Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **75**, 1–13; **http://www.math.u-szeged.hu/ejqtde/**

CAUCHY PROBLEMS FOR FUNCTIONAL EVOLUTION INCLUSIONS INVOLVING ACCRETIVE OPERATORS

TRAN DINH KE

Abstract. We study the existence and stability of solutions for a class of nonlinear functional evolution inclusions involving accretive operators. Our approach is employing the fixed point theory for multivalued maps and using estimates via the Hausdorff measure of noncompactness.

1. Introduction

Let X be a Banach space. We consider the following problem

$$u'(t) + Au(t) \ni f(t), t \in J := [0, T], \tag{1.1}$$

$$f(t) \in F(t, u(t), u_t), \tag{1.2}$$

$$u_t(s) = u(t + s),\ u(s) = \varphi(s), s \in [-h, 0], \tag{1.3}$$

where the state function u takes values in X, A is an m-accretive operator on X and F is a multivalued function defined on $J \times X \times C([-h, 0]; X)$.

It is known that system (1.1)–(1.3) is an abstract model of many problem involving retarded differential equations and inclusions. In the case when A is a linear operator and F is single-valued, there has been a great literature devoted to studying the global existence and asymptotic behavior of solutions. Regarding the latter objective, one of the most important and interesting problems is studying the stability of solutions to (1.1)–(1.3). For the stability theory for functional differential equations, see for instance the monographs of Driver [7], Halanay [9] and Hale [10]. Since the uniqueness for (1.1)–(1.3) is unavailable, the stability for this problem is a quite large subject. In the present paper, we will touch only the initial data dependence of the solution set and the exponential stability of the zero solution of problem (1.1)–(1.3) after proving its global solvability.

Nowadays, the evolution inclusions associated with m-accretive operators and nonlinear perturbations are getting more attractive. There are many works studying such problems with/without delays and subject to standard/nonlocal initial conditions. Let us quote in this note some significant results in [2, 12, 15, 17], among others. In most cases, the authors of the mentioned papers assume that $-A$ generates a compact semigroup. This assumption is then utilized to prove some compactness properties of the solution operator (whose fixed points are desired solutions). The reader is also referred to [5, 6] for some generalized cases of undelayed evolution problems with accretive operators. Precisely, in [5] some

2010 *Mathematics Subject Classification.* 35B35, 35R10, 47B44, 47H08, 47H10.
Key words and phrases. Exponential stability; fixed point; measure of noncompactness; MNC estimate.
The work was supported by the Ministry of Education and Training of Vietnam under grant No. B2013-17-42.

range conditions were imposed on A instead of m-accretive property while in [6], in dealing with reaction-diffusion systems, reaction terms (nonlinearities) were supposed to be merely measurable. In this paper, by using the techniques presented by Bothe in [4], we treat (1.1)–(1.3) in the case that the semigroup generated by $-A$ is equicontinuous only. The latter case, in particular, makes sense when A is in the form of the subdifferential of a proper, convex and lower semicontinuous functional Φ so that the level set $H_R = \{x \in X : ||x||^2 + \Phi(x) \leq R\}$ is not compact (see, e.g. [16]). To deal with the case of a noncompact semigroup, we impose a regular condition on the multivalued nonlinearity F expressed by measures of noncompactness (MNCs) in order to employ the technique of MNC estimates. Under this condition, we first prove that (1.1)–(1.3) is globally solvable for any $T > 0$ in Section 3. It is worth noting that, our existence result, in part, extends the one obtained by Bothe [4]. In Section 4, since the solution set is compact, we show that it depends semicontinuously on the initial data. Furthermore, under some additional assumptions, we prove that the zero solution of (1.1)–(1.3) is exponentially stable by using Halanay's inequality. In comparison with [4], the retarded case in our problem needs more sophisticated MNC estimates. The last section is an application of obtained results for a concrete problem, namely, the doubly nonlinear boundary problem with delays.

2. Preliminaries

Let $(X, \|.\|)$ be a Banach space and $\mathcal{P}(X)$ the collection of all nonempty subsets of X. For $x, y \in X, h \in \mathbb{R}\backslash\{0\}$, the following *product*

$$[x, y]_+ = \lim_{h \downarrow 0} \frac{\|x + hy\| - \|x\|}{h}$$

exists and satisfies (see [3])

(1) $|[x, y]_+| \leq \|y\|$;
(2) $[x, y + z]_+ \leq [x, y]_+ + [x, z]_+,\ \forall x, y, z \in X$.

An operator $A : D(A) \subset X \to \mathcal{P}(X)$ is called accretive if $[x_1 - x_2, y_1 - y_2]_+ \geq 0$ for all $(x_1, y_1), (x_2, y_2) \in A$. Here and in the sequel, we write $(x, y) \in A$ if $x \in D(A)$ and $y \in Ax$. An accretive operator A is said to be m-accretive if $R(I + \lambda A) = X$ for all $\lambda > 0$. If, in addition, $A - \omega I$ is accretive for $\omega \in \mathbb{R}$, we say that A is ω-m-accretive.

Consider the Cauchy problem

$$u'(t) + Au(t) \ni f(t),\ t \in J, \tag{2.1}$$

$$u(0) = u_0, \tag{2.2}$$

where $f \in L^1(J; X)$ and $u_0 \in \overline{D(A)}$ given. A function $u : J \to \overline{D(A)}$ is called an integral solution of problem (2.1)–(2.2) if $u \in C(J; X), u(0) = u_0$ and

$$\|u(t) - x\| \leq \|u(s) - x\| + \int_s^t [u(\tau) - x, f(\tau) + y]_+ d\tau,$$

for all $(x, y) \in A$ and $s, t \in J, s \leq t$.

Theorem 2.1 ([3, Theorem 4.1, p. 128]). *If A is an ω-m-accretive operator for some $\omega \in \mathbb{R}$, then there exists a unique integral solution $u = u(\cdot, u_0, f)$ to problem (2.1)–(2.2) for each $f \in L^1(J; X), u_0 \in \overline{D(A)}$. If $u = u(\cdot, u_0, f)$ and $v = v(\cdot, u_0, g)$ are two integral solutions of (2.1)–(2.2), then*

$$\|u(t) - v(t)\| \leq e^{-\omega(t-s)}\|u(s) - v(s)\| + \int_s^t e^{-\omega(t-\tau)}\|f(\tau) - g(\tau)\|d\tau, \tag{2.3}$$

for each $s, t \in J, s \leq t$.

Denote by $\{S(t)\}_{t\geq 0}$ the semigroup generated by $-A$, that is $S(t) : \overline{D(A)} \to \overline{D(A)}$, $S(t)u_0 = u(t, u_0, 0)$ being the integral solution of (2.1)–(2.2) with respect to $f = 0$. The semigroup $\{S(t)\}_{t\geq 0}$ is said to be compact if $S(t)$ is a compact operator for each $t > 0$. It is called equicontinuous if for each $0 < a < b$, $S(\cdot)D$ is an equicontinuous set in $C([a, b]; X)$ for any bounded set $D \subset X$.

We also denote by W the solution map for (2.1)–(2.2) with respect to f for fixed u_0. That is

$$W : L^1(J; X) \to C(J; X)$$
$$W(f)(t) = u(t, u_0, f).$$

If $\Omega \subset L^1(J; X)$ such that for all $f \in \Omega$, $\|f(t)\| \leq \nu(t)$ for a.e. $t \in J$, where $\nu \in L^1(J)$ then we say that Ω is integrably bounded.

Let $\mathcal{B}(X)$ be the collection of all bounded subsets of X. The following function defined on $\mathcal{B}(X)$,

$$\chi(D) = \inf\{\epsilon : D \text{ has a finite } \epsilon\text{-net}\},$$

is called the Hausdorff measure of noncompactness (MNC) on X.

Due to [4, Proposition 1, Lemma 4], we have the following assertion.

Lemma 2.2. *Let A be an m-accretive operator on X such that $-A$ generates an equicontinuous semigroup. Then we have*

(1) *If $\Omega \subset L^1(J; X)$ is integrably bounded then $W(\Omega)$ is an equicontinuous set in $C(J; X)$;*
(2) *If X^*, the dual space of X, is uniformly convex and $\{f_n\}_{n\in\mathbb{N}} \subset L^1(J; X)$ is integrably bounded then*

$$\chi(\{W(f_n)(t)\}) \leq \int_0^t \chi(\{f_n(s)\})ds, \ t \in J, \tag{2.4}$$

where χ is the Hausdorff MNC on X. In addition, if $f_n \rightharpoonup f$ (weakly) in $L^1(J; X)$ and $W(f_n) \to g$ (strongly) in $C(J; X)$ then $g = W(f)$.

Remark 2.1. *As mentioned in [4], if $-A$ generates a compact semigroup on X then W is a compact mapping in the sense that $W(\Omega)$ is compact in $C(J; X)$ provided that Ω is integrably bounded. In particular, we have*

$$\chi(W(\Omega)(t)) = 0, \text{ for all } t \in J.$$

Let $\mathcal{E}$ be a Banach space and Y a metric space.

Definition 2.1. *A multivalued map (multimap) $\mathcal{F} : Y \to \mathcal{P}(\mathcal{E})$ is said to be:*

(i) *upper semi-continuous (u.s.c) if* $\mathcal{F}^{-1}(V) = \{y \in Y : \mathcal{F}(y) \cap V \neq \emptyset\}$ *is a closed subset of* Y *for every closed set* $V \subset \mathcal{E}$*;*
(ii) *weakly upper semi-continuous (weakly u.s.c) if* $\mathcal{F}^{-1}(V)$ *is closed subset of* Y *for all weakly closed set* $V \subset \mathcal{E}$*;*
(iii) *closed if its graph* $\Gamma_{\mathcal{F}} = \{(y, z) : z \in \mathcal{F}(y)\}$ *is a closed subset of* $Y \times \mathcal{E}$*;*
(iv) *compact if its range* $\mathcal{F}(Y)$ *is relatively compact in* $\mathcal{E}$*;*
(v) *quasicompact if its restriction to any compact subset* $A \subset Y$ *is compact.*

We say that $\mathcal{F}$ *has contractible values if for* $u \in Y$, $C = \mathcal{F}(u)$ *there exists a continuous function* $h : [0, 1] \times C \to C$ *and* $z \in C$ *such that* $h(0, v) = z$ *and* $h(1, v) = v$ *for all* $v \in C$.

The following facts will be used.

Lemma 2.3 ([11, Theorem 1.1.12])**.** *Let* X *and* Y *be metric spaces and* $G : X \to \mathcal{P}(Y)$ *a closed quasi-compact multimap with compact values. Then* G *is u.s.c.*

Lemma 2.4 ([4, Proposition 2])**.** *Let* $\mathcal{E}$ *be a Banach space and* Ω *be a nonempty subset of another Banach space. Assume that* $G : \Omega \to \mathcal{P}(\mathcal{E})$ *is a multimap with weakly compact and convex values. Then* G *is weakly u.s.c iff* $\{x_n\} \subset \Omega$ *with* $x_n \to x_0 \in \Omega$ *and* $y_n \in G(x_n)$ *implies* $y_n \rightharpoonup y_0 \in G(x_0)$*, up to a subsequence.*

3. Existence result

Let us introduce the notations

$$\mathcal{P}_c(X) = \{D \in \mathcal{P}(X) : \ D \text{ is closed and convex}\},$$
$$\mathcal{C}_h = \{\phi : [-h, 0] \to \overline{D(A)}, \phi \in C([-h, 0]; X)\},$$
$$\mathcal{C}_\varphi = \{v : J \to \overline{D(A)}, v \in C(J; X), v(0) = \varphi(0)\},$$
$$\mathcal{D}_A = \overline{\text{conv}} D(A), \text{ the closure of convex hull of } D(A).$$

For $v \in \mathcal{C}_\varphi$ we define the function $v[\varphi] \in C([-h, T]; X)$ as follows

$$v[\varphi](t) = \begin{cases} v(t) & \text{if } t \geq 0, \\ \varphi(t) & \text{if } t < 0. \end{cases}$$

Let

$$\mathcal{P}_F(v) = \{f \in L^1(J; X) : f(t) \in F(t, v(t), v[\varphi]_t) \text{ for a.e. } t \in J\}, \ v \in \mathcal{C}_\varphi.$$

Definition 3.1. *A function* $u : [-h, T] \to \overline{D(A)}$ *is called an integral solution of problem* (1.1)–(1.3) *if* $u \in C([-h, T]; X), u(t) = \varphi(t)$ *for* $t \leq 0$ *and there exists* $f \in \mathcal{P}_F(u)$ *such that*

$$\|u(t) - x\| \leq \|u(s) - x\| + \int_s^t [u(\tau) - x, f(\tau) + y]_+ d\tau, \tag{3.1}$$

for all $(x, y) \in A$ *and* $s, t \in J, \ s \leq t$.

We now define the multioperator $\mathcal{F} : \mathcal{C}_\varphi \to \mathcal{P}(\mathcal{C}_\varphi)$ as follows

$$\mathcal{F} = W \circ \mathcal{P}_F, \tag{3.2}$$

where W is the solution map for (2.1)–(2.2). It easy to see that a function $u \in C([-h,T];X)$ is an integral solution for (1.1)–(1.3) iff $u|_{[-h,0]} = \varphi$ and $u|_J$ is a fixed point of $\mathcal{F}$.

In order to prove the existence result for problem (1.1)–(1.3), we make use of the following fixed point theorem (see e.g., [8]).

Lemma 3.1. *Let $\mathcal{E}$ be a Banach space and $D \subset \mathcal{E}$ be a nonempty compact convex subset. If the multivalued map $\mathcal{F} : D \to \mathcal{P}(D)$ is u.s.c with closed contractible values, then $\mathcal{F}$ has a fixed point.*

Concerning operator A and function F in problem (1.1)–(1.3), we assume that:

(**A**) *The operator A is an m-accretive operator such that $-A$ generates an equicontinuous semigroup.*

(**F**) *The multivalued function $F : \mathbb{R}^+ \times \mathcal{D}_A \times \mathcal{C}_h \to \mathcal{P}_c(X)$ is such that*

(1) *$F(\cdot, x, y)$ has a strongly measurable selection for fixed x, y and $F(t, \cdot, \cdot)$ is weakly u.s.c for fixed t;*

(2) *$\|F(t,x,y)\| = \sup\{\|\xi\| : \xi \in F(t,x,y)\} \le a(t)\|x\| + b(t)\|y\|_{\mathcal{C}_h} + c(t)$, for all $x \in \mathcal{D}_A, y \in \mathcal{C}_h$, where $a, b, c \in L^1_{loc}(\mathbb{R}^+)$ are nonnegative functions;*

(3) *there exist nonnegative functions $\alpha, \beta \in L^1_{loc}(\mathbb{R}^+)$ such that*

$$\chi(F(t,B,C)) \le \alpha(t)\chi(B) + \beta(t) \sup_{\tau \in [-h,0]} \chi(C(\tau)),$$

for all bounded subsets $B \subset \mathcal{D}_A, C \subset \mathcal{C}_h$.

By using the same arguments as in [4, Theorem 1], one gets the following results.

Proposition 3.2. *Let the hypotheses* (**A**), (**F**)(1) *and* (**F**)(2) *hold. Then the following assertions hold:*

(1) *If X^* is uniformly convex then the multioperator $\mathcal{F}$ is well-defined, that is $\mathcal{P}_F(u) \ne \emptyset$ for each $u \in \mathcal{C}_\varphi$. In addition, $\mathcal{P}_F : C(J;X) \to \mathcal{P}(L^1(J;X))$ is weakly u.s.c with weakly compact and convex values;*

(2) *The multioperator $\mathcal{F}$ has closed contractible values.*

We are in a position to state the main result of this section.

Theorem 3.3. *Let the hypotheses* (**A**)*and* (**F**) *hold. If X^* is uniformly convex then problem* (1.1)–(1.3) *has at least one integral solution for all initial data $\varphi \in \mathcal{C}_h$.*

Proof. Let $\{S(t)\}_{t\ge 0}$ be the semigroup generated by $-A$ and $v(t) = S(t)\varphi(0)$. Define

$$\mathcal{M}_0 = \{u \in \mathcal{C}_\varphi : \sup_{s \in [0,t]} \|u(s)\| \le \psi(t), t \in J\},$$

where ψ is the solution of the integral equation

$$\psi(t) = \sup_{t \in J} \|S(t)\varphi(0)\| + \|b\|_{L^1(J)}\|\varphi\|_{\mathcal{C}_h} + \|c\|_{L^1(J)} + \int_0^t [a(s) + b(s)]\psi(\tau)d\tau.$$

It is clear that $\mathcal{M}_0$ is a closed convex subset of $\mathcal{C}_\varphi$. We first show that $\mathcal{F}(\mathcal{M}_0) \subset \mathcal{M}_0$. Indeed, taking $u \in \mathcal{M}_0$ and $w \in \mathcal{F}(u)$, there exists $f \in \mathcal{P}_F(u)$ such that

$$\|w(t) - v(t)\| \le \int_0^t \|f(\tau)\|d\tau,$$

thanks to Theorem 2.1. This implies

$$\begin{aligned}
\|w(t)\| &\leq \sup_{t\in J}\|S(t)\varphi(0)\| + \int_0^t [a(s)\|u(s)\| + b(s)\|u_s\|_{\mathcal{C}_h} + c(s)]ds \\
&\leq \sup_{t\in J}\|S(t)\varphi(0)\| + \|c\|_{L^1(J)} \\
&\quad + \int_0^t [a(s)\|u(s)\| + b(s)\sup_{\tau\in[0,s]}\|u(\tau)\| + b(s)\|\varphi\|_{\mathcal{C}_h}]ds \\
&\leq \sup_{t\in J}\|S(t)\varphi(0)\| + \|b\|_{L^1(J)}\|\varphi\|_{\mathcal{C}_h} + \|c\|_{L^1(J)} \\
&\quad + \int_0^t [a(s)+b(s)]\sup_{\tau\in[0,s]}\|u(\tau)\|ds.
\end{aligned}$$

Noting that

$$\begin{aligned}
\|w(\rho)\| \leq &\sup_{t\in J}\|S(t)\varphi(0)\| + \|b\|_{L^1(J)}\|\varphi\|_{\mathcal{C}_h} + \|c\|_{L^1(J)} \\
&+ \int_0^t [a(s)+b(s)]\sup_{\tau\in[0,s]}\|u(\tau)\|ds
\end{aligned}$$

for all $\rho \leq t$, we obtain

$$\begin{aligned}
\sup_{\rho\in[0,t]}\|w(\rho)\| &\leq \sup_{t\in J}\|S(t)\varphi(0)\| + \|b\|_{L^1(J)}\|\varphi\|_{\mathcal{C}_h} + \|c\|_{L^1(J)} \\
&\quad + \int_0^t [a(s)+b(s)]\sup_{\tau\in[0,s]}\|u(\tau)\|ds \\
&\leq \sup_{t\in J}\|S(t)\varphi(0)\| + \|b\|_{L^1(J)}\|\varphi\|_{\mathcal{C}_h} + \|c\|_{L^1(J)} \\
&\quad + \int_0^t [a(s)+b(s)]\psi(s)ds \\
&= \psi(t).
\end{aligned}$$

Thus $w \in \mathcal{M}_0$.

Set

$$\mathcal{M}_{k+1} = \overline{\text{conv}}\mathcal{F}(\mathcal{M}_k),\ k = 0, 1, 2, \ldots$$

here the notation $\overline{\text{conv}}$ stands for the closure of convex hull of a subset in $\mathcal{C}_\varphi$.

We see that $\mathcal{M}_k$ is closed, convex and $\mathcal{M}_{k+1} \subset \mathcal{M}_k$ for all $k \in \mathbb{N}$. Let $\mathcal{M} = \cap_{k=0}^{\infty}\mathcal{M}_k$, then $\mathcal{M}$ is a closed convex subset of $\mathcal{C}_\varphi$ and $\mathcal{F}(\mathcal{M}) \subset \mathcal{M}$. We will show that $\mathcal{M}$ is compact. Indeed, for each $k \geq 0$, $\mathcal{P}_F(\mathcal{M}_k)$ is integrably bounded thanks to (**F**)(2). Then Lemma 2.2 ensures that $\mathcal{F}(\mathcal{M}_k) = W(\mathcal{P}_F(\mathcal{M}_k))$ is equicontinuous. It follows that $\mathcal{M}_{k+1}$ is equicontinuous for all $k \geq 0$. Thus $\mathcal{M}$ is equicontinuous as well. In order to apply the Arzelà–Ascoli theorem, we have to prove that $\mathcal{M}(t)$ is compact for each $t \geq 0$. This will be done if we show that $\mu_k(t) = \chi(\mathcal{M}_k(t)) \to 0$ as $k \to \infty$.

To verify the last claim, we make use of the fact that (see, e.g. [1]), for $\Omega \subset X, \epsilon > 0$, there exists a sequence $\omega_n \subset \Omega$ such that $\chi(\Omega) \leq 2\chi(\{\omega_n\}) + \epsilon$. Taking

$\{u_j\} \subset \mathcal{M}_{k+1}$ such that $\mu_{k+1}(t) \le 2\chi(\{u_j(t)\}) + \epsilon$, one can choose a sequence $v_j \in \mathcal{M}_k, f_j \in \mathcal{P}_F(v_j)$ such that $u_j = W(f_j)$. Obviously,

$$\begin{aligned} \chi(\{v_j(t)\}) &\le \chi(\mathcal{M}_k(t)) = \mu_k(t), \\ \chi(\{f_j(t)\}) &\le \alpha(t)\chi(\{v_j(t)\}) + \beta(t) \sup_{s\in[-h,0]} \chi(\{v_j[\varphi](t+s)\}) \\ &\le \alpha(t)\chi(\{v_j(t)\}) + \beta(t) \sup_{\tau\in[0,t]} \chi(\{v_j(\tau)\}), \end{aligned} \tag{3.3}$$

thanks to (F)(3). Hence, by Lemma 2.2, we obtain

$$\begin{aligned} \chi(\{u_j(t)\}) &\le \chi(\{W(f_j)(t)\}) \\ &\le \int_0^t \chi(\{f_j(s)\})ds \\ &\le \int_0^t [\alpha(s)\chi(\{v_j(s)\}) + \beta(s) \sup_{\tau\in[0,s]} \chi(\{v_j(\tau)\})]ds \\ &\le \int_0^t [\alpha(s) + \beta(s)] \sup_{\tau\in[0,s]} \chi(\{v_j(\tau)\})ds \\ &\le \int_0^t [\alpha(s) + \beta(s)] \sup_{\tau\in[0,s]} \mu_k(\tau)ds, \end{aligned}$$

thanks to (3.3). The last inequality implies

$$\mu_{k+1}(t) \le 2\chi(\{u_j(t)\}) + \epsilon \le 2\int_0^t [\alpha(s) + \beta(s)] \sup_{\tau\in[0,s]} \mu_k(\tau)ds + \epsilon.$$

Since ϵ is arbitrary, we have

$$\mu_{k+1}(t) \le 2\int_0^t [\alpha(s) + \beta(s)] \sup_{\tau\in[0,s]} \mu_k(\tau)ds.$$

Observing that the right term of the last inequality is non-decreasing in t, we can write

$$\nu_{k+1}(t) \le 2\int_0^t [\alpha(s) + \beta(s)]\nu_k(s)ds,$$

where $\nu_k(t) = \sup_{\tau\in[0,t]} \mu_k(\tau)$. Therefore

$$\nu_\infty(t) \le 2\int_0^t [\alpha(s) + \beta(s)]\nu_\infty(s)ds, \tag{3.4}$$

where $\nu_\infty(t) = \lim_{k\to\infty} \nu_k(t)$ for $t \in J$. Taking into account that $\mathcal{M}_k(0) = \{\varphi(0)\}$, one has $\mu_k(0) = 0$ and then $\nu_k(0) = 0$ for all $k \in \mathbb{N}$. This leads to $\nu_\infty(0) = 0$. Therefore, (3.4) deduces that $\nu_\infty(t) = 0$ for all $t \in J$. Now, since $0 \le \mu_k(t) \le \nu_k(t), t \in J$, we obtain

$$0 \le \mu_\infty(t) := \lim_{k\to\infty} \mu_k(t) \le \nu_\infty(t) = 0, t \in J.$$

So we have $\mathcal{M}(t)$ is compact as desired.

Now, consider $\mathcal{F} : \mathcal{M} \to \mathcal{P}(\mathcal{M})$. To apply the fixed point principle given by Lemma 3.1, it remains to show that $\mathcal{F}$ is u.s.c. By Lemma 2.3, this is the case if

$\mathcal{F}$ has closed graph. Let $\{u_n\} \subset \mathcal{M}$ with $u_n \to u^*$ and $v_n \in \mathcal{F}(u_n)$ with $v_n \to v^*$. Then, by the definition of $\mathcal{F}$, one can take $f_n \in \mathcal{P}_F(u_n)$ such that $v_n = W(f_n)$. Since $\mathcal{P}_F$ is weakly u.s.c with weakly compact and convex values (Proposition 3.2), one obtains $f_n \rightharpoonup f^* \in \mathcal{P}_F(u^*)$, up to a subsequence (Lemma 2.4). By virtue of Lemma 2.2, we have $v^* = W(f^*)$, and thus $v^* \in \mathcal{F}(u^*)$, which completes the proof. □

Remark 3.1. *In fact, the fixed point set of $\mathcal{F}$ is compact. Indeed, let $\Omega = \mathrm{Fix}(\mathcal{F})$, then $\Omega \subset \mathcal{F}(\Omega)$. Assume that $\{u_j\} \subset \Omega$, then one can choose $f_j \in \mathcal{P}_F(u_j)$ such that $u_j = W(f_j)$. By using similar estimates as in the proof of Theorem 3.3 for $\{u_j\}$, we obtain that $\{u_j\}$ is relatively compact.*

On the other hand, if $-A$ generates a compact semigroup on X, then one can drop assumption $(\mathbf{F})(3)$ due to the compactness of W. Indeed, since the subsets $\mathcal{M}_k, k \geq 1$ in the latter proof are compact, the set $\mathcal{M}$ is compact as well and we are able to obtain the conclusion of the Theorem easily.

4. Stability Results

The aim of this section is twofold. We first show that the solution set of (1.1)–(1.3) semicontinuously depends on the initial data. Then, under some additional conditions, we assert that the zero solution of (1.1)–(1.3) is exponentially stable in the sense of Lyapunov.

Let

$$\begin{aligned} &\Sigma : \ \mathcal{C}_h \to \mathcal{P}(C(J;X)) \\ &\Sigma(\phi) = \{u \in C(J;X) : \ u[\phi] \text{ is an integral solution of (1.1)–(1.3)}\}. \end{aligned} \tag{4.1}$$

Obviously,

$$\Sigma(\phi) \subset W \circ \mathcal{P}_F(\Sigma(\phi)). \tag{4.2}$$

Theorem 4.1. *Under assumptions $(\mathbf{A})$ and $(\mathbf{F})$, the solution map Σ defined by (4.1) is u.s.c.*

Proof. In view of Remark 3.1, Σ has compact values. By Lemma 2.3 it suffices to prove that Σ is quasi-compact and closed. We proceed with the proof in two steps. Let $\{\phi_n\} \subset \mathcal{C}_h$ be a convergent sequence such that $\phi_n \to \phi^*$ in $\mathcal{C}_h$.

Step 1. We show that $\Sigma(\{\phi_n\})$ is relatively compact. By $(\mathbf{F})(2)$ one can check that $\Sigma(\{\phi_n\})$ is a bounded set in $C(J;X)$. Then $\mathcal{P}_F(\Sigma(\{\phi_n\}))$ is integrably bounded. It follows that $W \circ P_F(\Sigma(\{\phi_n\}))$ is equicontinuous thanks to Lemma 2.2. Therefore $\Sigma(\{\phi_n\})$ is equicontinuous as well, in view of (4.2).

For $\epsilon > 0$, take a sequence $\{f_n\}$ such that $f_n \in \mathcal{P}_F(\Sigma(\phi_n))$ and

$$\chi(W \circ P_F(\Sigma(\{\phi_n\}))(t)) \leq 2\chi(\{W(f_n)(t)\}) + \epsilon.$$

Putting $\mu(t) = \chi(\Sigma(\{\phi_n\})(t))$, since $\Sigma(\phi_n) \subset W \circ \mathcal{P}_F(\Sigma(\phi_n))$, we find that

$$\begin{aligned}\mu(t) &\le 2\chi(\{W(f_n)(t)\}) + \epsilon \\ &\le 2\int_0^t \chi(\{f_n(s)\})ds + \epsilon \\ &\le 2\int_0^t [\alpha(s)\chi(\Sigma(\{\phi_n\})(s)) \\ &\quad + \beta(s) \sup_{\tau\in[-h,0]} \chi(\{v_n(s+\tau) : v_n \in \Sigma(\phi_n)[\phi_n]\})]ds + \epsilon,\end{aligned} \tag{4.3}$$

thanks to $(\mathbf{F})(3)$. Noting that

$$\Sigma(\phi_n)[\phi_n](\tau) = \phi_n(\tau), \text{ for } \tau \in [-h, 0],$$

and $\{\phi_n(\tau)\}, \tau \in [-h, 0]$, is compact, we get

$$\sup_{\tau\in[-h,0]} \chi(\{v_n(s+\tau) : v_n \in \Sigma(\phi_n)[\phi_n]\}) = \sup_{\rho\in[0,s]} \chi(\Sigma(\{\phi_n\})(\rho)).$$

Putting the last identity in (4.3) and noticing that ϵ is arbitrary, we have

$$\mu(t) \le 2\int_0^t [\alpha(s)\mu(s) + \beta(s) \sup_{\rho\in[0,s]} \mu(\rho)]ds \le 2\int_0^t [\alpha(s) + \beta(s)] \sup_{\rho\in[0,s]} \mu(\rho)ds.$$

Taking into account the fact that $\mu(0) = \chi(\Sigma(\{\phi_n\})(0)) = \chi(\{\phi_n(0)\}) = 0$, we deduce that $\mu(t) = 0$. Thus the application of the Arzelà–Ascoli theorem yields the relative compactness of $\Sigma(\{\phi_n\})$.
Step 2. Let $u_n \in \Sigma(\phi_n)$ such that $u_n \to u^*$. Then for $f_n \in \mathcal{P}_F(u_n)$ satisfying $u_n = W(f_n)$, one ensures that $f_n \rightharpoonup f^* \in \mathcal{P}_F(u^*)$ according to Proposition 3.2. Then by Lemma 2.2 we have

$$u^* = \lim_{n\to\infty} u_n = \lim_{n\to\infty} W(f_n) = W(f^*) \in W \circ \mathcal{P}_F(u^*).$$

Since $u_n(0) = \phi_n(0)$, one has $u^*(0) = \phi^*(0)$ and hence $u^*[\phi^*]$ is an integral solution of (1.1)–(1.3) with respect to the initial datum ϕ^*. Equivalently, $u^* \in \Sigma(\phi^*)$. The proof is complete. □

In what follows, we replace $(\mathbf{A})$ and $(\mathbf{F})$ by stronger assumptions:

- $(\mathbf{A}^*)$ The operator A is ω-m-accretive for $\omega > 0$, $(0, 0) \in A$ and $-A$ generates an equicontinuous semigroup;
- $(\mathbf{F}^*)$ The multimap F satisfies $(\mathbf{F})$ for $c = 0$ and a, b being bounded functions such that $a^* + b^* < \omega$, here $a^* = \sup_{t\ge0} a(t)$, $b^* = \sup_{t\ge0} b(t)$.

We need the following result (see [9, §4.5], or [18] for a generalized version).

Proposition 4.2 (Halanay's inequality)**.** *Let the function* $f : [t_0 - \tau, T) \to \mathbb{R}^+, 0 \le t_0 < T < +\infty$*, satisfy the functional differential inequality*

$$f'(t) \le -\gamma f(t) + \nu \sup_{s\in[t-\tau,t]} f(s),$$

for $t \ge t_0$*, where* $\gamma > \nu > 0$*. Then*

$$f(t) \le \kappa e^{-\ell(t-t_0)},\ t \ge t_0,$$

where $\kappa = \sup_{s\in[t_0-\tau,t_0]} f(s)$ *and* ℓ *is the solution of the equation* $\gamma = \ell + \nu e^{\ell\tau}$.

Using Halanay's inequality, we get the following result.

Theorem 4.3. *Let* u *be an integral solution of* (1.1)–(1.3). *If* $(\mathbf{A}^*)$ *and* $(\mathbf{F}^*)$ *hold, then*

$$\|u(t)\| \le \|\varphi\|_{\mathcal{C}_h} e^{-\ell t},\ \forall t > h,$$

where ℓ *is the solution of the equation* $\omega - a^* = \ell + b^* e^{\ell h}$. *That is, the zero solution of* (1.1)–(1.3) *is exponentially stable.*

Proof. Let u be an integral solution of (1.1)–(1.3). Then there exists $f \in \mathcal{P}_F(u)$ such that

$$\|u(t)\| \le e^{-\omega t}\|\varphi(0)\| + \int_0^t e^{-\omega(t-s)}\|f(s)\|ds, t \ge 0,$$

thanks to Theorem 2.1 and the assumption that $0 \in A0$. Hence using $(\mathbf{F})(2)$ one has

$$\|u(t)\| \le e^{-\omega t}\|\varphi(0)\| + \int_0^t e^{-\omega(t-s)}(a^*\|u(s)\| + b^*\|u_s\|_{\mathcal{C}_h})ds. \tag{4.4}$$

Put

$$z(t) = e^{-\omega t}\|\varphi(0)\| + \int_0^t e^{-\omega(t-s)}(a^*\|u(s)\| + b^*\|u_s\|_{\mathcal{C}_h})ds, t \ge 0,$$
$$z(t) = \|\varphi(t)\|,\ t \le 0.$$

Then it follows from (4.4) that

$$\begin{aligned} z'(t) &= -\omega z(t) + a^*\|u(t)\| + b^*\|u_t\|_{\mathcal{C}_h} \\ &\le -(\omega - a^*)z(t) + b^* \sup_{s\in[t-h,t]} z(s). \end{aligned}$$

By using Halanay's inequality, one obtains

$$\|u(t)\| \le z(t) \le \|\varphi\|_{\mathcal{C}_h} e^{-\ell t}, t \ge 0,$$

where ℓ is the solution of the equation $\omega - a^* = \ell + b^* e^{\ell h}$. □

Remark 4.1. *In the case when* A *is a linear operator such that* $-A$ *generates an exponentially stable semigroup and* F *depends on the time and the history state only, i.e.* $F = F(t, u_t)$, *our condition in* $(\mathbf{F}^*)$ *that* $a^* + b^* < \omega$ *reduces to the condition* $b^* < \omega$. *This is exactly the result by Travis and Webb* [14].

5. Application

Let Ω be a bounded set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the doubly nonlinear boundary value problem:

$$\frac{\partial u}{\partial t}(t,x) - \Delta_x u(t,x) + \partial\varphi(u(t,x)) + \lambda u(t,x) \ni f(t, u(t,x), u(t-h,x)), \quad x \in \Omega,\ t > 0, \tag{5.1}$$

$$\frac{\partial u}{\partial n}(t,x) + \partial\psi(u(t,x)) \ni 0,\ x \in \partial\Omega,\ t > 0, \tag{5.2}$$

$$u(x,s) = z(x,s),\ x \in \Omega,\ t > 0,\ s \in [-h, 0], \tag{5.3}$$

where λ is a positive number and $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are such that

- φ is proper, convex, lower semicontinuous, $\varphi(0) = 0$;
- ψ is a continuous and convex function and there is $C > 0$ such that

$$0 \le \psi(s) \le C(s^2 + 1), s \in \mathbb{R}.$$

Moreover, the real-valued function f defined on $\mathbb{R}^+ \times \mathbb{R}^2$ and $z \in C([-h, 0]; L^2(\Omega))$ are given.

Let $X = L^2(\Omega)$ with the norm $\|\cdot\|$. Denote

$$\Phi(v) = \int_\Omega \big(\varphi(v(x)) + \frac{\lambda}{2}|v(x)|^2\big)dx,$$

$$\Psi(v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla v(x)|^2 dx + \int_{\partial\Omega} \psi(v(x))ds, & v \in H^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then it is known that (see [13, Example 2.B, 2.E]) Φ and Ψ are proper, convex and lower semicontinuous functionals defined on X and

- $D(\Phi) = \{v \in L^2(\Omega), \varphi \circ v \in L^1(\Omega)\}$;
  $f \in \partial\Phi(v)$ if and only if
  $$v, f \in L^2(\Omega), f(x) \in \partial\varphi(v(x)), \text{ a.e. } x \in \Omega.$$
- $D(\Psi) = H^1(\Omega)$;
  $g \in \partial\Psi(v)$ if and only if
  $$-\Delta v = g \text{ in } L^2(\Omega) \text{ and } \frac{\partial v}{\partial n} + \partial\psi(v) \ni 0 \text{ in } L^2(\partial\Omega).$$

Furthermore, $\partial\Phi + \partial\Psi$ is m-accretive and equal to $\partial(\Phi + \Psi)$ (see [13, Example 2.F]). Let $A = \partial(\Phi + \Psi)$ with the domain $D(A) = D(\Phi) \cap D(\Psi)$. Then it is obvious that A is a λ-m-accretive operator in X.

Regarding the level set of A, one has

$$H_R = \{v \in L^2(\Omega) : \|v\|^2 + \Phi(v) + \Psi(v) \le R\} \text{ for } R > 0.$$

Since the sign of ψ is indefinite, the boundedness of H_R in $L^2(\Omega)$ is unavailable. Therefore H_R is noncompact, in general. So is the semigroup generated by $-A$. However, since A is in the form of a subdifferential, $-A$ generates an equicontinuous semigroup (see [16]). By this reason, $(\mathbf{A}^*)$ is fulfilled for (5.1)–(5.3).

As far as the nonlinearity f is concerned, we suppose that f is a Lipschitz-type function, i.e. there exists $\mu, \nu \in L^1_{loc}(\mathbb{R}^+)$ such that

$$|f(t, \xi_1, \eta_1) - f(t, \xi_2, \eta_2)| \le \mu(t)|\xi_1 - \xi_2| + \nu(t)|\eta_1 - \eta_2|, \text{ for all } \xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}.$$

By this setting, the function $F : \mathbb{R}^+ \times L^2(\Omega) \times C([-h, 0]; L^2(\Omega)) \to L^2(\Omega)$ given by

$$F(t, v, w)(x) = f(t, v(x), w(-h, x))$$

satisfies assumption $(\mathbf{F})$ with $a(t) = \alpha(t) = \sqrt{2}\mu(t), b(t) = \beta(t) = \sqrt{2}\nu(t)$ and $c(t) = |\Omega|.|f(t, 0, 0)|$ ($|\Omega|$ stands for the volume of Ω). Indeed, it is easy to check that

$$\|F(t, v_1, w_1) - F(t, v_2, w_2)\|^2 \le 2\mu^2(t)\|v_1 - v_2\|^2 + 2\nu^2(t)\|w_1(-h, \cdot) - w_2(-h, \cdot)\|^2. \quad (5.4)$$

Then, we see that $F(t,\cdot,\cdot)$ is continuous and $(\mathbf{F})(1)$ is evident. Taking $v_1 = v, w_1 = w, v_2 = 0$ and $w_2 = 0$ in (5.4), one gets

$$\begin{aligned}\|F(t,v,w)\| &\le \sqrt{2}\mu(t)\|v\| + \sqrt{2}\nu(t)\|w(-h)\| + |\Omega|.|f(t,0,0)| \\ &\le \sqrt{2}\mu(t)\|v\| + \sqrt{2}\nu(t)\sup_{s\in[-h,0]}\|w(s)\| + |\Omega|.|f(t,0,0)|\end{aligned}$$

and thus $(\mathbf{F})(2)$ is fulfilled.

For bounded subsets $B \subset L^2(\Omega), C \subset C([-h,0];L^2(\Omega))$ we have

$$\begin{aligned}\chi(F(t,B,C)) &\le \sqrt{2}\mu(t)\chi(B) + \sqrt{2}\nu(t)\chi(C(-h)) \\ &\le \sqrt{2}\mu(t)\chi(B) + \sqrt{2}\nu(t)\sup_{s\in[-h,0]}\chi(C(s))\end{aligned}$$

and $(\mathbf{F})(3)$ is testified. If, in addition, $f(t,0,0) = 0, \varphi(0) = \psi(0) = 0$, μ,ν are bounded and $\sqrt{2}\big(\sup_{t\ge 0}\mu(t) + \sup_{t\ge 0}\nu(t)\big) < \lambda$ then $(\mathbf{A}^*)$ and $(\mathbf{F}^*)$ are satisfied. Consequently, we have all conclusions of Theorem 3.3, 4.1 and 4.3.

Acknowledgement

The author would like to express his sincere thanks to anonymous reviewer for his/her careful reading and constructive comments and suggestions.

References

[1] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhäuser, Boston-Basel-Berlin, 1992.

[2] S. Aizicovici, V. Staicu, Multivalued evolution equations with nonlocal initial conditions in Banach spaces, Nonlinear Differ. Equ. Appl. 14 (2007) 361–376.

[3] V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, in: Springer Monographs in Mathematics, Springer-Verlag, 2010.

[4] D. Bothe, Multivalued Perturbations of m-Accretive Differential Inclusions, Israel J. Math. 108 (1998), 109–138.

[5] D. Bothe, Flow invariance for nonlinear accretive evolutions under range conditions, J. Evol. Equ. 5 (2005), 227–252.

[6] D. Bothe, P. Wittbold, Abstract reaction-diffusion systems with m-completely accretive diffusion operators and measurable reaction rates, Commun. Pure Appl. Anal. 11 (2012), 2239–2260.

[7] R. D. Driver, *Ordinary and Delay Differential Equations*, Springer-Verlag, New York Inc., 1977.

[8] L. Górniewicz, A. Granas and W. Kryszewski, Sur la méthode de l'homotopie dans la théorie des point fixes pour les applications multivoques. Partie 2: L'indice dans les ANRs compactes, Comptes Rendus de l'Académie des Sciences, Paris 308 (1989), 449–452.

[9] A. Halanay, *Differential Equations, Stability, Oscillations, Time Lags*, Academic Press, New York and London 1966.

[10] J. K. Hale, S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, 1993.

[11] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, in: de Gruyter Series in Nonlinear Analysis and Applications, vol. 7, Walter de Gruyter, Berlin, New York, 2001.

[12] A. Paicu, I. I. Vrabie, A class of nonlinear evolution equations subjected to nonlocal initial conditions, Nonlinear Anal. 72 (2010), 4091–4100.

[13] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society 1997.

[14] C. C. Travis, G. F. Webb, Existence and stability for partial functional differential equations, Trans. Amer. Math. Soc. 200 (1974), 395–418.

[15] I. I. Vrabie, Existence for nonlinear evolution inclusions with nonlocal retarded initial conditions, Nonlinear Anal. 74 (2011) 7047–7060.

[16] I. I. Vrabie, *Compactness methods for nonlinear evolutions*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 32, Longman Scientific & Technical, 1987.

[17] I. I. Vrabie, Existence in the large for nonlinear delay evolution inclusions with nonlocal initial conditions, J. Func. Anal. 262 (2012), 1363–1391.

[18] W. Wang, A Generalized Halanay Inequality for Stability of Nonlinear Neutral Functional Differential Equations, J. Ineq. Appl., Vol. 2010, ArtID 475019, 16 pages.

(Received July 20, 2012)

Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam

E-mail address: ketd@hnue.edu.vn