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Separation and the existence theorem for second order nonlinear differential equation[1]

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Abstract. Sufficient conditions for the invertibility and separability in $L_2(-\infty, +\infty)$ of the degenerate second order differential operator with complex-valued coefficients are obtained, and its applications to the spectral and approximate problems are demonstrated. Using a separability theorem, which is obtained for the linear case, the solvability of nonlinear second order differential equation is proved on the real axis.

Keywords: separability of the operator, complex-valued coefficients, completely continuous resolvent

Mathematics subject classifications: 34B40

1. Introduction and main results

A concept of the separability was introduced in the fundamental paper [1]. The Sturm-Liouville's operator

$$Jy = -y'' + q(x)y, \quad x \in (a, +\infty),$$

is called separable [1] in $L_2(a, +\infty)$, if $y, \ -y'' + qy \in L_2(a, +\infty)$ imply $-y'', qy \in L_2(a, +\infty)$. From this it follows that the separability of J is equivalent to the existence of the estimate

$$\|y''\|_{L_2(a,+\infty)} + \|qy\|_{L_2(a,+\infty)} \leq c\left(\|Jy\|_{L_2(a,+\infty)} + \|y\|_{L_2(a,+\infty)}\right), \quad y \in D(J), \qquad (1.1)$$

where $D(J)$ is the domain of J. In [1] (see also [2, 3]) some criteria of the separability depended on a behavior q and its derivatives has been obtained for J. Moreover, an example of non-separable operator J with non-smooth potential q was shown in this papers. Without differentiability condition on function q the sufficient conditions for the separability of J has been obtained in [4, 5]. In [6,7] so-called Localization Principle of the proof for the separability of higher order binomial elliptic operators was developed in Hilbert space. In [8,9] it was shown that local integrability and semiboundedness from below of q are enough for separability of J in $L_1(-\infty, +\infty)$. Valuation method of Green's functions [1-3,8,9] (see also [10]), parametrix method [4,5], as well as method of local estimates for the resolvents of some regular operators [6, 7] have been used in these works.

Sufficient conditions of the separability for the Sturm-Liouville's operator

$$y'' + Q(x)y$$

have been obtained in [11-15], where Q is an operator. A number of works were devoted to the separation problem for the general elliptic, hyperbolic and mixed-type operators.

An application of the separability estimate (1.1) in the spectral theory of J has been shown in [15-18], and it allows us to prove an existence and a smoothness of solutions of nonlinear differential equations in unbounded domains [11, 17-20]. Brown [21] has shown that certain properties of positive solutions of disconjugate second

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order differential expressions imply the separation. The connection of separation with concrete physical problems has been noted in [22].

We denote $L_2 := L_2(\mathbb{R})$, $\mathbb{R} = (-\infty, +\infty)$, the space of square integrable functions. Let l is a closure in L_2 of the expression $l_0 y = -y'' + r(x)y' + s(x)\bar{y}'$ defined in the set $C_0^\infty(\mathbb{R})$ of all infinitely differentiable and compactly sapported functions. Here r and s are complex - valued functions, and $\bar{y}$ is the complex conjugate to y.

In this report we investigate some problems for the operator l. Although the operator l, similarly to the Sturm-Liouville operator J, is a singular differential operator of second order, their properties are different. The theory of the Sturm-Liouville operator J, in contrast to the operator l, developing a long time, while the idea of research is often based on the positivity of the potential $q(x)$ (see, eg, [1-20]). Because of the coefficients r and s, are the methods developed for the Sturm-Liouville problems are often not applicable to the study of the operator l. The spectral properties for self-adjoint singular differential operators of second order, without the free term, have been to a certain extent investigated; a review of literature can be found in [23, 24]. Note that the differential operator l is used, in particular, in the oscillatory processes in the medium with resistance depended on velocity [25, pp. 111-116].

The operator l is said to be separable in L_2 if the following estimate holds:

$$\|y''\|_2 + \|ry'\|_2 + \|s\bar{y}'\|_2 \le c\left(\|ly\|_2 + \|y\|_2\right), \ y \in D(l),$$

where $\|\cdot\|_2$ is the L_2- norm. In the present communication the sufficient conditions for the invertibility and separability of the differential operator l are obtained. Moreover, spectral and approximate results for the inverse operator l^{-1} are achieved. Using a separation theorem, which is obtained for the linear case, the solvability of the degenerate nonlinear second order differential equation $-y'' + r(x, y)y' = F \ (x \in \mathbb{R})$ is proved.

Let's consider the degenerate differential equation

$$ly = -y'' + r(x)y' + s(x)\bar{y}' = f. \tag{1.2}$$

The function $y \in L_2$ is called a solution of (1.2) if there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - f\|_2 \to 0$ as $n \to +\infty$. If the operator l is separable, then the solution y of (1.2) belongs to the weighted Sobolev space $W_2^2(\mathbb{R}, |r| + |s|)$ with the norm $\|y''\|_2 + \|(|r| + |s|)y'\|_2$. So, the study of the qualitative behavior of solutions of (1.2) and spectral and approximative properties of l can be reduced to the investigation of embedding $W_2^2(\mathbb{R}, |r| + |s|) \hookrightarrow L_2$.

We denote

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,t)} \|1/h\|_{L_2(t,+\infty)} \, (t > 0), \quad \beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \|1/h\|_{L_2(-\infty,\tau)} \, (\tau < 0),$$

$$\gamma_{g,h} = \max\left(\sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau)\right),$$

where g and h are given functions. By $C_{loc}^{(1)}(\mathbb{R})$ we denote the set of functions f such that $\psi f \in C^{(1)}(\mathbb{R})$ for all $\psi \in C_0^\infty(\mathbb{R})$.

Theorem 1. *Let functions r and s satisfy the conditions*

$$r, s \in C_{loc}^{(1)}(\mathbb{R}), \quad Re\ r - |s| \ge \delta > 0, \quad \gamma_{1,Re\ r} < \infty. \tag{1.3}$$

Then l is invertible and l^{-1} is defined in all L_2.

Theorem 2. *Assume that functions r and s satisfy the conditions*

$$\begin{cases} r, s \in C_{loc}^{(1)}(\mathbb{R}),\ Re\ r - \rho[|Im\ r| + |s|] \geq \delta > 0,\ \gamma_{1,Re\ r} < \infty,\ 1 < \rho < 2, \\ c^{-1} \leq \frac{Re\ r(x)}{Re\ r(\eta)} \leq c \ \text{ at } \ |x - \eta| \leq 1,\ c > 1. \end{cases} \tag{1.4}$$

Then for $y \in D(l)$ the estimate

$$\|y''\|_2 + \|ry'\|_2 + \|s\bar{y}'\|_2 \leq c_l \|ly\|_2 \tag{1.5}$$

holds, i.e. the operator l is separable in L_2.

We use the statement of Theorem 2 for proof of the following Theorems 3-5.

Theorem 3. *Assume that functions r and s satisfy (1.4) and let $\lim\limits_{t\to+\infty} \alpha_{1,Re\,r}(t) = 0$, $\lim\limits_{\tau\to-\infty} \beta_{1,Re\,r}(\tau) = 0$. Then l^{-1} is completely continuous in L_2.*

We assume that the conditions of Theorem 3 hold, and consider a set

$$M = \{y \in L_2 : \|ly\|_2 \leq 1\}.$$

Let

$$d_k = \inf_{\Sigma_k \subset \{\Sigma_k\}} \sup_{y\in M} \inf_{w\in\Sigma_k} \|y - w\|_2 \quad (k = 0, 1, 2, ...)$$

be the Kolmogorov's widths of the set M in L_2. Here $\{\Sigma_k\}$ is a set of all subspaces Σ_k of L_2 whose dimensions are not greater than k. Through $N_2(\lambda)$ denote the number of widths d_k which are not smaller than a given positive number λ. Estimates of the width's distribution function $N_2(\lambda)$ are important in the approximation problems of solutions of the equation $ly = f$. The following statement holds.

Theorem 4. *Assume that the conditions of Theorem 3 be fulfilled, and let a function q satisfy $\gamma_{q,Re\ r} < \infty$. Then the following estimates hold:*

$$c_1\lambda^{-2}\mu\left\{x : |q(x)| \leq c_2^{-1}\lambda^{-1}\right\} \leq N_2(\lambda) \leq c_3\lambda^{-2}\mu\left\{x : |q(x)| \leq c_2\lambda^{-1}\right\},$$

where μ is a Lebesgue measure.

Example. Assume that $r = (1 + x^2)^\beta$ $(\beta > 0)$ and let $s = 0$. Then the conditions of Theorem 2 are satisfied if $\beta \geq 1/2$. If $\beta > 1/2$, then the conditions of Theorem 4 are satisfied and the following estimates hold:

$$c_4\lambda^{\frac{-2\beta+3}{2(2\beta-1)}} \leq N_2(\lambda) \leq c_5\lambda^{\frac{-2\beta+3}{2(2\beta-1)}}.$$

Consider the following nonlinear equation

$$Ly = -y'' + [r(x, y)]y' = f(x), \tag{1.6}$$

where $x \in \mathbb{R}$, r is a real-valued function and $f \in L_2$.

A function $y \in L_2$ is called a solution of equation (1.6), if there exists a sequence of twice continuously differentiable functions $\{y_n\}_{n=1}^\infty$ such that $\|\theta(y_n - y)\|_2 \to 0$, $\|\theta(Ly_n - f)\|_2 \to 0$ as $n \to \infty$ for any $\theta \in C_0^\infty(\mathbb{R})$.

Theorem 5. *Let the function r be continuously differentiable with respect to both arguments and satisfy the following conditions*

$$r \geq \delta_0\sqrt{1+x^2} \quad (\delta_0 > 0), \qquad \sup_{x,\eta\in\mathbb{R}:\,|x-\eta|\leq 1} \ \sup_{A>0} \ \sup_{|C_1|\leq A, |C_2|\leq A, |C_1-C_2|\leq A} \frac{r(x,C_1)}{r(\eta,C_2)} < \infty. \tag{1.7}$$

Then there exists a solution y of (1.6), and

$$\|y''\|_2 + \|[r(\cdot,y)]y'\|_2 < \infty. \tag{1.8}$$

2. Auxiliary statements

The next statement is a corollary of the well known Muckenhoupt's inequality [26].

Lemma 2.1. *Let functions g and h such that $\gamma_{g,h} < \infty$. Then for all $y \in C_0^\infty(\mathbb{R})$ the following inequality holds:*

$$\int_{-\infty}^{\infty} |g(x)y(x)|^2 dx \leq C \int_{-\infty}^{\infty} |h(x)y'(x)|^2 dx. \tag{2.1}$$

Moreover, if C is a smallest constant for which estimate (2.1) holds, then $\gamma_{g,h} \leq C \leq 2\gamma_{g,h}$.

The following lemma is a particular case of Theorem 2.2 [23].

Lemma 2.2. *Let the given function h satisfy conditions*

$$\lim_{x\to+\infty} \sqrt{x} \left(\int_x^{\infty} h^{-2}(t)dt \right)^{\frac{1}{2}} = 0,$$

$$\lim_{x\to-\infty} \sqrt{|x|} \left(\int_{-\infty}^{x} h^{-2}(t)dt \right)^{\frac{1}{2}} = 0.$$

Then the set

$$F_K = \left\{ y : y \in C_0^\infty(\mathbb{R}), \ \int_{-\infty}^{+\infty} |h(t)y'(t)|^2 dt \leq K \right\}, \quad K > 0,$$

is a relatively compact in $L_2(\mathbb{R})$.

Denote by L a closure in L_2-norm of the differential expression

$$L_0 z = -z' + rz + s\bar{z} \tag{2.2}$$

defined on the set $C_0^\infty(\mathbb{R})$.

Lemma 2.3. *Assume that functions r and s satisfy condition (1.3). Then the operator L is boundedly invertible in L_2.*

Proof. Let $L_\lambda = L + \lambda E$, where $\lambda \geq 0$, and E be the identity map of L_2 to itself. Note that L is separable if and only if $L_\lambda = L + \lambda E$ is separable for some λ. If z is a continuously differentiate function with a compact support, then

$$(L_\lambda z, z) = -\int_{\mathbb{R}} z'\bar{z}dx + \int_{\mathbb{R}} [(r+\lambda)|z|^2 + s\bar{z}^2]dx. \tag{2.3}$$

But

$$T := -\int_{\mathbb{R}} z'\bar{z}dx = \int_{\mathbb{R}} z\bar{z}'dx = -\bar{T}.$$

Therefore $Re\, T = 0$ and from (2.3) it follows that

$$Re\,(L_\lambda z, z) \geq c \int_{\mathbb{R}} [Re\ r + \lambda - |s|]|z|^2 dx. \tag{2.4}$$

We estimate the left-hand side of inequality (2.4) by using the Holder's inequality. Then by (1.3) we have $\|L_\lambda z\|_2 \geq \delta \|z\|_2$. This estimate implies that L_λ is invertible. Let us proof that L_λ^{-1} is defined in all L_2. Assume the contrary. Let $R(L_\lambda) \neq L_2$. Then there exists a non-zero element $z_0 \in L_2$ such that $z_0 \perp R(L_\lambda)$. According to operator's theory z_0 satisfies the equality

$$L_\lambda^* z_0 := z_0^{'} + (\bar{r}+\lambda)z_0 + s\bar{z}_0 = 0, \tag{2.5}$$

where L_λ^* is an adjoint operator.

Let $\theta \in C_0^\infty(\mathbb{R})$ is a real function. Denote $\psi = \theta z_0$. From (2.5) it follows that $z_0 \in W_{2,loc}^1(\mathbb{R})$, then $\psi \in D(L_\lambda^*)$. Using (2.5), we get $L_\lambda^* \psi = \theta' z_0$. Hence

$$(L_\lambda^* \psi, \psi) = \int_{\mathbb{R}} \theta'\theta |z_0|^2 dx. \tag{2.6}$$

On the other hand using the expression $L_\lambda^* \psi$ we have

$$Re\,(L_\lambda^* \psi, \psi) = \int_{\mathbb{R}} \theta^2 [Re\,(\bar{r}+\lambda)|z_0|^2 + Re\,(s\bar{z}_0^2)]dx \geq$$

$$\geq \int_{\mathbb{R}} \theta^2 [Re\,\bar{r} + \lambda - |s|]|z_0|^2 dx.$$

Hence by (2.6) the following estimate

$$\delta \int_{\mathbb{R}} \theta^2 |z_0|^2 dx \leq \int_{\mathbb{R}} \theta'\theta |z_0|^2 dx \tag{2.7}$$

holds. Choose the function θ such that

$$\theta(x) = \begin{cases} 1, & |x| \leq \xi \\ \\ 0, & |x| \geq \xi + 1, \end{cases}$$

$0 \leq \theta \leq 1$, $|\theta'| \leq C$. Here $\xi > 0$. Then it follows from (2.7)

$$\delta \int_{-\xi-1}^{\xi+1} \theta^2 |z_0|^2 dx \leq C \left[\int_{-\xi-1}^{-\xi} |z_0|^2 dx + \int_{\xi}^{\xi+1} |z_0|^2 dx \right].$$

Since $z_0 \in L_2$, passing to the limit as $\xi \to +\infty$ in the last inequality, we have $\|z_0\|_2 = 0$. Then $z_0 = 0$. We obtain the contradiction, which gives that $R(L_\lambda) = L_2$. The lemma is proved. □

Lemma 2.4. *Assume that functions r and s satisfy condition (1.4). Then L is separable in L_2 and for $z \in D(L)$ the following estimate holds:*

$$\|z'\|_2 + \|rz\|_2 + \|s\bar{z}\|_2 \leq c\,\|Lz\|_2\,. \tag{2.8}$$

Proof. From inequality (2.4) it follows that

$$\left\| \sqrt{Re\ r(\cdot) + \lambda} z \right\|_2 \leq c_1 \left\| \frac{1}{\sqrt{Re\ r(\cdot) + \lambda}} L_\lambda z \right\|_2. \tag{2.9}$$

It is easy to show that (2.9) holds for all z from $D(L_\lambda)$.

Let $\Delta_j = (j-1, j+1)$ $(j \in \mathbb{Z})$ and let $\{\varphi_j\}_{j=-\infty}^{+\infty}$ be a sequence of functions from $C_0^\infty(\Delta_j)$, such that

$$0 \leq \varphi_j \leq 1, \quad \sum_{j=-\infty}^{+\infty} \varphi_j^2(x) = 1.$$

We continue $r(x)$ and $s(x)$ from Δ_j to $\mathbb{R}$ so that its continuations $r_j(x)$ and $s_j(x)$ are bounded and periodic functions with period 2. Denote by $L_{\lambda,j}$ the closure in $L_2(\mathbb{R})$ of the differential operator $-z' + [r_j(x) + \lambda]z + s_j(x)\bar{z}$ defined on $C_0^\infty(R)$. Using the method which was applied for L_λ one can proof that $L_{\lambda,j}$ are invertible and $L_{\lambda,j}^{-1}$ are defined in all L_2. In addition, the following inequality

$$\left\| (Re\ r_j + \lambda)^{\frac{1}{2}} z \right\|_2 \leq c_2 \left\| (Re\ r_j + \lambda)^{-\frac{1}{2}} L_{\lambda,j} z \right\|_2, \quad z \in D(L_{\lambda,j}), \tag{2.10}$$

holds. From estimate (2.10) by (1.4) it follows

$$\|L_{\lambda,j} z\|_2 \geq c_3 \sup_{x \in \Delta_j} [Re\ r_j(x) + \lambda]\,\|z\|_2\,, \quad z \in D(L_{\lambda,j}). \tag{2.11}$$

Let us introduce the operators B_λ and M_λ:

$$B_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j'(x) L_{\lambda,j}^{-1} \varphi_j f, \quad M_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j(x) L_{\lambda,j}^{-1} \varphi_j f.$$

At any point $x \in \mathbb{R}$ the sums of the right-hand side in these terms contain no more than two summands, therefore B_λ and M_λ is defined on all L_2. It is easy to show that

$$L_\lambda M_\lambda = E + B_\lambda. \tag{2.12}$$

Using (2.11) and properties of φ_j $(j \in \mathbb{Z})$ we find that $\lim_{\lambda \to +\infty} \|B_\lambda\| = 0$, hence there exists a number λ_0 such that $\|B_\lambda\| \leq 0.5$ for all $\lambda \geq \lambda_0$. Then it follows from (2.12)

$$L_\lambda^{-1} = M_\lambda (E + B_\lambda)^{-1}, \quad \lambda \geq \lambda_0. \tag{2.13}$$

Using (2.13) and properties of $\varphi_j \ \ (j \in \mathbb{Z})$ we have

$$\left\|(Re\ r+\lambda)L_\lambda^{-1}f\right\|_2 \le c_4 \sup_{j\in\mathbb{Z}} \left\|(Re\ r_j+\lambda)L_{\lambda,j}^{-1}\right\|_{L_2\to L_2} \|f\|_2 . \tag{2.14}$$

Further, (1.4) and (2.11) imply that

$$\sup_{j\in\mathbb{Z}} \left\|(Re\ r_j+\lambda)L_{\lambda,j}^{-1}F\right\|_{L_2(\mathbb{R})} \le c_5 \frac{\sup\limits_{x\in\Delta_j}[Re\ r(x)+\lambda]}{\inf\limits_{t\in\Delta_j}[Re\ r(t)+\lambda]} \|F\|_{L_2(\mathbb{R})} \le$$

$$\le c_5 \sup_{x,z\in\mathbb{R}:|x-z|\le 2} \frac{Re\ r(x)+\lambda}{Re\ r(z)+\lambda} \|F\|_{L_2(\mathbb{R})} \le c_6 \|F\|_{L_2(\mathbb{R})} .$$

From the last inequalities and (2.14) we obtain $\|(Re\ r+\lambda)z\|_2 \le c_7 \|L_\lambda z\|_2, \ z \in D(L_\lambda)$, therefore it follows from condition (1.4)

$$\|z'\|_2 + \|(r+\lambda)z\|_2 + \|s\bar{z}\|_2 \le c_8 \|L_\lambda z\|_2 .$$

When $\lambda = 0$ from this inequality we have estimate (2.8). The lemma is proved. □

Lemma 2.5. *Assume that functions r and s satisfy condition (1.3). Then for $y \in D(l)$ the estimate*

$$\|y'\|_2 + \|y\|_2 \le c \|ly\|_2 . \tag{2.15}$$

holds.

Proof. Let $y \in C_0^\infty(\mathbb{R})$. Integrating by parts, we have

$$(ly, y') = -\int_{\mathbb{R}} y''\bar{y}'dx + \int_{\mathbb{R}} [r(x)|y'|^2 + s(x)(\bar{y}')^2]dx. \tag{2.16}$$

Since

$$A := -\int_{\mathbb{R}} y''\bar{y}'dx = \int_{\mathbb{R}} y'\bar{y}''dx = -\bar{A},$$

we see $Re\ A = 0$. Therefore, it follows from (2.16)

$$Re\ (ly, y') \ge \int_{\mathbb{R}} [Re\ r - |s|]|y'|^2 dx \ge \delta \|y'\|_2 .$$

Hence, using the Holder's inequality, the condition $\gamma_{1,Re\ r} < \infty$ in (1.3) and Lemma 2.1 we obtain (2.15) for any $y \in C_0^\infty(\mathbb{R})$. If y is an arbitrary element of $D(l)$, then there exists a sequence $\{y_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - ly\|_2 \to 0$ as $n \to \infty$. The estimate (2.15) holds for y_n. From (2.15) passing to the limit as $n \to \infty$ we obtain the same estimate for y. The lemma is proved. □

A function $y \in L_2$ is called a solution of the equation

$$ly \equiv -y'' + r(x)y' + s(x)\bar{y}' = f, \quad f \in L_2, \tag{2.17}$$

if there exists a sequence $\{y_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - f\|_2 \to 0$, $n \to \infty$.

Lemma 2.6. *If junctions r and s satisfy condition (1.3), then the equation (2.17) has a unique solution.*

Proof. From (2.15) it follows that the solution y of (2.17) is unique and belongs to $W_2^1(\mathbb{R})$. Lemma 2.3 shows that L^{-1} is defined in all L_2. Then by the construction (2.17) is solvable. The proof is complete. □

3. Proofs of Theorems 1-4

Proof of Theorem 1. From (1.3) and Lemma 2.6 we obtain that l is invertible and l^{-1} is defined in all L_2. □

Proof of Theorem 2. From Lemma 2.4 it follows that L is separated in L_2 under condition (1.4). And consequently, by construction $ly \equiv -y'' + r(x)y' + s(x)\bar{y}'$ is separated in L_2 and the estimate (1.5) holds. The theorem is proved. □

Proof of Theorem 3. The estimate (1.5) shows that l^{-1} maps L_2 into space $\tilde{W}_2^2(\mathbb{R})$ with the norm $\|y''\|_2 + \|ry'\|_2 + \|s\bar{y}'\|_2 + \|y\|_2$. By condition of the theorem Lemma 2.2 implies that $\tilde{W}_2^2(\mathbb{R})$ is compactly embedded into L_2. The proof is complete. □

Proof of Theorem 4. By Lemma 2.1 Theorem 2 implies that $\|y''\|_2 + \|qy\|_2 \le c\,\|ly\|_2\,,\ \ y \in D(l)$. Then Theorem 1 [27] gives the estimates in Theorem 4. □

Proof of Theorem 5. Let ϵ and A be positive numbers. We denote

$$S_A = \left\{ z \in W_2^1(\mathbb{R}) :\ \|z\|_{W_2^1(\mathbb{R})} \le A \right\}.$$

Let ν be an arbitrary element of S_A. Consider the following linear "perturbed" equation

$$l_{0,\nu,\epsilon} y \equiv -y'' + [r(x, \nu(x)) + \epsilon(1 + x^2)^2]y' = f(x). \tag{3.1}$$

Denote by $l_{\nu,\epsilon}$ the minimal closed operator in L_2 generated by expression $l_{0,\nu,\epsilon}y$. Since

$$r_\epsilon(x) := r(x, \nu(x)) + \epsilon(1 + x^2)^2 \ge 1 + \epsilon(1 + x^2)^2,$$

the function $r_\epsilon(x)$ satisfies condition (1.3). Further, if $|x - \eta| \le 1$ $(x,\, z \in \mathbb{R})$, then for $\nu \in S_A$ we have

$$|\nu(x) - \nu(\eta)| \le |x - \eta|\, \|\nu'\|_p \le |x - \eta|\, \|\nu\|_{W_2^1(\mathbb{R})} \le A. \tag{3.2}$$

It is easy to verify that

$$\sup_{x,\eta\in\mathbb{R}:|x-\eta|\le 1} \frac{(1 + x^2)^2}{(1 + \eta^2)^2} \le 9.$$

Now we assume that $\nu(x) = C_1,\ \ \nu(\eta) = C_2$. Then by (1.7) and (3.2) we obtain

$$\sup_{x,\eta\in\mathbb{R}:|x-\eta|\le 1} \frac{r_\epsilon(x)}{r_\epsilon(\eta)} \le \sup_{x,\eta\in\mathbb{R}:|x-\eta|\le 1}\ \sup_{A>0}\ \sup_{|C_1|\le A,\ |C_2|\le A, |C_1-C_2|\le A} \frac{r(x, C_1)}{r(\eta, C_2)} + 9\varepsilon < \infty.$$

Thus the coefficient $r_\epsilon(x)$ in (3.1) satisfies the conditions of Theorem 2. Therefore, (3.1) has a unique solution y and for y the estimate

$$\|y''\|_2 + \left\|[r(\cdot, \nu(\cdot)) + \epsilon(1 + x^2)^2]y'\right\|_2 \le C_3\, \|f\|_2 \tag{3.3}$$

holds (i.e. an operator $l_{\nu,\epsilon}$ is separated). By (1.7) and (2.1)

$$\|y\|_2 \le C_0\, \|ry'\|_2\,, \quad \left\|(1 + x^2)y\right\|_2 \le C_4 \left\|(1 + x^2)^2 y'\right\|_2. \tag{3.4}$$

Taking into account (3.4) from (3.3) we have

$$\|y''\|_2 + \frac{1}{2}\left\|(1+x^2)^2 y'\right\|_2 + \frac{1}{2C_0}\|y\|_2 + \frac{\epsilon}{C_4}\left\|(1+x^2)y\right\|_2 \le C_3\|f\|_2 .$$

Then for some $C_5 > 0$ the following estimate

$$\|y\|_W := \|y''\|_2 + \left\|(1+x^2)^2 y'\right\|_2 + \left\|[1+\epsilon(1+x^2)]y\right\|_2 \le C_5\|f\|_2 \tag{3.5}$$

holds. We choose $A = C_5\|f\|_2$, and denote $P(\nu,\epsilon) := L^{-1}_{\nu,\epsilon} f$. From estimate (3.5) it follows that the operator $P(\nu,\epsilon)$ maps $S_A \subset W_2^1(\mathbb{R})$ to itself. Moreover, $P(\nu,\epsilon)$ maps S_A into the set

$$Q_A = \{y:\ \|y''\|_2 + \left\|(1+x^2)^2 y'\right\|_2 + \left\|[1+\epsilon(1+x^2)]y\right\|_2 \le C_5\|f\|_2\}.$$

Q_A is the compact in Sobolcv's space $W_2^1(\mathbb{R})$. Indeed, if $y \in Q_A$, $h \ne 0$ and $N > 0$, then the following relations hold:

$$\|y(\cdot+h) - y(\cdot)\|^2_{W_2^1(\mathbb{R})} = \int_{-\infty}^{+\infty} [|y'(t+h) - y'(t)|^2 + |y(t+h) - y(t)|^2]dt =$$

$$= \int_{-\infty}^{+\infty}\left[\left|\int_t^{t+h} y''(\eta)d\eta\right|^2 + \left|\int_t^{t+h} y'(\eta)d\eta\right|^2\right] dt \le$$

$$\le |h| \int_{-\infty}^{+\infty}\left[\int_t^{t+h} |y''(\eta)|^2 d\eta + \int_t^{t+h} |y'(\eta)|^2 d\eta\right] dt =$$

$$= |h|^2 \int_{-\infty}^{+\infty} \left[|y''(\eta)|^2 + |y'(\eta)|^2\right] d\eta \le C_6\|f\|_2^2\,|h|^2, \tag{3.6}$$

$$\|y\|^2_{W_2^1(\mathbb{R}\setminus[-N,N])} = \int_{|\eta|\ge N} \left[|y'(\eta)|^2 + |y(\eta)|^2\right] d\eta \le$$

$$\le \int_{|\eta|\ge N} (1+\eta^2)^{-1}\left[|y''(\eta)|^2 + (1+\eta^2)^2|y'(\eta)|^2 + (1+\eta^2)|y(\eta)|^2\right] d\eta \le$$

$$\le C_7\|f\|_2^2\,(1+N^2)^{-1}. \tag{3.7}$$

Expressions in the right-hand side of (3.6) and (3.7) tend to zero as $h \to 0$ and as $N \to +\infty$, respectively. Then by Kolmogorov-Frechct's criterion the set Q_A is compact in $W_2^1(\mathbb{R})$. Hence $P(\nu,\epsilon)$ is a compact operator.

Let us show that $P(\nu, \epsilon)$ is continuous with respect to ν in S_A. Let $\{\nu_n\} \subset S_A$ be a sequence such that $\|\nu_n - \nu\|_{W_2^1(\mathbb{R})} \to 0$ as $n \to \infty$, and y_n and y such that $L_{\nu,\epsilon} y = f, \quad L_{\nu_n,\epsilon} y_n = f$. Then it is enough to show that the sequence $\{y_n\}$ converges to y in $W_2^1(\mathbb{R})$ - norm as $n \to \infty$. We have

$$P(\nu_n, \epsilon) - P(\nu, \epsilon) = y_n - y = L_{\nu_n,\epsilon}^{-1}[r(x, \nu_n(x)) - r(x, \nu(x))]y_n'.$$

The functions $\nu(x)$ and $\nu_n(x)$ $(n = 1, 2, ...)$ are continuous. Then by conditions of the theorem the difference $r(x, \nu_n(x)) - r(x, \nu(x))$ is also continuous with respect to x. Hence for each finite interval $[-a, a], \quad a > 0$, we have

$$\|y_n - y\|_{W_2^1(-a,a)} \leq c \max_{x \in [-a,a]} |r(x, \nu_n(x)) - r(x, \nu)| \cdot \|y_n'\|_{L_2(-a,a)} \to 0 \tag{3.8}$$

as $n \to \infty$. On the other hand, from Theorem 2 it follows that $\{y_n\} \in Q_A, \quad \|y_n\|_W \leq A, \quad y \in Q_A, \quad \|y\|_W \leq A$. Since the set Q_A is compact in $W_2^1(\mathbb{R})$, $\{y_n\}$ converges in the $W_2^1(\mathbb{R})$ - norm. Let z be the limit of $\{y_n\}$. By properties of $W_2^1(\mathbb{R})$

$$\lim_{|x| \to \infty} y(x) = 0, \quad \lim_{|x| \to \infty} z(x) = 0. \tag{3.9}$$

Since $L_{\nu,\epsilon}^{-1}$ is the closed operator, from (3.8) and (3.9) we obtain $y = z$. Then $\|P(\nu_n, \epsilon) - P(\nu, \epsilon)\|_{W_2^1(\mathbb{R})} \to 0$, as $n \to \infty$.

Summing up, we have that $P(\nu, \epsilon)$ is the completely continuous operator in $W_2^1(\mathbb{R})$ and maps S_A to itself. Then by Schauder's theorem $P(\nu, \epsilon)$ has a fixed point y $(P(y, \epsilon) = y)$ in S_A. And consequently, y is a solution of the equation

$$L_\epsilon y := -y'' + \left[r(x, y) + \epsilon(1 + x^2)^2\right] y' = f(x).$$

By (3.3) for y the estimate

$$\|y''\|_2 + \left\|\left[r(\cdot, y) + \epsilon(1 + x^2)^2\right] y'\right\|_2 \leq C_3 \|f\|_2$$

holds.

Now, suppose that $\{\epsilon_j\}_{j=1}^\infty$ is a sequence of positive numbers converged to zero. The fixed point $y_j \in S_A$ of $P(\nu, \epsilon_j)$ is a solution of the equation

$$L_{\epsilon_j} y_j := -y_j'' + \left[r(x, y_j) + \epsilon_j(1 + x^2)^2\right] y_j' = f(x).$$

For y_j the estimate

$$\left\|y_j''\right\|_2 + \left\|\left[r(\cdot, y_j(\cdot)) + \epsilon(1 + x^2)^2\right] y_j'\right\|_2 \leq C_3 \|f\|_2 \tag{3.10}$$

holds.

Suppose (a, b) is an arbitrary finite interval. From $\{y_j\}_{j=1}^\infty \subset W_2^2(a, b)$ one can select a subsequence $\left\{y_{\epsilon_j}\right\}_{j=1}^\infty$ such that $\left\|y_{\epsilon_j} - y\right\|_{L_2[a,b]} \to 0$ as $j \to \infty$. A direct verification shows that y is a solution of (1.6). In (3.10) passing to the limit as $j \to \infty$ we obtain (1.8). The theorem is proved. $\square$

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