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BOUNDED SOLUTIONS OF UNILATERAL PROBLEMS FOR STRONGLY NONLINEAR EQUATIONS IN ORLICZ SPACES

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Abstract. In this paper, we prove the existence of bounded solutions of unilateral problems for strongly nonlinear equations whose principal part having a growth not necessarily of polynomial type and a degenerate coercivity, the lower order terms do not satisfy the sign condition and appropriate integrable source terms. We do not impose the Δ_2-condition on the considered N-functions defining the Orlicz-Sobolev functional framework.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, and let M be an N-function. In this paper, we establish the existence of bounded solutions for the unilateral problem related to strongly nonlinear equations of the form

$$Au + g(x, u, \nabla u) = f, \tag{1.1}$$

in the subset Ω. The principal part A is a non everywhere defined elliptic differential operator in divergence form

$$Au = -\operatorname{div} a(x, u, \nabla u) \tag{1.2}$$

defined from its domain $D(A) := \Big\{u \in W_0^1 L_M(\Omega) : a(x, u, \nabla u) \in (L_{\overline{M}}(\Omega))^N\Big\}$ into $W^{-1}L_{\overline{M}}(\Omega)$ satisfying, among others, the following condition

$$a(x, s, \xi) \cdot \xi \geq \overline{M}^{-1}(M(h(|s|))M(|\xi|), \tag{1.3}$$

where $h : \mathbb{R} \to \mathbb{R}^+$ is a continuous decreasing function with unbounded primitive (for instance $h(t) = \frac{1}{(e+|t|\log(e+|t|))}$ and $a(x, s, \xi) = \overline{M}^{-1}(M(h(|s|)))\frac{\overline{M}^{-1}(M(|\xi|))}{|\xi|}\xi)$. The Hamiltonian $g(x, u, \nabla u)$ does not satisfy the sign condition (i.e. $g(x, s, \xi)s \geq 0$) but only grows at most like $M(|\nabla u|)$, precisely

$$|g(x, s, \xi)| \leq \beta(s)M(|\xi|), \tag{1.4}$$

where $\beta : \mathbb{R} \to \mathbb{R}^+$ is a continuous function, while the source term have a suitable summability. Let us note that when h is a nonzero constant and g satisfies the sign condition, Dirichlet problems having lower order terms that behave like $M(|\nabla u|)$ arise naturally in the Calculus of Variations. For example, if we consider the functional

$$I(u) = \int_\Omega \left(a(x, u) \int_0^{|\nabla u|} \overline{M}^{-1}(M(t))dt\right)dx - \int_\Omega f(x)u(x)dx,$$

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the Euler-Lagrange equation is

$$-\text{div}\left(a(x,u)\frac{\overline{M}^{-1}(M(|\nabla u|))}{|\nabla u|}\nabla u\right)+a'(x,u)\int_0^{|\nabla u|}\overline{M}^{-1}(M(t))dt=f.$$

Let $\psi:\Omega\to\overline{\mathbb{R}}$ be a measurable function such that $\mathcal{K}_\psi=\{v\in W_0^1L_M(\Omega):v\geq\psi$ a.e. in $\Omega\}$ is a nonempty set. In fact, we are interested in the existence of bounded solution for the following obstacle problem

$$\begin{cases} u\in\mathcal{K}_\psi,\ a(\cdot,u,\nabla u)\in(L_{\overline{M}}(\Omega))^N,\ g(\cdot,u,\nabla u)\in L^1(\Omega),\\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla(u-v)dx+\int_\Omega g(x,u,\nabla u)(u-v)dx\\ \displaystyle\leq\int_\Omega f(u-v)dx,\quad\forall v\in\mathcal{K}_\psi\cap L^\infty(\Omega),\end{cases}\tag{1.5}$$

When $M(t)=t^p$, $1<p<+\infty$, and h in (1.3) is a nonzero constant, existence of bounded solution for problem (1.5) have been obtained, using direct method, in [6] with $f\equiv 0$ and in[8] for quasilinear operators without lower order terms (i.e. $\beta=0$) and data satisfying

$$f\in L^m(\Omega),\ m>\frac{N}{2}$$

and then under smallness a condition on the data f in [11] with

$$f\in L^m(\Omega),\ m>\max\left(1;\frac{N}{p}\right)\tag{1.6}$$

using symmetrization methods.

In the non standard growth setting, existence basic works for variational inequalities (i.e. where $f\in W^{-1}E_{\overline{M}}(\Omega)$) were initiated by Gossez and Mustonen in [12] solving the obstacle problem (1.5) in the case $g(x,u,\nabla u)=g(x,u)$ by assuming some regularity conditions on the obstacle function ψ. Since, several papers were written on existence of solutions for problem like (1.5) either in the variational case see, for instance, [3] or with L^1-data see, for instance, [2, 4, 9]. In this latter case, solution is understood as meaning a function u such that

$$\begin{cases} T_k(u)\in W_0^1L_M(\Omega)\cap D(A),\ u\geq\psi \text{ a.e. in } \Omega,\\ a(\cdot,u,\nabla u)\in(L_{\overline{M}}(\Omega))^N,\ g(\cdot,u,\nabla u)\in L^1(\Omega),\\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx+\int_\Omega g(x,u,\nabla u)T_k(u-v)dx\\ \displaystyle\leq\int_\Omega fT_k(u-v)dx,\quad\forall v\in\mathcal{K}_\psi\cap L^\infty(\Omega),\ \forall k>0.\end{cases}$$

$T_k(s)=\max\{-k,\min\{k,s\}\}$, $k>0$, is the truncation function defined on $\mathbb{R}$.

It is our purpose, in this paper, to prove the existence of bounded solutions, for unilateral problem associated to (1.1) in the setting of the Orlicz-Sobolev spaces without assuming the Δ_2-condition on the N-function M. To this end, we use rearrangement techniques and conditions (3.6) and (3.7), (see [18]), on the source term covering (1.6) in the case of polynomial growth.

It is worth recalling here some difficulties we have found in dealing with this kind of problems. First of all, the operator (1.2) does not satisfy the 'coercivity' condition in the setting of Orlicz spaces (see [12]), this is due to the hypothesis (1.3) and the

fact that no positive lower bound is assumed on the function h when the unknown has large values. The second difficulty in proving the existence of a solution stems from the fact that $g(x, u, \nabla u)$ does not define a mapping from $W_0^1L_M(\Omega)$ into $W^{-1}E_{\overline{M}}(\Omega)$, but from $W_0^1L_M(\Omega) \cap L^\infty(\Omega)$ into $L^1(\Omega)$. The third one concerns the lower order term; it does not satisfy the well known sign condition (i.e. $g(x, s, \xi)s \geq 0$) and so appears the problem of getting the a priori estimates. This hindrance is overcome by using test functions of exponential type, the monotone convergence theorem and a comparison result.
As examples of equations (1.1) to which our result can be applied, we give

$$-\text{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) + \frac{\log(1+|u|)}{(1+|u|)^p}|\nabla u|^p = f,$$

here $M(t) = t^p$, $p > 1$, $h(t) = \dfrac{1}{(1+|u|)^\theta}$, $0 \leq \theta < 1$ and

$$-\text{div}\left(h(u)\exp\Big(|\nabla u| + h(u)\Big)\nabla u\right) + \frac{\exp\Big(|\nabla u| + h(u)\Big)}{(e+|u|)^3\log(e+|u|)}|\nabla u|^2 = f,$$

here $M(t) = t^2\exp(t)$ and $h(t) = \dfrac{1}{(e+|u|)\log(e+|u|)}$.

The paper is organized as follows: in Section 2 we give some preliminaries and auxiliary results. Section 3 contains the basic assumptions and the main result, while Section 4 is devoted to the proof of the main result.

2. Preliminaries

2.1 Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, ie. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. The N-function conjugate to M is defined as $\overline{M}(t) = \sup\{st - M(t), s \geq 0\}$. We will extend these N-functions into even functions on all $\mathbb{R}$. We recall that (see [1])

$$M(t) \leq t\overline{M}^{-1}(M(t)) \leq 2M(t) \quad \text{for all } t \geq 0 \tag{2.1}$$

and the Young's inequality: for all $s, t \geq 0$, $st \leq \overline{M}(s) + M(t)$. If for some $k > 0$,

$$M(2t) \leq kM(t) \qquad \text{for all } t \geq 0, \tag{2.2}$$

we said that M satisfies the Δ_2-condition, and if (2.2) holds only for t greater than or equal to $t_0 \geq 0$, then M is said to satisfy the Δ_2-condition near infinity. Let P and Q be two N-functions, the notation $P \ll Q$ means that P grows essentially less rapidly than Q, that is to say for all $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. That is the case if and only if $\frac{Q^{-1}(t)}{P^{-1}(t)} \to 0$ as $t \to \infty$.
2.2 Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence class of) real-valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))dx < \infty \quad \left(\text{resp.} \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < \infty \text{ for some } \lambda > 0\right).$$

Endowed with the norm

$$\|u\|_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx < \infty\right\},$$

$L_M(\Omega)$ is a Banach space and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. We define the Orlicz norm $\|u\|_{(M)}$ by

$$\|u\|_{(M)} = \sup \int_\Omega u(x)v(x)dx,$$

where the supremum is taken over all $v \in E_{\overline{M}}(\Omega)$ such that $\|v\|_{\overline{M}} \leq 1$, for which

$$\|u\|_M \leq \|u\|_{(M)} \leq 2\|u\|_M$$

holds for all $u \in L_M(\Omega)$ (see [17]). The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

2.3 The Orlicz-Sobolev space $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the norm closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that a sequence $\{u_n\}$ converges to u for the modular convergence in $W^1L_M(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) dx \to 0 \ \text{ for all } |\alpha| \leq 1;$$

this implies the convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.
If M satisfies the Δ_2-condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then the modular convergence coincides with norm convergence. Recall that the norm $\|Du\|_M$ defined on $W_0^1L_M(\Omega)$ is equivalent to $\|u\|_{1,M}$ (see [15]).

Let $W^{-1}L_M(\Omega)$ (resp. $W^{-1}E_M(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the usual quotient norm. Recall that an open domain $\Omega \subset \mathbb{R}^N$ has the segment property (see [15] p.167) if there exist a locally finite open covering $\{O_i\}$ of the boundary $\partial\Omega$ of Ω and corresponding vectors $\{y_i\}$ such that if $x \in \overline{\Omega} \cap O_i$ for some i, then $x + ty_i \in \Omega$ for $0 < t < 1$. If the open Ω has the segment property then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (see [15]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.
For an exhaustive treatment one can see for example [1, 17].

2.4 We will use the following lemma, (see[10]), which concerns operators of Nemytskii Type in Orlicz spaces. It is slightly different from the analogous one given in [17].

Lemma 2.1. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. let M, P and Q be N-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e.$x \in \Omega$ and for all $s \in \mathbb{R}$,*

$$|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_f, defined by $N_f(u)(x) = f(x,u(x))$, is strongly continuous from $\mathcal{P}(E_M, \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

We will also use the following technical lemma which can be found in [16].

Lemma 2.2. *If $\{f_n\} \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in Ω, f_n, $f \geq 0$ a.e. in Ω and $\displaystyle\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$, then $f_n \to f$ in $L^1(\Omega)$.*

2.5 We recall the definition of decreasing rearrangement of a real-valued measurable function u in a measurable subset Ω of $\mathbb{R}^N$ having finite measure. Let $|E|$ stands for the Lebesgue measure of a subset E of Ω. The distribution function of u, denoted by μ_u, is a map which informs about the content of level sets of u, that is

$$\mu_u(t) = |\{x \in \Omega : \ |u(x)| > t\}|, \quad t \geq 0.$$

The decreasing rearrangement of u is defined as the generalized inverse function of μ_u, that is the function $u^* : [0, |\Omega|] \to [0, +\infty]$, defined as

$$u^*(s) = \inf\{t \geq 0 : \ \mu_u(t) \leq s\}, \quad s \in [0, |\Omega|].$$

In other words, u^* is the (unique) non-increasing, right-continuous function in $[0, +\infty)$ equi-distributed with u. Furthermore, for every $t \geq 0$

$$u^*(\mu_t(t)) \leq t. \tag{2.3}$$

We also recall that (see [18])

$$u^*(0) = \operatorname{ess\,sup} |u|. \tag{2.4}$$

3. Basic assumptions and Main result

Through this paper Ω will be a bounded open subset in of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property and M is an N-function twice continuously differentiable and strictly increasing, and P is an N-function such that $P \ll M$. Let us consider the following convex set

$$\mathcal{K}_\psi = \{v \in W_0^1 L_M(\Omega) : v \geq \psi \quad \text{a.e. in } \Omega\} \tag{3.1}$$

where $\psi : \Omega \to \overline{\mathbb{R}}$ is a measurable function. On the convex $\mathcal{K}_\psi$ we assume that

- (A_1) $\psi^+ \in W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$,
- (A_2) for each $v \in \mathcal{K}_\psi \cap L^\infty(\Omega)$, there exists a sequence $\{v_j\} \subset \mathcal{K}_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that $v_j \longrightarrow v$ for the modular convergence.

Let $A : D(A) \subset W_0^1 L_M(\Omega) \to W^{-1} L_{\overline{M}}(\Omega)$ be the mapping (non-everywhere defined) given by

$$Au = -\text{div } a(x, u, \nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N (\xi \neq \eta)$, the following conditions

(A_3)

$$a(x,s,\xi)\cdot\xi \geq \overline{M}^{-1}(M(h(|s|))M(|\xi|), \tag{3.2}$$

where $h:\mathbb{R}^+\to\mathbb{R}^*_+$ is a continuous decreasing function such that: $h(0)\leq 1$ and its primitive $H(s)=\displaystyle\int_0^s h(t)dt$ is unbounded,

(A_4) there exist a function $c(x)\in E_{\overline{M}}(\Omega)$ and some positive constants k_1, k_2, k_3 and k_4 such that

$$|a(x,s,\xi)|\leq c(x)+k_1\overline{P}^{-1}M(k_2|s|)+k_3\overline{M}^{-1}M(k_4|\xi|), \tag{3.3}$$

(A_5)

$$(a(x,s,\xi)-a(x,s,\eta))\cdot(\xi-\eta)>0. \tag{3.4}$$

Let $g:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ be a Carathéodory function satisfying

(A_6) for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$ and for almost every $x\in\Omega$,

$$|g(x,s,\xi)|\leq\beta(s)M(|\xi|), \tag{3.5}$$

where $\beta:\mathbb{R}\to\mathbb{R}^+$ is a continuous function. We assume that the function $t\to\dfrac{\beta(t)}{\overline{M}^{-1}(M(h(|t|)))}$ belongs to $L^1(\mathbb{R})$. So that defining

$$\gamma(s)=\int_0^s\frac{\beta(t)}{\overline{M}^{-1}(M(h(|t|)))}\,dt,$$

for all $s\in\mathbb{R}$, we have that the function γ is bounded. For what concerns the right hand, we assume one of the following two assumptions: Either

$$f\in L^N(\Omega), \tag{3.6}$$

or

$$\begin{cases} f\in L^m(\Omega) & \text{with } m=\dfrac{rN}{r+1} \text{ for some } r>0,\\ \text{and} & \displaystyle\int_{\cdot}^{+\infty}\left(\frac{t}{M(t)}\right)^r dt<+\infty. \end{cases} \tag{3.7}$$

Remark 3.1. *If Ω has the segment property, assumption (A_2) is fulfilled if one of the following conditions is verified:*

1)- *There exists $\overline{\psi}\in\mathcal{K}_\psi\cap W_0^1E_M(\Omega)$ such that $\psi-\overline{\psi}$ is continuous on Ω (see* [12, Proposition 9]*).*
2)- *$\psi\in W^1E_M(\Omega)$ (see* [12, Proposition 10]*).*
3)- *The N-function M satisfies the Δ_2-condition.*
4)- *$\psi=-\infty$. In this case $\mathcal{K}_\psi=W_0^1L_M(\Omega)$, then (A_2) is a consequence of* [14, Theorem 4].

Our main result is the following

Theorem 3.2. *Suppose that the assumptions (A_1)–(A_6) and either (3.6) or (3.7) are fulfilled. Then, the following obstacle problem*

$$\begin{cases} u \in \mathcal{K}_\psi \cap L^\infty(\Omega),\ a(\cdot,u,\nabla u) \in \Big(L_{\overline{M}}(\Omega)\Big)^N,\ g(\cdot,u,\nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla(u-v)dx + \int_\Omega g(x,u,\nabla u)(u-v)dx \\ \displaystyle\le \int_\Omega f(u-v)dx, \quad \forall v \in \mathcal{K}_\psi \cap L^\infty(\Omega). \end{cases} \tag{3.8}$$

has at least one solution.

Before giving the proof of the previous result, the following remarks are in order.

Remark 3.3. *Observe that, in (3.8), we can not replace $v \in \mathcal{K}_\psi \cap L^\infty(\Omega)$ by only $v \in \mathcal{K}_\psi$, since in general the two integrals $\int_\Omega g(x,u,\nabla u)(u-v)dx$ and $\int_\Omega f(u-v)dx$ may have no meaning.*

Remark 3.4. 1)- *It is known (see* [12]*) that $\mathcal{K}_\psi$ is sequentially $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closed in $W_0^1L_M(\Omega)$.*

2)- *Observe that $K \cap W_0^1E_M(\Omega)$ is $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ dense in $\mathcal{K}_\psi$. This follows from assumption (A_2) and from the fact that for all $u \in \mathcal{K}_\psi$ one has $T_n(u) \to u$ for the modular convergence in $W^1L_M(\Omega)$.*

3)- *The assumption (A_1) is not a restriction on the obstacle function ψ, instead of it we can assume that $\mathcal{K}_\psi \cap W_0^1E_M(\Omega) \cap L^\infty(\Omega)$ is a nonempty set.*

Remark 3.5. *In light of Remark 3.1 and Remark 3.4, if $\psi = -\infty$ then $\mathcal{K}_\psi = W_0^1L_M(\Omega)$ and problem (3.8) will be reduced to an equation. Hence, our result extends to inequalities the one in* [5] *stated for equations and also these in* [6, 7, 8].

Remark 3.6. *Let $M(t)$ be an N-function. Consider the following equation*

$$-\mathrm{div}\,\left(a(x,u)\frac{\overline{M}^{-1}(M(|\nabla u|))}{|\nabla u|}\nabla u\right) + \beta(u)M(|\nabla u|) = f \quad \text{in } \Omega,$$

where $\beta : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $\frac{\beta(t)}{\overline{M}^{-1}(M(h(t)))}$ belongs to $L^1(\mathbb{R})$ and $a(x,u)$ is a Carathéodory function such that $\overline{M}^{-1}(M(h(u))) \le a(x,u) \le \alpha$, the function h is as above. Then, the assumptions (3.2), (3.3), (3.4), (3.5) of Theorem 3.2 are fulfilled.

In what follows, we will use the following real functions of a real variable $T_k(s) = \max(-k, \min(k,s))$, $k > 0$, $G_k(s) = s - T_k(s)$ and $\phi_\lambda(s) = s\exp(\lambda s^2)$, where λ is a positive real number. The following classical lemma turns out to be useful later

Lemma 3.7. *If c and d are positive real numbers such that $\lambda = (\frac{c}{2d})^2$ then*

$$d\phi'_\lambda(s) - c|\phi_\lambda(s)| \ge \frac{d}{2}, \quad \forall s \in \mathbb{R}.$$

4. Proof of Main result

The proof of Theorem 3.2 is divided into eight steps.

Step 1: Approximate problems. For $n \in \mathbb{N}^*$, Let us denote by m^* either N or m according as we assume (3.6) or (3.7). Define $f_n := T_n(f)$, $A_n u := -\text{div}\, a(x, T_n(u), \nabla u)$ and $g_n(x,s,\xi) := T_n(g(x,s,\xi))$. We can easily check that we have $|g_n(x,s,\xi)| \le |g(x,s,\xi)|$ and $|g_n(x,s,\xi)| \le n$. Let us consider the sequence of approximate problems,

$$\begin{cases} u_n \in \mathcal{K}_\psi \cap \mathcal{D}(A_n), \\ \displaystyle\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(u_n - v)dx + \int_\Omega g_n(x, u_n, \nabla u_n)(u_n - v)dx \\ \le \displaystyle\int_\Omega f_n(u_n - v)dx, \quad \forall v \in \mathcal{K}_\psi. \end{cases} \tag{4.1}$$

Let $\nu > 1$ be large enough. By (3.4) one has

$$\begin{aligned} -a(x, T_n(s), \xi) \cdot \nabla\psi^+ \ge & -\frac{1}{\nu} a(x, T_n(s), \xi) \cdot \xi - a(x, T_n(s), \nu\nabla\psi^+) \cdot \nabla\psi^+ \\ & - \overline{M}^{-1}(M(h(|T_n(s)|))) \frac{\nu - 1}{2\nu} \frac{|a(x, T_n(s), \nu\nabla\psi^+)|}{\overline{M}^{-1}(M(h(|T_n(s)|)))\frac{\nu-1}{2}} |\xi|. \end{aligned}$$

Then, Young's inequality enables us to get

$$\begin{aligned} -a(x, T_n(s), \xi) \cdot \nabla\psi^+ \ge & -\frac{1}{\nu} a(x, T_n(s), \xi) \cdot \xi - a(x, T_n(s), \nu\nabla\psi^+) \cdot \nabla\psi^+ \\ & - \overline{M}^{-1}(M(h(|T_n(s)|))) \frac{\nu - 1}{2\nu} \overline{M}\Big(\frac{|a(x, T_n(s), \nu\nabla\psi^+)|}{\overline{M}^{-1}(M(h(|T_n(s)|)))\frac{\nu-1}{2}} \Big) \\ & - \overline{M}^{-1}(M(h(|T_n(s)|))) \frac{\nu - 1}{2\nu} M(|\xi|). \end{aligned}$$

Let us define the positive real number $\rho_n := \overline{M}^{-1}(M(h(n)))\dfrac{\nu - 1}{2\nu}$ and the function γ_n by

$$\begin{aligned} \gamma_n(x) \quad := \quad & a(x, T_n(s), \nu\nabla\psi^+) \cdot \nabla\psi^+ \\ & + \overline{M}^{-1}(M(h(0))) \frac{\nu - 1}{2\nu} \overline{M}\Big(\frac{|a(x, T_n(s), \nu\nabla\psi^+)|}{\overline{M}^{-1}(M(h(n)))\frac{\nu-1}{2}} \Big). \end{aligned}$$

For each n in $\mathbb{N}$, the function γ_n belongs to $L^1(\Omega)$. Thus we have

$$a(x, T_n(s), \xi) \cdot (\xi - \nabla\psi^+) \ge \rho_n M(|\xi|) - \gamma_n(x),$$

By [12, Proposition 5] the operator A_n satisfies the conditions of [12, Proposition 1] with respect to ψ^+. So that in view of the Remark 3.4, by [12, Proposition 1] the variational inequality (4.1) has at least a solution u_n.

Step 2: Preliminary results.

Lemma 4.1. *Let u_n be a solution of (4.1). For all t, ϵ in $\mathbb{R}_+^*$ with $t > \|\psi^+\|_\infty$, one has the following inequality:*

$$\begin{aligned} & \int_{\{t < u_n \le t + \varepsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n e^{\gamma(u_n^+)} dx \\ & \le \int_{\{u_n > t\}} f_n^+ e^{\gamma(u_n^+)} T_\varepsilon(G_{t - \|\psi^+\|_\infty}((u_n^+ - \|\psi^+\|_\infty)^+)) dx. \end{aligned} \tag{4.2}$$

Proof. Let ε, t, k in $\mathbb{R}_+^*$ with $t > \|\psi^+\|_\infty$. Define

$$v = u_n - \eta e^{\gamma(T_k(u_n^+))} T_\varepsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))$$

where $w_n = (u_n^+ - \|\psi^+\|_\infty)^+$ and $\eta = e^{-\gamma(k)}$. Thanks to [13, lemma 2], the function v belongs to $\mathcal{K}_\psi$. Thus, using v as test function in (4.1) and then (3.2) we get

$$\begin{aligned}
&\int_\Omega \beta(T_k(u_n^+)) M(|\nabla T_k(u_n^+)|) e^{\gamma(T_k(u_n^+))} T_\varepsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))dx \\
&+ \int_{\{t-\|\psi^+\|_\infty < T_k(w_n) \le t-\|\psi^+\|_\infty+\varepsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(w_n) e^{\gamma(T_k(u_n^+))} dx \\
&+ \int_\Omega g_n(x, u_n, \nabla u_n) e^{\gamma(T_k(u_n^+))} T_\varepsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))dx \\
&\le \int_\Omega f_n e^{\gamma(T_k(u_n^+))} T_\varepsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))dx.
\end{aligned} \tag{4.3}$$

Now, we will pass to the limit as k tends to $+\infty$ in (4.3). Observe that the second integral in the left-hand side of (4.3) reads as

$$\begin{aligned}
&\int_{\{t-\|\psi^+\|_\infty < T_k(w_n) \le t-\|\psi^+\|_\infty+\varepsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(w_n) e^{\gamma(T_k(u_n^+))} dx \\
&= \int_{\{t<u_n^+ \le t+\varepsilon\} \cap \{0<u_n^+ - \|\psi^+\|_\infty < k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n^+ e^{\gamma(T_k(u_n^+))} dx.
\end{aligned}$$

It follows by applying the monotone convergence theorem, that

$$\begin{aligned}
&\int_{\{t-\|\psi^+\|_\infty < T_k(w_n) \le t-\|\psi^+\|_\infty+\varepsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(w_n) e^{\gamma(T_k(u_n^+))} dx \\
&\to \int_{\{t<u_n^+ \le t+\varepsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n^+ e^{\gamma(u_n^+)} dx,
\end{aligned}$$

as $k \to +\infty$. In the first integral in the left-hand side of (4.3) the integrand function is nonnegative, so that Fatou's lemma allows us to get

$$\begin{aligned}
&\int_\Omega \beta(u_n^+) M(|\nabla u_n^+|) e^{\gamma(u_n^+)} T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))dx \\
&\le \liminf_{k\to\infty} \int_\Omega \beta(T_k(u_n^+)) M(|\nabla T_k(u_n^+)|) e^{\gamma(T_k(u_n^+))} T_\varepsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))dx,
\end{aligned}$$

while for the remaining terms in (4.3), being g_n and f_n bounded, we apply the Lebesgue's dominated convergence theorem. Consequently, letting k tends to $+\infty$ in (4.3) we obtain

$$\begin{aligned}
&\int_\Omega \beta(u_n^+) M(|\nabla u_n^+|) e^{\gamma(u_n^+)} T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))dx \\
&+ \int_{\{t<u_n^+ \le t+\varepsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n e^{\gamma(u_n^+)} dx \\
&+ \int_\Omega g_n(x, u_n, \nabla u_n) e^{\gamma(u_n^+)} T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))dx \\
&\le \int_\Omega f_n e^{\gamma(u_n^+)} T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))dx.
\end{aligned} \tag{4.4}$$

Due to the fact that $u_n^+ \geq \psi^+$, the function w_n vanishes if $u_n \leq 0$. By virtue of (3.5) we get

$$\begin{aligned}
&\int_\Omega \beta(u_n^+)M(|\nabla u_n^+|)e^{\gamma(u_n^+)}T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))dx \\
&+\int_\Omega g_n(x,u_n,\nabla u_n)e^{\gamma(u_n^+)}T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))dx \\
&=\int_\Omega \beta(u_n^+)M(|\nabla u_n^+|)e^{\gamma(u_n^+)}T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))dx \\
&+\int_{\{0<u_n\}} g_n(x,u_n,\nabla u_n)e^{\gamma(u_n^+)}T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))dx \ \geq 0.
\end{aligned}$$

Hence, (4.4) is reduced to

$$\begin{aligned}
&\int_{\{t<u_n^+\leq t+\varepsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx \\
&\leq \int_\Omega f_n e^{\gamma(u_n^+)}T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))dx.
\end{aligned}$$

Since $T_\varepsilon(G_{t-\|\psi^+\|_\infty}(w_n))$ is different from zero only on the subset

$$\{w_n > t-\|\psi^+\|_\infty\} = \{u_n^+ > t\}$$

and $f_n \leq f_n^+$ we finally have

$$\begin{aligned}
&\int_{\{t<u_n\leq t+\varepsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx \\
&\leq \int_{\{u_n>t\}} f_n^+ e^{\gamma(u_n^+)}T_\varepsilon(G_{t-\|\psi^+\|_\infty}((u_n^+-\|\psi^+\|_\infty)^+))dx.
\end{aligned}$$

□

Lemma 4.2. *Let u_n be a solution of (4.1). For all t, ϵ in $\mathbb{R}_+^*$, one has the following inequality:*

$$\begin{aligned}
&\int_{\{-t-\epsilon\leq u_n<-t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx \\
&\leq \int_{\{u_n<-t\}} f_n^- e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx.
\end{aligned} \tag{4.5}$$

Proof. For all $k>0$ the function $v = u_n + e^{\gamma(T_k(u_n^-))}T_\varepsilon(G_t(T_k(u_n^-)))$ belongs to $\mathcal{K}_\psi$. Thus, the choice of v as test function in (4.1), yields

$$\begin{aligned}
&-\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)e^{\gamma(T_k(u_n^-))}T_\varepsilon(G_t(T_k(u_n^-))) \\
&\times\frac{\beta(T_k(u_n^-))}{\overline{M}^{-1}(M(h(|T_k(u_n^-)|)))}dx \\
&-\int_{\{t<T_k(u_n^-)\leq t+\varepsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)e^{\gamma(T_k(u_n^-))}dx \\
&-\int_\Omega g_n(x,u_n,\nabla u_n)e^{\gamma(T_k(u_n^-))}T_\varepsilon(G_t(T_k(u_n^-)))dx \\
&\leq -\int_\Omega f_n e^{\gamma(T_k(u_n^-))}T_\varepsilon(G_t(T_k(u_n^-)))dx.
\end{aligned} \tag{4.6}$$

The first integral in the left-hand side of (4.6) is written as

$$
\begin{aligned}
&-\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)e^{\gamma(T_k(u_n^-))}T_\varepsilon(G_t(T_k(u_n^-)))\\
&\times\frac{\beta(T_k(u_n^-))}{\overline{M}^{-1}(M(h(|T_k(u_n^-)|)))}dx\\
&=\int_{\{-k<u_n\le 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}T_\varepsilon(G_t(u_n^-))\\
&\times\frac{\beta(u_n^-)}{\overline{M}^{-1}(M(h(|u_n^-|)))}dx.
\end{aligned}
$$

By the monotone convergence theorem, we have

$$
\begin{aligned}
&-\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)e^{\gamma(T_k(u_n^-))}T_\varepsilon(G_t(T_k(u_n^-)))\\
&\times\frac{\beta(T_k(u_n^-))}{\overline{M}^{-1}(M(h(|T_k(u_n^-)|)))}dx\\
&\to\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}T_\varepsilon(G_t(u_n^-))\frac{\beta(u_n^-)}{\overline{M}^{-1}(M(h(|u_n^-|)))}dx,
\end{aligned}
$$

as $k\to+\infty$. For the seconde integral in the left-hand side of (4.6), we write

$$
\begin{aligned}
&-\int_{\{t<T_k(u_n^-)\le t+\varepsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)e^{\gamma(T_k(u_n^-))}dx\\
&=\int_{\{t<T_k(u_n^-)\le t+\varepsilon\}\cap\{-k<u_n<0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx\\
&=\int_{\{-t-\varepsilon\le u_n<-t\}\cap\{-k<u_n<0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx.
\end{aligned}
$$

Applying again the monotone convergence theorem, we obtain

$$
\begin{aligned}
&-\int_{\{t<T_k(u_n^-)\le t+\varepsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)e^{\gamma(T_k(u_n^-))}dx\\
&\to\int_{\{-t-\varepsilon\le u_n<-t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx,
\end{aligned}
$$

as $k\to+\infty$. Since g_n, f_n and γ are bounded, we apply the Lebesgue's dominated convergence theorem for the remaining integrals in (4.6). Hence, letting k tend to $+\infty$ in (4.6), we get

$$
\begin{aligned}
&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}T_\varepsilon(G_t(u_n^-))\frac{\beta(u_n^-)}{\overline{M}^{-1}(M(h(|u_n^-|)))}dx\\
&\int_{\{-t-\varepsilon\le u_n<-t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx\\
&-\int_\Omega g_n(x,u_n,\nabla u_n)e^{\gamma(u_n^-)}T_\varepsilon(G_t(u_n^-))dx\\
&\le-\int_\Omega f_n e^{\gamma(u_n^-)}T_\varepsilon(G_t(u_n^-))dx.
\end{aligned}
$$

Since $u_n^- = |u_n|$ on the set $\{x \in \Omega : \ u_n(x) \le 0\}$, using (3.2) and (3.5) we obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n e^{\gamma(u_n^-)} T_\varepsilon(G_t(u_n^-)) \frac{\beta(u_n^-)}{\overline{M}^{-1}(M(h(|u_n^-|)))} dx$$
$$- \int_\Omega g_n(x, u_n, \nabla u_n) e^{\gamma(u_n^-)} T_\varepsilon(G_t(u_n^-)) dx \ge 0.$$

Observing that $-f_n \le f_n^-$ and $\{u_n^- > t\} \cap \{u_n \le 0\} = \{u_n < -t\}$, we have

$$\int_{\{-t-\varepsilon \le u_n < -t\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n e^{\gamma(u_n^-)} dx$$
$$\le \int_{\{u_n < -t\}} f_n^- e^{\gamma(u_n^-)} T_\varepsilon(G_t(u_n^-)) dx.$$

□

Lemma 4.3. *Let u_n be a solution of (4.1). There exists a constant c_0, not depending on n, such that for almost every $t > \|\psi^+\|_\infty$*

$$-\frac{d}{dt} \int_{\{|u_n| > t\}} \overline{M}^{-1}(M(h(|u_n|))) M(|\nabla u_n|) dx \le c_0 \int_{\{|u_n| > t\}} |f_n| dx. \tag{4.7}$$

Proof. Being γ bounded, summing up both inequalities (4.2) and (4.5), there is a constant c_0 not depending on n, such that for all $t > \|\psi^+\|_\infty$ and all $\epsilon > 0$

$$\int_{\{t < |u_n| \le t + \epsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n dx \le \epsilon c_0 \int_{\{|u_n| > t\}} |f_n| dx.$$

Using (3.2), dividing by ϵ and then letting ϵ tends to 0^+ we obtain (4.7). □

Inequality (4.7) allows us to obtain the following comparison result, proved in [5], which is the starting point to obtain uniform estimation in L^∞ for solutions of approximate equations (4.1).

Lemma 4.4. *Let $K(t) = \frac{M(t)}{t}$ and $\mu_n(t) = |\{x \in \Omega : |u_n(x)| > t\}|$, for all $t > 0$. We have for almost every $t > \|\psi^+\|_\infty$:*

$$h(t) \le \frac{2M(1)(-\mu_n'(t))}{\overline{M}^{-1}(M(1)) N C_N^{\frac{1}{N}} \mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0 \int_{\{|u_n| > t\}} |f_n| dx}{\overline{M}^{-1}(M(1)) N C_N^{\frac{1}{N}} \mu_n(t)^{1-\frac{1}{N}}} \right). \tag{4.8}$$

where C_N stands for the measure of the unit ball in $\mathbb{R}^N$ and c_0 is the constant which appears in (4.7).

Proof. The hypotheses made on the N-function M, allow to affirm that the function $C(t) = \dfrac{1}{K^{-1}(t)}$ is decreasing and convex (see [18]). Hence, Jensen's inequality yields

$$C\left(\frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}\right)$$

$$= C\left(\frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} K(|\nabla u_n|)\overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}\right)$$

$$\le \frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}$$

$$\le \frac{\overline{M}^{-1}(M(h(t)))(-\mu_n(t+k)+\mu_n(t))}{\overline{M}^{-1}(M(h(t+k)))\displaystyle\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|dx}.$$

Taking into account that $\overline{M}^{-1}(M(h(t))) \le \overline{M}^{-1}(M(1))$, using the convexity of C and then letting $k \to 0^+$, we obtain for almost every $t > \|\psi^+\|_\infty$

$$\frac{\overline{M}^{-1}(M(1))}{\overline{M}^{-1}(M(h(t)))}\, C\left(\frac{\displaystyle -\frac{d}{dt}\int_{\{|u_n|>t\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx}{\displaystyle \overline{M}^{-1}(M(1))(-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n|dx)}\right)$$
$$\le \frac{-\mu_n'(t)}{\displaystyle -\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n|dx}.$$

Recall the following inequality, (see for instance [18]):

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n|dx \ge N C_N^{\frac{1}{N}} \mu_n(t)^{1-\frac{1}{N}} \quad \text{for almost every } t > 0. \tag{4.9}$$

The monotonicity of the function C, (4.7) and (4.9) yield

$$\frac{1}{\overline{M}^{-1}(M(h(t)))}$$
$$\le \frac{-\mu_n'(t)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}}\, K^{-1}\left(\frac{\displaystyle c_0\int_{\{|u_n|>t\}} |f_n|dx}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}}\right).$$

Using (2.1) and the fact that $0 < h(t) \leq 1$, we obtain (4.8). $\square$

Step 3: Uniform L^∞-estimation. If we assume (3.6), by the Hölder's inequality one has

$$\int_{\{|u_n|>t\}} |f_n| dx \leq \|f\|_N \mu_n(t)^{1-\frac{1}{N}}.$$

Then for almost every $t > \|\psi^+\|_\infty$, inequality (4.8) becomes

$$h(t) \leq \frac{2M(1)(-\mu_n'(t))}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right).$$

Then, integrating between $\|\psi^+\|_\infty$ and s, we get

$$\int_{\|\psi^+\|_\infty}^s h(t)dt \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right)\int_{\|\psi^+\|_\infty}^s \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}}dt.$$

So that one has

$$\begin{aligned} H(s) \quad &\leq \int_0^{\|\psi^+\|_\infty} h(t)dt \\ &+ \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right)\int_{\|\psi^+\|_\infty}^s \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}}dt. \end{aligned}$$

Hence, a change of variables yields

$$H(s) \leq \|\psi^+\|_\infty + \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right)\int_{\mu_n(s)}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

By (2.3) we get

$$H(u_n^*(\sigma)) \leq \|\psi^+\|_\infty + \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right)\int_{\sigma}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

So that

$$H(u_n^*(0)) \leq \|\psi^+\|_\infty + \frac{2M(1)|\Omega|^{\frac{1}{N}}}{\overline{M}^{-1}(M(1))C_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right).$$

Thanks to (2.4) and the fact that $\lim_{s\to+\infty} H(s) = +\infty$, we conclude that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Moreover, if we denote by H^{-1} the inverse function of H, one has:

$$\|u_n\|_\infty \leq H^{-1}\left(\|\psi^+\|_\infty + \frac{2M(1)|\Omega|^{\frac{1}{N}}}{\overline{M}^{-1}(M(1))C_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right)\right). \tag{4.10}$$

We now assume that (3.7) is filled. Then, using again Hölder's inequality we have

$$\int_{\{|u_n|>t\}} |f_n| dx \leq \|f\|_m \mu_n(t)^{1-\frac{1}{m}}.$$

For almost every $t > \|\psi^+\|_\infty$, inequality (4.8) becomes

$$h(s) \le \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{\frac{1}{m}-\frac{1}{N}}} \right).$$

Integrating between $\|\psi^+\|_\infty$ and s, we get

$$H(s) \le \|\psi^+\|_\infty + \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} \int_{\|\psi^+\|_\infty}^{s} \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{\frac{1}{m}-\frac{1}{N}}} \right) dt.$$

Then, a change of variables gives

$$H(s) \le \|\psi^+\|_\infty + \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} \int_{\mu_n(s)}^{|\Omega|} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\sigma^{\frac{1}{m}-\frac{1}{N}}} \right) \frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

By virtue of (2.3) we get

$$H(u_n^*(\tau)) \le \|\psi^+\|_\infty + \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} \int_{\tau}^{|\Omega|} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\sigma^{\frac{1}{m}-\frac{1}{N}}} \right) \frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

Then, by (2.4) we obtain

$$H(\|u_n\|_\infty) \le \|\psi^+\|_\infty + \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} \int_{0}^{|\Omega|} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\sigma^{\frac{1}{m}-\frac{1}{N}}} \right) \frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

A change of variables gives

$$H(\|u_n\|_\infty) \le \|\psi^+\|_\infty + \frac{2M(1)c_0^r\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}} \int_{\lambda}^{+\infty} rt^{-r-1}K^{-1}(t)dt,$$

where $\lambda = \dfrac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}|\Omega|^{\frac{1}{rN}}}$. Then, by an integration by parts we obtain that

$$H(\|u_n\|_\infty) \le \|\psi^+\|_\infty + \frac{2M(1)c_0^r\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}} \left(\frac{K^{-1}(\lambda)}{\lambda^r} + \int_{K^{-1}(\lambda)}^{+\infty} \left(\frac{s}{M(s)}\right)^r ds \right).$$

The assumption made on H guarantees that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Indeed, denoting by H^{-1} the inverse function of H, one has

$$\|u_n\|_\infty \le H^{-1}\left(\|\psi^+\|_\infty + \frac{2M(1)c_0^r\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}} \left(\frac{K^{-1}(\lambda)}{\lambda^r} + \int_{K^{-1}(\lambda)}^{+\infty} \left(\frac{s}{M(s)}\right)^r ds \right) \right). \tag{4.11}$$

Consequently, in both cases the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$, so that in the sequel, we will denote by c_∞ the constant appearing either in (4.10) or in (4.11), that is :

$$\|u_n\|_\infty \le c_\infty. \tag{4.12}$$

Step 4: Estimation in $W_0^1 L_M(\Omega)$. It's easy to see that the function $v_n = u_n - \eta\phi_\lambda(u_n - \psi^+)$, where $\eta = e^{-\lambda(c_\infty + \|\psi^+\|_\infty)^2}$, belongs to $\mathcal{K}_\psi$ and can be used as test function in (4.1), giving

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla(u_n - \psi^+)\phi'_\lambda(u_n - \psi^+)dx \\ &+ \int_\Omega g_n(x, u_n, \nabla u_n)\phi_\lambda(u_n - \psi^+)dx \\ &\le \int_\Omega f_n\phi_\lambda(u_n - \psi^+)dx. \end{aligned} \tag{4.13}$$

Let now $\nu > 1$ be large enough. By (3.4) one has

$$\begin{aligned} &-a(x, u_n, \nabla u_n) \cdot \nabla\psi^+ \\ &\ge -\frac{1}{\nu}a(x, u_n, \nabla u_n) \cdot \nabla u_n - a(x, u_n, \nu\nabla\psi^+) \cdot \nabla\psi^+ \\ &-\overline{M}^{-1}(M(h(|u_n|)))\frac{\nu - 1}{2\nu}\frac{|a(x, u_n, \nu\nabla\psi^+)|}{\overline{M}^{-1}(M(h(|u_n|)))\frac{\nu-1}{2}}|\nabla u_n|. \end{aligned}$$

Then, Young's inequality enables us to get

$$\begin{aligned} &-a(x, u_n, \nabla u_n) \cdot \nabla\psi^+ \\ &\ge -\frac{1}{\nu}a(x, u_n, \nabla u_n) \cdot \nabla u_n - a(x, u_n, \nu\nabla\psi^+) \cdot \nabla\psi^+ \\ &-\overline{M}^{-1}(M(h(|u_n|)))\frac{\nu - 1}{2\nu}\overline{M}\Big(\frac{|a(x, u_n, \nu\nabla\psi^+)|}{\overline{M}^{-1}(M(h(|u_n|)))\frac{\nu-1}{2}}\Big) \\ &-\overline{M}^{-1}(M(h(|u_n|)))\frac{\nu - 1}{2\nu}M(|\nabla u_n|). \end{aligned}$$

Let us define the positive real number $\rho := \overline{M}^{-1}(M(h(c_\infty)))\dfrac{\nu - 1}{2\nu}$ and the function γ_n by

$$\gamma_n(x) := a(x, u_n, \nu\nabla\psi^+) \cdot \nabla\psi^+ + \overline{M}^{-1}(M(h(0)))\frac{\nu - 1}{2\nu}\overline{M}\left(\frac{|a(x, u_n, \nu\nabla\psi^+)|}{\overline{M}^{-1}(M(h(c_\infty)))\frac{\nu-1}{2}}\right).$$

It is clear to see that $\|\gamma_n\|_{L^1(\Omega)}$ is uniformly bounded in $L^1(\Omega)$, this stems from (4.12) and the fact that ψ^+ belongs to $W_0^1 E_M(\Omega)$. Therefore, we obtain

$$a(x, u_n, \nabla u_n) \cdot \nabla(u_n - \psi^+) \ge \rho M(|\nabla u_n|) - \gamma_n(x).$$

Being β continuous, thanks to (4.12) the sequence $\{\beta(u_n)\}$ is uniformly bounded. Thus, there exists a constant β_0 such that

$$\|\beta(u_n)\|_\infty \le \beta_0. \tag{4.14}$$

In view of (3.5), we can rewrite (4.13) as

$$\int_\Omega M(|\nabla u_n|)\Big[\rho\phi'_\lambda(u_n-\psi^+)-\beta_0|\phi_\lambda(u_n-\psi^+)|\Big]dx$$
$$\le \int_\Omega |f_n||\phi_\lambda(u_n-\psi^+)|dx+\int_\Omega \gamma_n\phi'_\lambda(u_n-\psi^+)dx.$$

Applying now Lemma 3.7 with $c=\beta_0$, $d=\rho$ and $\lambda=(\frac{\beta_0}{2\rho})^2$, we get

$$\begin{aligned}&\int_\Omega M(|\nabla u_n|)dx\\ &\le \frac{2}{\rho}\Big(\|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}}\phi_\lambda(c_\infty+\|\psi^+\|_\infty)+\|\gamma_n\|_{L^1(\Omega)}\phi'_\lambda(c_\infty+\|\psi^+\|_\infty)\Big),\end{aligned} \tag{4.15}$$

where m^* stands for either N or m according as we assume (3.6) or (3.7). It follows that the sequence $\{u_n\}$ is bounded in $W_0^1L_M(\Omega)$. Consequently, there exist a subsequence on $\{u_n\}$, still denote by $\{u_n\}$, and a function $u\in W_0^1L_M(\Omega)$ such that

$$u_n \rightharpoonup u \ \text{ weakly in } W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega),\Pi E_{\overline{M}}(\Omega)) \tag{4.16}$$

$$u_n\to u \ \text{ in } E_M(\Omega) \text{ strongly and a.e. in } \Omega. \tag{4.17}$$

Step 5: Almost every where convergence of the gradients. Let us begin by the following lemma which will be used in the sequel.

Lemma 4.5. *The sequence $\{a(x,T_n(u_n),\nabla u_n)\}_n$ is uniformly bounded in the space $(L_{\overline{M}}(\Omega))^N$.*

Proof. We will use the dual norm of $(L_{\overline{M}}(\Omega))^N$. Let $\varphi\in(E_M(\Omega))^N$ such that $||\varphi||_M=1$, by (3.4) we have

$$\Big(a(x,T_n(u_n),\nabla u_n)-a\Big(x,T_n(u_n),\frac{\varphi}{k_4}\Big)\Big)\cdot\Big(\nabla u_n-\frac{\varphi}{k_4}\Big)\ge 0.$$

Then we can write

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\varphi dx\le k_4\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n dx$$
$$-k_4\int_\Omega a\Big(x,T_n(u_n),\frac{\varphi}{k_4}\Big)\cdot\nabla u_n dx+\int_\Omega a\Big(x,T_n(u_n),\frac{\varphi}{k_4}\Big)\cdot\varphi dx.$$

Let $\lambda=1+k_1+k_3$. Using (3.3), (4.12) and the Young inequality, we obtain

$$\begin{aligned}&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\varphi dx\\ &\le k_4\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n dx+k_4\lambda\frac{c_\infty c_0\|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}}}{\overline{M}^{-1}M(h(c_\infty))}\\ &+(1+k_4)\int_\Omega\overline{M}(|a_0(x)|)dx+k_1(1+k_4)\overline{MP}^{-1}M(k_2c_\infty)|\Omega|+k_3(1+k_4)+\lambda.\end{aligned}$$

To end the proof it is sufficient to show that $\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n dx$ can be estimated by a constant which does not depend on n. To do this, let us use u_n as test function in (4.1) and then (3.5) obtaining

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n dx \le\le c_\infty\|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}} + c_\infty\beta_0\int_\Omega M(|\nabla u_n|)dx.$$

So that by (4.15) we get the desired result. □

From (4.12), (4.16) and (4.17) one deduces that $u\in\mathcal{K}_\psi\cap L^\infty(\Omega)$. So that by [14, Theorem 4], there exists a sequence $\{v_j\}$ in $\mathcal{D}(\Omega)$ such that $v_j\to u$ in $W_0^1L_M(\Omega)$, as $j\to\infty$, for the modular convergence and almost everywhere in Ω. Moreover

$$\|v_j\|_\infty \le (N+1)\|u\|_\infty.$$

Hence, we have $v_j\ge\psi$ a.e. in Ω. For $s>0$, we denote by χ_j^s the characteristic functions of the two subsets $\Omega_j^s=\{x\in\Omega : |\nabla v_j(x)|\le s\}$ and $\Omega^s=\{x\in\Omega : |\nabla u(x)|\le s\}$ respectively. Define $v=u_n-\eta\phi_\lambda(u_n-v_j)$ with $\eta=e^{-\lambda(N+2)^2c_\infty^2}$. It's clear that $v\in\mathcal{K}_\psi$. Let $n>c_\infty$. Using $v=u_n-\eta\phi_\lambda(u_n-v_j)$ as test function in (4.1) we get

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla(u_n-v_j)\phi'_\lambda(u_n-v_j)dx\\&+\int_\Omega g_n(x,u_n,\nabla u_n)\phi_\lambda(u_n-v_j)dx \le \int_\Omega f_n\phi_\lambda(u_n-v_j)dx.\end{aligned}\tag{4.18}$$

In what follows, $\epsilon_i(n,j)$ $(i=0,1,2,...)$ denote various sequences of real numbers which converge to 0 when n and $j\to\infty$, i.e.

$$\lim_{j\to\infty}\lim_{n\to\infty}\epsilon_i(n,j)=0.$$

In view of (4.12) and (4.17), we have

$$\phi_\lambda(u_n-v_j)\to\phi_\lambda(u-v_j)\text{ weakly in }L^\infty(\Omega)\text{ for }\sigma^*(L^\infty,L^1)\text{ as }n\to\infty.$$

So that one has

$$\int_\Omega f_n\phi_\lambda(u_n-v_j)dx\to\int_\Omega f\phi_\lambda(u-v_j)dx\text{ as }n\to\infty.$$

Since $u-v_j\to 0$ weakly in $L^\infty(\Omega)$ for $\sigma^*(L^\infty,L^1)$ as $j\to\infty$, we obtain $\int_\Omega f\phi_\lambda(u-v_j)dx\to 0$ as $j\to\infty$. Hence,

$$\int_\Omega f_n\phi_\lambda(u_n-v_j)dx=\varepsilon_0(n,j).$$

For the first term in the left-hand side of (4.18), we write

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla(u_n-v_j)\phi'_\lambda(u_n-v_j)dx\\&=\int_\Omega\left(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\right)\cdot\left(\nabla u_n-\nabla v_j\chi_j^s\right)\phi'_\lambda(u_n-v_j)dx\\&+\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot(\nabla u_n-\nabla v_j\chi_j^s)\phi'_\lambda(u_n-v_j)dx\\&-\int_{\Omega\setminus\Omega_j^s}a(x,u_n,\nabla u_n)\cdot\nabla v_j\phi'_\lambda(u_n-v_j)dx.\end{aligned}$$

As a consequence of Lemma 4.5, there exists $l \in (L_M(\Omega))^N$ such that

$$a(x, u_n, \nabla u_n) \rightharpoonup l \quad \text{weakly in } (L_M(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M).$$

Since $\nabla v_j \chi_{\Omega \setminus \Omega_j^s} \in (E_M(\Omega))^N$, we have

$$\int_{\Omega \setminus \Omega_j^s} a(x, u_n, \nabla u_n) \cdot \nabla v_j \phi'_\lambda(u_n - v_j)dx \ \rightarrow \ \int_{\Omega \setminus \Omega_j^s} l \cdot \nabla v_j \phi'_\lambda(u - v_j)dx$$

as $n \to \infty$, then the modular convergence of $\{v_j\}$ gives

$$\int_{\Omega \setminus \Omega_j^s} l \cdot \nabla v_j \phi'_\lambda(u - v_j)dx \rightarrow \int_{\Omega \setminus \Omega^s} l \cdot \nabla u dx$$

as $j \to \infty$. So that

$$\int_{\Omega \setminus \Omega_j^s} a(x, u_n, \nabla u_n) \cdot \nabla v_j \phi'_\lambda(u_n - v_j)dx = \int_{\Omega \setminus \Omega^s} l \cdot \nabla u dx + \epsilon_1(n, j).$$

Since $a(x, u_n, \nabla v_j \chi_{\Omega \setminus \Omega_j^s})\phi'_\lambda(u_n - v_j) \ \rightarrow \ a(x, u, \nabla v_j \chi_{\Omega \setminus \Omega_j^s})\phi'_\lambda(u - v_j)$ strongly in $(E_{\overline{M}}(\Omega))^N$ as $n \to \infty$ by Lemma 2.1 and $\nabla u_n \rightarrow \nabla u$ weakly in $(L_M(\Omega))^N$ by (4.16), we obtain

$$\begin{aligned}&\int_\Omega a(x, u_n, \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s)\phi'_\lambda(u_n - v_j)dx \\ &\rightarrow \int_\Omega a(x, u, \nabla v_j \chi_j^s) \cdot (\nabla u - \nabla v_j \chi_j^s)\phi'_\lambda(u - v_j)dx\end{aligned}$$

as $n \to \infty$, and since $\nabla v_j \chi_j^s \ \rightarrow \ \nabla u \chi^s$ strongly in $(E_M(\Omega))^N$ as $j \to \infty$, we get

$$\int_\Omega a(x, u, \nabla v_j \chi_j^s) \cdot (\nabla u - \nabla v_j \chi_j^s)\phi'_\lambda(u - v_j)dx \ \rightarrow \ 0$$

as $j \to \infty$. Thus,

$$\int_\Omega a(x, u_n, \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s)\phi'_\lambda(u_n - v_j)dx \ = \ \epsilon_2(n, j).$$

Hence, (4.18) becomes

$$\begin{aligned}&\int_\Omega \left(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla v_j \chi_j^s)\right) \cdot \left(\nabla u_n - \nabla v_j \chi_j^s\right) \phi'_\lambda(u_n - v_j)dx \\ &+ \int_\Omega g_n(x, u_n, \nabla u_n)\phi_\lambda(u_n - v_j)dx \ = \ \int_{\Omega \setminus \Omega^s} l \cdot \nabla u dx \ + \ \epsilon_3(n, j).\end{aligned} \tag{4.19}$$

Let $\sigma_0 = \frac{\beta_0}{\overline{M}^{-1} M(h(c_\infty))}$ where c_∞ is the constant in (4.12). We now turn to evaluate the second term in the left-hand side of (4.19). Using (3.5), (3.2) and (4.12), we have

$$
\begin{aligned}
&\left|\int_\Omega g_n(x,u_n,\nabla u_n)\phi_\lambda(u_n-v_j)dx\right| \\
&\le \int_\Omega \beta(u_n)M(|\nabla u_n|)|\phi_\lambda(u_n-v_j)|dx \\
&\le \int_\Omega \frac{\beta(u_n)}{\overline{M}^{-1}M(h(|u_n|))}a(x,u_n,\nabla u_n)\cdot\nabla u_n|\phi_\lambda(u_n-v_j)|dx \\
&\le \sigma_0\int_\Omega \left(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\right)\cdot\left(\nabla u_n-\nabla v_j\chi_j^s\right)|\phi_\lambda(u_n-v_j)|dx \\
&+\sigma_0\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot(\nabla u_n-\nabla v_j\chi_j^s)|\phi_\lambda(u_n-v_j)|dx \\
&+\sigma_0\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v_j\chi_j^s|\phi_\lambda(u_n-v_j)|dx.
\end{aligned}
$$

In a similar way as above, we have

$$
\begin{aligned}
\sigma_0\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot(\nabla u_n-\nabla v_j\chi_j^s)|\phi_\lambda(u_n-v_j)|dx &= \epsilon_4(n,j), \\
\sigma_0\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v_j\chi_j^s|\phi_\lambda(u_n-v_j)|dx &= \epsilon_5(n,j).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left|\int_\Omega g_n(x,u_n,\nabla u_n)\phi_\lambda(u_n-v_j)dx\right| \\
&\le \sigma_0\int_\Omega \Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\Big)\cdot\Big(\nabla u_n-\nabla v_j\chi_j^s\Big)|\phi_\lambda(u_n-v_j)|dx \\
&+\epsilon_6(n,j).
\end{aligned}
$$

This inequality enables us to write (4.19) under the forme

$$
\begin{aligned}
&\int_\Omega \Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\Big)\cdot\Big(\nabla u_n-\nabla v_j\chi_j^s\Big) \\
&\times\Big(\phi'_\lambda(u_n-v_j)-\sigma_0|\phi_\lambda(u_n-v_j)|\Big)dx \\
&\le \int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx \ + \ \epsilon_7(n,j).
\end{aligned}
$$

Applying now Lemma 3.7 with $c=\sigma_0$, $d=1$ and $\lambda=(\frac{\sigma_0}{2})^2$, we get

$$
\begin{aligned}
&\int_\Omega \Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\Big)\cdot\Big(\nabla u_n-\nabla v_j\chi_j^s\Big)dx \\
&\le \ 2\int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx \ + \ 2\epsilon_7(n,j).
\end{aligned}
\tag{4.20}
$$

On the other hand

$$\begin{aligned}
&\int_\Omega \Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u\chi^s)\Big)\cdot\Big(\nabla u_n-\nabla u\chi^s\Big)dx\\
&=\int_\Omega \Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\Big)\cdot\Big(\nabla u_n-\nabla v_j\chi_j^s\Big)dx\\
&+\int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla v_j\chi_j^s-\nabla u\chi^s)dx\\
&-\int_\Omega a(x,u_n,\nabla u\chi^s)\cdot(\nabla u_n-\nabla u\chi^s)dx\\
&+\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot(\nabla u_n-\nabla v_j\chi_j^s)dx.
\end{aligned}$$

Similarly as above we have

$$\begin{aligned}
\int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla v_j\chi_j^s-\nabla u\chi^s)dx &= \epsilon_8(n,j),\\
\int_\Omega a(x,u_n,\nabla u\chi^s)\cdot(\nabla u_n-\nabla u\chi^s)dx &= \epsilon_9(n,j),\\
\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot(\nabla u_n-\nabla v_j\chi_j^s)dx &= \epsilon_{10}(n,j).
\end{aligned}\tag{4.21}$$

These estimates together with inequality (4.20) allow us to get

$$\begin{aligned}
&\int_\Omega \Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u\chi^s)\Big)\cdot\Big(\nabla u_n-\nabla u\chi^s\Big)dx\\
&\le 2\int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx+\epsilon_{11}(n,j).
\end{aligned}$$

Let now $r\le s$, we write

$$\begin{aligned}
0 &\le \int_{\Omega^r}\Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)\Big)\cdot\Big(\nabla u_n-\nabla u\Big)dx\\
&\le \int_{\Omega^s}\Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)\Big)\cdot\Big(\nabla u_n-\nabla u\Big)dx\\
&= \int_{\Omega^s}\Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u\chi^s)\Big)\cdot\Big(\nabla u_n-\nabla u\chi^s\Big)dx\\
&\le \int_{\Omega}\Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u\chi^s)\Big)\cdot\Big(\nabla u_n-\nabla u\chi^s\Big)dx\\
&\le 2\int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx+\epsilon_{11}(n,j).
\end{aligned}$$

Since $l\cdot\nabla u\in L^1(\Omega)$, letting $s\to\infty$, we get

$$\int_{\Omega^r}\Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)\Big)\cdot\Big(\nabla u_n-\nabla u\Big)dx\to 0 \tag{4.22}$$

as $n\to\infty$. Let D_n be defined by

$$D_n=\Big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)\Big)\cdot\Big(\nabla u_n-\nabla u\Big).$$

As a consequence of (4.22), one has $D_n\to 0$ strongly in $L^1(\Omega^r)$, extracting a subsequence, still denoted by $\{u_n\}$, we get

$$D_n \quad\to\quad 0 \quad \text{a.e. in } \Omega^r.$$

Then, there exists a subset Z of Ω^r, of zero measure, such that: $D_n \to 0$ for all $x \in \Omega^r \setminus Z$. Fixe $x \in \Omega^r \setminus Z$, we obtain by using (3.2) and (3.3)

$$\begin{aligned} D_n(x) \quad \geq \quad & \overline{M}^{-1} M(h(c_\infty)) M(|\nabla u_n(x)|) \\ & - c_1(x)\Big(1 + \overline{M}^{-1} M(k_4|\nabla u_n(x)|) + |\nabla u_n(x)|\Big) \end{aligned}$$

where c_∞ is the constant which appears in (4.12) and $c_1(x)$ is a constant which does not depend on n. Thus, the sequence $\{\nabla u_n(x)\}$ is bounded in $\mathbb{R}^N$, then for a sequence $\{u_{n'}(x)\}$, we have

$$\nabla u_{n'}(x) \to \xi \quad \text{ in } \mathbb{R}^N,$$

and

$$(a(x, u(x), \xi) - a(x, u(x), \nabla u(x))) \cdot (\xi - \nabla u(x)) = 0.$$

Since $a(x, s, \xi)$ is strictly monotone, we have $\xi = \nabla u(x)$ and then $\nabla u_n(x) \to \nabla u(x)$ for the whole sequence. It follows that

$$\nabla u_n \to \nabla u \quad \text{ a.e. in } \Omega^r.$$

Consequently, as r is arbitrary, one can deduce that

$$\nabla u_n \to \nabla u \quad \text{ a.e. in } \Omega. \tag{4.23}$$

It follows from Lemma 4.5, (4.17), (4.23) and [17, Theorem 14.6] that

$$a(x, T_n(u_n), \nabla u_n) \rightharpoonup a(x, u, \nabla u) \in (L_{\overline{M}}(\Omega))^N \text{ weakly for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{4.24}$$

Step 6: Modular convergence of the gradients. Let $n > c_\infty$. Going back to (4.20) we can write

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \quad \leq \quad & \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v_j \chi_j^s dx \\ & + \int_\Omega a(x, u_n, \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s) dx \\ & + 2 \int_{\Omega \setminus \Omega^s} a(x, u, \nabla u) \cdot \nabla u dx \; + \; 2\epsilon_7(n, j). \end{aligned}$$

Then by (4.21), we get

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \quad \leq \quad & \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v_j \chi_j^s dx \\ & + 2 \int_{\Omega \setminus \Omega^s} a(x, u, \nabla u) \cdot \nabla u dx \; + \; \epsilon_{12}(n, j). \end{aligned}$$

Passing to the limit superior over n and then to the limit over j in both sides of this inequality, we obtain

$$\begin{aligned} \limsup_{n \to \infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \quad \leq \quad & \int_\Omega a(x, u, \nabla u) \cdot \nabla u \chi^s dx \\ & + 2 \int_{\Omega \setminus \Omega^s} a(x, u, \nabla u) \cdot \nabla u dx, \end{aligned}$$

Then by letting $s \to \infty$, one has

$$\limsup_{n\to\infty} \int_\Omega a(x,u_n,\nabla u_n)\cdot \nabla u_n dx \quad \leq \quad \int_\Omega a(x,u,\nabla u)\cdot \nabla u dx.$$

Finally, Fatou's Lemma allows us to get

$$\lim_{n\to\infty} \int_\Omega a(x,u_n,\nabla u_n)\cdot \nabla u_n dx = \int_\Omega a(x,u,\nabla u)\cdot \nabla u dx.$$

Hence, by Lemma 2.2, we deduce that

$$a(x,u_n,\nabla u_n)\cdot \nabla u_n \quad \to \quad a(x,u,\nabla u)\cdot \nabla u \quad \text{strongly in } L^1(\Omega). \tag{4.25}$$

By (3.2) and (4.12) and the convexity of the N-function M, we can write

$$\begin{aligned} M\left(\frac{|\nabla u_n - \nabla u|}{2}\right) \quad &\leq \quad \frac{1}{2\overline{M}^{-1}(M(h(c_\infty)))}\overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|) \\ &\quad + \frac{1}{2\overline{M}^{-1}(M(h(c_\infty)))}\overline{M}^{-1}(M(h(|u|)))M(|\nabla u|) \\ &\leq \quad \frac{1}{2\overline{M}^{-1}(M(h(c_\infty)))}\, a(x,u_n,\nabla u_n)\cdot \nabla u_n \\ &\quad + \frac{1}{2\overline{M}^{-1}(M(h(c_\infty)))}\, a(x,u,\nabla u)\cdot \nabla u. \end{aligned}$$

Then, by (4.25) and Vitali's theorem we conclude that

$$u_n \to u \quad \text{in } W_0^1 L_M(\Omega) \text{ for the modular convergence.}$$

Step 7: Compactness of the nonlinearities. We need to show that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega). \tag{4.26}$$

To this end, we use Vitali's theorem. Thanks to (4.17) and (4.23), one has

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{a.e. in } \Omega.$$

It remains to show that the the sequence $\{g(x,u_n,\nabla u_n)\}$ is uniformly equi-integrable. By (3.5) and (4.14), we have

$$|g_n(x,u_n,\nabla u_n)| \leq \beta_0 M(|\nabla u_n|).$$

Let E be a measurable subset of Ω. Thanks to (3.2), (4.12) and (4.14), we have

$$\int_E |g_n(x,u_n,\nabla u_n)| dx \leq \frac{\beta_0}{\overline{M}^{-1} M(h(c_\infty))} \int_E |a(x,u_n,\nabla u_n)\cdot \nabla u_n| dx.$$

Thus, the equi-integrability of the sequence $\{g(x,u_n,\nabla u_n)\}$ follows from (4.25) and Vitali's theorem. So that (4.26) is proved.

Step 8: Passing to the limit. Let $v \in \mathcal{K}_\psi \cap L^\infty(\Omega)$. By (A_2) there is a sequence $\{v_j\} \subset \mathcal{K}_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that $v_j \longrightarrow v$ for the modular convergence in $W_0^1 L_M(\Omega)$. For all $n > c_\infty$, inserting v_j as a test function in (4.1) yields

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla(u_n - v_j) dx + \int_\Omega g_n(x, u_n, \nabla u_n)(u_n - v_j) dx \\ \leq \int_\Omega f_n(u_n - v_j) dx.$$

Since $\nabla v_j \in (E_M(\Omega))^N$, by (4.24) one has

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v_j dx \to \int_\Omega a(x, u, \nabla u) \cdot \nabla v_j dx$$

as $n \to \infty$. So that by (4.25) we get

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla(u_n - v_j) dx \to \int_\Omega a(x, u, \nabla u) \cdot \nabla(u - v_j) dx.$$

Using (4.12) and (4.26), we can pass to the limit as $n \to +\infty$ obtaining

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla(u - v_j) dx + \int_\Omega g(x, u, \nabla u)(u - v_j) dx \leq \int_\Omega f(u - v_j) dx.$$

As we have, up to a subsequence still indexed by j, $v_j \to v$ *a.e.* in Ω and weakly for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$, we can pass to the limit as $j \to \infty$ to obtain

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla(u - v) dx + \int_\Omega g(x, u, \nabla u)(u - v) dx \leq \int_\Omega f(u - v) dx.$$

By virtue of (4.17) we have $u \in \mathcal{K}_\psi \cap L^\infty(\Omega)$. This completes the proof of Theorem 3.2.

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