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# Fixed Points and Differential Equations with Asymptotically Constant or Periodic Solutions

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ABSTRACT. Cooke and Yorke developed a theory of biological growth and epidemics based on an equation $x'(t) = g(x(t)) - g(x(t-L))$ with the fundamental property that $g$ is an arbitrary locally Lipschitz function. They proved that each solution either approaches a constant or $\pm\infty$ on its maximal right-interval of definition. They also raised a number of interesting questions and conjectures concerning the determination of the limit set, periodic solutions, parallel results for more general delays, and stability of solutions. Although their paper motivated many subsequent investigations, the basic questions raised seem to remain unanswered.

We study such equations with more general delays by means of two successive applications of contraction mappings. Given the initial function, we explicitly locate the constant to which the solution converges, show that the solution is stable, and show that its limit function is a type of "selective global attractor." In the last section we examine a problem of Minorsky in the guidance of a large ship. Knowledge of that constant to which solutions converge is critical for guidance and control.

## 1. Introduction.

In this study we focus on a celebrated paper on epidemics by Cooke and Yorke [7] who presented three models of considerable interest for the growth of a population. Our analysis of these problems is by means of contraction mappings which offer a very simple, quick, and effective way of treating many qualitiative behavior problems in functional differential equations.

The models of Cooke and Yorke on which we focus are:

(a) $x'(t) = g(x(t)) - g(x(t-L)), \quad L > 0,$

(b) $x'(t) = g(x(t-L_1)) - g(x(t-L_1-L_2)), \quad L_i > 0,$

and

(c) $x'(t) = g(t, x(t)) - g(t, x(t-L)), \quad g(t+L, x) = g(t, x).$

These equations share three fundamental properties:

(A) $g$ is **any** locally Lipschitz function.

(B) Every constant function is a solution of each of them.

(C) Each equation has a first integral.

They prove that every solution of (a) satisfies either:

(I) $x(t)$ tends to a constant,

or

(II) $x(t)$ tends to $+\infty$,

or

(III) $x(t)$ tends to $-\infty$

on its maximal right-interval of definition.

They claim that (b) is too difficult for their methods and they do not attempt (c).

Later, Kaplan, Sorg, and Yorke [13] proved the same three conclusions for a more general equation than (a) which still enjoyed (A) and (B), but not (C). A vast number of papers followed treating equations more general than (a) which fail to allow (A) and (C), but strengthen the conclusion to (I) alone. Recently, Arino and Pituk [1] considered a very general equation with finite delay along the same lines, asking only a type of global Lipschitz condition, and used fixed point theory to prove that solutions tend exponentially to a constant solution and that the constant solution is uniformly stable.

The literature will show that (A) will eliminate one point of uncertainty in biological investigations. It was a brilliant contribution of Cooke and Yorke, and is the center of our focus.

Here, we note a tacit assumption of Cooke and Yorke which suggests a contraction mapping approach. Moreover, Cooke and Yorke suggest asking $|g(x)| \leq K|x|$ for large $|x|$ in order to ensure that all solutions can be continued for all future time. If we adopt a variant of these two conditions, we can show that:

(i) Every solution approaches a constant.

(ii) The limit constant can be known in advance.

(iii) Each solution is stable and the limit constant is a "selective global attractor," attracting all solutions having initial functions with the same "average value."

(iv) All the work can be done in exactly the same way for very general delays, including pointwise, distributed, infinite, and combinations. All are illustrated here.

(v) The problem (b) is not harder than (a); in fact, the analysis is the same.

(vi) The periodic case (c) is handled in the same way. But it does not yield the result they had conjectured in the form of a periodic solution to which all others converged. That behavior is not promoted by such equations.

(vii) Neutral equations can also be handeled in the same way.

(viii) Higher order equations possessing properties (A), (B), and (C) can be treated similarly.

(ix) The general construction of $g(x(t)) - g(x(t-L))$ in an equation can result in extremely stable behavior of solutions. It is very useful in designing control problems in which it is simple to specify a target and initial conditions to achieve that target.

All of this is done with great simplicity using contractions. Fixed point theory can be very effective in dealing with problems for which we can invert the equation in such a way that a mapping is produced which will map the set of desired candidates for solutions into itself. We can not, however, treat the more general problems of Kaplan, Sorg, and Yorke [13] because we have been unable to invert them in a usable way.

It should be mentioned early on that if $\int_0^x g(s)ds \to -\infty$ as $|x| \to \infty$ then the trivial Liapunov function will show that all solutions of (a) are bounded and (I) holds. The Liapunov functional is

$$V(x(\cdot)) = -2\int_0^x g(s)ds + \int_{t-L}^t g^2(x(s))ds$$

and its derivative along solutions of (a) satisfies

$$\begin{aligned} V'_{(a)} &= -2g^2(x) + 2g(x)g(x(t-L)) + g^2(x) - g^2(x(t-L)) \\ &= -g^2(x) + 2g(x)g(x(t-L) - g^2(x(t-L)) \\ &= -(g(x) - g(x(t-L))^2 \end{aligned}$$

Thus, $V(x) \to \infty$ as $|x| \to \infty$ and $V' \leq 0$ yields all solutions bounded, the maximal interval of definition is $[0, \infty)$, and both (II) and (III) are impossible. When solutions depend continuously on initial conditions, then a classical result by Krasovskii [14; p. 153] and Hale [11] shows that every bounded solution approaches the set where $V' = 0$. That set consists of those functions where $g(x(t)) = g(x(t - L))$ and so $x'(t) = 0$ and $x(t)$ is constant. The particular proof given by Krasovskii bears more study. It seems likely that the required continuous dependence of solutions on initial conditions might be reduced.

## 2. The fundamental assumption.

In 1973 Cooke and Yorke [7] introduced a delay-differential equation as a proposed model for an epidemic. In fact, Cooke [5] had proposed the model some years earlier, but the 1973 paper contained substantial analysis. The model itself, if not the biological application, generated enormous interest for thirty years. The purpose of this work is to point out that fixed point theory provides an excellent means of attack which preserves the outstanding feature of the Cooke-Yorke problem, a feature lost in most of the other attacks to be found in the literature.

In many population problems there is endless speculation on the form of the functions generating growth and decline. The classic monograph by Maynard-Smith [16] devotes Chapters 2 and 3 to describing several choices for growth functions with less than strong reason for choosing one over the other. Pielou [18; p. 20] suggests a mechanical choice by picking the simplest part of a Tayor series which will generate an S-shaped curve; this yields $g(x) = ax - bx^2$ with $a$ and $b$ positive constants. It is to be noted that in so many of the papers motivated by the Cooke-Yorke paper, monotonicity and sign conditions on $g$ are assumed. Thus, those papers seem outside the general framework of interest here and will not be discussed. The interested reader may pick more than fifty references to such work from the bibliographies of Arino and Pituk [1], Haddock [8], and Krisztin [15].

Cooke and Yorke by-passed all those questions. Their discussion is worth repeating and the next paragraph is a quote of their work from [7; pp. 76-77].

"Thus, if $x(t)$ denotes the number of individuals in a population at time $t$, the number of births $B(t)$ is some function of $x(t)$, say $B(t) = g(x(t))$. Let us assume at first that every individual has life span $L$, a constant. Inasmuch as every individual dies at age $L$, the number of deaths per unit time at time $t$ is $g(x(t-L))$. Since the difference $g(x(t))-g(x(t-L))$ is net change in population per unit time, the growth of the population is governed by the equation

$$x'(t) = g(x(t)) - g(x(t-L)). \tag{1}$$

This is one of the equations to be analyzed here. Models of this type were suggested by one of the authors in [5]. **We must emphasize that $g$ is allowed to be any differentiable function in our results.**"

The bold emphasis is ours and it is really the critical part of their paper, putting it far ahead of so many similar investigations, both before and after publication of that paper. It should be noted that Cooke continued the investigation in [6]. So many of the subsequent investigators focused on solutions approaching constants, but seemed to ignore this all-important lack of condition on $g$.

## 3. Limits, first integrals, tacit assumptions.

To specify a solution of (1) we require an initial function on an initial interval. Typically, we need a continuous function $\psi : [-L, 0] \to R$ and obtain a continuous function $x(t, 0, \psi)$ with $x(t, 0, \psi) = \psi(t)$ on $[-L, 0]$, while $x$ satisfies (1) for $t > 0$. Although $x$ is continuous, the derivative may fail at $t = 0$. Equation (1) is autonomous so we lose nothing by starting the solution at zero. At times we will consider non-autonomous equations and may still start at zero for simple convenience. Existence theory can be found in Chapter 3 of [4], for example.

In [7; p. 78], Cooke and Yorke note that (1) has a first integral

$$x(t) = \int_{t-L}^{t} g(x(s))ds + c \tag{2*}$$

where $c$ is a constant of integration. They claim that "to have a correct biological interpretation we must have $c = 0$." In fact, for a given initial function $\psi$, then

$$c = \psi(0) - \int_{-L}^{0} g(\psi(s))ds. \tag{3}$$

Thus, we focus on ($2^*$) with $c$ defined by (3) and we denote it by

$$x(t) = \psi(0) - \int_{-L}^{0} g(\psi(s))ds + \int_{t-L}^{t} g(x(s))ds. \tag{2}$$

The discussion of Cooke and Yorke [7] suggests that they have two tacit assumptions in mind. We find both of those assumptions fundamental for the investigation. By formalizing and strengthening those assumptions we are able to very simply answer the questions posed in the paper using contraction mappings. First, to understand what the aforementioned condition $c = 0$ may mean for the problem, we note that Cooke and Yorke [7; p. 83] assume a local Lipschitz condition and suggest that the reader may wish to insert the requirement that there is a constant $K$ with

$$|g(x)| \leq K|x| \quad \text{for all large} \quad x.$$

Such a condition ensures that solutions can be continued for all future time. It is well-known that if $g(x) > 0$ on an interval $[a, \infty)$ with $\int_a^\infty \frac{ds}{g(s)} < \infty$, then solutions can not be continued for all future time. Thus, if we want to continue solutions for all future time and if we want to allow $g$ to be free of sign restricitons, then some kind of growth condition must be assumed. Our investigation rests squarely on a growth restriction in the form of a global Lipschitz condition.

The situation discussed above is sufficiently critical that an example should be given. Suppose we have $g(x) = x - x^2$. With that $g$ in (1), there will be solutions with finite escape time whenever the initial function is negative. Such initial functions seldom, if ever, have any biological significance. However, in [7; p. 87] Cooke and Yorke ask that we consider a $g(t, x)$ with $g$ periodic in $t$, representing seasonal changes in the growth. Thus, if we take $g(t, s) = p(t)(x - x^2)$ with $p$ continuous and periodic, even if the initial function is positive it is readily shown that there can be finite escape time if $p$ is negative at one point.

Next, Cooke and Yorke consistently state that $c$ should be zero for correct biological applications. They also focus on the fact that every constant function is a solution. If we were to ask $c = 0$, ask for a constant solution $x(t) = k \neq 0$, and ask that $g(x) = Kx$, then (3) would become

$$0 = k - KkL$$

so that

$$KL = 1 \tag{4}$$

since we ask for a solution $x = k \neq 0$. The choice of $c = \psi(0) - \int_{-L}^{0} g(\psi(s))ds$ permits all constants to be solutions of (2). We allow all constants in this discussion. But we also need to control $KL$ just a bit more strongly than Cooke and Yorke would with (4). We need $\alpha < 1$ with $KL \leq \alpha$.

## 4. Some solutions of the problems of Cooke and Yorke.

We will assume that there is a fixed positive constant $K$ such that $x, y \in R$ implies that

$$|g(x) - g(y)| \leq K|x - y|, \tag{5}$$

a global Lipschitz condition. Next, we suppose there is a positive constant $\alpha < 1$ with

$$KL \leq \alpha. \tag{6}$$

**Theorem 1.** Let (5) and (6) hold and let $\psi : [-L, 0] \to R$ be a given continuous function. Then there is a unique constant $k$ satisfying

$$k = \psi(0) + g(k)L - \int_{-L}^{0} g(\psi(s))ds \tag{7}$$

such that the unique solution of (2) with initial function $\psi$ satisfies $x(t, 0, \psi) \to k$ as $t \to \infty$.

*Proof.* Let us first show that (7) has a unique solution. Define a mapping $Q : R \to R$ by $k \in R$ implies that

$$Qk = \psi(0) + g(k)L - \int_{-L}^{0} g(\psi(s))ds.$$

Then for $k, d \in R$ and $| \cdot |$ denoting absolute value we have

$$|Qk - Qd| \leq L|g(k) - g(d)| \leq LK|k - d| \leq \alpha|k - d|,$$

so $Q$ is a contraction on the complete metric space $(R, | \cdot |)$. Thus $Q$ has a unique fixed point $k$.

Next, let $(M, \|\cdot\|)$ be the complete metric space of bounded continuous functions $\phi : [-L, \infty) \to R$ with $\phi(t) = \psi(t)$ on $[-L, 0]$, $\phi(t) \to k$ as $t \to \infty$, and $\|\cdot\|$ is the supremum metric.

Define $P : M \to M$ by $\phi \in M$ implies that $(P\phi)(t) = \psi(t)$ on $[-L, 0]$ and for $t > 0$ let

$$(P\phi)(t) = \psi(0) - \int_{-L}^{0} g(\psi(s))ds + \int_{t-L}^{t} g(\phi(s))ds; \tag{8}$$

we have used (2) to define $P$ and a fixed point will solve (1) and (2). Notice that since $\phi(t) \to k$ we have $\int_{t-L}^{t} g(\phi(s))ds \to g(k)L$ as $t \to \infty$. Using this, (8), and then (7) we see that $(P\phi)(t) \to k$ as $t \to \infty$. Thus, $P : M \to M$. To see that $P$ is a contraction we note that for $\phi, \eta \in M$ then

$$\begin{aligned} |(P\phi)(t) - (P\eta)(t)| &\le \int_{t-L}^{t} |g(\phi(s)) - g(\eta(s))|ds \\ &\le LK\|\phi - \eta\| \end{aligned}$$

so $P$ is a contraction with unique fixed point $\phi \in M$. By the way (8) was constructed, $\phi$ satisfies (2).

**Remark 1.** This theorem tells us precisely what the limit of each solution will be, based on its initial function. We note that in [7; p. 84] Cooke and Yorke show that if $g$ of a solution stays above a certain value for $L$ time units, then $g$ of it will never go below that value. They also show that if $g$ of the solution stays below a certain value for $L$ time units, then $g$ of it will stay below that value forever. These results can require care in interpretation unless $g(x)$ has the sign of $x$, a condition which is studiously avoided here. Our result is much sharper than that of Cooke and Yorke; given an initial function, we know exactly the limiting value of the corresponding solution. There is no need at all for the solution to stay above a certain value over an interval of length $L$; it merely needs to do so on some well-defined type of average. While the aforementioned papers of Kaplan, Sorg, and York [13] and Arino and Pituk [1] deal with general Lipschitz equations and show that solutions approach a constant, they do not seem to be able to identify that constant directly from the initial function. Moreover, both are restricted to finite delay, while we will illustrate that any of our problems can be extended to infinite delay equations.

**Remark 2.** Examine Equation (7) which determines the unique constant $k$ to which the solution $x(t,0,\psi)$ converges. Notice that $\psi$ enters as

$$\psi(0)-\int_{-L}^{0}g(\psi(s))ds.$$

If $\eta:[-L,0]\to R$ is continuous and satisfies

$$\eta(0)-\int_{-L}^{0}g(\eta(s))ds=\psi(0)-\int_{-L}^{0}g(\psi(s))ds$$

then Theorem 1 will also show that $x(t,0,\eta)\to k$. For a given $\psi$, there is an infinite set of functions $\eta$ which qualify and $\|\eta\|$ is unbounded. We can think of $k$ as being a "selective global attractor". The same observation can be made in all of the subsequent problems.

**Remark 3.** Equation (1) is also an extremely stable control problem. Given a desired target $k$, solve (7) for

$$\int_{-L}^{0}g(\psi(s))ds-\psi(0)=g(k)L-k.$$

The right-hand-side is a fixed constant. Pick any $\psi$ satisfying that equation. Use the chosen $\psi$ as the initial function. The resulting solution will approach the desired target, $k$. This will have significant application in a second order control problem in the last section of this paper.

Theorem 1 is in the way of a stability result. Continual dependence of solutions on initial functions tells us that solutions which start close will remain close on finite intervals. But under conditions of Theorem 1 they remain close forever and their asymptotic constants are close.

This is the only stability result we will state, but parallel work can be done for all the equations considered here.

**Theorem 2.** Under the conditions of Theorem 1, every continuous initial function is stable: for each $\epsilon>0$ there is a $\delta>0$ such that $\|\psi_1-\psi_2\|<\delta$ implies that $|x(t,0,\psi_1)-x(t,0,\psi_2)|<\epsilon$ for $t\geq 0$. In particular, if $x(t,0,\psi_1)\to k_1$ and $x(t,0,\psi_2)\to k_2$ then $|k_1-k_2|<\epsilon$.

*Proof.* We will use the notation of the proof of Theorem 1 and we have also denoted the supremum of initial functions $\psi$ on $[-L,0]$ by $\|\psi\|$, even though we also use that as the supremum metric on $[-L, \infty)$. Let $\psi_1$ be fixed and let $\psi_2$ be any other continuous initial function. Then by Theorem 1 there are unique $\phi_1, \phi_2, k_1, k_2$ such that if $P_i$ is the mapping defined with $\psi_i$ then

$$P_1\phi_1 = \phi_1, \quad P_2\phi_2 = \phi_2, \quad \phi_1(t) \to k_1, \quad \phi_2(t) \to k_2.$$

Notice that since $\phi_i = \psi_i$ on $[-L, 0]$ we have

$$\begin{aligned}
|\phi_1(t) - \phi_2(t)| &= |(P_1\phi_1)(t) - (P_2\phi_2)(t)| \\
&\leq |\psi_1(0) - \psi_2(0)| + \int_{-L}^{0} |g(\psi_1(s)) - g(\psi_2(s))|ds \\
&+ \int_{t-L}^{t} |g(\phi_1(s)) - g(\phi_2(s))|ds \\
&\leq |\psi_1(0) - \psi_2(0)| + KL\|\psi_1 - \psi_2\| + KL\|\phi_1 - \phi_2\|
\end{aligned}$$

or

$$\|\phi_1 - \phi_2\| \leq \frac{KL+1}{1-KL}\|\psi_1 - \psi_2\| < \epsilon$$

provided that

$$\|\psi_1 - \psi_2\| < \frac{\epsilon(1-KL)}{1+KL} =: \delta.$$

This proves the first part.

For the second part, we have $|\phi_i(t) - k_i| \to 0$ as $t \to \infty$ and so

$$\begin{aligned}
|k_1 - k_2| &= |k_1 - \phi_1(t) + \phi_1(t) - \phi_2(t) + \phi_2(t) - k_2| \\
&\leq |k_1 - \phi_1(t)| + \|\phi_1 - \phi_2\| + |\phi_2(t) - k_2|
\end{aligned}$$

and the last term tends to $\|\phi_1 - \phi_2\| < \epsilon$. This completes the proof.

Cooke and Yorke [7; p.85] show that for (1) solutions tend to a constant or to $\pm\infty$. They remark that periodicity in $g$ might yield a periodic solution (for each initial function) to which other solutions might converge. Their idea is that the growth is seasonally affected. Thus, it appears that they would divide $L$ into a fixed number of periods so that $L$, itself, would be a period, although probably not the smallest period. We will now show that simple inspection reveals that the only solutions of period $L$ are constant functions. As in Theorem 1, for a given $\psi$ there is a constant to which the solution converges.

Consider the equation

$$x'(t) = f(t, x(t)) - f(t, x(t - L)) \tag{9}$$

where $f$ is continuous and

$$f(t + L, x) = f(t, x) \tag{10}$$

for all $x$. By (10) we can write (9) as

$$\begin{aligned} x'(t) =& f(t, x(t)) - f(t - L, x(t - L)) \\ =& \frac{d}{dt}\int_{t-L}^{t} f(s, x(s))ds. \end{aligned} \tag{11}$$

**Theorem 3.** If (10) is satisfied and if (9) has a periodic solution of period $L$, then that solution is constant.

*Proof.* If $x(t)$ is a solution of (9) with period $L$, then we can integrate (11) and write

$$x(t) = x(0) + \int_{t-L}^{t} f(s, x(s))ds - \int_{-L}^{0} f(s, x(s))ds. \tag{12}$$

As $x(t + L) = x(t)$, it follows from (10) that

$$F(t) := f(t, x(t))$$

satisfies $F(t+L) = F(t)$. Thus, the integral of $F$ over any period of length $L$ has the same constant value. It then follows from (12) that $x(t) = x(0)$ for all $t$. This completes the proof.

Continuing with their question about periodicity, if $L$ is a period of $f$, then we can again find asymptotic limits of solutions. We will need a counterpart of (5). Suppose there is a constant $K$ such that $t, x, y \in R$ implies that

$$|f(t, x) - f(t, y)| \leq K|x - y|. \tag{13}$$

**Theorem 4.** Let (6), (10), and (13) hold for (9) and let $\psi : [-L, 0] \to R$ be a given continuous function. Then there is a unique constant $k$ satisfying

$$k = \psi(0) + \int_{t-L}^{t} f(s,k)ds - \int_{-L}^{0} f(s,\psi(s))ds \tag{14}$$

and the unique solution of (9) with this initial function satisfies $x(t, 0, \psi) \to k$ as $t \to \infty$.

Here is a brief sketch. The proof proceeds exactly as that of Theorem 1 when we notice in (14) that $f(s,k)$ has period $L$ in $s$ so that $\int_{t-L}^{t} f(s,k)ds$ is constant; thus, $Q$ has a fixed point. Defining $P$ from (12) and $M$ as before, it readily follows that $\phi(t) \to k$ implies that $(P\phi)(t) \to k$. We will see more detail of this type in later, more difficult, theorems.

Cooke and Yorke [7; p. 87] continue the study and propose a model in which they postulate a time lag $L_1$ between conception and birth. Then the number of births at time $t$ is $g(x(t - L_1))$ and (1) is replaced by

$$x'(t) = g(x(t - L_1)) - g(x(t - L_1 - L_2)) \tag{15}$$

or in integrated form

$$x(t) = c + \int_{t-L_1-L_2}^{t-L_1} g(x(s))ds. \tag{16}$$

They state that the integral in (16) is the number of individuals born in the past generation $[t - L_2, t]$ (conceived in $[t - L_1 - L_2, t - L_1]$). Their view is that $c$ must be zero for correct biological interpretation.

It is worth taking a look at their statement in [7; p. 87] concerning (1) and (16) in order to see what fixed point theory can do for this type of study. They state that analysis of (15) is much more difficult than that of (1) and they state that they have no result for (16). By contrast, we show that with fixed point theory analysis of (1) and (16) is the same. Moreover, they state that they expect a wider range of behavior of solutions of (16) than of (1) and, indeed, expect (16) to have periodic solutions. Fixed point theory shows that the behavior of solutions of (1) and (16) is the same and that there are no periodic solutions, except constants.

To fully describe (16) we need a continuous initial function $\psi : [-L_1 - L_2, 0] \to R$. Under (5) this will yield a unique solution $x(t, 0, \psi)$ of (15) satisfying

$$x(t) = \psi(0) + \int_{t-L_1-L_2}^{t-L_1} g(x(s))ds - \int_{-L_1-L_2}^{-L_1} g(\psi(s))ds \tag{17}$$

for $t > 0$. In this problem, $L$ is replaced by $L_2$ in (6) so we ask that there exist $\alpha < 1$ with

$$KL_2 \leq \alpha. \tag{18}$$

**Theorem 5.** Let (5) and (18) hold and let $\psi : [-L_1-L_2, 0] \to R$ be a given continuous function. Then there is a unique constant $k$ satisfying

$$k = \psi(0) + L_2 g(k) - \int_{-L_1-L_2}^{-L_1} g(\psi(s))ds$$

and the solution $x(t, 0, \psi)$ of (15) converges to $k$ as $t \to \infty$.

*Proof.* The mapping $Q$ of the proof of Theorem 1, adapted to (18), is a contraction with unique solution $k$. Let $(M, \|\cdot\|)$ be the complete metric space with the supremum metric and with

$$M = \{\phi : [-L_1 - L_2, \infty) \to R | \phi \in C, \phi(t) = \psi(t) \quad \text{on} \quad [-L_1 - L_2, 0], \quad \lim_{t\to\infty} \phi(t) = k\}.$$

Define $P : M \to M$ by $\phi \in M$ implies that $(P\phi)(t) = \psi(t)$ on $[-L_1 - L_2, 0]$, and for $t \geq 0$ then

$$(P\phi)(t) = \psi(0) + \int_{t-L_1-L_2}^{t-L_1} g(\phi(s))ds - \int_{-L_1-L_2}^{L_1} g(\psi(s))ds.$$

If $\phi, \eta \in M$ then we readily find that

$$|(P\phi)(t) - (P\eta)(t)| \leq L_2 K \|\phi - \eta\|,$$

so $P$ is a contraction. Since $\phi(t) \to k$ as $t \to \infty$, it follows that

$$(P\phi)(t) \to \psi(0) + L_2 g(k) - \int_{-L_1-L_2}^{-L_1} g(\psi(s))ds = k.$$

This completes the proof.

## 5. Jehu's periodic theory.

Jehu [12] considers an equation in the spirit of (9), although he does not mention the Cooke-Yorke work. His equation is

$$x'(t) = -f(t, x(t)) + f(t, x(t-1)) + h(t)$$

in which $f(t+1, x) = f(t, x), h(t+1) = h(t)$, $\int_{-1}^{0} h(s)ds = 0$, and $f(t, x)$ is strictly increasing in $x$ for fixed $t$. The equation does not have the property that each constant is a solution, unless $h(t) = 0$. He shows that each solution approaches a periodic solution. It is to be noted that his work does not require uniqueness and it does yield a periodic solution which is not necessarily constant. Jehu's work is totally different from ours, but there is one marked similarity. We have explicitly found the constant to which solutions converge, and Jehu explicitly finds the periodic function to which solutions converge.

Our purpose in this section is to show that $f$ can be replaced with a function which is simply Lipschitz, instead of being strictly increasing, but the Lipschitz constant must be restricted. That will force solutions to be unique as well.

Consider the equation

$$x'(t) = -f(t, x(t)) + f(t, x(t-L)) + h(t) \tag{19}$$

where (6), (10), and (13) hold, $h$ is continuous, $h(t+L) = h(t)$, and $\int_{-L}^{0} h(s)ds = 0$. This equation will have a first integral since it can be written as

$$x(t) = -\frac{d}{dt}\int_{t-L}^{t} f(s, x(s))ds + \frac{d}{dt}\int_{0}^{t} h(s)ds$$

and $\int_0^t h(s)ds$ is also $L-$periodic since $h$ has mean value zero.

For a given continuous initial function $\psi : [-L, 0] \to R$ we have

$$x(t) = -\int_{t-L}^{t} f(s, x(s))ds + \int_{0}^{t} h(s)ds + \psi(0) + \int_{-L}^{0} f(s, \psi(s))ds \tag{20}$$

and we define

$$c := \psi(0) + \int_{-L}^{0} f(s, \psi(s))ds. \tag{21}$$

**Theorem 6**. If (6), (10), and (13) hold, then there is an $L$-periodic function $\gamma$ satisfying

$$\gamma(t) = c - \int_{t-L}^{t} f(s, \gamma(s))ds + \int_{0}^{t} h(s)ds$$

and the unique solution $x(t, 0, \psi)$ of (2) converges to $\gamma$ as $t \to \infty$.

*Proof.* Let $(Y, \|\cdot\|)$ be the Banach space of continuous $L$-periodic functions $\phi : R \to R$ with the supremum norm and define a mapping $Q : Y \to Y$ by $\phi \in Y$ implies that

$$(P\phi)(t) = c - \int_{t-L}^{t} f(s, \phi(s))ds + \int_{0}^{t} h(s)ds.$$

Because of (10) and the fact that $h$ has mean value zero, $Q$ does map $Y$ into $Y$. Because of (6) and (13), $Q$ is a contraction and there is a unique fixed point $\gamma$ in $Y$.

Notice that $\gamma$ does satisfy (19) and for each constant $c$ there is such a periodic solution of (19).

Let $(X, \|\cdot\|)$ be the complete metric space of continuous functions $\phi : [-L, \infty) \to R$ satisfying $\phi(t) = \psi(t)$ on $[-L, 0]$, $\phi(t) \to \gamma(t)$ as $t \to \infty$. Define $P : X \to X$ by $\phi \in X$ implies that $(P\phi)(t) = \psi(t)$ on $[-L, 0]$ and

$$(P\phi)(t) = -\int_{t-L}^{t} f(s, \phi(s))ds + \int_{0}^{t} h(s)ds + c.$$

Since $(P\phi)(0) = \psi(0)$, $P$ does map $X \to X$. By (6) and (13), $P$ is a contraction. Moreover, if $-J \leq \gamma(t) \leq J$ for some $J > 0$, since $f$ is uniformly continuous on $[0, L] \times [-2J, 2J]$, since $f$ is periodic in $t$, and since $\phi(t) \to \gamma(t)$, it follows that

$$|\int_{t-L}^{t} f(s, \phi(s))ds - \int_{t-L}^{t} f(s, \gamma(s))ds| \to 0;$$

thus, $(P\phi)(t) \to \gamma(t)$. This completes the proof.

## 6. Some generalizations of the delay: periodicity.

Brilliant as was the idea of Cooke and Yorke to let $g$ be an arbitrary differentiable function, it was not matched by the assumption that the number of deaths would be represented by $g(x(t - L))$. The deaths of those born at time $t$ would be distributed all along the time period $[t, t + L]$, and certainly a few beyond $t + L$.

A seemingly more appropriate model was offered by Haddock and Terjeki [10] (see also [8]) in the form of

$$x'(t) = -g(x(t)) + \int_{-L}^{0} p(s)g(x(t+s))ds \tag{23}$$

with

$$\int_{-L}^{0} p(s)ds = 1$$

but was removed from the Cooke-Yorke class by the additional assumptions of $g(0) = 0$ and $g$ strictly increasing. They also considered

$$x'(t) = F[-x(t) + ax(t-L) + b\int_{-\infty}^{0} p(s)x(t+s)ds] \tag{24}$$

with $F$ increasing, $F(0) = 0, a + b = 1, \int_{-\infty}^{0} p(s)ds = 1$. The linearity inside $F$ and the monotonicity of $F$ seem critical.

We will examine what can be said for these kinds of distributed delays, without the monotonicity assumptions, using contraction mappings. Thus, we consider

$$x'(t) = g(x(t)) - \int_{t-L}^{t} p(s-t)g(x(s))ds \tag{25}$$

with $p$ continuous,

$$\int_{-L}^{0} p(s)ds = 1, \tag{26}$$

and assume that there is a constant $K$ such that for all real $x, y$ we have

$$|g(x) - g(y)| \le K|x - y|. \tag{27}$$

For the contraction condition we will need $\alpha < 1$ such that

$$K\int_{-L}^{0} |p(s)|(-s)ds \le \alpha. \tag{28}$$

**Remark 4.** If $p(t) = 1/L$ so that there is a uniform distribution of deaths over an interval of length $L$ then (26) is satisfied, while (28) yields $KL/2 \le \alpha$, a weaker requirement than in the constant delay case.

We can write (25) as

$$x'(t) = \frac{d}{dt}\int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))duds. \tag{29}$$

Then to specify a solution we need a continuous function $\psi : [-L, 0] \to R$ so that we can write (25) as

$$x(t) = \psi(0) + \int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))duds - \int_{-L}^{0} p(s)\int_{s}^{0} g(\psi(u))duds. \tag{30}$$

This theorem is actually a corollary to Theorem 8, but the proof is so simple it seems wrong to embed it in that framework with a much more complicated proof.

**Theorem 7.** Suppose that (26), (27), and (28) hold. For the given initial function $\psi$ there is a unique constant $k$ satisfying

$$k = \psi(0) + g(k)\int_{-L}^{0} p(s)(-s)ds - \int_{-L}^{0} p(s)\int_{s}^{0} g(\psi(u))duds \tag{31}$$

and the unique solution $x(t, 0, \psi)$ of (3) converges to $k$ as $t \to \infty$.

*Sketch of proof.* Use (31) to define a mapping $Q$ as we did in the proof of Theorem 1. The mapping will be a contraction because of (28). We then define

$$M = \{\phi : [-L, 0] \to R | \phi \in C, \phi(t) = \psi(t) \quad \text{on} \quad [-L, 0], \lim_{t\to\infty} \phi(t) = k\}. \tag{32}$$

With the supremum metric, $M$ is a complete metric space. Use (30) to define a mapping $P$ of that space into itself. In particular, we note that $\phi \in M$ implies $(P\phi)(t) \to k$ as $t \to \infty$. Also, $P$ will be a contraction because of (28) with unique fixed point. This completes the proof.

Consider the scalar equation

$$x'(t) = g(t, x(t)) - \int_{t-L}^{t} p(s-t)g(s, x(s))ds \tag{33}$$

where $g$ and $p$ are continuous, there is a $K > 0$ such that $x, y \in R$ implies

$$|g(t, x) - g(t, y)| \le K|x - y|, \tag{34}$$

$$g(t+L,x)=g(t,x), \tag{35}$$

and $p$ satisfies (26).

We can write (33) as

$$x'(t)=\frac{d}{dt}\int_{-L}^{0}p(s)\int_{t+s}^{t}g(u,x(u))duds. \tag{36}$$

Then for a continuous initial function $\psi$ we can write (36) as

$$x(t)=\int_{-L}^{0}p(s)\int_{t+s}^{t}g(u,x(u))duds+\psi(0)-\int_{-L}^{0}p(s)\int_{s}^{0}g(u,x(u))duds.$$

In the next result we could also prove that the solutions are stable following the ideas in the proof of Theorem 2.

**Theorem 8.** If (26), (28), (34), and (35) hold, then for a given continuous function $\psi:[-L,0]\to R$ there is a unique periodic solution $\gamma$ of (33) having period $L$ and the unique solution $x(t,0,\psi)$ of (33) tends to $\gamma$ as $t\to\infty$. Also, $\gamma$ is constant only if there is a $k\in R$ with

$$g(t,k)=\int_{-L}^{0}p(s)g(t+s,k)ds.$$

*Proof.* Let $c=\psi(0)-\int_{-L}^{0}p(s)\int_{s}^{0}g(u,\psi(u))duds$ and let $(X,\|\cdot\|)$ be the Banach space of continuous $L$-periodic functions with the supremum norm. Define $Q:X\to X$ by $\phi\in X$ implies that

$$(Q\phi)(t)=c+\int_{-L}^{0}p(s)\int_{t+s}^{t}g(u,\phi(u))duds.$$

Note that

$$\begin{aligned}(Q\phi)(t+L)&=c+\int_{-L}^{0}p(s)\int_{t+L+s}^{t+L}g(u,\phi(u))duds\\&=c+\int_{-L}^{0}p(s)\int_{t+s}^{t}g(u+L,\phi(u+L))duds\\&=(Q\phi)(t).\end{aligned}$$

To see that $Q$ is a contraction, if $\phi,\eta\in X$ then

$$|(Q\phi)(t)-(Q\eta)(t)|\le\int_{-L}^{0}|p(s)|\int_{t+s}^{t}Kduds\|\phi-\eta\|\le\alpha\|\phi-\eta\|.$$

Hence, $Q$ has a unique fixed point $\gamma \in X$, an $L$-periodic solution of (33).

Now, let $(M, \|\cdot\|)$ be the complete metric space of continuous functions $\phi$ with $\phi(t) = \psi(t)$ on $[-L, 0]$ and $\phi(t) \to \gamma(t)$ as $t \to \infty$. We take the supremum metric.

Define a mapping $P : M \to M$ by $\phi \in M$ implies that $(P\phi)(t) = \psi(t)$ on $[-L, 0]$, and for $t > 0$ define

$$\begin{aligned}(P\phi)(t) = \psi(0) &- \int_{-L}^{0} p(s) \int_{s}^{0} g(u, \psi(u))duds \\ &+ \int_{-L}^{0} p(s) \int_{t+s}^{t} g(u, \phi(u))duds \\ &= c + \int_{-L}^{0} p(s) \int_{t+s}^{t} g(u, \phi(u))duds.\end{aligned}$$

We must show that $(P\phi)(t) \to \gamma(t)$. From the definition of $\gamma$ and $P$ for $t > L$ we have

$$\begin{aligned}|(P\phi)(t) - \gamma(t)| &= |\int_{-L}^{0} p(s) \int_{t+s}^{t} g(u, \phi(u))du - \int_{-L}^{0} p(s) \int_{t+s}^{t} g(u, \gamma(u))duds| \\ &\leq \int_{-L}^{0} |p(s)| \int_{t+s}^{t} K|\phi(u) - \gamma(u)|duds \\ &\leq \int_{-L}^{0} |p(s)| \int_{t-L}^{t} K|\phi(u) - \gamma(u)|duds\end{aligned}$$

and the last term tends to zero as $t \to \infty$ since $|\phi(u) - \gamma(u)| \to 0$ as $u \to \infty$ and $p$ is continuous (so is bounded).

The details for showing that $P$ is a contraction are identical to that of showing that $Q$ is a contraction. Hence, $P$ has a unique fixed point in $M$ and that fixed point does converge to $\gamma$.

If (33) has a constant solution $k$, then $x'(t) = k' = 0$ so in (36)

$$\int_{-L}^{0} p(s)[g(t, k) - g(t + s, k)]ds = 0$$

or

$$g(t, k) \int_{-L}^{0} p(s)ds = \int_{-L}^{0} p(s)g(t + s, k)ds$$

and since $\int_{-L}^{0} p(s)ds = 1$ we have

$$g(t, k) = \int_{-L}^{0} p(s)g(t + s, k)ds.$$

This completes the proof.

**Remark 5.** It is worth thinking about the proof for a moment. One is tempted to say that $P$ and $Q$ are the same, but they are not because $P$ is defined separately on $[-L, 0]$. The solution $\phi$ will be periodic only if one is so fortunate as having guessed an initial function $\psi$ whose periodic extension to all of $R$ is a fixed point of $Q$.

## 7. Our preferred model.

Cooke and Yorke produced the model (15) and provided the rationale given with it. We have just considered the model of Haddock and Terjeki (23). And we have yet to consider their model (24). Careful consideration of these three models leads us to suggest the model

$$x'(t) = \int_{t-L}^{t} p(s-t)g(x(s))ds - \int_{-\infty}^{t} q(s-t)g(x(s))ds \tag{37}$$

as the one embracing the most realistic properties from each of the models. The first term on the right takes into account the ideas from (15) in a more general form. The second term takes into account the deaths distributed over all past times. Here, we suppose that

$$\int_{-L}^{0} p(s)ds = 1 \tag{38}$$

while

$$\int_{-\infty}^{0} q(s)ds = 1 \quad \text{and} \quad \int_{-\infty}^{0}\int_{-\infty}^{v} |q(u)|dudv \quad \text{exists} . \tag{39}$$

Moreover, the equation offers an interesting challenge in that it does not have a ready first integral. There is a simple fixed point solution, but there is a price to pay. Write (37) as

$$x'(t) = \int_{t-L}^{t} p(s-t)g(x(s))ds - g(x(t)) + g(x(t)) - \int_{-\infty}^{t} q(s-t)g(x(s))ds \tag{40}$$

and then write

$$x'(t) = -\frac{d}{dt}\int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))duds + \frac{d}{dt}\int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)dug(x(s))ds. \tag{41}$$

If $c$ is any constant then

$$x(t) = c - \int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))duds + \int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)dug(x(s))ds \tag{42}$$

is a solution of (41).

Let $\psi : (-\infty, 0] \to R$ be a bounded continuous function and write (41) as

$$x(t) = -\int_{-L}^{0} p(s) \int_{t+s}^{t} g(x(u))duds + \int_{-\infty}^{t} \int_{-\infty}^{s-t} q(u)dug(x(s))ds$$

$$\psi(0) + \int_{-L}^{0} p(s) \int_{s}^{0} g(\psi(u))duds - \int_{-\infty}^{0} \int_{-\infty}^{s} q(u)dug(\psi(s))ds. \tag{43}$$

It will be convenient to denote

$$c := \psi(0) + \int_{-L}^{0} p(s) \int_{s}^{0} g(\psi(u))duds - \int_{-\infty}^{0} \int_{-\infty}^{s} q(u)dug(\psi(s))ds. \tag{44}$$

We suppose there is a $K > 0$ such that $x, y \in R$ implies that

$$|g(x) - g(y)| \leq K|x - y| \tag{45}$$

and there is an $\alpha < 1$ such that

$$\int_{-L}^{0} |p(s)|(-s)ds + \int_{-\infty}^{0} \int_{-\infty}^{v} |q(u)|dudv \leq \alpha/K. \tag{46}$$

**Theorem 9.** Let (38), (39), (45), and (46) hold. Then there is a unique constant $k$ satisfying

$$k = c - g(k) \int_{-L}^{0} p(s)(-s)ds + g(k) \int_{-\infty}^{0} \int_{-\infty}^{v} q(u)dudv \tag{47}$$

where $c$ is defined in (44). The unique solution $x(t, 0, \psi)$ of (43) approaches $k$ as $t \to \infty$.

*Proof.* Use (47) to define a mapping $Q : R \to R$ by

$$Qk = c - g(k) \int_{-L}^{0} p(s)(-s)ds + g(k) \int_{-\infty}^{0} \int_{-\infty}^{v} q(u)dudv$$

so that for $k, d \in R$ we have

$$\begin{aligned} |Qk - Qd| &\leq K|k - d| \int_{-L}^{0} |p(s)s|ds + K|k - d| \int_{-\infty}^{0} \int_{-\infty}^{v} |q(u)|dudv \\ &\leq \alpha|k - d| \end{aligned}$$

by (46). This yields the unique $k$.

For the given $\psi$ and the fixed point $k$, define $M$ as the set of continuous functions $\phi : (-\infty, \infty) \to R$ with $\phi(t) = \psi(t)$ on $(-\infty, 0]$ and $\phi(t) \to k$ as $t \to \infty$. Use the supremum metric. Define $P : M \to M$ by $\phi \in M$ implies that $(P\phi)(t) = \psi(t)$ on $(-\infty, 0]$ and for $t \geq 0$ then use (43) and (44) to define

$$(P\phi)(t) = c - \int_{-L}^{0} p(s) \int_{t+s}^{t} g(\phi(u))duds + \int_{-\infty}^{t} \int_{-\infty}^{s-t} q(u)dug(\phi(s))ds.$$

We must show that $(P\phi)(t) \to k$ as $t \to \infty$.

For a fixed $\phi \in M$, let $\epsilon > 0$ be given and find positive numbers $J$ and $T$ such that $|g(\phi(t)) - g(k)| \leq J$ for all $t$ and $|\phi(t) - k| < \epsilon$ if $T \leq t < \infty$.

First, it is clear that

$$\int_{-L}^{0} p(s) \int_{t+s}^{t} g(\phi(u))duds \to g(k) \int_{-L}^{0} p(s)(-s)ds$$

as $t \to \infty$. Next,

$$\begin{aligned}
&|\int_{-\infty}^{t} \int_{-\infty}^{s-t} q(u)du(g(\phi(s)) - g(k))ds| \\
&\leq J \int_{-\infty}^{T} \int_{-\infty}^{s-t} |q(u)|duds + \int_{T}^{t} \int_{-\infty}^{s-t} |q(u)|duK\epsilon ds \\
&= J \int_{-\infty}^{T-t} \int_{-\infty}^{s} |q(u)|duds + K\epsilon \int_{-\infty}^{t} \int_{-\infty}^{s-t} |q(u)|duds.
\end{aligned}$$

In the last line, the first term tends to zero as $t \to \infty$ because of the assumed convergence in (46). The second term is bounded by $K\epsilon\alpha/K = \alpha\epsilon$ by (46). Hence,

$$\int_{-\infty}^{t} \int_{-\infty}^{s-t} q(u)dug(\phi(s))ds \to g(k) \int_{-\infty}^{t} \int_{-\infty}^{s-t} q(u)duds$$

as $t \to \infty$. Comparing these results with (47) shows that $(P\phi)(t) \to k$.

To see that $P$ is a contraction, if $\phi, \eta \in M$ then

$$\begin{aligned}
&|(P\phi)(t) - (P\eta)(t)| \\
&\leq K\|\phi - \eta\| \int_{-L}^{0} |p(s)s|ds + K\|\phi - \eta\| \int_{-\infty}^{t} \int_{-\infty}^{s-t} |q(u)|duds \\
&\leq \alpha\|\phi - \eta\|.
\end{aligned}$$

There is a fixed point in $M$ which meets the conditions in the theorem.

**Remark 6.** This is the only inifinite delay problem we will do here, but the interested reader will see that all of our delays could be changed to infinite delays and the analysis would be parallel to that just given. We could also follow the proof of Theorem 8 and introduce periodicity in $t$ in $g(t,x)$. Finally, solutions are stable and each limit constant is a selective global attractor.

## 9. Neutral equations

Considerable effort has gone into extending the theory of equations where each constant is a solution to neutral equations. And there is good reason for doing so. The problem has its roots in mathematical biology, a subject in which neutral-type behavior is ubiquitous. Every parent and every gardener has observed a living organism displaying ordinary or sub-ordinary growth. Suddenly, growth accelerates and acceleration gives birth to more acceleration until the observer may claim to actually see the growth taking place. This is typical of neutral growth. Present growth rate depends not only on the past state, but on the past growth rate. A typical example in our context is

$$\frac{d}{dt}(x(t) - h(x(t-L_1))) = g(x(t)) - g(x(t-L_2)). \tag{48}$$

Clearly, any constant function is a solution. We suppose that $g, h : R \to R$ and that there are positive constants $K_1, K_2$ so that for all $x, y \in R$ we have

$$|h(x) - h(y)| \leq K_1|x - y| \tag{49}$$

and

$$|g(x) - g(y)| \leq K_2|x - y| \tag{50}$$

with an $\alpha < 1$ such that

$$K_1 + L_2K_2 \leq \alpha. \tag{51}$$

If $L = \max(L_1, L_2)$ and if $\psi : [-L, 0] \to R$ is continuous, then we can write our equation with that initial function as

$$x(t) = h(x(t-L_1)) + \int_{t-L_2}^{t} g(x(s))ds + \psi(0) - h(\psi(-L_1)) - \int_{-L_2}^{0} g(\psi(s))ds. \tag{52}$$

**Theorem 10.** Let the above conditions hold and let $\psi$ be a given continuous function. There is a unique constant $k$ satisfying

$$k = h(k) + g(k)L_2 + \psi(0) - h(\psi(-L_1)) - \int_{-L_2}^{0} g(\psi(s))ds \tag{53}$$

and the unique solution $x(t, 0, \psi)$ of (52) tends to $k$ as $t \to \infty$.

*Proof.* Use (53) to define a mapping $Q$ as before. By (51) it will be a contraction with unique fixed point. For the given $\psi$, let $(M, \|\cdot\|)$ be the complete metric space of continuous functions $\phi : [-L, \infty) \to R$ with $\phi(t) = \psi(t)$ on $[-L, 0]$ and $\phi(t) \to k$ as $t \to \infty$. Define $P : M \to M$ by $\phi \in M$ implies that $(P\phi)(t) = \psi(t)$ on $[-L, 0]$ and for $t \geq 0$ then

$$(P\phi)(t) = h(\phi(t - L_1)) + \int_{t-L_2}^{t} g(\phi(s))ds + \psi(0) - h(\psi(-L_1)) - \int_{-L_2}^{0} g(\psi(s))ds.$$

We may note that $(P\phi)(0) = \psi(0)$ and that since $\phi(t) \to k$ we have $(P\phi)(t) \to k$ as $t \to \infty$. Moreover, $P$ is a contraction since for $\phi, \eta \in M$ we have

$$\begin{aligned} |(P\phi)(t) - (P\eta)(t)| &\leq K_1|\phi(t - L_1) - \eta(t - L_1)| + \int_{t-L_2}^{t} K_2|\phi(s) - \eta(s)|ds \\ &\leq (K_1 + K_2L_2)\|\phi - \eta\|. \end{aligned}$$

This completes the proof.

There are two results in Haddock *et al* [9] which compare loosely with the one just given. Theirs requires sign conditions and linearity, while ours does not. Our results specify precisely the limit of each solution, while theirs does not. We strongly control the Lipschitz constant and the delay, while they do not.

Arino and Pituk [1] use a condition closely akin to (51) and Krasnoselskii's fixed point theorem to prove that solutions exponentially approach constants and that those constants are uniformly stable. We obtain more in that we find the constants, our technique of Theorem 2 would prove each solution stable, we could show that the constant limit is a selective global attractor, and our techniques work equally well in the infinite delay case. Periodicity can be considered as before. Their problem is much more abstract and their analysis is deep. The contraction argument is quick and totally elementary.

## 10. Higher order equations.

Several authors have investigated this problem for systems, but the central condition of Cooke and Yorke that $g$ be an arbitrary differentiable function, possibly bounded by $K|x|$, seems to have been pushed aside, except in [1]. Atkinson and Haddock [2] and Atkinson, Haddock, and Staffans [3] have looked at problems involving a system $x' = P(t)[x(t) - x(t-h)]$ and have found a variety of conditions, such as $P \in L^2[0, \infty)$ to ensure that solutions approach constants.

In this section we offer a different kind of example which we feel is closer to the original problem posed by Cooke and Yorke.

Minorsky designed an automatic steering device for the large ship the New Mexico which was governed by an equation

$$x''(t) + cx'(t) + g(x(t-h)) = 0.$$

Here, the rudder of the ship has angular position $x(t)$ and there is friction force proportional to the velocity. There is a direction indicating instrument which points in the actual direction of motion and there is an instrument pointing in the desired direction. These two are connected to a device which activates an electric motor producing a certain force to move the rudder so as to bring the ship onto the desired course. There is a time lag of amount $h$ between the time the ship gets off course and the time the electric motor activates the restoring force. The object is to give conditions ensuring that $x(t)$ will stay near zero so that the ship closely follows its proper course. One may find a bit about the problem in Minorsky [17; p. 517] and in Burton [4; p. 149].

These kinds of control devices tend to over-correct the errors. The Cooke-Yorke type equation may do a much better job. If Minorsky had applied the opposite force $L$-time units later, say $g(x(t-h-L))$, then his ship would have eventually gone in a straight line, but on a course parallel to his desired course, provided that $g$ satisfies a Lipschitz condition and that $h$ and $L$ are not too large.

Instead of Minorsky's equation we look at

$$x'' + cx'(t) + g(x(t-h)) - g(x(t-h-L)) = 0 \tag{54}$$

which we write as

$$x'' + cx' = -\frac{d}{dt}\int_{t-h-L}^{t-h} g(x(s))ds. \tag{55}$$

Given an initial function $\psi : [-h-L, 0] \to R$ and $x'(0)$ we can write (55) as

$$x'(t) + cx(t) = x'(0) + c\psi(0) + \int_{-h-L}^{-h} g(\psi(s))ds - \int_{t-h-L}^{t-h} g(x(s))ds$$

$$=: \Psi - \int_{t-h-L}^{t-h} g(x(s))ds. \tag{56}$$

and then as

$$x(t) = \psi(0)e^{-ct} + \int_0^t e^{-c(t-s)}[\Psi - \int_{s-h-L}^{s-h} g(x(u))du]ds. \tag{57}$$

Following the proof of Theorem 5, we ask that there is $K > 0$ such that for all real $x, y$ we have

$$|g(x) - g(y)| \leq K|x - y| \tag{58}$$

and there is an $\alpha < 1$ with

$$KL \leq \alpha c. \tag{59}$$

**Theorem 11.** Let (58) and (59) hold, let $\psi$ be a given continuous initial function on $[-h-L, 0]$, let $x'(0)$ be given, and let $\Psi$ be defined in (56). Then there is a unique constant $k$ satisfying

$$ck = \Psi - g(k)L \tag{60}$$

and the unique solution $x(t, 0, \psi, x'(0))$ of (57) converges to $k$ as $t \to \infty$.

*Proof.* Clearly, the mapping $Q : R \to R$ defined by $k \in R$ implies

$$Qk := (\Psi - g(k)L)/c \tag{61}$$

is a contraction with unique fixed point. Let $(M, \|\cdot\|)$ be the complete metric space of continuous functions $\phi : [-h-L, \infty) \to R$ satisfying $\phi(t) \to k$ as $t \to \infty$, $\phi(t) = \psi(t)$ on $[-h-L, 0]$. Define $P : M \to M$ by $\phi \in M$ implies that $(P\phi)(t) = \psi(t)$ on $[-h-L, 0]$ and for $t > 0$ then

$$(P\phi)(t) = \psi(0)e^{-ct} + \int_0^t e^{-c(t-s)}[\Psi - \int_{s-h-L}^{s-h} g(\phi(u))du]ds. \tag{62}$$

We have

$$(P\phi)(t) = \psi(0)e^{-ct} + \int_0^t e^{-c(t-s)}\Psi ds$$
$$- \int_0^t e^{-c(t-s)} \int_{s-h-L}^{s-h} [g(\phi(u)) - g(k) + g(k)]duds.$$

Now, examine part of the last term. Let $T$ be a large positive number and let

$$\|\phi - k\|_{[T,\infty)} = \sup_{t \geq T-h-L} |\phi(u) - k|.$$

Then

$$|H(t)| := |\int_0^t e^{-c(t-s)} \int_{s-h-L}^{s-h} [g(\phi(u) - g(k)]duds|$$
$$\leq \int_0^T e^{-c(t-s)} KL\|\phi - k\|ds + \int_T^t e^{-c(t-s)} KL\|\phi - k\|_{[T,\infty)} ds.$$

The last term can be made as small as we please by taking $T$ large since $\phi(t) \to k$. The other term tends to zero; hence, this whole term, $H(t)$, can be made as small as we please by taking $t$ large.

Thus, we can say that

$$(P\phi)(t) = \psi(0)e^{-ct)} + \frac{\Psi}{c}[1 - e^{-ct}]$$
$$- \frac{g(k)L}{c}[1 - e^{-ct}] + H(t)$$

which tends to

$$\frac{\Psi}{c} - \frac{g(k)L}{c} = k$$

as $t \to \infty$.

Moreover, $P$ is a contraction since $\phi, \eta \in M$ imply that

$$|(P\phi)(t) - (P\eta)(t)| \leq \int_0^t e^{-c(t-s)} LK\|\phi - \eta\|ds \leq \frac{LK}{c}\|\phi - \eta\|.$$

**Remark 6.** We now remind the reader of Remark 3. Given a target $k$, we can solve (60) for $\Psi$. If we then select any initial function having the value in (56) of this number $\Psi$

and manually steer the ship in the pattern of that initial function, then the solution will converge to $k$. This is a type of control problem.

Next, we consider a problem having an asymptotically periodic solution.

Let $a$ be a positive constant, $g(t, x)$ a continuous scalar function satisfying

$$|g(t,x) - g(t,y)| \leq K|x - y| \tag{63}$$

for all $x, y \in R$ and some $K > 0$. Assume that there is an $L > 0$ with

$$g(t + L, x) = g(t, x) \tag{64}$$

for all $t$ and $x$, while there is an $\alpha < 1$ with

$$KL/a \leq \alpha. \tag{65}$$

Consider the scalar equation

$$x'' + ax' + g(t, x(t)) - g(t, x(t - L)) = 0. \tag{66}$$

Every constant function is a solution and the equation does have a first integral since

$$x'' + ax' + \frac{d}{dt}\int_{t-L}^{t} g(s, x(s))ds = 0. \tag{67}$$

We finish with a simple artificial model containing a periodic term, following the standard spring-mass-dashpot system found in elementary differential equations texts. Distributed and infinite delays, as well as stability, would present no more difficulties than in earlier first order problems.

There is a unit mass which is free to move along the $x$-axis. It has a rod attached to it which leads to the dashpot exerting a force of $-ax'(t)$ on the mass. There is also a rod attached to the mass which exerts a force $-g(t, x)$ on the mass when the mass is in position $x(t)$ at time $t$. But exactly $L$ time units later the last mentioned rod also exerts a force of $g(t, x(t - L))$ on the mass. We have

$$x'' = -ax'(t) - g(t, x(t)) + g(t, x(t - L)). \tag{68}$$

The mass is subjected to an initial velocity and initial position $\psi$. What will the limiting position of the mass be?

Given a continuous initial function $\psi : [-L, 0] \to R$ and an $x'(0)$ we can write (67) as

$$x'(t) + ax(t) = x'(0) + a\psi(0) + \int_{-L}^{0} g(s, \psi(s))ds - \int_{t-L}^{t} g(s, x(s))ds$$

$$(69) \qquad =: \Psi - \int_{t-L}^{t} g(s, x(s))ds$$

and then as

$$(70) \qquad x(t) = \psi(0)e^{-at} + \int_{0}^{t} e^{-a(t-s)}[\Psi - \int_{s-L}^{s} g(u, x(u))du]ds.$$

**Theorem 11.** Let (63)-(65) hold and let $\psi : [-L, 0] \to R$ be continuous with

$$\Psi = x'(0) + a\psi(0) + \int_{-L}^{0} g(s, \psi(s))ds.$$

Then there is an $L$-periodic function $\gamma$ satisfying

$$\gamma(t) = \int_{-\infty}^{t} e^{-a(t-s)}[\Psi - \int_{s-L}^{s} g(u, \gamma(u))du]$$

and the unique solution $x(t, 0, \psi)$ of (70) converges to $\gamma(t)$ as $t \to \infty$.

*Sketch of proof.* For that given initial function we can find a periodic solution to which the solution $x(t, 0, \psi)$ converges as $t \to \infty$. Instead of integrating as we did just now, write

$$(xe^{at})' = e^{at}[\Psi - \int_{t-L}^{t} g(s, x(s))ds].$$

Formally integrate from $-\infty$ to $t$ and obtain

$$(71) \qquad x(t) = \int_{-\infty}^{t} e^{-a(t-s)}[\Psi - \int_{s-L}^{s} g(u, x(u))du]ds.$$

If we use this last equation to define a mapping $Q$ from the space of continuous $L$ periodic functions with the supremum norm into itself, then we can show that $Q$ is a contraction with a unique fixed point $\gamma$.

Next, we define $M$ as the set of continuous functions $\psi : [-L, \infty) \to R$ which agree with $\psi$ on the initial interval and which satisfy $\phi(t) \to \gamma(t)$ as $t \to \infty$. We then use (70) to define a mapping $P$ of $M$ into itself and show that it is a contraction under the supremum metric. We then show that $(P\phi)(t) \to \gamma(t)$ using the classical proof that the convolution of an $L^1-$function with a function tending to zero does, itself, tend to zero.

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