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A fixed point theorem for multivalued mappings

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Abstract

A generalization of the Leray-Schauder principle for multivalued mappings is given. Using this result, an existence theorem for an integral inclusion is obtained.

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1. Introduction

The Schauder Fixed Point Theorem is, undoubtedly, one of the most important theorems of nonlinear analysis.

Theorem 1.1 *(Schauder) Let M be a nonempty closed bounded convex subset of a Banach space X. Suppose that $F : M \to M$ is a continuous operator and $F(M)$ is a relatively compact set in X. Then F admits fixed points.*

By using this theorem one can prove the following result due to Leray-Schauder (see [6], p. 245).

Theorem 1.2 *(Leray-Schauder) Let X be a Banach space and $F : X \to X$ an operator. Suppose that:*

(i) F is a continuous operator which maps every bounded subset of X into a relatively compact set;

(ii) (a priori estimate) there exists an $r > 0$ such that if $x = \lambda F(x)$, with $\lambda \in (0,1)$, then $\|x\| \le r$.

Then F has fixed points.

If we set

$$A := \{x \in X,\ (\exists)\ \lambda \in (0,1),\ x = \lambda F(x)\},$$

then hypothesis (ii) can be written under an alternative form, i.e.

(ii)$_a$ *either the set* A *is unbounded, or the equation* $x = \lambda F(x)$ *has solutions for* $\lambda = 1$.

Having this statement, the Leray-Schauder Theorem has been generalized in the case of locally convex spaces by H. Schaefer (see [5]).

The Schauder's Theorem has been extended in different ways and directions. One of these directions is the one when instead of a mapping one considers a multivalued mapping F. One of the most representative theorems for this direction is the Bohnenblust-Karlin Theorem (see [6], p. 452).

Theorem 1.3 *(Bohnenblust-Karlin) Let* M *be a closed and convex subset of the Banach space* X *and* $F : M \to \mathcal{P}(M)$ *a multivalued mapping. Suppose that:*

(i) the set $F(M)$ *is relatively compact;*

(ii) the multivalued mapping F *is upper semi-continuous on* M;

(iii) the set $F(x)$ *is nonempty closed and convex for all* $x \in M$.

Then there is $x \in M$ *such that* $x \in F(x)$.

In the present Note we shall give a generalization of Theorem 1.3 in the sense of Theorem 1.2. Also we shall give an application in the case of an integral inclusion.

2. Preliminaries

In what follows we shall enumerate some classical notions and results regarding the multivalued mappings. Although many of these are available in a more general framework, we shall mention them only in the form we need in the present Note.

Let $(X, \|\cdot\|)$ be a Banach space and $M \subset X$; set

$$\mathcal{P}(M) := \{N,\ N \subset M,\ N \neq \emptyset\}.$$

We call **multivalued mapping** (or **multi-function**) defined on M every application $F : M \to \mathcal{P}(X)$; denote

$$\begin{aligned} F(M) &:= \bigcup_{x \in M} F(x),\ F^{-1}(M) := \{x \in M,\ F(x) \cap M \neq \emptyset\}, \\ B(a,r) &:= \{x \in X,\ \|x-a\| < r\},\ B[a,r] := \{x \in X,\ \|x-a\| \leq r\}. \end{aligned}$$

We call $F : M \to \mathcal{P}(M)$ **upper semi-continuous in** x_0 (in brief u.s.c.) if for all U open subset of X, with $F(x_0) \subset U$, there exists $\eta > 0$ such that for all $x \in B(x_0, \eta)$ we have $F(x) \subset U$.

We call F **u.s.c. on** M if it is u.s.c. in each point of M. In particular, $F : M \to \mathcal{P}(X)$ is upper semicontinuous on M if and only if for each closed subset $N \subset X$, $F^{-1}(N)$ is closed in M.

Another important category of multivalued mappings is the closed multivalued mapping. We call the multivalued mapping $F : M \to \mathcal{P}(X)$ **closed on** M if for every $x_0 \in M$ and for every sequence $(x_n)_n \subset M$, with $x_n \to x_0$ and for every sequence $(y_n)_n \subset F(x_n)$, with $y_n \to y_0$, one has $y_0 \in F(x_0)$.

If F is closed on M, then for every $x \in M$, $F(x)$ is a closed subset of X.

If F is u.s.c. on M and $F(x)$ is closed and bounded for all $x \in M$, then F is closed on M. The converse is not true. But, if $F : M \to \mathcal{P}(X)$ is closed and $F(M)$ is relatively compact, then F is u.s.c. on M.

We call $F : X \to \mathcal{P}(X)$ **compact** if for every M bounded subset of X, $F(M)$ is relatively compact.

From the above definitions it follows that if $F : X \to \mathcal{P}(X)$ is a compact and closed operator, then F is u.s.c. on X. Indeed, for every $x \in X$, there exists $r > 0$ such that $x \in B[0, r]$ and, since F is closed on $B[0, r]$, it follows that F is u.s.c. on $B[0, r]$. We remark that by the hypotheses made, it follows that $F(x)$ is compact for every x.

3. Main result

Consider the operator $F : X \to \mathcal{P}(X)$, where $(X, \|\cdot\|)$ is a Banach space. Set

$$A = \{x \in X, \ (\exists) \lambda \in (0, 1), \ x \in \lambda F(x)\}$$

and

$$B_r = B[0, r].$$

Theorem 3.1 *Suppose that:*

(a) for all $x \in X$, $F(x)$ is a closed and convex set;

(b) F is u.s.c. on X;

(c) $F : X \to \mathcal{P}(X)$ is a compact multivalued mapping.

Then either the set A is unbounded or there exists $x \in X$ such that $x \in F(x)$.

Proof. Suppose that the set A is bounded. The set $F(A)$ being relatively compact, it will be also bounded; it follows that there exists $r > 0$ such that

$$F(A) \subset B_r. \tag{3.1}$$

Set

$$B := B_{2r},\ K := \sup_{y \in F(B_{2r})} \{\|y\|\},\ k := \max\{K, 2r+1\}.$$

Define a multivalued mapping $G : B \to \mathcal{P}(X)$ through

$$G(x) = \begin{cases} F(x) \cap B, \text{ if } F(x) \cap B \neq \emptyset \\ \frac{2r}{k} F(x), \text{ if } F(x) \cap B = \emptyset \end{cases}. \tag{3.2}$$

We shall prove that G fulfills the hypotheses of Theorem 1.3.

Step 1. $G(B) \subset B$. In the first case of (3.2), this inclusion is immediate. In the second case of (3.2), the same inclusion follows by the fact that for every $y \in F(x)$ one has $\frac{1}{k}\|y\| \leq 1$.

Step 2. $G(x)$ is a convex set. This is an immediate consequence of the fact that $F(x)$ is convex.

Step 3. G is closed multivalued mapping on B. So, let $x \in B$ and $(x_n)_n \subset B$ such that $x_n \to x$. Let $y_n \in G(x_n)$ such that $y_n \to y$. We consider two cases, namely there is

$$\text{a subsequence } (y_{n_k})_{n_k} \text{ of } (y_n)_n \text{ such that } \|y_{n_k}\| \leq 2r \tag{3.3}$$

and

$$\text{a subsequence } (y_{n_p})_{n_p} \text{ of } (y_n)_n \text{ such that } \|y_{n_p}\| > 2r. \tag{3.4}$$

In the case (3.3) we have $y_{n_k} \in F(x_{n_k}) \cap B = G(x_{n_k})$ and, consequently, $y \in F(x) \cap B$, since F is closed.

In the case (3.4) we have $y_{n_p} \in \frac{2r}{k} F(x_{n_p})$, i.e. $\frac{k}{2r} y_{n_p} \in F(x_{n_p})$ and hence $y \in \frac{2r}{k} F(x) = G(x)$.

Step 4. $G(x)$ is closed for all $x \in B$. This assertion follows from Step 3, by setting $x_n \equiv x$.

Step 5. $G(B)$ is relatively compact. This assertion is proved by using the reason from Step 3 and hypothesis (c).

The results contained in Steps 3 and 5 allow us to conclude that G is u.s.c. on B.

By applying Theorem 1.3 to operator G, it follows that

$$(\exists)\, x \in B,\, x \in G(x). \tag{3.5}$$

Two cases are possible. If $F(x) \cap B \neq \emptyset$, then $x \in F(x)$ and the proof is complete.

The case $F(x) \cap B = \emptyset$ is impossible. Indeed, in this case we have

$$(x \in G(x)) \Longrightarrow \left(x \in \frac{2r}{k} F(x)\right)$$

and so,

$$x = \frac{2r}{k} y,\ y \in F(x),\ \ \|y\| > 2r. \tag{3.6}$$

But, since $2r/k < 2r/(2r+1) < 1$, it follows that $x \in A$, and so $\|y\| \leq r$, taking into account (3.1), which contradicts (3.6). The proof is complete. □

4. An existence result for an integral inclusion

We start by recalling certain things related to the theory of multivalued integrals. We shall refer only to the concrete case in which we shall work, although the notions and the properties are available in a more general framework.

Let $\Phi : J \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued mapping with the property that $\Phi(t)$ is a closed set for every t, where $J = [0, T]$.

We call Φ **measurable** if for every closed set M, $\Phi^{-1}(M)$ is measurable.

We call Φ **integrable in Aumann's sense** if there exists an integral selection for Φ, that is, if there exists $\varphi \in L^1(J, \mathbb{R}^n)$ with $\varphi(t) \in \Phi(t)$ for a.e. $t \in J$; we set

$$\int_0^t \Phi(s)\, ds := \left\{ \int_0^t \varphi(s)\, ds,\ \varphi \in L^1,\ \varphi(t) \in \Phi(t),\ \text{a.e. } t \in J \right\}. \tag{4.1}$$

We call Φ **integrably bounded** if there exists $\alpha \in L^1(J, \mathbb{R})$, $\alpha(t) \geq 0$ a.e., **such that** $\|\Phi(t)\| \leq \alpha(t)$ a.e. on J, where

$$\|\Phi(t)\| = \sup_{\varphi(t) \in \Phi(t)} \{|\varphi(t)|\}, \tag{4.2}$$

$|\cdot|$ being a norm in $\mathbb{R}^n$.

A classical result states that for every measurable and bounded multivalued mapping Φ for which $\Phi(t)$ is a compact and convex set for a.e. $t \in J$, the set appearing in (4.1) is nonempty.

Consider the integral inclusion

$$x(t) \in \mathcal{H}(t) + \int_0^t \mathcal{K}(t,s) \cdot \mathcal{G}(s, x(s))\, ds, \tag{4.3}$$

where $\mathcal{H} : J \to \mathcal{P}(\mathbb{R}^n)$, $\mathcal{K} : J \times J \to \mathcal{M}_n(\mathbb{R})$, $\mathcal{G} : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$.

For $x = (x_i)_{i=\overline{1,n}} \in \mathbb{R}^n$, we set

$$|x| := \max_{1 \le i \le n} \{|x_i|\},$$

and for $A = (a_{ij})_{i,j\in\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$, we set

$$|A| := \max_{1 \le i \le n} \left\{ \sum_{j=1}^n |a_{ij}| \right\}.$$

Admit the following hypotheses.

H$_1$) $\mathcal{H}$ is l.s.c. on J (lower semi-continuous), i.e. $\mathcal{H}^{-1}(U)$ is open for all open sets U;

H$_2$) $\mathcal{K} : J \times J \to \mathcal{M}_n(\mathbb{R})$ is a continuous mapping;

G$_1$) $\mathcal{G}(t,x)$ is compact and convex for all (t,x), and $\mathcal{G}(t,0) = 0$;

G$_2$) $\mathcal{G}(\cdot, x)$ is measurable for every $x \in \mathbb{R}^n$;

G$_3$) $\mathcal{G}(t, \cdot)$ is u.s.c. for a.e. $t \in J$;

G$_4$)

$$\|\mathcal{G}(t_1,x) - \mathcal{G}(t_2,x)\| \le |\alpha(t_1) - \alpha(t_2)| \cdot \beta(|x|), \ \forall t_1, t_2 \in J, \ \forall x \in \mathbb{R}^n,$$

where $\alpha \in L^1(J, \mathbb{R}_+)$, $\beta \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha(0) = 0$, $\beta(t)$ increasing,

$$\|\mathcal{G}(t,x)\| := \sup\{|g(t,x)|, \ g(t,x) \in \mathcal{G}(t,x)\}.$$

The following result holds.

Theorem 4.1 *Assume that hypotheses $H_1)-H_2)$ and $G_1)-G_4)$ are fulfilled. If*

$$\int^\infty \frac{dt}{\beta(t)} = \infty, \tag{4.4}$$

then the inclusion (4.3) *admits solutions.*

Proof. We sketch the proof of Theorem 4.1. Denote

$$X := C(J, \mathbb{R}^n), \ Y := L^1(J, \mathbb{R}^n),$$

with the usual norms

$$\|x\| := \sup_{t \in J} \{|x(t)|\}, \ \|y\| := \int_0^T |y(t)| \, dt.$$

By G_4) it follows that

$$\|\mathcal{G}(t, x)\| \leq \alpha(t) \cdot \beta(|x|). \tag{4.5}$$

Then, for every $x \in X$ we have $\mathcal{G}(t, x(t))$ is measurable and

$$\|\mathcal{G}(t, x(t))\| \leq m_x \cdot \alpha(t),$$

where $m_x = \beta(\|x\|)$. Therefore, $\mathcal{G}(t, x(t))$ is measurable and integrably bounded, and consequently the set

$$S_G(x) := \{g \in Y, \ g(t) \in G(t, x(t)) \ \text{a.e.}\} \tag{4.6}$$

in nonempty, for all $x \in X$.

Define on X the operator

$$F(x) := h(\cdot) + \int_0^{(\cdot)} \mathcal{K}(\cdot, s) \, \mathcal{G}(s, x(s)) \, ds, \tag{4.7}$$

where h is a fixed continuous selection of $\mathcal{H}$, whose the existence is assured by the Michael's Theorem (see, e.g., [6], p. 466).

Relation (4.6) defines a multivalued mapping S_G from X to $\mathcal{P}(Y)$. One remarks firstly that S_G is closed. Indeed, let x_n, $x_0 \in X$, g_n, $g_0 \in Y$, $x_n \to x_0$ in X, $g_n \to g_0$ in Y. There exists a subsequence g_{n_k} which converges a.e. on J to g_0. Let $t \in J$ be fixed; one has

$$g_{n_k}(t) \in G(t, x_{n_k}(t)). \tag{4.8}$$

On the other hand, by hypotheses on $\mathcal{G}$, it follows that for a.e. t, $\mathcal{G}(t, \cdot)$ is closed. From (4.8) we deduce then

$$g_0(t) \in \mathcal{G}(t, x_0(t)),$$

which proves that S_G is closed.

In addition, by hypothesis G_4), taking into account the Riesz's compactity criterion in L^1, it follows that for every $B_r \subset X$, the set $S_G(B_r)$ is relatively compact in L^1. Since S_G is closed, it follows that $S_G(B_r)$ is a compact set. Let us check the hypotheses of Theorem 3.1 to F given by (4.7).

First, from the hypotheses and the properties of the integral, it is obvious that $F : X \to \mathcal{P}(X)$.

Let us show that F is a closed multivalued operator. So, let $x_m, y_m \in X$, such that $y_m \in F(x_m)$, $m \geq 1$ and $x_m \to x_0$ in X, $y_m \to y_0$ in Y. We have

$$y_m(t) = h(t) + \int_0^t \mathcal{K}(t,s)\, g_m(s)\, ds, \tag{4.9}$$

$$g_m \in S_G(x_m). \tag{4.10}$$

By the properties of S_G established above, it follows that g_m admits a subsequence g_{m_k} convergent in L^1 to an $g_0 \in S_G(x_0)$. But then, from (4.9), taking into account the dominated convergence Theorem, $y_{m_k}(t)$ converges for each t to $h(t) + \int_0^t \mathcal{K}(t,s)\, g_0(s)\, ds$, which means that $y_0 \in F(x_0)$.

By using the Arzela's Theorem, one can establish, through a classical reason, that $F(B_r)$ is relatively compact in X. Hence, taking into account that F is also closed, we conclude that F is u.s.c. on X.

It remains to check that the set A is not unbounded. So, let $(x, \lambda) \in X \times (0,1)$, such that

$$x \in \lambda F(x).$$

Therefore,

$$x(t) = \lambda h(t) + \lambda \int_0^t \mathcal{K}(t,s) \cdot g_x(s)\, ds$$

and so,

$$|x(t)| \leq c_0 + c_1 \int_0^t \alpha(s)\, \beta(|x(s)|)\, ds,$$

where c_0, c_1 are positive constants.

Set

$$w(t) := \sup_{0 \leq s \leq t \leq T} \{|x(s)|\}.$$

Obviously, w is increasing and

$$|x(t)| \leq w(t).$$

Denote

$$u(t) = c_0 + c_1 \int_0^t \alpha(s)\, \beta(w(s))\, ds.$$

We have

$$w(t) \leq u(t),$$

and so,

$$|x(t)| \leq u(t).$$

But,

$$\dot{u}(t) = c_1 \alpha(t)\, \beta(w(t)) \ \text{ a.e.}$$

and so,

$$\dot{u}(t) \leq c_1 \alpha(t)\, \beta(u(t)) \ \text{ a.e.}$$

hence,

$$\int_{c_0}^{u(t)} \frac{ds}{\beta(s)} \leq c_1 \int_0^t \alpha(s)\, ds,\ t \in J. \tag{4.11}$$

If A would be unbounded, it would follow by (4.11) that the integral $\int_0^t \alpha(s)\, ds$ is unbounded, which contradicts the hypothesis $\alpha \in L^1(J, \mathbb{R})$.

All hypotheses of Theorem 3.1 being satisfied, the conclusion of Theorem 4.1 follows. □

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