \end{aligned}$$

Omitting the first two terms on the left-hand side of the last inequality and using **P2**, we find

$$\|u(t)\|_{1,\Omega}^2 \leq \frac{2\sqrt{2}L}{\alpha\beta_0}\int_0^t \|u(s)\|_{1,\Omega}^2 ds, \quad \forall t \in I.$$

Thanks to Lemma 2.1, we conclude that

$$\|u(t)\|_{1,\Omega} = 0, \quad \forall t \in I,$$

which implies the uniqueness of the solution. ∎

Theorem 12. *Let properties **P1-P2** be fulfilled. Moreover, let $u(x,t)$ and $u^*(x,t)$ be two solutions of problem (1.1)-(1.4) corresponding to (u_0, f) and (u_0^*, f^*), respectively. If there exists a continuous nonnegative function $K(t)$ and a positive constant L' such that the following estimate*

$$\begin{aligned} & \|f(t,u,p) - f^*(t,u^*,p^*)\|_{0,\Omega} \qquad (6.9) \\ \leq \; & K(t) + L'\left(\|u-u^*\|_{0,\Omega} + \|p-p^*\|_{0,\Omega}\right) \end{aligned}$$

for all $t \in I$ and takes pace for all u, u^*, p and $p^* \in \mathbb{L}^2(\Omega)$, then

$$\|u-u^*\|^2_{C(\overline{I},V)} \leq C_{11}\left(\int_0^t K^2(s)ds + \|u_0-u_0^*\|^2_{1,\Omega}\right), \tag{6.10}$$

where C_{11} is a positive constant independent on u and u^*.

Proof. Considering the variational formulation of problem (1.1)-(1.4) written for u, subtracting from it the same integral identity written for $u^*$and setting $v = u(t) - u^*(t)$, we obtain after integrating the obtained identity over $(0,t) \subset I$

$$\begin{aligned} & \|u(t)-u^*(t)\|^2_{0,\Omega} + 2\int_0^t a\left(u(s)-u^*(s), u(s)-u^*(s)\right)ds \\ & +\alpha a\left(u(t)-u^*(t), u(t)-u^*(t)\right) \qquad (6.11) \\ = \; & 2\int_0^t \left(f\left(s,u(s),\nabla u(s)\right)\right. \\ & \qquad \left.-f^*\left(s,u^*(s),\nabla u^*(s)\right), u(s)-u^*(s)\right)_{0,\Omega} ds \\ & +\|u_0-u_0^*\|^2_{0,\Omega} + \alpha a\left(u_0-u_0^*, u_0-u_0^*\right). \end{aligned}$$

Invoking properties **P1-P2** and (6.9), we get, in consequence of the positivity of the first two terms on the left-hand side of (6.10) and of the application of the elementary inequalities $2ab \leq a^2+b^2$ and $(a+b+b)^2 \leq 3(a^2+b^2+c^2)$ to the right-hand side, after some re-arrangement

$$\begin{aligned} & \|u(t)-u^*(t)\|^2_{1,\Omega} \\ \leq \; & \max\left(\frac{3}{\alpha\beta_0}, \frac{1+\alpha\kappa_0}{\alpha\beta_0}\right)\left(\int_0^t K^2(s)ds + \|u_0-u_0^*\|^2_{1,\Omega}\right) \\ & +\frac{3L'^2+1}{\alpha\beta_0}\int_0^t \|u(s)-u^*(s)\|^2_{1,\Omega} ds. \end{aligned}$$

Thanks to Lemma 2.1 we get inequality (6.10), with

$$C_{11} := \max\left(\frac{3}{\alpha\beta_0}, \frac{1+\alpha\kappa_0}{\alpha\beta_0}\right)\exp\left(\frac{(3L'^2+1)T}{\alpha\beta_0}\right).$$

∎

7. Error estimate

Theorem 13. *Under the assumption* ***A4*** *and properties* ***P1-P2****, the error of the approximation*

$$\left\|u^{(n)}-u\right\|_{C(\overline{I},V)} \leq C_{12}h_n \tag{7.1}$$

is valid for all $n \in \mathbb{N}^*$.

Proof. Considering the difference between (5.1) and (3.4), and putting $v = u^{(n)}(t) - u(t)\ (\in V)$, we have

$$\begin{aligned}
& \left(\frac{d}{dt}\left(u^{(n)}(t)-u(t)\right), u^{(n)}(t)-u(t)\right)_{0,\Omega} \\
& +a\left(\overline{u}^{(n)}(t)-u(t), u^{(n)}(t)-u(t)\right) \\
& +\alpha a\left(\frac{d}{dt}\left(u^{(n)}(t)-u(t)\right), u^{(n)}(t)-u(t)\right) \\
= \ & \left(\overline{f}^{(n)}\left(t, \tau_{h_n}\overline{u}^{(n)}(t), \nabla\tau_{h_n}\overline{u}^{(n)}(t)\right) - f\left(t, u\left(t\right), \nabla u\left(t\right)\right), u^{(n)}(t)-u(t)\right)_{0,\Omega},
\end{aligned}$$

from which we obtain

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left\|u^{(n)}(t)-u(t)\right\|_{0,\Omega}^2 + a\left(u^{(n)}(t)-u(t), u^{(n)}(t)-u(t)\right) \\
& +\alpha a\left(\frac{d}{dt}\left(u^{(n)}(t)-u(t)\right), u^{(n)}(t)-u(t)\right) \\
= \ & \left(\overline{f}^{(n)}\left(t, \tau_{h_n}\overline{u}^{(n)}(t), \nabla\tau_{h_n}\overline{u}^{(n)}(t)\right) - f\left(t, u\left(t\right), \nabla u\left(t\right)\right), u^{(n)}(t)-u(t)\right)_{0,\Omega} \\
& +a\left(u^{(n)}(t)-\overline{u}^{(n)}(t), u^{(n)}(t)-u(t)\right), \qquad a.e.\ t \in I.
\end{aligned}$$

Owing to properties **P1-P2** and the Schwarz and Cauchy inequalities, it yields

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left\|u^{(n)}(t)-u(t)\right\|_{0,\Omega}^2 + \beta_0\left\|u^{(n)}(t)-u(t)\right\|_{1,\Omega}^2 \\
& +\frac{1}{2}\alpha\beta_0\frac{d}{dt}\left\|u^{(n)}(t)-u(t)\right\|_{1,\Omega}^2 \\
\leq \ & \frac{1}{2}\left\|\overline{f}^{(n)}\left(t, \tau_{h_n}\overline{u}^{(n)}(t), \nabla\tau_{h_n}\overline{u}^{(n)}(t)\right) - f\left(t, u\left(t\right), \nabla u\left(t\right)\right)\right\|_{0,\Omega}^2 \\
& +\frac{1}{2}\left\|u^{(n)}(t)-u(t)\right\|_{0,\Omega}^2 \\
& +\kappa_0\left\|\overline{u}^{(n)}(t)-u^{(n)}(t)\right\|_{1,\Omega}\left\|u^{(n)}(t)-u(t)\right\|_{1,\Omega}
\end{aligned} \tag{7.2}$$

Therefore, performing similar calculations as in Section 6, it comes

$$\left\| \overline{f}^{(n)} \left(t, \tau_{h_n} \overline{u}^{(n)}, \nabla \tau_{h_n} \overline{u}^{(n)}\right) - f\left(t, u\left(t\right), \nabla u\left(t\right)\right) \right\|_{0,\Omega}^{2} \tag{7.3}$$
$$\leq \; 3L^2 \left(\left(M' + 6C_2^2\right) h_n^2 + \left\| u^{(n)}\left(t\right) - u\left(t\right) \right\|_{1,\Omega}^{2} \right).$$

Inserting (7.3) into (7.2), applying the Cauchy inequality and using (5.5)-(5.6), we get

$$\frac{1}{2}\frac{d}{dt}\left\|u^{(n)}(t)-u(t)\right\|_{0,\Omega}^{2}+\beta_0\left\|u^{(n)}(t)-u(t)\right\|_{1,\Omega}^{2} \tag{7.4}$$
$$+\frac{1}{2}\alpha\beta_0\frac{d}{dt}\left\|u^{(n)}(t)-u(t)\right\|_{1,\Omega}^{2}$$
$$\leq \; \frac{1}{2}\left(3L^2\left(M'+6C_2^2\right)+\kappa_0^2C_2^2\right)h_n^2$$
$$+\left(\frac{3L^2}{2}+1\right)\left\|u^{(n)}\left(t\right)-u\left(t\right)\right\|_{1,\Omega}^{2}.$$

Integrating (7.4) over $(0,t)$ by taking into account the fact that $u^{(n)}(0) = u(0) = u_0$, and neglecting the first term on the left hand-side, we find

$$\int_0^t \left\|u^{(n)}(s)-u(s)\right\|_{1,\Omega}^{2} ds + \left\|u^{(n)}(t)-u(t)\right\|_{1,\Omega}^{2}$$
$$\leq \; \frac{T}{\beta_0 \min\left(2,\alpha\right)}\left(3L^2\left(M'+6C_2^2\right)+\kappa_0^2C_2^2\right)h_n^2$$
$$+\frac{\left(3L^2+2\right)}{\beta_0\min\left(2,\alpha\right)}\int_0^t\left\|u^{(n)}(s)-u(s)\right\|_{1,\Omega}^{2}ds, \qquad \forall t\in\overline{I}.$$

It then follows, by means of Lemma 2.2

$$\int_0^t \left\|u^{(n)}(s)-u(s)\right\|_{1,\Omega}^{2} ds + \left\|u^{(n)}(t)-u(t)\right\|_{1,\Omega}^{2} \tag{7.5}$$
$$\leq \; \frac{T}{\beta_0 \min\left(2,\alpha\right)}\left(3L^2\left(M'+6C_2^2\right)+\kappa_0^2C_2^2\right)h_n^2\exp\left(\frac{\left(3L^2+2\right)}{\beta_0\min\left(2,\alpha\right)}T\right),$$

for all $t\in\overline{I}$. Hence, in the left hand-side of (7.5), taking the upper bound with respect to t, we obtain

$$\left\|u^{(n)}-u\right\|_{L^2(I,V)}^{2}+\left\|u^{(n)}-u\right\|_{C\left(\overline{I},V\right)}^{2}$$
$$\leq \; \frac{T}{\beta_0 \min\left(2,\alpha\right)}\left(3L^2\left(M'+6C_2^2\right)+\kappa_0^2C_2^2\right)h_n^2\exp\left(\frac{\left(3L^2+2\right)}{\beta_0\min\left(2,\alpha\right)}T\right),$$

from which we get estimate (7.1), with

$$C_{12} = \sqrt{\frac{T\left(3L^2\left(M' + 6C_2^2\right) + \kappa_0^2 C_2^2\right)}{\beta_0 \min(2,\alpha)} h_n^2 \exp\left(\frac{(3L^2+2)}{\beta_0 \min(2,\alpha)} T\right)}.$$

∎

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