Electronic Journal of Qualitative Theory of Differential Equations
Proc. 7th Coll. QTDE, 2004, No. **4** 1-12;
http://www.math.u-szeged.hu/ejqtde/

ISOPERIMETRIC INEQUALITIES FOR SOME NONLINEAR EIGENVALUE PROBLEMS

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Abstract. In this paper we intend to review many of the known inequalities for eigenvalues of the Laplacian in Euclidean plane. Our aim is to show that we can generalize some results for the pseudo-Laplacian. We focus on isoperimetric inequalities for the first eigenvalue of the Dirichlet eigenvalue problem.

1. The nonlinear eigenvalue problem

We seek the eigenfunctions u_j and corresponding eigenvalues $\lambda_j \quad (j = 1, 2, ...)$ of the following nonlinear eigenvalue problem

$$(D) \qquad -Q_p = \lambda \left|u\right|^{p-1} u \quad \text{in} \quad \Omega,$$

where the nonlinear operator Q_p is the pseudo-Laplacian defined by

$$Q_p = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-1}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{p-1}\frac{\partial u}{\partial y}\right) \quad \text{for} \quad 0 < p < \infty.$$

The boundary condition corresponding to the problem is

$$u|_{\partial\Omega} = 0.$$

Here Ω denotes a two-dimensional body with boundary $\partial\Omega$. The boundary value problem in which equation (D) is to be solved is called Dirichlet problem.

For $p = 1$ the problem (D) describes the vibration of an elastic membrane with fixed boundary:

$$(L) \qquad \begin{aligned} \Delta u + \lambda u = 0 \quad &\text{in} \ \ \Omega, \\ u = 0 \quad &\text{on} \ \ \partial\Omega. \end{aligned}$$

The function $u \in W_0^{1,p+1}(\Omega)$ is a generalized or weak solution of (D) if for every $v \in W_0^{1,p+1}(\Omega)$

$$\int_\Omega \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\left|\frac{\partial u}{\partial x}\right|^{p-1} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\left|\frac{\partial u}{\partial y}\right|^{p-1}\right) dx = \lambda \int_\Omega vu\left|u\right|^{p-1} dx.$$

It is known (see [3]) that there exist countably many number of distinct normalized eigenfunctions in $W_0^{1,p+1}(\Omega)$ with associated eigenvalues to the eigenvalue problem (D). For the eigenvalues $\lambda_k(p)$ of the Dirichlet eigenvalue problem (D)

1991 *Mathematics Subject Classification.* 35J20, 35P15.

Key words and phrases. eigenvalue, pseudo-Laplace operator, isoperimetric inequality, Faber-Krahn inequality.

This paper is in final form and no version of it will be submitted for publication elsewhere.

the relation $\lambda_k(p) \to \infty$ holds when $k \to \infty$. Every eigenvalue is positive. Here, the first eigenfunction has also many special properties. The first eigenfunction does not change sign and the corresponding first eigenvalue λ_1 is simple [19].

The main objective of the paper is to give lower bounds for the first eigenvalue to the eigenvalue problem (D).

2. THE CLASSICAL ISOPERIMETRIC INEQUALITY

The classical isoperimetric inequality after which all such inequalities are named states that the circle encloses the largest area of all plane curves of given perimeter. Our aim is to show that many Euclidean results remain valid in Minkowski plane with the norm $\|x\|_p := \left(|x_1|^{p+1} + |x_2|^{p+1}\right)^{1/(p+1)}$. Of course, for $p = 1$ it is the usual Euclidean norm on $\mathbf{R}^2$.

Let Ω be a simply connected convex domain. We denote the Minkowski arc length of $\partial\Omega$ by L_p, the usual area of Ω by A, and the radius of the greatest inscribed curve c_ϱ of Ω by ϱ, when domain Ω is bounded by curve c_ϱ

$$(c_\varrho) \qquad |x|^{\frac{1}{p}+1} + |y|^{\frac{1}{p}+1} = \varrho^{\frac{1}{p}+1}, \quad \varrho \in \mathbf{R}^+.$$

Curve c_ϱ is significant because many Euclidean results are preserved if the unit-sphere is replaced by the solution of the isoperimetric problem which is called isoperimetrix. The Minkowski length of curve c_ϱ is

$$(2.1) \quad L_p(c_\varrho) = 4\int\limits_{x=0}^{\varrho} \sqrt[p+1]{1 + |y'|^{p+1}}\,dx = 4\int\limits_{x=0}^{\varrho} \sqrt[p+1]{1 + \frac{|x|^{\frac{p+1}{p}}}{\rho^{\frac{p+1}{p}} - |x|^{\frac{p+1}{p}}}}\,dx = 2P\varrho,$$

(see [25]) and the area of the domain bounded by this curve is

$$(2.2) \qquad A(c_\varrho) = 4\int\limits_{x=0}^{\varrho} [\varrho^{\frac{p+1}{p}} - |x|^{\frac{p+1}{p}}]^{\frac{p}{p+1}}\,dx = P\varrho^2,$$

where

$$(2.3) \qquad P = 2\frac{p}{p+1}\,\mathrm{B}\left(\frac{p}{p+1}, \frac{p}{p+1}\right)$$

and $\mathrm{B}\left(\frac{p}{p+1}, \frac{p}{p+1}\right)$ is a Beta function (see [13]). If $p = 1$, then $P = \pi$.

G. D. Chakerian [8] proved and applied the Bonnesen inequality

$$(2.4) \qquad (L_p)\,\varrho \geq A + P\varrho^2$$

in the Minkowski plane for any convex n-gon (and consequently for any convex curve). This inequality was proved by L. Fejes Tóth [12] for nonconvex curves in the Euclidean plane. This proof can be generalized without difficulty to such Minkowski geometry where the "circle" is any centrally symmetric convex curve. So the Bonnesen inequality (2.4) is valid for non-convex curves in Minkowski geometry.

In the case $p = 1$ the inequality (2.4) is reduced to the Bonnesen inequality valid on the Euclidean plane.

From the Bonnesen inequality for a simply connected convex domain G. D. Chakerian [8] showed that the isoperimetric inequality in the Minkowski metric for a simply connected convex domain Ω has the form

$$(L_p)^2 - 4PA \geq 0. \tag{2.5}$$

In (2.5) equality holds if and only if domain Ω is bounded by curve c_ϱ.

3. THE ISOPERIMETRIC INEQUALITY IN "BROADER SENSE"

There are several interesting and important geometrical and physical quantities depending on the shape and size of a curve: e.g., the length of its perimeter, the area included, the moment of inertia, with respect to the centroid, of a homogeneous plate bounded by the curve, the torsional rigidity of an elastic beam the cross-section of which is bounded by the given curve, the principal frequency of a membrane of which the given curve is the rim, the electrostatic capacity of a plate of the same shape and size, and several other quantities.

By the help of the isoperimetric inequalities we can estimate physical quantities on the basis of easily accessible geometrical data.

3.1. **Bounds for eigenvalues.** Isoperimetric inequalities are also useful in the derivation of explicit a priori inequalities employed in the determination of a priori bounds in various types of initial or boundary value problems.

In the linear case (L) many papers and books were published on the estimation of the first eigenvalue (see [22],.[1], [23] or [18]). Such bounds are based on geometrical data of the domain.

For the case of (L) Lord Rayleigh conjectured that for all membranes with given area A the circle yields the minimum value of λ_1. This property can be expressed by the inequality

$$\lambda_1 \geq \frac{\pi j_0^2}{A} \tag{3.1}$$

with equality only for the circle and where j_0 is the first positive zero of the Bessel function of the first kind $J_0(x)$. G.Faber [11] and E.Krahn [16] found independently essentially the same proof of Rayleigh's conjecture. Another proof was given by G.Pólya and G.Szegõ [22] by using Steiner symmetrization.

We know that for the domain Ω with sufficiently smooth boundary $\partial\Omega$ the first eigenvalue in the fixed membrane problem (D) admits the following characterization:

$$\lambda_1 = \min_{u \in W_0^{1,p+1}(\Omega)} \frac{\int\limits_\Omega \left(|u_x|^{p+1} + |u_y|^{p+1} \right) \, dx}{\int\limits_\Omega |u|^{p+1} \, dx}.$$

This characterization gives us an upper bound for λ_1, i.e., for any $v \in W_0^{1,p+1}(\Omega)$

$$\lambda_1 \leq \frac{\int\limits_\Omega \left(|v_x|^{p+1} + |v_y|^{p+1} \right) \, dx}{\int\limits_\Omega |v|^{p+1} \, dx}.$$

The equality sign will always hold for some choice of v.

3.2. **A lower bound for the first eigenvalue.** As it was shown in the introduction many papers were published on the estimation of the first eigenvalue for the linear eigenvalue problem.

Let the domain Ω be a simply connected convex domain. We define the curve (c_{ρ_0}) by

$$|x-x_0|^{\frac{1}{p}+1}+|y-y_0|^{\frac{1}{p}+1}=\rho_0^{\frac{1}{p}+1}, \quad \rho_0\in\mathbf{R}^+, \quad (x_0,y_0)\in\Omega. \tag{3.2}$$

In this part let us denote the radius of the greatest inscribed curve (c_{ρ_0}) of Ω by ρ. We shall consider (x_0,y_0) as the origin of the new coordinate system.

Theorem 1. *Let* λ_1 *be the smallest eigenvalue of the eigenvalue problem (D) such that* $u_1\geq 0$ *in the simply connected convex domain* $\Omega\in\mathbf{R}^2$. *Then the inequality*

$$\lambda_1\geq\left[\frac{A+\sigma}{(p+1)\rho A}\right]^{p+1} \tag{3.3}$$

holds, where A *is the area of* Ω, ρ *is the radius of the greatest inscribed isoperimetrix* c_ρ *of* Ω, *and* $\sigma=P\,\varrho^2$ *is the area of the isoperimetrix of radius* ρ.

Proof. As in [22] and [15] we shall reconstruct the first eigenfunction $u_1(x,y)$ from its level sets $\Omega(\tau)=\{(x,y)\in\Omega|\ u_1(x,y)\geq c\}$, where the area of $\Omega(\tau)$ is τ, so $0\leq\tau\leq A$ and $\Omega(A)=\Omega$ (see [5]). The domain $\Omega(\tau)$ may be disconnected but each of its components is simply connected because u_1 does not have local minima in Ω. Instead of coordinates x,y we introduce the new coordinates τ and s in the domain Ω, where s is the arc length in Minkowski metric of the level line which bounds $\Omega(\tau)$. Therefore $0\leq s\leq L(\tau)$, where $L(\tau)$ is the total length of the level line.

Let the function $\varphi(\tau)$ be defined as

$$\varphi(\tau)=u_1(x,y)\quad\text{on}\quad\partial\Omega(\tau).$$

Evidently $\varphi(A)=0$ and $\varphi(\tau)$ is monotonically decreasing when $0\leq\tau\leq A$. We have for the derivatives

$$\frac{\partial u_1}{\partial\tau}=\frac{\partial\varphi}{\partial\tau}=\varphi'(\tau)=\frac{\partial u_1}{\partial x}\frac{\partial x}{\partial\tau}+\frac{\partial u_1}{\partial y}\frac{\partial y}{\partial\tau},$$

$$\frac{\partial u_1}{\partial s}=\frac{\partial\varphi}{\partial s}=0=\frac{\partial u_1}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u_1}{\partial y}\frac{\partial y}{\partial s}.$$

Since

$$\left|\frac{\partial x}{\partial s}\right|^{p+1}+\left|\frac{\partial y}{\partial s}\right|^{p+1}=1.$$

therefore we get

$$\left|\frac{\partial u_1}{\partial x}\right|^{p+1}+\left|\frac{\partial u_1}{\partial y}\right|^{p+1}=\frac{|\varphi'(\tau)|^{p+1}}{|\triangle|^{p+1}}, \tag{3.4}$$

where the Jacobian $\triangle$ is

$$\triangle=\begin{vmatrix}\dfrac{\partial x}{\partial\tau} & \dfrac{\partial y}{\partial\tau}\\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s}\end{vmatrix}$$

(similarly as in [17] for the linear case $p = 1$). The Rayleigh quotient can be expressed by the new coordinates τ and s:

$$\lambda_1 = \frac{\int\limits_{\Omega}\left(\left|\frac{\partial u_1}{\partial x}\right|^{p+1} + \left|\frac{\partial u_1}{\partial y}\right|^{p+1}\right) dx\ dy}{\int\limits_{\Omega}|u_1|^{p+1} dx\ dy} = \frac{\int\limits_{\tau=0}^{A}\left[|\varphi'(\tau)|^{p+1}\int\limits_{s=0}^{L(\tau)}\frac{ds}{|\triangle|^p}\right] d\tau}{\int\limits_{\tau=0}^{A}\left[|\varphi(\tau)|^{p+1}\int\limits_{s=0}^{L(\tau)}|\triangle| ds\right] d\tau}. \tag{3.5}$$

Since

$$\tau = \int\limits_{\Omega(\tau)} dx\ dy = \int\limits_{t=0}^{\tau}\int\limits_{s=0}^{L(t)} |\triangle|\ ds\ dt$$

we obtain

$$\int\limits_{s=0}^{L(\tau)} |\triangle|\ ds = 1. \tag{3.6}$$

Applying the Hölder inequality we get

$$\left(\int\limits_{s=0}^{L(\tau)} |\triangle|\ ds\right)^{\frac{p}{p+1}} \left(\int\limits_{s=0}^{L(\tau)} \frac{ds}{|\triangle|^p}\right)^{\frac{1}{p+1}} \geq L(\tau).$$

Hence by (3.6)

$$\int\limits_{s=0}^{L(\tau)} \frac{ds}{|\triangle|^p} \geq [L(\tau)]^{p+1}.$$

Now from (3.5) we have

$$\lambda_1 \geq \frac{\int\limits_{\tau=0}^{A} |\varphi'(\tau)L(\tau)|^{p+1} d\tau}{\int\limits_{\tau=0}^{A} |\varphi(\tau)|^{p+1} d\tau}.$$

Let us denote by $\rho(\tau)$ the radius of the greatest inscribed curve (c_{ρ_0}) of $\Omega(\tau)$. Now we have the inequality $\rho(\tau_1) \leq \rho(\tau_2)$ when $\tau_1 < \tau_2$. We shall use the isoperimetric inequality (2.5)

$$L(\tau) \geq \sqrt{4P\tau}, \quad P = 2\ \frac{p}{p+1}\ B\left(\frac{p}{p+1}, \frac{p}{p+1}\right)$$

and the Bonnesen inequality (2.4)

$$L(\tau) \geq \frac{\tau}{\varrho(\tau)} + P\varrho(\tau).$$

For $\sigma \leq \tau \leq A$ where

$$\sigma = P\varrho^2$$

we show that $L(\tau) \geq \dfrac{\tau}{\varrho} + P\varrho$. Because the function $f(t) = t + \frac{1}{t}$ is increasing if $t > 1$ and

$$\frac{\sqrt{\tau}}{\sqrt{P}\varrho(\tau)} \geq \frac{\sqrt{\tau}}{\sqrt{P}\varrho} \geq \frac{\sqrt{\sigma}}{\sqrt{P}\varrho} =1$$

then

$$L(\tau) \geq \frac{\tau}{\varrho(\tau)} + P\varrho(\tau) = \sqrt{P\tau}\left[\frac{\sqrt{\tau}}{\sqrt{P}\varrho(\tau)} + \frac{1}{\dfrac{\sqrt{\tau}}{\sqrt{P}\varrho(\tau)}}\right] \geq$$

$$\geq \sqrt{P\tau}\left[\frac{\sqrt{\tau}}{\sqrt{P}\varrho} + \frac{1}{\dfrac{\sqrt{\tau}}{\sqrt{P}\varrho}}\right] = \frac{\tau}{\varrho} + P\varrho.$$

We define the function $X(\tau)$ as follows

$$X(\tau) = \begin{cases} \sqrt{4P\tau} & \text{if } \ 0 \leq \tau < \sigma, \\ \dfrac{\tau}{\rho} + P\rho & \text{if } \ \sigma \leq \tau \leq A, \end{cases}$$

and we have the relation

$$X(\tau) \leq L(\tau) \quad (0 \leq \tau \leq A) \tag{3.7}$$

Applying (3.7) we obtain

$$\lambda_1 \geq \frac{\int\limits_{\tau=0}^{A} |\varphi'(\tau)X(\tau)|^{p+1}d\tau}{\int\limits_{\tau=0}^{A} |\varphi(\tau)|^{p+1}d\tau}.$$

For the function $X(\tau)$ we find

$$\lim_{\tau\to\sigma-0} X(\tau) = \lim_{\tau\to\sigma+0} X(\tau) = 2P\varrho,$$

$$\lim_{\tau\to\sigma-0} X'(\tau) = \lim_{\tau\to\sigma+0} X'(\tau) = \frac{1}{\varrho}$$

therefore $X(\tau) \in C^1(0, A)$. Now we introduce the function

$$Y(\tau) = \frac{A+\sigma}{A\varrho}\tau$$

for which $Y(0) = X(0), Y(A) = X(A)$. For $0 \leq \tau < \sigma$ we get

$$\sqrt{4P\tau} \geq \sqrt{4\ P\rho^2\frac{\tau}{\rho^2}} = \sqrt{4\ \sigma\frac{\tau}{\rho^2}} \geq \sqrt{4\ \frac{\tau^2}{\rho^2}} \geq \sqrt{\left(1+\frac{\sigma}{A}\right)^2\ \frac{\tau^2}{\rho^2}} = \frac{A+\sigma}{A\varrho}\tau.$$

If $\sigma \leq \tau \leq A$ then

$$\frac{\tau}{\varrho}+P\varrho \geq \frac{\tau}{\rho} + \frac{P\rho}{A}\tau = \frac{\tau}{\rho} + \frac{\sigma}{A\rho}\tau = \frac{A+\sigma}{A\varrho}\tau,$$

therefore $Y(\tau) \leq X(\tau)$ when $0 \leq \tau \leq A$.

Using the function $Y(\tau)$ we get

$$\lambda_1 \geq \left[\frac{A+\sigma}{A\varrho}\right]^{p+1} \frac{\int\limits_{\tau=0}^{A} |\tau\varphi'(\tau)|^{p+1} d\tau}{\int\limits_{\tau=0}^{A} |\varphi(\tau)|^{p+1} d\tau}.$$

For reals X, Y and $p > 0$ inequality

$$|X|^{p+1} + p\,|Y|^{p+1} - (p+1)X\;Y\;\;|Y|^{p-1} \geq 0 \tag{3.8}$$

holds, with equality if and only if $X = Y$.

Making use of inequality (3.8) with $X = -(p+1)\tau\varphi'$ and $Y = \varphi$ we can write

$$|(p+1)\tau\varphi'|^{p+1} + p|\varphi|^{p+1} + (p+1)^2\tau\varphi'\varphi^{\overset{*}{p}} \geq 0, \quad \varphi^{\overset{*}{p}} = |\varphi|^p sgn\ \varphi,$$

and therefore

$$|(p+1)\tau\varphi'|^{p+1} - |\varphi|^{p+1} = |(p+1)\tau\varphi'|^{p+1} + p|\varphi|^{p+1} - (p+1)|\varphi|^{p+1} \geq$$
$$\geq -(p+1)[(p+1)\tau\varphi'\varphi^{\overset{*}{p}} + |\varphi|^{p+1}].$$

Integrating by parts we have

$$\int\limits_{\tau=0}^{A} [|(p+1)\tau\varphi'|^{p+1} - |\varphi|^{p+1}]d\tau \geq$$

$$\geq -(p+1) \int\limits_{\tau=0}^{A} [(p+1)\tau\varphi'\varphi^{\overset{*}{p}} + |\varphi|^{p+1}]d\tau =$$
$$= -(p+1)\ [|\varphi|^{p+1}\tau]_{\tau=0}^{A} = 0$$

since $\varphi(A) = 0$. Hence

$$\frac{\int\limits_{\tau=0}^{A} |\tau\varphi'(\tau)|^{p+1} d\tau}{\int\limits_{\tau=0}^{A} |\varphi(\tau)|^{p+1} d\tau} \geq \frac{1}{(p+1)^{p+1}},$$

thus for the first eigenvalue λ_1 we obtain the inequality (3.3). ▮

In the case $p = 1$ the lower bound for the smallest eigenvalue is reduced to the lower bound for λ_1 given by Makai in [17].

3.3. **Application of symmetrization.** For the nonlinear eigenvalue problem (D) let us consider the bounded domain Ω and the continuous function $u(x,y) \in W_0^{1,p+1}(\Omega)$ on $\overline{\Omega}$ satisfying $u_{|\partial\Omega} = 0$. We introduce the level set Ω_c of u for $c \in \mathbf{R}$ by

$$\Omega_c = \{(x,y) \in \overline{\Omega} :\ u(x,y) \geq c\}.$$

The most frequently used type of symmetrization is the Schwarz symmetrization which is centrally symmetric. We define the Schwarz symmetrization $\Omega_c^{(0)}$ of $\Omega_c \subset \mathbf{R}^2$ by

$$\Omega_c^{(0)} := \begin{cases} \text{bounded by curve } (c_\rho) \text{ with the same area as } \Omega_c & \text{if } \Omega_c \neq \emptyset, \\ \emptyset & \text{if } \Omega_c = \emptyset. \end{cases}$$

For u the level sets Ω_c are replaced by concentric curves (c_ρ) centered at zero. Therefore the Schwarz symmetrization of u is defined by

$$u^{(0)} := \sup \left\{ c \in \mathbf{R} | \quad (x,y) \in \Omega_c^{(0)} \right\}.$$

Let us consider the bounded domain Ω and the continuous function $u(x,y)$ on $\overline{\Omega}$ satisfying $u_{|\partial\Omega} = 0$. We introduce the level set Ω_c of u for $c \in \mathbf{R}$ by

$$\Omega_c = \{(x,y) \in \bar{\Omega} : \; u(x,y) \geq c\}.$$

Clearly

$$\Omega_{c'} \supset \Omega_{c''} \quad \text{if} \quad u_{min} \leq c' < c'' \leq u_{max},$$

and

$$\begin{aligned} \Omega_c = \Omega \quad &\text{if} \quad c < u_{min}, \\ \Omega_c = \emptyset \quad &\text{if} \quad c > u_{max}. \end{aligned}$$

By these properties we can replace function u by a related function $u^{(0)}$ which has some desired properties (see [15] p.21).

Theorem 2. *Let λ_1 be the smallest eigenvalue and u_1 be the corresponding eigenfunction, respectively, to the eigenvalue problem (D) on the domain Ω and $\lambda_1^{(0)}$ be the smallest eigenvalue with the corresponding eigenfunction $u_1^{(0)}$ on the domain $\Omega^{(0)}$. Then $\lambda_1 > \lambda_1^{(0)}$ unless $\partial\Omega$ is a curve (c_ϱ).*

Proof. Our proof is based on the proof given by G.Faber [11] and E. Krahn [16] in the linear case (L). First of all we give the differences between the linear and nonlinear cases. We consider the level sets of the first eigenfunction u_1

$$\Omega_c = \{(x,y) \in \bar{\Omega} : \quad u_1(x,y) \geq c\}, \quad c \in \mathbf{R}.$$

Instead of coordinates x, y we introduce the new coordinates w, and s. The intersection of the plane $w = c$ with the surface of $w = u_1$ gives the level sets Ω_c. Therefore we get $0 \leq w \leq a$, where a is the maximum value of u_1. Let the coordinate s be the arc length of the level line from 0 the total length $L\,(w)$ of $\partial\Omega_c$ We have the following connections:

$$\frac{\partial u_1}{\partial w} = 1, \qquad \frac{\partial u_1}{\partial s} = 0$$

that is

$$\begin{aligned} \frac{\partial u_1}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial u_1}{\partial y}\frac{\partial y}{\partial w} &= 1, \\ \frac{\partial u_1}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u_1}{\partial y}\frac{\partial y}{\partial s} &= 0, \end{aligned}$$

and

$$\left|\frac{\partial x}{\partial s}\right|^{p+1}+\left|\frac{\partial y}{\partial s}\right|^{p+1}=1.$$

For the new coordinates we have the Jacobian

$$\Delta=\left|\begin{array}{cc}\frac{\partial x}{\partial w} & \frac{\partial x}{\partial s}\\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial s}\end{array}\right|.$$

Hence we obtain

$$\left|(u_1)_x\right|^{p+1}+\left|(u_1)_y\right|^{p+1}=\frac{1}{|\Delta|^{p+1}}$$

so

$$\int\limits_{\Omega}\left[\left|(u_1)_x\right|^{p+1}+\left|(u_1)_y\right|^{p+1}\right]dx\ dy=\int\limits_{w=0}^{a}\int\limits_{s=0}^{L(w)}\frac{1}{|\Delta|^{p}}ds\ dw.$$

We denote by $A\left(w\right)$ the area of Ω_c that is

$$A\left(w\right)=\int\limits_{D_c}dx=-\int\limits_{\widetilde{w}=a}^{w}\int\limits_{s=0}^{L(\widetilde{w})}|\Delta|\ ds\ d\widetilde{w}$$

and $A\left(0\right)=$mes $\Omega,\ \ A\left(a\right)=0$. Therefore we have

$$A'\left(w\right)=-\int\limits_{s=0}^{L(w)}|\Delta|\,ds, \tag{3.9}$$

moreover by the Hölder inequality and by inequality (2.5) we obtain

$$\int\limits_{s=0}^{L(w)}|\Delta|\,ds\left[\int\limits_{s=0}^{L(w)}\frac{1}{|\Delta|^{p}}ds\right]^{\frac{1}{p}}\geq|L(w)|^{\frac{1}{p}+1}\geq[4PA(w)]^{\frac{p+1}{2p}}. \tag{3.10}$$

By equation (3.9) we get

$$\int\limits_{s=0}^{L(w)}\frac{1}{|\Delta|^{p}}ds\geq|-A'(w)|^{-p-1}\left[-A'(w)\right][4PA(w)]^{\frac{p+1}{2}}$$

therefore

$$\int\limits_{\Omega}\left[\left|(u_1)_x\right|^{p+1}+\left|(u_1)_y\right|^{p+1}\right]dx\ dy\geq \tag{3.11}$$

$$(4P)^{\frac{p+1}{2}}\int\limits_{w=0}^{a}|-A'(w)|^{-p-1}\left[-A'(w)\right][A(w)]^{\frac{p+1}{2}}dw.$$

Since $u_1^{(0)}$ is symmetric function we can write

$$u_1^{(0)}(x,y) = v(\rho).$$

We obtain

$$\int\limits_\Omega \left[\left| \left(u_1^{(0)} \right)_x \right|^{p+1} + \left| \left(u_1^{(0)} \right)_y \right|^{p+1} \right] dx\ dy = \int\limits_{w=0}^{a} \int\limits_{s=0}^{L(w)} \left| \frac{dv}{d\rho} \right|^p ds\ dw =$$

$$= \int\limits_{w=0}^{a} 2P\rho \left| \frac{dv}{d\rho} \right|^p dw = 2^{p+1} P \int\limits_{w=0}^{a} \rho^{p+1} \left| \frac{dw}{d(\rho^2)} \right|^p dw. \tag{3.12}$$

Since $A(w) = P\rho^2$,

$$\rho = \left(\frac{A(w)}{P} \right)^{\frac{1}{2}}$$

thus

$$\left| \frac{dw}{d(\rho^2)} \right| = -\frac{P}{A'(w)},$$

and from (3.12)

$$\int\limits_\Omega \left[\left| \left(u_1^{(0)} \right)_x \right|^{p+1} + \left| \left(u_1^{(0)} \right)_y \right|^{p+1} \right] dx\ dy = \tag{3.13}$$

$$= (4P)^{\frac{p+1}{2}} \int\limits_{w=0}^{a} |-A'(w)|^{-p-1} [-A'(w)] [A(w)]^{\frac{p+1}{2}} dw.$$

Comparing with (3.11) we obtain

$$\int\limits_\Omega \left[|(u_1)_x|^{p+1} + \left| (u_1)_y \right|^{p+1} \right] dx\ dy \geq \int\limits_\Omega \left| \left(u_1^{(0)} \right)_x \right|^{p+1} + \left| \left(u_1^{(0)} \right)_y \right|^{p+1} dx\ dy.$$

Using the property *ii.)* from [15] we get

$$\|u_1\|_{L^{p+1}}^{p+1} = \int\limits_\Omega \left| \left(u_1^{(0)} \right) \right|^{p+1} dx\ dy. \tag{3.14}$$

Therefore (3.14) yields the estimate

$$\lambda_1 = \frac{\int\limits_\Omega \left[|(u_1)_x|^{p+1} + \left| (u_1)_y \right|^{p+1} \right] dx\ dy}{\|u_1\|_{L^{p+1}}^{p+1}} \geq \frac{\int\limits_\Omega \left[\left| \left(u_1^{(0)} \right)_x \right|^{p+1} + \left| \left(u_1^{(0)} \right)_y \right|^{p+1} \right] dx\ dy}{\int\limits_\Omega \left| \left(u_1^{(0)} \right) \right|^{p+1} dx\ dy} \geq$$

$$\geq \inf_{v\in F(\Omega^{(0)})} \frac{\int\limits_{\Omega} \left(|v_x|^{p+1} + |v_y|^{p+1} \right) \, dx \; dy}{\int\limits_{\Omega} |v|^{p+1} \, dx \; dy} = \lambda_1^{(0)}.$$

From the isoperimetric inequality (2.5) it follows that in (3.11) we have equality (also in (3.10), respectively) if the level lines are curves (c_ϱ). Thus $\lambda_1 > \lambda_1^{(0)}$ unless $\partial\Omega$ is a curve (c_ϱ). ∎

Corollary 1. *In the case when the domain Ω is bounded by the curve (c_ϱ), then the first eigenvalue of (D) can be given by*

$$\lambda_1^{(0)} = \left(\frac{h_0}{\rho}\right)^{p+1} = \frac{P^{\frac{p+1}{2}} h_0^{p+1}}{A^{\frac{p+1}{2}}},$$

where h_0 is the first positive zero of the generalized nonlinear Bessel function $H_0(x)$ satisfying the nonlinear ordinary differential equation

$$\frac{d}{dx}\left[\left|\frac{dH_0}{dx}\right|^{p-1}\frac{dH_0}{dx}\right] + \frac{1}{x}\left|\frac{dH_0}{dx}\right|^{p-1}\frac{dH_0}{dx} + \lambda\,|H_0|^{p-1} H_0 = 0 \tag{3.15}$$

with conditions

$$H_0(1) = 0 \tag{3.16}$$

and

$$\frac{dH_0}{dx}(0) = 0,$$

$\lambda = 1$ *and* $P = 2\frac{p}{p+1} B\left(\frac{p}{p+1}, \frac{p}{p+1}\right)$ *(see [6]). Hence for any domain on the plane with area A we get a lower bound for the first eigenvalue:*

$$\lambda_1 \geq \frac{P^{\frac{p+1}{2}} \, h_0^{p+1}}{A^{\frac{p+1}{2}}}. \tag{3.17}$$

Inequality (3.17) is the generalization of the Faber-Krahn inequality (3.1) to the nonlinear problem (D).

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(Received September 30, 2003)

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